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Two-dimensional models

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Abstract

It is not possible to compactly review the overwhelming literature on two-dimensional models in a meaningful way without a specific viewpoint; I have therefore tacitly added to the above title the words "as theoretical laboratories for general quantum field theory".

I dedicate this contribution to the memory of J. A. Swieca with whom I have shared the passion of exploring 2-dimensional models for almost one decade.

A shortened version of this article is intended as a contribution to the project "Encyclopedia of mathematical physics" and comments, suggestions and critical remarks are welcome.

1 History and motivation

Local quantum physics of systems with infinitely many interacting degrees of freedom leads to situations whose understanding often requires new physical intuition and mathematical concepts beyond that acquired in quantum mechanics and perturbative constructions in quantum field theory. In this situation two-dimensional soluble models turned out to play an important role. On the one hand they illustrate new concepts and sometimes remove misconceptions in an area where new physical intuition is still in process of being formed. On the other hand rigorously soluble models confirm that the underlying physical postulates are mathematically consistent, a task which for interacting systems with infinite degrees of freedom is mostly beyond the capability of pedestrian methods or brute force application of hard analysis on models whose natural invariances has been mutilated by a cut-off.

In order to underline these points and motivate the interest in 2-dimensional QFT, let us look at the history and the physical significance of the three oldest two-dimensional models of relevance for statistical mechanics and relativistic particle physics namely the Lenz-Ising model, Jordan's model of bosonization/fermionization and the Schwinger model (QED_2).

The Lenz-Ising (L-I) model was proposed in 1920 by Wihelm Lenz [1] as the simplest discrete statistical mechanics model with a chance to go beyond the P. Weiss phenomenological Ansatz involving long range forces and instead explain ferromagnetism in terms of non-magnetic short range interactions. Its one-dimensional version was solved 4 years later by his student Ernst Ising. In his 1925 university of Hamburg[1] thesis, Ising [3] not only showed that his chain solution could not account for ferromagnetism, but he also proposed some (as it turned out much later) not entirely correct intuitive arguments to the extend that this situation prevails to the higher dimensional lattice version. His advisor Lenz as well as Pauli (at that time Lenz's assistant) accepted these reasonings and as a result there was considerable disappointment among the three which resulted in Ising's decision (despite Lenz's high praise of Ising's thesis) to look for a carrier outside of research. For many years a reference by Heisenberg [4] (to promote his own proposal to explain ferromagnetism) to Ising's negative result was the only citation; the situation begun to change when Peierls [5] drew attention to "Ising's solution" and the results of Kramers and Wannier [6] cast doubts on Ising's intuitive arguments beyond the chain solution. The rest of this fascinating episode i.e. Lars Onsager's rigorous two-dimensional solution exhibiting ferromagnetic phase transition, Brucia Kaufman's simplification which led to conceptual and mathematical enrichments (as well as later contributions by many other illustrious personalities) hopefully remains a well-known part of mathematical physics history even beyond my own generation.

This work marks the beginning of applying rigorous mathematical physics methods to solvable two-dimensional models as the ultimate control of intuitive arguments in statistical mechanics and quantum field theory. The L-I model continued to play an important role in the shaping of ideas about universality classes of critical behavior; in the hands of Leo Kadanoff it became the key for the development of the concepts about order/disorder variables (The microscopic version of the famous Kramers-Wannier duality) and operator product expansions which he proposed as a concrete counterpart to the more general field theoretic setting of Ken Wilson. Its massless version and its derivative the Coulomb gas became a role model in the Belavin-Polyakov-Zamolodchikov [7] setting of minimal chiral models and it remained up to date the only model realizing non-abelian braid group (plektonic) statistics for which the n-point correlators can be written down explicitly in terms of elementary functions [8]. Chiral theories confirmed the pivotal role of "exotic" statistics [9][10] in low dimensional QFT by exposing the appearance of braid group statistics as a novel manifestation of Einstein causality [8]. As free field theories in higher dimensions are fixed by their mass, spin and internal symmetries, the structure of chiral theories is almost entirely determined by their "plektonic" data i.e. they are as free (of genuine interactions, or as kinematical) as possible under the condition that they realize nontrivial braid group statistics[2] which is

[1] The university of Hamburg was founded in 1919, but the town fathers deemed it unnessary to have a separate chair for theoretical physics. The newly appointed mathematicians (Artin, Blaschke, Hecke) had the splendid idea to invite Einstein (who already enjoyed general fame) for a talk open to the public. The subsequent public pressure on the city government led to change of their decision [2]. Einstein indirect role as a catalyzer was vindicated since the theoretical physics activities at Hamburg university immediately led to the important discovery of the exclusion principle and the construction of one of the most fruitful 2-dimensional model.

[2] There are also massive "anyon" models in d=1+2 which are "free" in this sense (one likely attribute: zero cross section) explained in [11], but their unavoidable string localization make their explicit analytic construction difficult.

incompatible with a linear equation of motion.

Another conceptually rich model which also lay dormant for almost two decades as the result of a misleading speculative higher dimensional generalization by its protagonist is the bosonization/fermionization model first proposed by Pascual Jordan [12]. This model establishes a certain equivalence between massless two-dimensional Fermions and Bosons; the possibility of such a conversion is the basis of the solution of the much later proposed models of Luttinger [13] and Thirring[3] [15]. One reason why even nowadays hardly anybody knows Jordan's contribution (besides some lack of comprehension of its content on the part of his readers) is certainly the ambitious but unfortunate title "The neutrino theory of light" under which he published a series of papers; besides some not entirely justified criticism of content, the reaction of his contemporaries consisted in a good-humored carnivalesque Spottlied (mockery song) about its title [16]. Its mathematical content, namely the realization that the current of a free zero mass Dirac fermion in d=1+1 is linear in canonical Boson creation/annihilation operators ("Bosonization") and that such zero mass Fermions (Jordan's "neutrinos") permit a formal representation in terms of ordered exponential expressions in a free Boson field ("Fermionization") turned out to be a cornerstone in the context of two-dimensional conformal theories and their chiral decomposition; it also furnished the simplest nontrivial illustration [32] of the Doplicher-Haag-Roberts (DHR) theory of superselection sectors and the origin of statistics (spacelike commutation relations) and internal symmetries. Furthermore it simplified the presentation of the Thirring model[4]. The massive version of the Thirring model became the role model of integrable relativistic QFT and shed additional light on two-dimensional bosonization [23].

Both discoveries demonstrate the usefulness of having controllable low-dimensional models; at the same time they also illustrate the danger of rushing to premature "intuitive" conclusions about extensions to higher dimensions. The search for the appropriate higher dimensional analogon of a 2-dimensional observation is an extremely subtle endeavour. In the aforementioned two historical examples the true physical message of those models only became clear through hard mathematical work and profound conceptual analysis by other authors many years after the discovery of the original model.

