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Wigner Particle Theory and Local Quantum Physics

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Abstract

Wigner's irreducible positive energy representations of the Poicaré group are often used to give additional justifications for the Lagrangian quantization formalism of standard QFT. Here we study another more recent aspect. We explain in this paper modular concepts by which we are able to construct the local operator algebras for all standard positive energy representations directly i.e. without going through field coordinatizations. In this way the artificial emphasis on Lagrangian field coordinates is avoided from the very beginning. These new concepts allow to treat also those cases of "exceptional" Wigner representations associated with anyons and the famous Wigner "spin tower"which have remained inaccessible to Lagrangian quantization. Together with the d=1+1 factorizing models (whose modular construction has been studied previously), they form an interesting family of theories with a rich vacuum-polarization structure (but no on shell real particle creation) to which the modular methods can be applied for their explicit construction. We explain and illustrate the algebraic strategy of this construction.

We also comment on possibilities of formulating the Wigner theory in a setting of a noncommutative spacetime substrate. This is potentially interesting in connection with recent unitarity- and Lorentz invariance- preserving results of the special nonlocality caused by this kind of noncommutativity.

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1 The setting of the problem

The algebraic framework of local quantum physics shares with the standard textbook approach to QFT the same physical principles but differs in concepts and tools used for their implementation. Whereas the standard approach is based on "field-coordinatizations" in terms of pointlike fields (without which the canonical- or functional integral- quantization is hardly conceivable), the algebraic framework permits to formulate local quantum physics directly in terms of a net of local operator algebras i.e. without the intervention of the rather singular pointlike field coordinates whose indiscriminate use is the potential source of ultraviolet divergencies. Among the many advantages is the fact that the somewhat artistic[2] standard scheme is replaced by a conceptually better balanced setting.

The advantages of such an approach [1][2][3] were in the eyes of many particle physicist offset by its constructive weaknesses of which even its protagonists (who used it mainly for structural investigations as TCP, Spin&Statistics and alike) were well aware [3]. In particular even those formulations of renormalized perturbation theory which were closest in spirit to the algebraic approach namely the causal perturbation theory and its recent refinements [4] uses a coordinatization of algebras in terms of fields at some stage. The underlying "Bogoliubov-axiomatics" [5] in terms of an off-shell generating "S-matrix" S(g) suffers apparently from the same ultraviolet limitations as any other pointlike field formulation.

However there are signs of change which are not only a consequence of the lack of promised success of many popular attempts in post standard model particle theory. Rather it is also becoming slowly but steadily clear that the times of constructive nonperturbative weakness of the algebraic approach (AQFT) are passing and the significant conceptual investments are beginning to bear fruits for the actual construction of models.

The constructive aspects of these gains are presently most clearly visible in situations in which there is no real (on-shell) particle creation but for which, different from free field theories, the vacuum-polarization structure remains very rich. It is not possible in those models to locally generate one-particle states from the vacuum without accompanying vacuum-polarization clouds. Besides the well-known d=1+1 factorizing models, this includes the QFTs associated with exceptional Wigner representations i.e. d=1+2 "anyonic" spin and the d=1+3 "spin towers" (Wigner's famous exceptional zero mass representations with an infinite number of interlinked helicity states). In both cases the absence of compact localization renders the theories more noncommutative and in turn less accessible to Lagrangian quantization methods. The main content of this paper deals with constructive aspects of such models.

The historical roots of the algebraic approach date back to the 1939 famous Wigner paper [6] whose aim was to obtain an intrinsic conceptual understanding of particles avoiding the ambiguous wave equation method and the closely related Lagrangian quantization so that a physical equivalence of different Lagrangian descriptions could be easily recognized. In fact it was precisely this fundamental intrinsic appeal and the unicity of Wigner's approach that some authors felt compelled to present this theory as a kind of additional partial justification for the the Lagrangian (canonical- or functional-) quantization [15]. Since the late 50s there has been a dream about a royal path into nonperturbative particle physics which starts from Wigner's representation-theoretic particle setting and introduces interactions in a maximally

[2] The postulated canonical or functional representation requirement is known to get lost in the course of the calculations and the physical (renormalized) result only satisfies the more general causality/locality properties.

intrinsic and invariant way i.e. by using concepts which avoid doing computations in terms of the standard singular field coordinationations and lean instead on the unitary and crossing symmetric scattering operator and the associated spaces of formfactors. It is well-known that this dream in its original form failed, and that some of the old ideas were re-processed and entered string theory via Veneziano's dual model. In the following we will show that certain aspects of that old folklore (which certainly does not include that of a "Theory of Everything"), if enriched with new concepts, can have successful applications for the above mentioned class of models.

According to Wigner, particles should be described by irreducible positive energy representation of the Poincaré group. In fact they are the indecomposable building blocks of those multi-localized asymptotically stable objects in terms of which each state can be interpreted and measured in counter-coincidence arrangements in the large time limit. This raises the question what localization properties particles should be expected to have, and which positive energy representations permit what kind of localization.

There are two localization concepts. One is the "Born-localization" taken over from Schroedinger theory which is based on probabilities and associated projectors projecting onto compactly supported subspaces of spatially localized wave functions at a fixed time (which in the relativistic context also bears the name "Newton-Wigner" localization). The incompatibility of this localization with relativistic covariance and Einstein causality was already noted and analyzed by its protagonists [7]. Covariance as well as macro-causality are however satisfied in the asymptotic region and therefore the covariance and the cluster separability of the Moeller operators and the S-matrix are not effected by the use of this less than perfect quantum mechanical localization. On the other hand there exists a fully relativitic covariant localization which is intimately related to the characteristic causality- and vacuum polarization-properties of QFT; in the standard formulation of QFT it is that localization which is encoded in the position of the dense subspace obtained by applying smeared fields (with a fixed test function support) to the vacuum. Since in the field-free formulation of local quantum physics this localization turns out to be inexorably linked to the Tomita-Takesaki modular theory of operator algebras, it will be shortly referred to as "modular localization". Its physical content is less obvious and its consequences are less intuitive and therefore we will take some care in its presentation.