A review of the early historical benchmarks of conceptual progress through the study of solvable two-dimensional models would be incomplete without mentioning Schwinger's proposed solution of two-dimensional quantum electrodynamics, afterwards referred to as the Schwinger model. Schwinger used this model in order to argue that gauge theories are not necessarily tied to zero mass vector particles; in opposition to the rest of the world (including Pauli[5]), he thought that it is conceivable that there may exist a strong coupling regime of a QED-like gauge coupling which converts the massless photons into a massive vectormeson and he used the soluble massless QED_2 to illustrate his point that gauge theory setting does not exclude massive vectormesons. His solution in terms of indefinite metric correlation functions [20] was however quite far removed from the interpretation of its physical aspects. Some mathematical physics work was necessary [21] to unravel its physical content with the result that the would-be charge of that QED_2 model was "screened" and its would-be chiral symmetry broken i.e. the model exists only in the so-called Schwinger-Higgs phase with massive free particles accounting for its physical content[6]. Enriching the abelian gauge model by adding a SU(2) flavor, the difference of screening versus confinement appeared in a sharper focus; whereas screening consisted in a radical re-arrangement of degrees of freedom leading to color-neutral carriers (infraparticles) of fundamental SU(2) flavor, confinement amounted to the absence of fundamental flavor-carrying objects in the physical Hilbert space [23]. The latter phenomenon was observed in the massive version of the model at generic values of the so-called θ-angle. One believes that the basic features of this difference between the Schwinger-Higgs screening mechanism versus fundamental flavor confinement continue to apply in the 4-dimensional standard model but, the lack of an intrinsic

[3]In its original massless and conformally invariant version it reached its mathematical perfection in the work of Klaiber's Boulder lecture [14]. Klaiber also enriched the model by an additional parameter which allows a realization of the Thirring model in a anyonic statistics mode with the expected spin-statistics relation between the anomalous Lorentz spin and the anyonic commutation relation.

[4]However Thirring's motivation for studying 2-dimensional couplings of 4 Fermions was not the Fermion-Boson equivalence but rather Heisenberg's "nonlinear spinor theory" (an ill-fated attempt initially supported by Pauli which preceded the idea of quarks and their QCD interactions).

[5]For a long time it was thought that the use of abelian or nonabelian gauge theories [18][19] in particle physics of massive vectormesons was not possible.

[6]As an illustration of historical prejudices against Schwinger's ideas it is interesting to note that Swieca apparently was not able to convince Peierls (on a visit to Brazil) about the possibility of having massive vectormesons in a gauge theory (private communication by J. A. Swieca around 1975).

meaning of notions of spin as well as statistics in d=1+1 prevent simple-minded analogies[7]. However not each 2-dim. model which entered the literature with another name is really intrinsically distinct. The so called "chiral" Schwinger model is just a different way of presenting the same operator content by selecting a different field coordinatization (it would have been extremely surprising if Schwinger would have missed a model with a different physical content so close to his own discovery).

Thanks to its property of being superrenormalizable, the Schwinger model also served as a useful testing ground for the Euclidean integral formulation in the presence of Atiyah-Singer zero modes and their role in the Schwinger-Higgs chiral symmetry breaking [24]. These classical topological aspects of the functional integral formulation attracted a lot of attention beginning in the late 70s and through the 1980s but, as most geometrical aspects of the Euclidean functional integral representations, their intrinsic physical significance remained questionable. Even in those superrenormalizable 2-dim models, where the measure theory underlying Euclidean functional integration can be mathematically controlled [25], there is no good reason why within this measure theoretical setting (outside of quasiclassical approximations [26]) topological properties derived from continuity requirements should assert themselves. This is no problem in the operator algebra approach where no topological or differential geometrical property is imposed but certain geometric structures (spacetime- and internal- symmetry properties) are encoded in the causality and spectral principles of observable algebras.

In passing it is worthwhile to mention that Schwinger's idea on charge screening found a rigorous formulation in a structural theorem which links the issue of charge versus charge screening to the spectral property near zero of the mass operator [28]. Mass generation via charge screening in 4-dimensional *perturbation* theory is not possible without additional physical (Higgs) degrees of freedom [27].

There are several books and review articles [23][29][30][31] on d=1+1 conformal as well a on massive factorizing models. To the extend that they use concepts and mathematical structures which permit no extension to higher dimensions (Kac-Moody algebras, loop groups, integrability, presence of an infinite number of conservation laws) or treat chiral QFT as a tool of string theory, their approach will not be followed in this report since our primary interest will be the use of two-dimensional models of QFT as "theoretical laboratories" of general QFT. Our aim is two-fold; on the one hand we intend to illustrate known principles of general QFT in a mathematically controllable context and on the other hand we want to identify new concepts whose adaptation to QFT in d=1+1 lead to their solvability. We will not review the methods of so-called "constructive field theory" which basically is a quantization method supplemented by quantum mechanical kind of hard analysis[25]; the reason is not only that these methods are very different from the ones used in this article but also that their ultraviolet requirement limits their application to superrenormalizable models.

The best use the reader can make of these notes is to re-work the literature by using the present viewpoint as an ordering device for the overwhelming amount of detailed accumulated knowledge about 2-dim. models.

Although this article deals with problems of mathematical physics, the style of presentation is less on the mathematical physics and more on a narrative side so that physicists not familiar with e.g. the mathematics of operator algebras can still understand the results and decide whether they want to look up the proofs and more detailed arguments in the literature.

2 General concepts and their two-dimensional manifestation

The general framework of QFT to which the rich world of controllable two-dimensional models contributes as a theoretical laboratory exists in two quite different but nevertheless closely related formulations: the setting in terms of pointlike covariant fields due to Wightman, and the more algebraic setting initiated by Haag and Kastler based on spacetime-indexed operator algebras which developed over a long period of time with contributions of many other authors. Whereas the Wightman approach, whose framework has been basically abstracted from the renormalized Lagrangian quantization setting by placing its salient features into a rigorous mathematical setting of operator-valued distribution, aims directly at the (not necessarily observable) quantum fields, the operator algebraic setting is more ambitious. It starts from

[7]Massive particles in d=1+1 are statistical "schizons" [22] i.e. the commutation relation of their interpolating fields together with their Lorentz-spin can be changed freely by passing (nonlocally) to other operators in the same Hilbert space. In the massless scale invariant limit this is prevented as the result of the emergence of new superselection rules.

physically motivated assumptions about the algebraic structure of local observables and reconstructs the full field theory (including the operators carrying the superselected charges) in the spirit of a local representation theory of the assumed structure of the local observables. This has the advantage that the somewhat mysterious concept of an inner symmetry (as opposed to a spacetime (outer) symmetry) can be traced back to its physical roots which is the representation theoretical structure of the local observable algebra. In the Lagrangian quantization approach the inner symmetry is part of the input (the multiplicity index of field components on which subgroups of SU(n) or SO(n) act linearly) and hence it is not possible to even formulate this fundamental question. Whenever the sharp separation (O'Rafarteigh) of inner versus outer symmetry becomes blurred as a result of the appearance of braid group statistics in low spacetime dimensions, the Lagrangian quantization setting becomes inappropriate and even the Wightman framework has to be extended.

2.1 The general framework

The most important physical properties which are shared between the Wightman approach (WA) and the operator algebra (AQFT) setting are the spacelike locality (often referred to as Einstein causality) and the existence positive energy representations of the Poincaré group implementing the covariance of the Wightman fields resp. the local observable algebras.

- Spacelike commutativity:

$$\begin{aligned} &[\psi(x),\varphi(y)]_{\mp}=0,\ (WA) \\ &\mathcal{A}(\mathcal{O}')\subseteq\mathcal{A}(\mathcal{O})',\ \mathcal{O}\ open\ nbhd.\ (AQFT) \end{aligned} \tag{1}$$

- Positive energy reps. of the Poincaré group $\mathcal{P}$:

$$\begin{aligned} &U(a,\Lambda)\psi(x)U(a,\Lambda)^*=D^{-1}(\Lambda)\psi(\Lambda x+a),\ (WA) \\ &U(a,\Lambda)\mathcal{A}(\mathcal{O})U(a,\Lambda)^*=\mathcal{A}(\mathcal{O}_{(a,\Lambda)}),\ (AQFT) \\ &U(a)=e^{iPa},\ specP\subseteq V_+,\ P\Omega=0 \end{aligned} \tag{2}$$

Here $\psi(x),\varphi(x)$...are (singular) field operators (operator-valued distributions) in a Wightman QFT which are assumed to either commute (+) or anticommute (-) for spacelike distances and a structural theorem [32] ties (through the use of the positive energy requirement) the commutator relation to finite dimensional representations of $\mathcal{P}$ whereas the anticommutator has to be used for projective representations which turn out to be usual representations of the two-fold covering $\widetilde{\mathcal{P}}$. The observable algebra consists of a family of (weakly closed) operator algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ indexed by a family of convex causally closed spacetime regions $\mathcal{O}$ (with $\mathcal{O}'$ denoting the spacelike complement and $\mathcal{A}'$ the von Neumann commutant) which act in one common Hilbert space. Certain properties cannot be naturally formulated in the pointlike field setting (vis. Haag duality for convex regions $\mathcal{A}(\mathcal{O}')=\mathcal{A}(\mathcal{O})'$[8]). The connection between the two formulations of local quantum physics is quite close; in particular in case of two-dimensional theories there are convincing arguments that one can pass between the two without imposing additional technical requirements.

The two above requirements are often (depending on what kind of structural properties one wants to derive) complemented by additional impositions which, although not carrying the universal weight of principles, represent natural assumptions whose violation even though not prohibited by the principles, would cause paradigmatic attention. Examples are "weak additivity", "Haag duality" and "the split property". Weak additivity i.e. the requirement $\vee\mathcal{A}(\mathcal{O}_i)=\mathcal{A}(\mathcal{O})$ if $\mathcal{O}=\cup\mathcal{O}_i$ which expresses the "collection from parts" aspect of Wightman fields. Haag duality is the statement that the commutant not only contains the algebra of the causal complement (Einstein causality) but is exhausted it $\mathcal{A}(\mathcal{O}')=\mathcal{A}(\mathcal{O})'$; its violation in the vacuum sector for convex causally complete regions signals spontaneous symmetry breaking in the associated charge-carrying (D-R, see below) field algebra and its always possible enforcement

[8]Haag duality holds for for observable algebras in the vacuum sector in the sense that any violation can be explained in terms of a spontaneously broken symmetry; in local theories it always can be enforced by dualization and the resulting Haag dual algebra has a charge superselection structure associated with the unbroken subgroup.

by the dualization extension and its violation for multi-local region reveals the possible charge content of the model by charge-anticharge splittings of the neutral observable algebra [33]. The split property $\mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) \simeq \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$ is a result of the adaptation of the "finiteness of phase space" property of QM to QFT which amounts to the so-called "nuclearity property". Related to the Haag duality is the local version of the "time slice property" (the QFT counterpart of the classical causal dependency property) sometimes referred to as "strong Einstein causality" $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})''$ [65].

One of the most astonishing achievements of the algebraic approach is the theory of superselection sectors i.e. the realization that the structure of charged (nonvacuum) representations and the spacetime properties of the carriers of these generalized charges including their spacelike commutation relation (leading to the particle statistics) and their internal symmetry properties are already encoded in the structure of the Einstein causal observable algebra. The intuitive basis of this remarkable result (whose prerequisite is locality) is that on can generate charged sectors by spatially separating charges in the vacuum (neutral) sector and disposing of the unwanted charges at spatial infinity [34].

An important concept which especially in d=1+1 has considerable constructive clout is "modular localization". It is a consequence of the above algebraic setting if either the net of algebras have pointlike field generators [35] or if the one-particle masses are separated by spectral gaps so that the formalism of time dependent scattering holds [36]. It rests on the basic observation that the Tomita-Takesaki modular theory applied to the standard[9] pair of wedge algebra and the vacuum state $(\mathcal{A}(W), \Omega)$ leads to two canonically associated modular objects with geometric-physical significance. The two modular objects are defined through the polar decomposition of the closable unbounded antilinear involutive Tomita S-operator

$$SA\Omega = A^*\Omega, \;\; S = J\Delta^{\frac{1}{2}} \tag{3}$$

$$J_W = U(j_W) = S_{scat}J_0, \; \Delta^{it} = U(\Lambda_W(2\pi t)) \tag{4}$$

where the second line (which identifies the Tomita reflection (the "angular" part) with the antiunitary implementor of reflection along the edge of the wedge and the radial part with the implementor of the W-preserving boost at $t = -i$) holds specifically for the standard QFT pair $(\mathcal{A}(W), \Omega)$. Within the 2-dim. setting of QFT this theory leads to particular powerful constructive results.

2.2 Boson/Fermion equivalence and superselection theory

The simplest demonstration is obtained by using a 2-dim. massless Dirac current and showing that it may be expressed in terms of scalar canonical Bose creation/annihilation operators

$$j_\mu =: \overline{\psi}\gamma_\mu\psi := \partial_\mu\phi, \; \phi := \int_{-\infty}^{+\infty} \{e^{ipx}a^*(p) + h.c.\}\frac{dp}{2\,|p|} \tag{5}$$

Although the potential of the current $\phi(x)$ is (as a result of its infrared behavior) not an operator valued distribution in the Fock space of the $a(p)^{\#}$[10], the formal exponential

$$: e^{i\alpha\phi(x)} :\equiv \lim_{f\rightarrow\chi(-\infty,x)} : e^{i\alpha j(f)} : \tag{6}$$

which results by dumping one charge at the end of a finite j-string to -∞ while keeping the opposite charge at x. Performing this limit inside vacuum expectation values and imposing the boundedness of the limit (by renormalizing), one easily checks that the only nonvanishing expectations are those with the same number of positive and negative (Hermitian conjugate) charges fields: $e^{\pm i\alpha\phi(x)}$:. So the notation for the resulting "exponential field" is to be taken with a grain of salt; contrary to the exponential of the smeared current this one-sided limit creates states in a charge superselection sector which are orthogonal to the neutral current sector[11]. The resulting model is conformally invariant and decomposes into two chiral QFT associated with the lightray decomposition of the current $j_\pm(x_\pm)$; no information is lost

[9] Standardness means that the algebra of the pair $(\mathcal{A}, \Omega)$ act cyclic and separating on Ω.

[10] It becomes an operator after smearing with test functions whose Fourier transform vanishes at p=0.