In fact the remaining part of this introductory section is used to contrast the Newton-Wigner localization with the modular localization. This facilitates the understanding of both concepts.

The use of Wigner's group theory based particle concept for the formulation of what has been called[3] "direct interactions" in relativistic mutiparticle systems can be nicely illustrated by briefly recalling the arguments which led to this relativistic form of macro-causal quantum mechanics. Bakamjian and Thomas [8] observed as far back as 1953 that it is possible to introduce an interaction into the tensor product space describing two Wigner particles by keeping the additive form of the total momentum $\vec{P}$, its canonical conjugate $\vec{X}$ and the total angular momentum $\vec{J}$ and by implementing interactions through an additive change of the invariant free mass operator M_0 by an interaction v (with only a dependence on the relative c.m. coordinates $\vec{p}_{rel}$) which then leads to a modification of the 2-particle Hamiltonian H

[3]This name was chosen in [9] in order to distinguish it from the field-mediated interactions of standard QFT.

with a resulting change of the boost $\vec{K}$ according to

$$M = M_0 + v, \;\; M_0 = 2\sqrt{\vec{p}_{rel}^{\,2} + m^2} \tag{1}$$
$$H = \sqrt{\vec{P}^2 + M^2}$$
$$\vec{K} = \frac{1}{2}(H\vec{X} + \vec{X}H) - \vec{J} \times \vec{P}(M+H)^{-1}$$

The commutation relations of the Poincaré generators are maintained, provided the interaction operator v commutes with $\vec{P}, \vec{X}$ and $\vec{J}$. For short range interactions the validity of the time-dependent scattering theory is easily established and the Moeller operators $\Omega_\pm(H, H_0)$ and the S-matrix $S(H, H_0)$ are Poincaré invariant in the sense of independence on the L-frame

$$O(H, H_0) = O(M, M_0), \;\; O = \Omega_\pm, S \tag{2}$$

and they also fulfill the cluster separability

$$s - \lim_{\delta \to \infty} O(H, H_0)T(\delta) \to \mathbf{1} \tag{3}$$

where the T operation applied to a 2-particle vector separates the particle by an additional spatial distance δ. The subtle differences to the non-relativistic case begin to show up for 3 particles [9]. Rather than adding the two-particle interactions one has to first form the mass operators of the e.g. 1-2 pair with particle 3 as a spectator and define the 1-2 pair-interaction operator in the 3-particle system

$$M(12,3) = \left(\left(\sqrt{M(12)^2 + \vec{p}_{12}^{\,2}} + \sqrt{m^2 + \vec{p}_3^{\,2}} \right)^2 - (\vec{p}_{12} + \vec{p}_3)^2 \right)^{\frac{1}{2}} \tag{4}$$
$$V^{(3)}(12) \equiv M(12,3) - M(1,2,3), \;\; M(1,2,3) \equiv M_0(123)$$

where the notation speaks for itself (the additive operators carry a subscript labeling and the superscript in the interaction $V^{(3)}(12)$ operators remind us that the interaction of the two particles within a 3-particle system is not identical to the original two-particle $v \equiv V^{(2)}(12)$ operator in the two-particle system). Defining in this way $V^{(3)}(ij)$ for all pairs, the 3-particle mass operator and the corresponding Hamiltonian are given by

$$M(123) = M_0(123) + \sum_{i<j} V^{(3)}(ij) \tag{5}$$
$$H(123) = \sqrt{M(123)^2 + p_{123}^2}$$

and lead to a L-invariant and cluster-separable 3-particle Moeller operator and S-matrix, where the latter property is expressed as a strong operator limit

$$S(123) \equiv S(H(123), H_0(123)) = S(M(123), M_0(123)) \tag{6}$$
$$s\text{-}\lim_{\delta \to \infty} S(123)T(\delta_{13}, \delta_{23}) = S(12) \times \mathbf{1}$$

with the formulae for other clusterings being obvious. By iteration and the use of the framework of rearrangement collision theory (which introduces an auxiliary Hilbert space of bound fragments), this can be generalized to n-particles including bound states [10].

As in nonrelativistic scattering theory, there are many different relativistic direct particle interactions which lead to the same S-matrix. As Sokolov showed, this freedom to modify off-shell operators (e.g. $H, \vec{K}$ as functions of the single particle variables $\vec{p}_i, \vec{x}_i, \vec{j}_i$ and the interaction v) may be used to construct to each system of the above kind a "scattering-equivalent" system in which the interaction-dependent generators $H, \vec{K}$ restricted to the images of the fragment spaces become the sum of cluster Hamiltonians (or boosts) with interactions between clusters being switched off [10]. Using these interaction-dependent equivalence transformations, the cluster separability can be made manifest. It is also possible to couple channels in order to describe particle creation, but this channel coupling "by hand" does not define a natural mechanism for interaction-induced vacuum polarization.

Even though such direct interaction models between relativistic particles can hardly have fundamental significance, their very existence as relativistic theories (i.e. consistent with the physically indispensible macro-causality) help us rethink the position of micro-causal and local versus nonlocal but still macro-causal relativistic theories.

Since our intuition on theses matters is notoriously unreliable and ridden by prejudices, it is very useful to have such illustrations. This is of particular interest in connection with recent attempts to implement nonlocality through noncommutativity of the spacetime substrate (see the last section). But even some old piece of QFT folklore, which claimed that the construction of unitary relativistic invariant and cluster-separable S-matrices can only be achieved through local QFT, are rendered incorrect.

It turns out that if one adds crossing symmetry to the list of S-matrix properties it is possible to show that if the on-shell S-matrix originates at all from a local QFT, it determines its local system of operator algebras uniquely [11]. This unicity of local algebras is of course the only kind of uniqueness which one can expect since individual fields are analogous to coordinates in differential geometry (in the sense that passing to another locally related field cannot change the S-matrix).