[11] Correspondingly the exponential notation is not meant in the literal sense as a power series, even though the Wick-contraction formalism augmented by the charge selection rule gives the correct result for expectation values.

by studying one of the isomorphic copies (thus forgetting the $\pm$ indexing). Chiral conformal QFT has an extension to the compactified line $\dot{R} = S^1$ which permits the encoding of the charge conservation rules of the exponential field into an attached QM (the "zero mode" QM) so that they are manifestly built into the formalism in form of a pre-exponential charge conserving quantum mechanical factor[12] and the remaining exponential field has its standard meaning in the Fock space generated from the vacuum by the rotational Fourier components of the current. These observations together with the increase of knowledge about the particle/field relation (for an up-to-date exposition see [63]) reveal that the correct extension of Jordan's mathematical observation is not an imagined "neutrino theory of light" but rather an early special illustration of the DHR theory [34][37] of superselection sectors and the exotic statistics which in low dimensions goes with generic values of charges. The general result of that theory valid in higher spacetime dimensions is that starting from the causal localization structure of the observable algebra (always bosonic by definition) the superselection theory allows to extend this algebra in a natural way to the "field" algebra whose charge-carrying operators may include operators obeying Fermi statistics; vice versa the charge neutral subalgebra obtained by averaging the field algebra (which contains possibly Fermion operators) over the compact global gauge group is always bosonic. For the higher dimensional case there is no alternative to having a product structure which sharply separated spacetime from inner symmetries (O'Rafartaigh) whereas for low-dimensional theories in the presence of braid group statistics this separation becomes blurred. Two dimensional theories (and in particular chiral theories) are somewhat different in that their superselected charges may have a continuous spectrum as it is the case in the above free current model. In that case the total Hilbert space which incorporates all charges would be non-separable; this can be avoided by "maximally" extending the current algebra [38]. The Weyl algebra formulation permits an alternative formulation which encodes the particularity of the zero mass field into a non-regular vacuum state.

In passing the following comment is instructive. In the course of reconstructing the full field algebra from its "observable shadow" (the analog of Mark Kac's inverse problem "how to hear the shape of a drum") one runs across a "small" subalgebra which incorporates all the data about statistics&internal symmetries and played an important role in the DHR theory [34]. Contrary to the "big" QFT algebras it has a natural trace state (with the so-called Markov property, a crucial tool of subfactor theory introduced by Vaughn Jones and baptized with this extremely well chosen name) and it does not admit any spacetime localization concept. It becomes especially rich in low-dimensional QFTs in the presence of braid group statistics where it encodes invariants of 3-manifolds. Witten made the surprising observation [39] that such combinatorial (from the viewpoint of algebraic QFT) data can be obtained from quasiclassical approximations of functional integrals of the Chern-Simons type. This explains their name "topological field theories" since they describe the topological "bones" of actual field theories without their spacetime "flesh" which is encoded in localization[13]. As a result of absence of localization, these theories do not "live" on 3-manifolds, rather those invariants (and more generally representations of the mapping class group) are inexorably linked to the braid group statistics of Minkowski space QFTs. It is not unreasonable to think that in a scattering theory of massive d=1+2 plektons whose interpolating fields are string-localized these objects could appear (Gedankenexperiment, future condensed matter physics?).

The intuitive idea of a charge split which is at the heart of the superselection theory allows a particular beautiful explicit model-independent formulation in the operator-algebraic setting of chiral theories. Instead of constructing the superselection sectors by disposing an unwanted charge at infinity as in the DHR theory, one looks as mentioned above at the operator algebraic aspects of an algebra $\mathcal{A}(I \vee J) \simeq \mathcal{A}(I) \otimes \mathcal{A}(J)$ localized on two spatially separated intervals on the circle. In that case one can show that the presence of charge sectors manifests itself in a defect of the before mentioned Haag duality which passes to an inclusion $\mathcal{A}(I \vee J) \subseteq \mathcal{A}(I \vee J)''$ where the equality only holds for chiral observable algebras for which the vacuum representation is the only finite energy representation[14]. The selfdual cases of maximally extended multicomponent abelian current algebras are easily constructed and furnish nice illustrations which include the famous example of the "moonshine" model [41].

[12] Jordan (who writes the formula in the old notation used in the 30s [12]) also had a pre-exponential charge-carrying factor.

[13] This is related to impossibility of having localizable theories in $d \geq 1{+}2$ which are invariant under infinite diffeomorphism groups.

[14] These chiral models are called "holomorphic" in most of the literature; a terminology which in the present context would be misleading.

There is an unexpected deep relation between the Schwinger model whose charges are screened, and the Jordan model which has (liberated) charge sectors. Since the Lagrangian formulation is a gauge theory, the analog of the 4-dim. asymptotic freedom wisdom would suggest the possibility of charge liberation in the short distance limit of the Schwinger model. This seems to contradict the statement that the intrinsic content of the Schwinger model is the QFT of a free massive Bose field and such a simple free field is not expected to contain subtle information about asymptotic charge liberation. Well, the massless limit as the potential of the free abelian current really does contain this information as the above analysis of the Jordan model suggests and as one can demonstrate in detail [42]. This statement is intrinsic since it refers to the screened phase unlike the 4-dimensional asymptotic freedom statement which is based on the perturbative phase instead of the physical quark confinement phase (and hence is the result of a consistency check falling short of a mathematical theorem). There are other properly renormalizable (i.e. not superrenormalizable as the Schwinger model) two-dimensional models in which one can prove the validity of asymptotic freedom in the physical phase, but the poor state of nonperturbative knowledge in d=1+3 is hampering an understanding in realistic cases.

As a result of the peculiar nature of the zero mass limit of the derivative of the massive free field Jordan's model is closely related to the massless Thirring model (and the closely related Luttinger model for an interacting one-dimensional electron gas) whose massive version is in the class of factorizing models (see later section)[15]. Together with the massive version of the L-I QFT it shows two new (interconnected) properties: the absence of an intrinsic meaning of statistics and the emergence of dual order-disorder variables explained below.

The Thirring model is a special case in a vast class of "generalized" (multi-coupling) multi-component Thirring models i.e. 4-Fermion interactions. Under this name they were studied in the early 70s [43][44] with the particular aim to identify massless subtheories for which the currents have a chiral decomposition and form current algebras. There was the question whether in addition to the abelian componentwise bosonization one could think of a more intrinsic nonabelian bosonization. Witten [45] found the positive answer to this question. There is even a Lagrangian description in form of the WZW Lagrangian, but since it has a "topological" boundary contribution in a 3-dimensional Euclidean setting (which is not intrinsic in the sense in which we use this terminology in this article) it does not follow the usual patterns of perturbative Lagrangian quantization. Whereas in geometrically directed mathematical investigations this Lagrangian bosonization is quite useful, the computations are easier in the representation theoretical setting of chiral current algebras and the introduction of the Fermions or their bosonized version as intertwiners between inequivalent representations.

It is interesting to also consider the massive version of the Thirring model. The counterpart of the potential of the conserved Dirac current is the Sine-Gordon field, i.e. a composite field which in the attractive regime of the Thirring coupling again obeys the so-called Sine-Gordon equation of motion. Coleman [17] gave an argument which however does not reveal the limitation in the size of the coupling[16]. A rigorous confirmation of these facts was recently given in [47]. Models which have a continuous or discrete internal symmetry have "disorder" fields which are local fields which implement half-space-symmetry-implementing transformation on the charge-carrying field (acting as the identity in the other half). These are pointlike bosonic fields which live in the same Hilbert space as the charged field but only create the neutral part from the vacuum. Multiplication (leading short distance contribution of the disorder operator product with the charged field) generates "order" fields which act cyclically on the vacuum. The order/disorder fields have an interesting connection with phase transitions (for discrete symmetries). Whereas in the lattice version the correlation functions [48] of the L-I model the system undergoes a second order phase transition as the temperature passes through the critical value, the mass parameter represents only the slope of temperature at criticality and lost its role of connecting two phases; the only memory of the different phases in the QFT resulting in the scaling limit consists in the presence of a pair of order/disorder variables whose interchange in passing from one phase to another has to be decreed as an additional rule. The resulting n-point order/disorder correlation functions of the L-I field theory can be represented in terms of order/disorder variables of a free Majorana field or as the (suitable defined) square root of the exponential disorder field of a free Dirac Fermion, both in the massive [46]

[15]Another structural consequence of this peculiarity leads to Coleman's theorem [40] which connects the Mermin-Wagner no-go theorem for two-dimensional spontaneous continuous symmetry breaking with these zero mass peculiarities.

[16]The current potential of the free massive Dirac Fermion (g=0) does not obey the Sine-Gordon equation [46].

as well as in the massless limit [51]. They are scalar Bose fields with a Z_2 "half-space" commutation relation between them. Whereas the massive scaling limit fields still have correlation functions which are order/disorder unsymmetrical, the conformally invariant zero leads to a symmetric situation where both variables carry superselected charges. The emergence of new charges in connection with the appearance of critical exponents of order/disorder fields in 2-dim. QFT is actually the content of a general theorem [49].

2.3 The conformal setting, chiral theories

Chiral theories play a special role within the setting of conformal quantum fields. Conformal theories have observable algebras which live on compactified Minkowski space (S^1 in the case of chiral models) and fulfill the Huygens principle, which in an even number of spacetime dimension means that the commutator is only nonvanishing for lightlike separation of the fields. The fact that this rule breaks down for non-observable "would be" conformal fields (vis. the massless Thirring field) was noticed at the beginning of the 70s and at that time considered paradoxical ("reverberation" in the timelike (Huygens) region). Its resolution led 1974/75 to two differently formulated but basically equivalent concepts about globally causal objects on the universal covering spacetime and the other (operator-valued distributional) "sections" (conformal blocks) on the (compactified) Minkowski spacetime. The connection is given by a decomposition formula into irreducible conformal blocks with respect to the center $\mathbf{Z}$ of the covering group $\widetilde{SO(2,n)}$

$$A(x) = \sum A_{\alpha,\beta}(x),\ A_{\alpha,\beta}(x) = P_\alpha A(x) P_\beta,\ Z = \sum e^{id_\alpha} P_\alpha \tag{7}$$
$$A_{\alpha,\beta}(x) B_{\beta,\gamma}(y) = \sum_{\beta'} R^{(\alpha,\gamma)}_{\beta,\beta'} B_{\alpha,\beta'}(y) A_{\beta',\gamma}(x),\ x > y$$

where α, β are labels for the eigenspaces of the generating unitary Z of the abelian center $\mathbf{Z}$ and $A(x)$ is a global Luescher-Mack [52] field which lives on the universal Dirac-Weyl covering where it fulfills the I. Segal conformal causality [53] i.e. the global causality adapted to the hells and heavens of the covering. The irreducible components $A_{\alpha\beta}$ on the other hand are operator-valued sections on compactified Minkowski spacetime of the conformal block decomposition in [54]. The causal ordering structure in timelike direction $(x \gtrless y)$ together with the associativity of the composition forces the timelike commutation relation to be of the Lie-field type with the coefficients $R^{(\alpha,\gamma)}_{\beta,\beta'}$ fulfilling braid group relations [55]. The problem of classifying conformal fields which are simultaneously bosonic for spacelike and plektonic for timelike separation is difficult and a solution is presently not known (for some analytic attempts concerning the construction of observable fields see [56]). Fortunately the problem simplifies significantly for d=1+1 where the fields decompose into two chiral components (following the factorization of the 6-parametric Moebius group) and for each chiral component space- and time- like coalesces to just one light-like commutation structure. A similar idea of classifying Lie fields i.e. fields with specified commutation relation in space- and time-like directions already appeared as far back as [57]. The use of 2-dim. conformal QFT for critical phenomena was first pointed out in the beginnings of the 70s by Polyakov [59].

For a long time the exponential charged field of the previous section was the only illustration of the block decomposition theory. The situation changed radically with the discovery of the very nontrivial family of "minimal models" by Belavin, Polyakov and Zamolodchikov [7] which correspond to irreducible representation of the energy-momentum tensor field at specific admissable values (by locality&positivity) of the normalization of its two-point function. Although computations of this object for free Dirac fields showed that its Fourier components were related to an algebraic structure found in parts[17] before in a different context by Virasoro (part of what in mathematics was known as the Witt algebra), the BPZ work marks a turning point in 2-dimensional model building. It became clear that the Lie-field commutation structure is a general structural aspect of chiral theories. In this way the rich mathematics of Kac-Moody algebras and loop groups as well the deeper structures of complex function theory and

[17] Virasoro's algebraic structure was afterwards completed by explicit computations of the commutator of the Sugawara energy momentum tensor obtained from free fermion currents. Structural arguments establishing the "Lie-field structure" of the energy-momentum tensor in the general conformal setting first appeared in [58][29].

Riemann surfaces became working tools of many physicists and on the other hand mathematicians begun to appreciate the formalism and the open problems of two-dimensional models of QFT as an inspiration for mathematical innovations. The price payed for this impressive gain of detailed knowledge from special structures which only exist in low spacetime dimensions has often been a separation from the mainstream of QFT (see the position of conformal QFT in the table of content of this encyclopedia). The adaptation of concepts from general QFT (local origin of charge sectors, statistics and symmetry, modular localization,...) to low dimensions as presented here tries to counteract this tendency. These field theoretic concepts, which where introduced by Doplicher Haag and Roberts in the beginnings of the 1970s (but remained little known outside a small circle) in order to explain the mysterious notion of internal symmetries (in terms of local endomorphisms and inclusions of observable algebras) turned out to be strongly related to the mathematical inclusion theory of Vaughan Jones. A brief exposition of the adaptation of that mathematical theory to the QFT setting with a profound application to the problem of amalgamating two chiral theories into one 2-dim. local QFT can be found in [60].

Most chiral models which have been constructed up to now are in one way or the other related to (a multi-component extension) of the above abelian current model by various technically involved but conceptually and mathematically clear steps: reduced tensor-products (to get to level k nonabelian current algebras), coset constructions, "orbifolds" (averaging over subsymmetries) and generating Casimir invariants. There is however a powerful new method which produces new theories from old in a systematic way: the "alpha-induction" method [60] (the analog of the "basic Jones extension"). In a later section we will yet meet another method namely holographic projection from factorizing massive particle theories. Whether any of these methods (or all taken together) is exhaustive remains to be seen.

2.4 QFT in terms of modular positioning of copies of a "monade algebra"

QFT has been enriched by a powerful new concept which promises to revolutionize the task of (nonperturnative) classification and construction of models: modular localization. For a description of its history and aims, the reader is referred to the literature [61][62][63]. It had been known for some time that the localized operator algebras $\mathcal{A}(\mathcal{O})$ of AQFT are isomorphic to an algebra which belongs to a class which already appeared in the famous classification of factor algebras by Murray and von Neumann and whose special role was highlighted by Connes and Haagerup. It is somewhat surprising that the full richness of QFT can be encoded into the relative position of a finite number of copies of this "monade"[18] within a common Hilbert space [64]. Chiral conformal field theory offers the simplest theoretical laboratory in which the emergence of the spacetime symmetry of the vacuum (the Moebius group) and net of spacetime indexed (intervals on the compactified lightray) algebras can be analyzed by starting from a fixed modular position of two local subalgebras (the quarter circle position), but the modular positioning works (with an increasing but finite number of relative positioned monades) also in higher spacetime dimensions. This research gave rise to interesting new implementations [60] of the inclusion theory of Vaughan Jones (closely related to the DHR theory of superselection sectors leading to compact group symmetry) as well to impressive new inclusion concepts (modular inclusions, modular intersections [64]).