The new concept which implements the desired crossing property and also insures the principle of "nuclear democracy"[4] (both properties are not compatible with the above relativistic QM) is modular localization. In contrast to the quantum mechanical Newton Wigner localization, it is not based on projection operators (which project on quantum mechanical subspaces of wave functions with support properties) but rather is reflected in the Einstein causal behavior of expectation values of local variables in modular localized state vectors. Modular localization in fact relates off-shell causality, interaction-induced vacuum polarization and on-shell crossing in an inexorable manner and in particular furnishes the appropriate setting for causal propagation properties (see next section). Since it allows to give a completely intrinsic definition of interactions in terms of the vacuum polarization clouds which accompany locally generated one-particle states without reference to field coordinates or Lagrangians, one expects that it serves as a constructive tool for nonperturbative investigations. This is borne out for those models considered in this paper.

It is interesting to note that both localizations are preempted in the Wigner theory. Used in the Bakamjian-Thomas-Coester spirit of QM of relativistic particles with the Newton-Wigner localization, it leads to relativistic invariant scattering operators which obey cluster separability properties and hence are in perfect harmony with macro-causality. On the other hand used as a starting point of modular localization one can directly pass to the system of local operator algebras and relate the notion of inter-

[4] Every particle may be interpreted as bound of all others whose fused charge is the same. An explicit illustration is furnished by the bootstrap properties of d=1+1 factorizing S-matrices [14].

action (and exceptional statistics) inexorably with micro-causality and vacuum polarization clouds which accompany the local creation of one particle states. Perhaps the conceptually most surprising fact is the totally different nature of the local algebras from quantum mechanical algebras.

In the second section we will present the modular localization structure of the standard halfinteger spin Wigner representation in the first subsection and that of the exceptional (anyonic, spin towers) representations in the second subsection.

The subject of the third section is the functorial construction of the local operator algebras associated with the modular subspaces of the standard Wigner representations. The vacuum polarization aspects of localized particle creation operators associated with exceptional Wigner representations are treated in the fourth section. In section 5 we explain our strategy for the construction of theories which have no real particle creation but (different from free fields) come with a rich vacuum polarization structure in the context of d=1+1 factorizing models.

Apart from the issue of anyons, the most interesting and unexplored case of QFTs related to positive energy Wigner representations is certainly that of the massless d=1+3 "Wigner spin towers". This case is in several aspects reminiscent of structures of string theory. It naturally combines all (even, odd, supersymmetric) helicities into one indecomposable object. If it would be possible to introduce interactions into this tower structure, then the standard argument that any consistent interacting object which contains spin 2 must also contain an (at least a quasiclassical) Einstein-Hilbert action (which is used by string theorist in order to link strings with gravity) applies as well here [5].

Recently there has been some interest in the problem whether the Wigner particle structure can be consistent with a noncommutative structure of spacetime where the minimal consistency is the validity of macro-causality. We will have some comments in the last section.

2 Modular aspects of positive energy Wigner representations

In this in the next subsection we will briefly sketch how one obtains the interaction-free local operator algebras directly from the Wigner particle theory without passing through pointlike fields. The first step is to show that there exist a relativistic localization which is different from the non-covariant Newton-Wigner localization.

2.1 The standard case: halfinteger spin

For simplicity we start from the Hilbert space of complex momentum space wave function of the irreducible $(m, s = 0)$ representation for a neutral (selfconjugate) scalar particle. In this case we only need to remind the reader of published results [12][13][18][35].

$$H_{Wig} = \left\{ \psi(p) \middle| \int |\psi(p)|^2 \frac{d^3p}{2\sqrt{p^2+m^2}} < \infty \right\} \tag{7}$$
$$\left(\mathfrak{u}(\Lambda, a)\psi\right)(p) = e^{ipa}\psi(\Lambda^{-1}p)$$

[5] In this connection it appears somewhat ironic that the infinite spin tower Wigner representation is often dismissed as "not used by nature" without having investigated its physical potential.

For the construction of the real subspace $H_R(W_0)$ of the standard t-z wedge $W_0 = (z > |t|, x, y$ arbitrary) we use the $z-t$ Lorentz boost $\Lambda_{z-t}(\chi) \equiv \Lambda_{W_0}(\chi)$

$$\Lambda_{W_0}(\chi) : \begin{pmatrix} t \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \tag{8}$$

which acts on H_{Wig} as a unitary group of operators $\mathfrak{u}(\chi) \equiv \mathfrak{u}(\Lambda_{z-t}(\chi), 0)$ and the z-t reflection $r : (z,t) \rightarrow (-z,-t)$ which, since it involves time reflection, is implemented on Wigner wave functions by an unti-unitary operator $\mathfrak{u}(r)$ [35][18]. One then forms (by the standard functional calculus) the unbounded[6] "analytic continuation" in the rapidity $\mathfrak{u}(\chi \rightarrow i\chi)$ which leads to unbounded positive operators. Using a notation which harmonizes with that of the modular theory (see appendix A), we define the following operators in H_{Wig}

$$\begin{aligned} \mathfrak{s} &= \mathfrak{j}\delta^{\frac{1}{2}} \qquad (9) \\ \mathfrak{j} &= \mathfrak{u}(r) \\ \delta^{it} &= \mathfrak{u}(\chi = -2\pi t) \\ (\mathfrak{s}\psi)(p) &= \psi(-p)^* \qquad (10) \end{aligned}$$

Note that all the operators are functional-analytically extended geometrically defined objects within the Wigner theory; in particular the last line is the action of an unbounded involutive $\mathfrak{s}$ on Wigner wave functions which involves complex conjugation as well as an "analytic continuation" into the negative mass shell. Note that $\mathfrak{u}(r)$ is apart from a π-rotation around the x-axis the one-particle version of the TCP operator. The last formula for $\mathfrak{s}$ would look the same even if we would have started from another wedge $W \neq W_0$. This is quite deceiving since physicists are not accustomed to consider the domain of definition as an integral part of the definition of the operator. If the formula would describe a bounded operator the formula would define the operator uniquely but in the case at hand $dom\mathfrak{s} \equiv dom\mathfrak{s}_{W_0} \neq dom\mathfrak{s}_W$ for $W_0 \neq W$ since the domains of δ_{W_0} and δ_W are quite different; in fact the geometric positions of the different $W's$ can be recovered from the $\mathfrak{s}'s$. All Tomita S-operators are only different in their domains but not in their formal appearance; this makes modular theory a very treacherous subject.