All chiral models constructed in the above indicated way as components of 2-dim conformal QFTs come with a an energy momentum tensor whose Fourier components are the generators of an projective representation of an infinite dimensional Lie group of $Diff(S^1)$[19]. This nonvacuum-preserving symmetry and a traceless energy-momentum tensor is also suggested by analogy to classical two-dimensional conformal invariant field theory. But chiral theories also result from lightfront holography where in constradistiction to the chiral decomposition of conformal d=1+1 there is no physical reason for the presence of a chiral energy-momentum tensor. So the question arises if the algebraic formulation of chiral QFT

[18] We borrow this terminology from the mathematician and philosopher Gottfried Ephraim Leibnitz; in addition to its intended philosophical content it has the advantage of being much shorter than "hyperfinite type III_1 Murray-von Neumann factor".

[19] There is a widespread folklore in articles and textbooks claiming that the spatial symmetry group of chiral conformal QFT is the group of all analytic transformations $z \to f(z)$. This is incorrect, the only group of locally analytic transformations of the (compactified) plane is the Moebius group wheras the analytic functions in a bounded region do not even form a group (nor admit a Lie algebra structure). The correct symmetry group (of which only the Moebius subgroup preserves the vacuum and deserves the name analytic) is rather Diff(S)=$\cup Diff_n(S)$ where $Diff_n(S)$ is a *quasisymmetric* subgroup of *quasiconformal* transformations [66]. which have a finite distance from the Moebius transformations (in a suitably topology defined in terms of the Schwartz derivative of f).

can also lead to a more intrinsic understanding (i.e. not using "classical crutches", nor postulating the presence of an energy-momentum tensor) of the appearance of unitary ray representations of $Diff(S^1)$. Here one should remember that diffeomorphisms played an absolutely crucial role in Einstein's covariance principle of general relativity. This principle was recently incorporated into the algebraic setting of QFT in curved spacetime [65] and it was shown that the above axioms of AQFT (the Haag-Kastler setting) are a consequence of a more general setting in which the Poincare covariance and locality are replaced by a quantum "local covariance principle". The adaptation of this principle immediately leads to the existence of an automorphic action of $Diff(S^1)$ on the net of local algebras[20]. In order to show that this automorphism group is implemented by unitary operators of modular origin (as is the Moebius group), one must identify localized subalgebras and construct special states on them which form a standard pair, so that the associated modular groups are generating subgroups of $Diff(S^1)$. As an example let us consider the subgroup $Dil_2(t) \subset Diff(S^1)$ which is the embedding of the two-fold covering of the Moebius dilation $Dil(t)$ into $Diff(S^1)$ [67][68][69]. It acts on the circular variable $z = e^{i\varphi}$ as

$$z \rightarrow g_2(z) = \left(\frac{\alpha z^2 + \beta}{\gamma z^2 + \delta}\right)^{\frac{1}{2}}, \quad \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \tag{8}$$

and has 4 fixpoints $(1, i, -1, -i)$ which fits the prerequisites for a geometric modular group for the doubly localized algebra $\mathcal{A}(I) \vee \mathcal{A}(J)$ with $I = (0, \frac{\pi}{2})$, $J = (\pi, \frac{3\pi}{2})$ being the two opposite quarter circles of the first and third quadrant. The problem here is to find a modular group invariant state so that the modular group of this doubly-localized algebra with respect to that state is $Dil_2(t)$. It is easy to see that the global algebra has no invariant state under $Dil_2(t)$ and hence the restriction to the doubly-localized algebra is essential. The trick to get a partially invariant state is to use the split property i.e. the isomorphism of the double localized algebra with the tensor product $A(I) \otimes A(J)$ and use a KMS state under the action of the local action of the Dil_2 automorphism on the tensor product. As a result one obtains a partial[21] geometric modular action. The assumption of the so-called "strong additivity property" (stating that the removal of points does not affect the algebra) allows to establish the unitary implementation of the Diff(S) automorphisms [70] (its uniqeness was already established before [71]) This intrinsic approach resulting in the theorem that chiral theories (in the sense of a Moebius invariant vacuum state) which are strongly additive and fulfill the split property lead automatically to a unitarily implemented $\mathrm{Diff(S^1)}$, gives much deeper insight than can be obtained through Lagrangian quantization

2.5 Euclidean chiral theory, the modular analog of the Osterwalder-Schrader setting

Euclidean theory associated with certain real time QFTs is a subject whose subtlety and restrictive nature has been lost in many contemporary publications as a result of the "banalization" of the Wick rotation (for some pertinent critical remarks see Lesson.4 in [72]). The mere analytic continuation without establishing the subtle reflection positivity (which is necessary to derive the real time spacelike commutativity as well as the Hilbert space structure) is hardly of any physical use. The statement that a real time bosonic theory whose correlation functions admit the analytic continuation into a large region (the Bargmann-Hall-Wightman region [32]) can be completely characterized in terms of reflection positivity of its Euclidean Schwinger function (with relation to statistical mechanics) is highly nontrivial and its validity is by no means guarantied by being able to baptize a theory by writing down a functional integral involving a local classical action. The operator algebra aspects which distinguish commutative Euclidean statistical mechanics from noncommutative causal real time QFT are encoded in the reflection positivity property. In the setting of chiral theories modular operator theory provides a powerful tool for an operational connection between the real and the imaginary aspects of the theory. In this case modular theory provides an operator formalism for the Wick rotation. In fact there are two Wick rotations, one connected to the Moebius translation and one to the Moebius rotation. The formal aspects are reflected in the two formulas

[20] This result is in agreement with the recently established uniqueness of the extension of the Moebius invariance to the Diff(S) [71].

[21] This modular group does not act geometrically on the commutant.

$$A(ix) = \Delta^{\frac{1}{4}} A(x) \Delta^{-\frac{1}{4}} = U(ix) A(0) U(ix)^{-1} \qquad (9)$$
$$'U(x)' = U(ix) = \Delta^{\frac{1}{4}} U(x) \Delta^{-\frac{1}{4}} = e^{-'P'x}$$

$$A(\varphi)|_{\varphi=2\pi it} \equiv \Delta^{\frac{1}{4}} \widetilde{\Delta}^{it} A(0) \widetilde{\Delta}^{-it} \Delta^{-\frac{1}{4}} \qquad (10)$$
$$'U_{rot}(2\pi it)' \equiv \Delta^{\frac{1}{4}} \widetilde{\Delta}^{it} \Delta^{-\frac{1}{4}} = e^{-2\pi t' L_0'}$$

where the $\widetilde{\Delta}^{it}$ is the modular-parametrized Moebius transformation which fixes the interval (-1,1) (or $(-\frac{\pi}{2}, \frac{\pi}{2})$ in the angular parametrization) and in the second line we abused the notation by maintaining the same field symbol for the angular field parametrization. Although these relation are would-be operator versions of relations within the complex extension of the Moebius fractional transformation group, they are meaningless as they stand as operator relations[22] in the original setting. However modular theory attributes a well-defined algebraic meaning by changing the operator algebra. One starts from the half-space (upper circle) algebra in the vacuum representation $(\mathcal{A}(0,\pi), \Omega)$ which defines a standard pair of operator modular theory leading to modular objects Δ^{it}, J. The Hilbert space is the closure of the linear vector space $L_{\mathcal{A}} = \{A\Omega \,|\, A \in \mathcal{A}(\prime,\pi)\}$ in the standard inner product. By using the modular group in the definition of a new inner product as (apologies for being unduly technical at this point)

$$\langle \Xi(A\Omega), \Xi(B\Omega) \rangle := (A\Omega, \Delta^{\frac{1}{2}} B\Omega) = (\Omega, JAJB\Omega), \;\; A\Omega, B\Omega \in L_{\mathcal{A}} \qquad (11)$$
$$\Xi(A)\Xi(B\Omega) \equiv \Xi(AB\Omega) = \Xi(AB)\widehat{\Omega}, \;\; \widehat{\Omega} \equiv \Xi(\Omega), \;\; \Xi(A) \in \hat{\mathcal{A}} \qquad (12)$$

one reprocesses the original algebra $\mathcal{A}(O)$ into a new operator algebra $\Xi(\mathcal{A}(O))$ acting on a Hilbert space $\hat{\mathrm{H}}$ which is the new closure of $L_{\mathcal{A}}$. As its stands this algebra is neither an algebra of bounded operators nor a *-algebra in the original meaning of conjugation. The following estimate shows how to select a subalgebra of $\Xi(A)$ consisting of bounded operators

$$\|\Xi(A)\Xi(B\Omega)\|^2_{\widehat{H}} = \left\|\Delta^{\frac{1}{4}} AB\Omega\right\|^2_H \le \left\|\Delta^{\frac{1}{4}} A \Delta^{-\frac{1}{4}}\right\|^2_H \|\Xi(B)\|^2_{\widehat{H}}$$

i.e. the boundedness in $\widehat{H}$ requires the restriction of operators in $\Xi(\mathcal{A}(\prime,\pi))$ to the operator domain $\Xi(D(\sigma_{-\frac{i}{4}}))$

$$D(\sigma_{-\frac{i}{4}}) \equiv \left\{A \in \mathcal{A} \,|\, \Delta^{\frac{1}{4}} A \Delta^{-\frac{1}{4}} \in B(H)\right\} \qquad (13)$$
$$\curvearrowright \Xi(D(\sigma_{-\frac{i}{4}})) \subset B(\widehat{H})$$

It is important to realize that this subalgebra on the Euclidean side is not a star subalgebra i.e. $\Xi(D(\sigma_{-\frac{i}{4}}))^\dagger \neq \Xi(D(\sigma_{-\frac{i}{4}}))$ since $\Xi(A)^\dagger = \Xi(\sigma_{-\frac{i}{2}}(A)^\dagger)$ for $A \in D(\sigma_{-\frac{i}{4}})$ where $\dagger$ is the star with respect to the Euclidean inner product. Therefore one defines the Euclidean operator algebra $\widehat{\mathcal{A}} \subset B(\widehat{H})$ to be the smallest von Neumann algebra generated by $\Xi(D(\sigma_{-\frac{i}{4}}))$.i.e. the von Neumann double commutant in $B(\widehat{H})$. .

$$\widehat{\mathcal{A}} \equiv \Xi(D(\sigma_{-\frac{i}{4}}))'' \qquad (14)$$

It remains to be shown that $(\widehat{\mathcal{A}}, \widehat{H}, \widehat{\Omega})$ has the standardness property ($\widehat{\mathcal{A}}$ has a cyclic and separating action on $\widehat{\Omega}$) which leads to a "Euclidean modular theory". Both theories share dense (non star) subalgebras which contains pointlike field generators. The following properties can be derived by straightforward computations:

[22] The fact that they become meaningful if restricted to a certain dense subset in the vacuum sector does not justify their operator algebraic use in the original setting.

- $(\widehat{\mathcal{A}}, \widehat{H}, \widehat{\Omega})$ is a standard triple and its modular objects are $\widehat{\Delta}^{it} = \Delta^{it}$, $\widehat{J} \sim \mathbf{C}$ where $\mathbf{C}$ denotes the unbounded (charge) conjugation defined by the star in B(H) which according to the following formula induces a antiunitary operator on $\widehat{H}$; in fact the charge conjugation operator and the Tomita reflection J interchange their role. The two different algebras share a certain subalgebra of analytic elements (with respect to the action of the modular group) but the extensions to the real lightray chiral algebra and its Euclidean counterpart are different in their notion of star and of boundedness. The chiral Wick rotation is a shortcut for describing this transition. The one-sided original translation is a contracting semigroup acting as a one-sided compression (t>0) in the Euclidean theory and vice versa.
- The previously introduced modular unitary $\widetilde{\Delta}^{it}$ acts as a two-sided compression ($U(x)$ acts as a one-sided compression) on $\mathcal{A}(\prime, \pi)$. This takes the form of a contractive (for t>0) imaginary rotation $e^{-2\pi \widehat{L}_0 t}$ and defines a generator of rotations $\widehat{L}_0$ on the Euclidean Hilbert space $\widehat{H}$. The associated rotation group $e^{2\pi i \widehat{L}_0 t}$ may be used to augment the algebra $\widehat{\mathcal{A}}$ to the universal algebra $\widehat{\mathcal{A}}_{uni}$ [73].

The change of conjugation (star) and the existence of a shared analytic operator structure is of course the hallmark of the link of the Wightman setting with the associated Osterwalder-Schrader formulation. An important consequence is the so-called Nelson-Symanzik symmetry which in the thermal setting of 2-dim. models amounts to theory in a periodic box of length L at temperatur β being the same as that obtained by interchanging space and imaginary time periodicity.

In the present chiral case, the Wick rotation is implemented by modular operators which re-process a joint analytic core algebra into a generating set for two different C*algebra extensions. The universal algebra generated by the action of the Euclidean rotation contains a charge measuring operator, whereas the original $A(0, \pi)$ algebra has charge-changing endomorphisms. Since the modular Euclidization interchanges charge-creation into charge measuring and the $\widehat{L}_0$ thermal Gibbs state is automatically 2π-periodic, the natural chiral counterpart of the 2-dim Nelson-Symanzik symmetry should be a relation in which the transformation matrix from “charge changers to charge meassuers” enters the relation between real and imaginary lightray theory (i.e. the thermal modularity relations passes to the Nelson-Symanzik duality if there are no charges sectors). A lattice analogy would be the famous and still somewhat mysterious corner transfer matrix.