The content of (9) is nothing but an adaptation of the spatial version of the Bisognano-Wichmann theorem to the Wigner one-particle theory [35][18]. The former is in turn a special case of Rieffel's and van Daele's spatial generalization [16] of the operator-algebraic Tomita-Takesaki modular theory (see appendix A). Since the antiunitary t-z reflection commutes with the t-z boost δ^{it}, it inverts the unbounded $(\delta^i)^{-i} = \delta$ i.e. $\mathfrak{j}\delta = \delta^{-1}\mathfrak{j}$. As a result of this commutation relation, the unbounded antilinear operator $\mathfrak{s}$ is involutive on its domain of definition i.e. $\mathfrak{s}^2 \subset 1$ so that it may be used to define a real subspace (closed in the real sense i.e. its complexification is not closed) as explained in the appendix. The definition of $H_R(W_0)$ is in terms of $+1$ eigenvectors of $\mathfrak{s}$

$$\begin{aligned} H_R(W_0) &= clos\left\{\psi \in H_{Wig} | \, \mathfrak{s}\psi = \psi\right\} \qquad (11) \\ &= clos\left\{\psi + \mathfrak{s}\psi | \, \psi \in dom\mathfrak{s}\right\} \\ \mathfrak{s}\mathfrak{i}\psi &= -\mathfrak{i}\psi, \;\; \psi \in H_R(W_0) \end{aligned}$$

[6]The unboundedness is of crucial importance since the domain of definition is the only distinguishing property of the involution (10) into which geometric properties (causally closed regions in Minkowski space) are encoded.

The +1 eigenvalue condition is equivalent to analyticity of $\delta^{it}\psi$ in $-\frac{1}{2} < Imt < 0$ (and continuity on the boundary) together with a reality property relating the two boundary values on this strip. The localization in the opposite wedge i.e. the $H_R(W^{opp})$ subspace turns out to correspond to the symplectic (or real orthogonal) complement of $H_R(W)$ in H_{Wig} i.e.

$$\mathrm{Im}(\psi, H_R(W_0)) = 0 \Leftrightarrow \psi \in H_R(W_0^{opp}) \equiv \mathfrak{j}H_R(W_0) = H_R(rW_0) \tag{12}$$

One furthermore finds the following properties for the subspaces called "standardness"

$$\begin{aligned} &H_R(W_0) + iH_R(W_0) \; is \; dense \; in \; H_{Wig} \\ &H_R(W_0) \cap iH_R(W_0) = \{0\} \end{aligned} \tag{13}$$

For completeness we sketch the proof. The closedness of the densely defined $\mathfrak{s}$ leads to the following decomposition of the domain $dom\mathfrak{s}$

$$\begin{aligned} dom(\mathfrak{s}) &= \left\{ \psi \in H_{Wig} | \, \psi = \frac{1}{2}(\psi + \mathfrak{s}\psi) + \frac{i}{2}(\psi - \mathfrak{s}\psi) \right\} \\ &= H_R(W_0) + iH_R(W_0) \end{aligned} \tag{14}$$

On the other hand from $\psi \in H_R(W_0) \cap iH_R(W_0)$ one obtains

$$\begin{aligned} \psi &= \mathfrak{s}\psi \\ i\psi &= \mathfrak{s}\mathfrak{i}\psi = -\mathfrak{i}\mathfrak{s}\psi = -\mathfrak{i}\psi \end{aligned} \tag{15}$$

from which $\psi = 0$ follows. In the appendix it was shown that vice versa the standardness of a real subspace H_R leads to the modular objects $\mathfrak{j}, \delta$ and $\mathfrak{s}$.

Since the Poincaré group acts transitively on the $W's$ and carries the W_0-affiliated $\mathfrak{u}(\Lambda_{W_0}(\chi)), \mathfrak{u}(r_{W_0})$ into the corresponding W-affiliated L-boosts and reflections, the subspaces $H_R(W)$ have the following covariance properties

$$\begin{aligned} \mathfrak{u}(\Lambda, a)H_R(W_0) &= H_R(W = \Lambda W_0 + a) \\ \mathfrak{s}_W &= \mathfrak{u}(\Lambda, a)s_{W_0}\mathfrak{u}(\Lambda, a)^{-1} \end{aligned} \tag{16}$$

where the Poincaré-transformation is only determined up to those transformations which fix the two wedges.

Having arrived at the wedge localization spaces, one may construct localization spaces for smaller spacetime regions by forming intersections over all wedges containing this region $\mathcal{O}$

$$H_R(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} H_R(W) \tag{17}$$

These spaces are again standard and covariant. They have their own "pre-modular" (see the appendix on the spatial theory, the true Tomita modular operators appear in the next section) object $\mathfrak{s}_{\mathcal{O}}$ and the radial and angular part $\delta_{\mathcal{O}}$ and $\mathfrak{j}_{\mathcal{O}}$ in their polar decomposition (9), but this time their action cannot be described in terms of spacetime diffeomorphisms since for massive particles the action is not implemented by a geometric transformation in Minkowski space. To be more precise, the action of $\delta_{\mathcal{O}}^{it}$ is only local in the sense that $H_R(\mathcal{O})$ and its symplectic complement $H_R(\mathcal{O})' = H_R(\mathcal{O}')$ are transformed onto themselves (whereas $\mathfrak{j}$ interchanges the original subspace with its symplectic complement), but for massive Wigner particles there is no geometric modular transformation (in the massless case there is a modular diffeomorphism of the compactified Minkowski space). Nevertheless the modular transformations $\delta_{\mathcal{O}}^{it}$ for $\mathcal{O}$ running through all double cones and wedges (which are double cones "at infinity") generate the action of an infinite dimensional Lie group. Except for the finite parametric Poincaré group (or conformal group in the case of zero mass particles) the action is partially "fuzzy" i.e. not implementable by a diffeomorphism on Minkowski spacetime but still being the product of modular group action where each factor respects the causal closure (causal "horizon") of a region $\mathcal{O}$ (more precisely: it is asymptotically gemometric near the horizon). The emergence of these *fuzzy acting Lie groups is a pure quantum phenomenon*; there is no analog in classical physics. They describe hidden symmetries [22][23] which the Lagrangian formalism does not expose.