For the L_0 partition functions $Z(\beta)$ such a “modular” (with a different meaning of this word) relation was proposed by Verlinde and proved by Poisson resummation for all models for which the $Z(\beta)$ had been computed. Recently a more structural model independent proof was given within the vertex operator setting [75]. For a quantum field theorist, who wants to see all the properties of chiral QFT to emerge from a common stock of principles (rather than from special prescriptions which have no analogon in the general setting of QFT), these derivations are not completely satisfactory. notwithstanding their mathematical rigor. A quantum field theorist would prefer to see the Verlinde analytic modular relation (and a field theoretic version of the corner transfer matrix relation) to be the attribute of an operationally understood Wick rotation in the chiral context. The original form of the N-S symmetry for 2-dimensional conformal QFT should then re-appears via the modular invariant product of right/left chiral components. In short the analytic meaning of the word modular should subjugate itself in a natural way as a manifestation of the adaptation of the Tomita-Takesaki modular operator theory in the context of local QFT so that what presently looks like a verbal accident would acquire a deeper meaning (I hope to return to this interesting scenario in a separate publication).

Modular operator theory is also expected to play an important role in bridging the still existing gap between the Cardy Euclidean type boundary value problem setting [74] and those in the recent real time operator algebra formulation by Longo and Rehren [76].

3 “PFG” and lightfront holography

In this last section we want to draw the attention to two recent ideas which address the problem of (nonperturbative) classification and construction of 2-dim. massive models in a rather systematic fashion. Let us restrict our interest to theories with a mass gap; this is the standard situation for the validity of the LSZ scattering theory. Let us in addition assume that the Fock space of asymptotic multi-particle states is equal to the total Hilbert space (asymptotic completeness). To keep the notation simple we imagine that we are dealing with an interacting theory of just kind of particle. Let $G\eta\mathcal{A}(\mathcal{O})$ be a (generally unbounded) operator affiliated with the local algebra $\mathcal{A}(\mathcal{O})$. We call G a vacuum-**p**olarization-**f**ree **g**enerator (PFG) if the state vector $G\Omega$ (with Ω the vacuum) is a one-particle state without any vacuum polarization admixture. PFGs are by definition on-shell operators and it is well-known that the existence of a *subwedge-localized* PFG forces the theory to be free, i.e. the local algebras are generated by free fields. However, and this is the surprising fact, this link between PFG and freeness breaks down in wedge-localized algebras. Although the existence of well-behaved PFGs in that case does not exclude interactions, the possibilities are severely limited to factorizable models in 2-dimensions. In fact it turns out that the rapidity-parametrized Fourier transformation of PFGs are creation/annihilation like operators which fulfill the Zamolodchikov-Faddeev algebra commutation relations [63].

The recognition that the knowledge of the position of a wedge-localized subalgebra $\mathcal{A}(W)$ with $\mathcal{A}(W)' = \mathcal{A}(W')$ within the full Fock space algebra B(H) (including the action of the representation of the Poincaré group in $B(H)$ on the $\mathcal{A}(W)$) in principle determines the full net of algebras $\mathcal{A}(\mathcal{O})$ via intersections

$$\mathcal{A}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) \tag{15}$$

is actually independent of spacetime dimensions and factorizability. But only in d=1+1 within the setting of factorizable models one finds simple generators for $\mathcal{A}(W)$ which permit the computation of intersections. The generating operators of the intersection algebras turn out to be representable as infinite series in the Z-F momentum space creation/annihilation operators (with coefficient functions which are identical to the general formfactors of those operators).

There is another constructive idea based on modular inclusion and intersections which does not require the very restrictive presence of wedge-localized PFGs.. This is the holographic projection to the lightfront. In d=1+1 it maps a massive (non-conformal) QFT to a chiral theory on the lightfront (lightray) $x_- = 0$ in such a way that the global original algebra on Minkowski spacetime $\mathcal{A}(M) = B(H)$ and its global holographic lightfront projection $\mathcal{A}(LF) = B(H)$ coalesce[23], but the local substructure (the spacetime indexed net) is radically different, (apart from wedge-localized algebras which are identical to the algebras of their upper lightfront boundary $\mathcal{A}(W) = \mathcal{A}(LF(W))$). Using concepts of modular theory (modular inclusions and modular intersections of wedge algebras) one can construct the local structure [63] (the local algebraic net) and identify the subgroup $G(LF)$ of the Poincaré group which is the symmetry group of the holographic lightfront projection. Whereas some of the ambient Poincaré symmetries are evidently lost, the holographic projection is also symmetry-enhancing in the sense that the infinite dimensional Diff(S^1) group of the compactified lightray in the lightfront becomes a unitary implemented automorphism group of the holographic image (which turns out to be a transversely extended chiral theory [63]). This symmetry is already present in the ambient theory but it is not noticed because it acted in a nonclassical fuzzy manner and hence escapes the standard quantization approach. Whereas the holographic lightfront projection exists in every spacetime dimension, the setting of d=1+1 factorizing models presents a nice theoretical laboratory to study the intricate but exact relation between massive models and their (in this case bona fide) chiral projection in the context of mathematically controllable models. Those chiral observables which appear as the holographic projection of factorizable massive models have the property of admitting generators with simple Z-F algebraic creation/annihilation properties and a covariant transformation property under the full 2-dim Poincaré group. It is clear that a chiral theory specified in terms of such P-covariant operators admits (in analogy to free fields) a unique natural *holographic inversion* which leads from a chiral theory to a massive 2-dim. ambient theory. But without this additional knowledge, the relation of ambient theories to their holographic projection is not expected to be one-to-one. As in the case of the canonical equal time formalism, one rather expects that the specification of a kind of *Hamiltonian propagating in the x_- direction* is necessary for a unique holographic inversion.

The holographic relation between chiral models and factorizing theories is different from the renormalization flow relation in Zamolodchikov's perturbative identification of factorizing theories starting from chiral models. In the latter case there are composite fields (e.t. in the massive L-I QFT obtained by Wick multiplying short distance minimal order/disorder operators with local field functions) which simply vanish in the scale-invariant limit i.e. there is no algebraic one-to-one correspondence. In holography

[23] Conformally invariant theories in d=1+1 are the only exception to this equality.

on the other hand there is no loss of algebraic structure, only a radical change of spacetime indexing of the algebraic substrate.

4 Concluding remarks

In order to present 2-dimensional models as theoretical laboratories for the still unfinished project of QFT which was initiated more than three quarters of a century ago by Pascual Jordan's "Quantelung der Wellenfelder" [77], we have used the three oldest models proposed by Jordan, Lenz-Ising and Schwinger as paradigmatic role models. The conceptual messages they reveal allow to analyze and structure the vast contemporary literature on low dimensional QFT and expose the achievements as well as the unsolved problems in a reader-friendly manner without compromising their depth and complexity (which even their protagonists were not aware of). Among the many unsolved structural problems there is the need for a more profound understanding of the real-imaginary time (Wick rotation) as consequences of operator-algebraic properties which is a prerequisite for the understanding of Verlinde relations as special manifestations of Nelson-Symanzik symmetries adapted to the superselection sector setting of chiral theories and for the relation of Euclidean boundary problems (Cardy) with its real time operator setting (Longo-Rehren). The general setting of lightfront holography highlights chiral models in a new way and poses the interesting question which of them (all?) appear as holographic images of factorizing models and which non-factorizing models are contained in one holographic equivalence class of one factorizing theory. The realization that on-shell quantities (S-matrix, formfactors) in factorizing models admit an analyticity interval around zero coupling (whereas there exist arguments that this property is lost in correlation functions) raises the general question of whether on-shell quantities could have better convergence properties in the coupling parameters than off-shell objects. In short, the number of unanswered questions about 2-dim. models today is much larger than at the times of Ising, Jordan and Schwinger.

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