Note also that the modular formalism characterizes the localization of subspaces. In fact for the present $(m, s = 0)$ Wigner representations the spaces $H_R(\mathcal{O})$ have a simple description in terms of Fourier transforms of spacetime-localized test functions. In the selfconjugate case one finds

$$H_R(\mathcal{O}) = rclos\left\{\psi = E_m\tilde{f}\,|f \in \mathcal{D}(\mathcal{O}), f = f^*\right\} \tag{18}$$

where the closure is taken within the real subspace i.e. one imposes the reality condition $f = f^*$ in the mass-shell restriction corresponding to a projector E_m acting on the Fourier transform i.e. $(E_m\tilde{f})(p) = \left(E_m\tilde{f}\right)^*(-p)$, $p^2 = m^2, p_0 > 0$. This space may also be characterized in terms of a closure of a space of entire functions with a Paley-Wiener asymptotic behaviour. From these representations (1718) it is fairly easy to conclude that the inclusion-preserving maps $\mathcal{O} \to H_R(\mathcal{O})$ are maps between orthocomplemented lattices of causally closed regions (with the complement being the causal disjoint) and modular localized real subspaces (with the simplectic or real orthogonal complement). In particular one finds $H_R(\mathcal{O}_1 \cap \mathcal{O}_2) = H_R(\mathcal{O}_1) \cap H_R(\mathcal{O}_2)$. The complement of this relation is called the additivity property which is an indispensible requirement if the Global is obtained by piecing together the Local.

The dense subspace $H(W) = H_R(W) + iH_R(W)$ of H_{Wig} changes its position within H_{Wig} together with W. If one would close it in the topology of H_{Wig} one would loose all this subtle geometric information encoded in the $\mathfrak{s}$-domains. One must change the topology in such a way that the dense subspace $H(W)$ becomes a Hilbert space in its own right. This is achieved in terms of the graph norm of $\mathfrak{s}_W$ (for the characterization of the $H_R(\mathcal{O})$ in terms of test function (18) one did not need the $\mathfrak{s}$-operator

$$(\psi, \psi)_{G\mathfrak{s}} \equiv (\psi, \psi) + (\mathfrak{s}\psi, \mathfrak{s}\psi) < \infty \tag{19}$$

This topology is simply an algebraic way of characterizing a Hilbert space which consists of localized vectors only. It is easy to write down a modified measure in which the $\mathfrak{s}$ becomes a bounded operator

$$(\psi, \psi)_{ther} = \int \psi^*(\theta, p_\perp)\frac{1}{\delta - 1}\psi(\theta, p_\perp)d\theta \tag{20}$$

$$\psi(\theta, p_\perp) = \psi(p),\ p = (m_{eff}\cosh\theta, p_\perp, m_{eff}\sinh\theta)$$

Clearly $\delta = \mathfrak{s}^*\mathfrak{s}$ and $1 + \delta$ are bounded in this norm. Defining the Fourier transform

$$f(\theta) = \frac{1}{\sqrt{2\pi}}\int \tilde{f}(\kappa)e^{i\kappa\theta} \tag{21}$$

The modification takes on the appearance of a *thermal* Bose factor at temperature $T = 2\pi$ with the role of the Hamiltonian being played by the Lorentz boost generator K in $\delta = e^{-2\pi K}$ (which is the reason for using the subscript *ther*). In fact the Wigner one-particle theory preempts the fact that the associated free field theory in the vacuum state restricted to the wedge becomes thermal i.e. satisfies the KMS condition and the thermal inner product becomes related to the two-point-function of that wedge restricted QFT. We have taken a wedge because then the modular Hamiltonian K has a geometric interpretation in terms of the L-boost, but the modular Hamiltonian always exists; if not in a geometric sense then as a fuzzy transformation which fixes the localization region and its causal complement. Hence for any causally closed spacetime region $\mathcal{O}$ and its nontrivial causal complement $\mathcal{O}'$ there exists such a thermally closed Hilbert space of localized vectors and for the wedge W this preempts the Unruh-Hawking effect associated with the geometric Lorentz boost playing the role of a Hamiltonian (in case of ($m=0, s=$halfinteger) representations this also holds for double cones since they are conformally equivalent to wedges).

After having obtained some understanding of modular localization it is helpful to highlight the difference between N-W and modular localization by a concrete illustration. Consider the energy momentum density in a one-particle wave function of the form $\psi_f = E_m f \in H_R(\mathcal{O})$ where $suppf \subset \mathcal{O}$, f real

$$\begin{aligned} t_{\mu\nu}(x,\psi) &= \partial_\mu \psi_f(x) \partial_\nu \psi_f(x) + \frac{1}{2} g_{\mu\nu}(m^2 \psi_f(x)^2 - \partial^\nu \psi_f(x) \partial_\nu \psi_f(x)) \\ &= \langle f, c \,|\!: T_{\mu\nu}(x) :\!|\, f, c\rangle \,,\ |f,c\rangle \equiv W(f)\,|0\rangle \end{aligned} \tag{22}$$

where on the right hand side we used the standard field theoretic expression for the expectation value of the energy-momentum density in a coherent state obtained by applying the Weyl operator corresponding to the test function f to the vacuum. Since $\psi_f(x) = \int \Delta(x-y,m) f(y) d^4y$ we see that the one-particle expectation (22) complies with Einstein causality (no superluminal propagation outside the causal influence region of $\mathcal{O}$), but there is no way to affiliate a projector with the subspace $H_R(\mathcal{O})$ or with coherent states (the real projectors appearing in the appendix are really unbounded operators in the complex sense). We also notice that as a result of the analytic properties of the wave function in momentum space the expectation value has crossing properties, i.e. it can be analytically continued to a matrix element of T between the vacuum and a modular localized two-particle two-particle state. This follows either by explicit computation or by using the KMS property on the field theoretic interpretation of the expectation value. A more detailed investigation shows that the appearance of this crossing (vacuum polarization) structure and the absence of localizing projectors are inexorably related. This property of the positive energy Wigner representations preempts a generic property of local quantum physics: *relativistic localization cannot be described in terms of (complex) subspaces and projectors, rather this must be done in terms of expectation values of local observables in modular localized states which belong to real subspaces.*

The use of the inappropriate localization concept is the prime reason why there have been many misleading papers on “superluminal propagation” in which Fermi’s result that the classical relativistic propagation inside the forward light cone continues to hold in relativistic QFT was called into question (for a detailed critical account see [19]).

On the more formal mathematical level this absence of localizing projectors is connected to the absence of pure states and minimal projectors in the local operator algebras. The standard framework of QM and the concepts of “quantum computation” simply do not apply to the local operator algebras since the latter are of von Neumann type III_1 hyperfinite operator algebras and not of the quantum mechanical type I. Therefore it is a bit misleading to say that local quantum physics is just QM with the nonrelativistic Galilei group replaced by Poincaré symmetry; these two requirements would lead to the relativistic QM mentioned in the previous section whereas QFT is characterized by micro-causality of observables and modular localization of states. To avoid any misunderstanding, projectors in compact causally closed local regions $\mathcal{O}$ of course exist, but they necessarily describe fuzzy (non sharp) localization within $\mathcal{O}$ [20] and the vacuum is necessarily a highly entangled temperture state if restriceted via this projector (in QM spatial restrictions only create isotopic representations i.e. enhanced multiplicities but do not cause genuine entanglement or thermal behavior).

It is interesting that the two different localization concepts have aroused passionate discussions in philosophical circles as evidenced e.g. from bellicose sounding title as “Reeh-Schlieder defeats Newton-Wigner” in [21]. As it should be clear from our presentation particle physics finds both very useful, the first for causal (non-superluminal) propagation and the second for scattering theory where only asymptotic covariance and causality is required.

After having made pedagogical use of the simplicity of the scalar neutral case in order to preempt some consequences of the modular aspects of QFT on the level of the Wigner one-particle theory, it is now easy to add the modifications which one has to make for charged scalar particles and those with nonzero spin. The Wigner representation of the connected part of a Poincaré group describes only one particle, so in order to incorporate the antipartice which has identical Poincaré properties one just doubles the Wigner space and defines the $\mathfrak{j}$ and the $\mathfrak{s}$ as follows (still spin-less)

$$
\begin{aligned}
(\mathfrak{j}\psi)(p) &= \psi^c(rp),\ (\mathfrak{s}\psi)(p) = \psi^c(-p) \qquad (23)\\
\psi(p) &= \begin{pmatrix} \psi_1(p) \\ \psi_2(p) \end{pmatrix},\ \psi^c(p) = \begin{pmatrix} \psi_2(p)^* \\ \psi_1(p)^* \end{pmatrix} \\
\psi^c(p) &= C\psi(p)^*,\ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

It is then easy to see that $\mathfrak{s}$ has a polar decomposition as before in terms of j and a Lorentz boost $\mathfrak{s} = \mathfrak{j}\delta$. The real subspaces resulting from closed +1 eigenstates of $\mathfrak{s}$ are

$$
H_R(W) = rclos\left\{\psi(p) + \psi^c(-p) |\, \psi \in dom\mathfrak{s}\right\} \qquad (24)
$$

where the real closure is taken with respect to real linear combinations. Again the subspaces $H_R(\mathcal{O})$ defined by intersection as in (17) admit a representation in terms of real closures of (mass shell projected, two-component, C-conjugation-invariant) $\mathcal{O}$-supported test function spaces as in (18).

However it would be misleading to conclude from this spinless example that modular localization in positive energy Wigner representations theory is always quite that simple. For nontrivial halfinteger spin massive particles the 2s+1 component wave function transform according to

$$
\begin{aligned}
\left(\mathfrak{u}(\tilde{\Lambda}, a)\psi\right)(p) &= e^{iap} D^{(s)}(\tilde{R}(\Lambda, p))\psi(\Lambda^{-1}p) \qquad (25)\\
\tilde{R}(\Lambda, p) &= \alpha(L(p))\alpha(\Lambda)\alpha(L^{-1}(\Lambda^{-1}p)) \\
\alpha(L(p)) &= \sqrt{\frac{p^\mu \sigma_\mu}{m}}
\end{aligned}
$$

Here α denotes the SL(2.C) covering (transformation of undotted fundamental spinors) and $\tilde{R}(\Lambda, p)$ is an element of the (covering of the) “little group” which is the fixed point subgroup[7] of the chosen reference

[7] We will use the letter R even in the massless case when the little group becomes the noncompact Euclidean group.

vector $p_R = (m, 0, 0, 0)$ on the $(m > 0, s)$ orbit. $L(p)$ is the chosen family of boosts which transform p_R into a generic p on the orbit. The fixed point group for the case at hand is the quantum mechanical rotation group i.e. $\tilde{R}(\Lambda, p) \in SU(2)$ and the D-operators are representation matrices $D^{(s)}$ of $SU(2)$ obtaines by symmetrizing the 2s-fold SU(2) tensor products.

For $s = \frac{n}{2}, n\,\mathrm{odd}$, the Wigner matrices $\tilde{R}(\Lambda_{W_0}(-2\pi t), p)$ enter the definition of the operator $\mathfrak{s}$ and they generally produce a square-root cut in the analytic strip region. As a representative case of halfinteger spin we consider the case of a selfdual massive $s = \frac{1}{2}$ particle. The fact that the $SU(2)$ Wigner rotation is only pseudo-real i.e. that the conjugate representation (although being $i\sigma_2$-equivalent to the defining one, there is no equivalence transformation which makes them identical) forces us to double order deal with selfconjugate Wigner transformation matrices

$$\psi_d := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \psi_1 \\ i\sigma_2\psi_2 \end{pmatrix}, \tag{26}$$
$$\psi_d \to D_d\psi_d,\ D_d = \begin{pmatrix} \mathrm{Re}\, D & \mathrm{Im}\, D \\ -\mathrm{Im}\, D & \mathrm{Re}\, D \end{pmatrix}$$

where D denote the original $SU()$-valued Wigner transformation matrices. Therefore the representation space will be represented by 4×2 component spinor

$$\Psi(p) = \begin{pmatrix} \psi_d^{(1)}(p) \\ \psi_d^{(2)}(p) \end{pmatrix} \xrightarrow{C} \Psi^C(p) = \begin{pmatrix} \psi_d^{(2)}(p) \\ \psi_d^{(1)}(p) \end{pmatrix} \tag{27}$$

so that the definition for the spatial Tomita operator

$$\mathfrak{s}\Psi(p) = \Psi^C(-p) \tag{28}$$
$$H_R(W) = \{\Psi(p) |\, \mathfrak{s}\Psi(p) = \Psi(p)\} \curvearrowright \psi_d^{(1)}(p) = \psi_d^{(2)}(-p)^*$$

complies with the conjugacy properties of the Wigner transformations. For selfconjugate (Majorana) particles one has in addition $\psi_1 = \psi_2$.

The original Wigner transformation D (25) contains the t-dependent 2×2 matrix which in Pauli matrix notation reads

$$\frac{1}{\sqrt{m}}\left(\cosh 2\pi t \cdot p^0\mathbf{1} - \sinh 2\pi t \cdot p^1\sigma_1 + p^2\sigma_2 + p^3\sigma_3\right)^{\frac{1}{2}} \tag{29}$$

which in the analytic continuation $t \to z$ develops a square root cut in the would-be analytic strip $-\frac{1}{2} < Imz < 0$. This square root cut in D_d complicates the description of the domain $dom\mathfrak{s}$.

The only way to retain strip analyticity in the presence of the Wigner transformation law is to have a compensating singularity in the transformed wave function $\Psi(\Lambda_{W_0}(-2\pi t)p)$ as t is continued into the strip. This is achieved by factorizing the Wigner wave function in terms of intertwiners α. Let us make the following ansatz for the original 2-component Wigner wave function

$$\psi(p) = \alpha(L(p))\,(E_m\Phi)\,(p) \tag{30}$$
$$\alpha(L(p)) = \sqrt{\frac{p^\mu\sigma_\mu}{m}}$$
$$\tilde{R}(\Lambda, p)\alpha(L(\Lambda^{-1}p)) = \alpha(L(p))\alpha(\Lambda)$$

where in the last line we wrote the intertwining relation for the intertwining matrix $\alpha(L(p))$. $\Phi_\alpha(x) \in \mathcal{D}(W_0), \alpha = 1, 2$ is a two-component space of test functions with support in the standard wedge W_0. Such test functions whose associated Fourier transformed wave functions projected onto the mass shell $(E_m\Phi)(p)$ obviously fulfill the strip analyticity are interpreted as (undotted) spinors i.e. they are equipped with the transformation law

$$\Phi(x) \to \alpha(\Lambda)\Phi(\Lambda x),\ \alpha(\Lambda) \in SL(2, C) \tag{31}$$

The covariant (undotted) spinorial transformation law[8] changes the support in a geometric way. As a consequence of group theory, the spinor wave function defined by (with E_m a mass shell projector as before and $\mathfrak{u}(p)$ intertwiner matrix $\mathfrak{u}(p) =$ transforms according to Wigner as

$$\psi(p) \to \alpha(\tilde{R}(\Lambda, p))\mathfrak{u}(\Lambda^{-1}p)(E_m\Phi)(\Lambda^{-1}p) = \mathfrak{u}(p)\alpha(\Lambda)\psi(\Lambda^{-1}p) \tag{32}$$

where in the second line we wrote the intertwining law of $\mathfrak{u}(p) = \alpha(L(p))$ of which the first line is a consequence. We see that the product Ansatz $\psi = uE_m\Phi$ solves the problem of the strip analyticity since the $\mathfrak{u}(p)$ factor develops a square root cut which compensates that of the Wigner rotation and $E_m\Phi$ is analytic from the wedge localization of Φ. The test function space provides a dense set in $H_R(W)$ so by adding limits, one obtains all of $H_R(W)$ i.e. all the full $+1$ eigenspace of $\mathfrak{s}$. In fact this Ansatz avoids the occurance of singular pre-factor for any causally complete localization region $\mathcal{O}$; in the compact case the closure of the test function space turns out to be a space of entire functions with an appropriate Paley-Wiener-Schwartz asymptotic behaviour reflecting the size of the double cones $\mathcal{O}$. Although our analyticity discussion was done on the original Wigner representation, it immediatly carries over to the doubled version which we have used for the construction of the real modular subspaces $H_R(W)$. Again $H(W) = H_R(W) + iH_R(W)$ will be dense in H_{Wig} for the same reason as in the cases before. To obtain the solution for arbitrary halfinteger spin one only has to use symmetrized tensor representations of $SL(2, C)$ and its $SU(2)$ subgroup.

If we now try to represent our $\mathfrak{s}$-operator as $j\Delta^{\frac{1}{2}}$in terms of geometrically defined reflections and boosts we encounter a surprise; the geometrically defined object is different by a phase factor i. This factor results from the analytically continued Wigner rotation in the boost parameter for all halfinteger spins. The only way to compensate it consistent with the polar decomposition is to say that the j deviates from the geometric j_0 by a phase factor t

$$j = tj_0,\ t = i \tag{33}$$

It turns out that this also happens for the exeptional Wigner representations; for d=1+2 anyons one obtains a phase factor related to the spin of the anyon whereas for the d>1+3 spin towers t is an operator in the infinite tower space related to the analytically continued infinite dimensional Wigner matrix. These cases are characteized by the failure of compact modular localization (see below).

The modular localization in the massless case is similar as long as the helicity stays finite (trivially represented Euclidean "translations") is similar. The concrete determination of the Λ, p-dependent $\tilde{R}$

[8] Since here we have to distinguish between undotted and dotted spinors, we use the notation $\alpha(\Lambda)$ and $\beta(\Lambda) = \overline{\alpha(\Lambda)}$ instead of the previous $\tilde{\Lambda}$.

requires a selection of a family of boosts i.e. of Lorentz transformations $\tilde{L}(p)$ which relate the reference vector p_R uniquely a general p on the respective orbit. The natural choice for the associated 2×2 matrices in case of d=1+3 is (we use α for the $SL(2,C)$ representation)

$$\alpha(\tilde{L}(p)_0) = \frac{1}{\sqrt{p_0+p_3}} \begin{pmatrix} p_0+p_3 & p_1-ip_2 \\ 0 & 1 \end{pmatrix}, m=0 \tag{34}$$

with the associated little groups being $SU(2)$ or for m=0 $\tilde{E}(2)$ (the 2-fold covering of the 2-dim. Euclidean group)

$$\tilde{E}(2): \begin{pmatrix} e^{i\frac{\varphi}{2}} & z=a+ib \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix}, \quad m=0 \tag{35}$$

For the standard (halfinteger helicity) massless representations the "z-translations" are mapped into the identity. As a result of the projection property of the reference vector there exists a projected form of the intertwining relation ($\alpha(\tilde{L}(p))$ as in (34))

$$\begin{aligned} p_R\tilde{R}(\Lambda,p) &= p_R\tilde{R}(\Lambda,p)_{11} \\ \tilde{R}(\Lambda,p) &= \alpha(\tilde{L}(p))\tilde{\Lambda}\alpha(\tilde{L}^{-1}(\Lambda^{-1}p)) \end{aligned} \tag{36}$$

This projection allows to incorporate the one-component formalism into the SL(2,C) matrix formalism. In fact this embedding permits to use the same mass independent W-supported test function spaces as before, one only has to replace the E_m projectors by projectors on the zero mass orbit. Again the definition of j generally demands a further doubling of the test function. At the end one obtains a representation of modular localization spaces $H_R(W)$ (and more generally $H_R(\mathcal{O})$ for double cones $\mathcal{O}$) in terms of W or $\mathcal{O}$ supported spinorial test function spaces whose nontriviality is secured by the classical Schwartz distribution theory.

It is easy to see that the modular formalism also works for halfinteger spin in d=1+2 dimensions. In this case one can work with the same 2×2 matrix model, we only have to restrict $SL(2,R)$ to $SL(2,R)\simeq SU(1,1)$ which is conveniently done by omitting the σ_2 Pauli matrix. Choosing again the rest frame reference vector we obtain

$$\begin{aligned} \tilde{L}(p) &= +\sqrt{\frac{p^\mu\sigma_\mu}{m}},\ m>0,\ \sigma_2 \ \mathit{omitted} \\ \tilde{L}(p) &= \frac{1}{\sqrt{p_0+p_3}} \begin{pmatrix} p_0+p_3 & p_1 \\ 0 & 1 \end{pmatrix}, m=0 \end{aligned} \tag{37}$$

with the little group G_l being the abelian rotation or the abelian "translation" group respectively.

$$\begin{aligned} gp_Rg^* &= p_R \\ G_l: g &= \begin{pmatrix} \cos\frac{1}{2}\Omega & \sin\frac{1}{2}\Omega \\ \sin\frac{1}{2}\Omega & \cos\frac{1}{2}\Omega \end{pmatrix}, m>0 \\ G_l: g &= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, m=0 \end{aligned} \tag{38}$$

In order to preserve the analogy in the representations, we take halfinteger spin representations in the first case and trivial representation of the little group in the massless case. Whereas the massless case has a modular wedge structure like the scalar case, the modular structure of the (m,s) case is solved by a u-intertwiner as in the previous d=1+3 case. We have and will continue to refer to these representations with finite (half)integer finite spin as "standard". Their modular localization spaces $H_R(\mathcal{O})$ can be described in terms of classical $\mathcal{O}$-supported test functions. The remaining cases, here called "exceptional", will be treated in the next subsection. They include the d=1+2 "anyonic" spin of massive particles as well as massless cases with faithful representations of the little group in any spacetime dimension $d \geq 1+2$. For $d \geq 1+3$ they are identical to the famous Wigner spin towers where infinitely many spins (like in a dynamical string) are combined in one irreducible representation.

We will see that for these exceptional representations the best possible modular localization is noncompact and generally not susceptible to a classical description in terms of support properties of functions. This preempts the more noncommutative properties of the associated QFTs which are outside of Lagrangian quantization.

2.2 Exceptional cases: anyons and infinite "spin towers"

The special role of d=1+2 spacetime dimensions for the existence of braid group statistics is due to the fact that the universal covering is infinite sheeted and not two-fold as considered in the previous section. The fastest way to obtain a parametrization of the latter is to use the Bargmann [25] parametrization

$$\{(\gamma,\omega)\,|\,\gamma\in\mathbb{C}, |\gamma|<1,\, \omega\in\mathbb{R}\} \tag{39}$$

for the two-fold matrix covering

$$\frac{1}{\sqrt{1-\gamma\bar{\gamma}}}\begin{pmatrix} e^{i\frac{\omega}{2}} & \gamma e^{i\frac{\omega}{2}} \\ \bar{\gamma}e^{-i\frac{\omega}{2}} & e^{-i\frac{\omega}{2}} \end{pmatrix} \tag{40}$$

It is then easy to abstract the multiplication law for the universal covering from this matrix model

$$\begin{aligned} (\gamma_2,\omega_2)\,(\gamma_1,\omega_1) &= (\gamma_3,\omega_3) \\ \gamma_3 &= \frac{\left(\gamma_1+\gamma_2 e^{-i\frac{\omega}{2}_1}\right)}{\left(1+\gamma_2\bar{\gamma}_1 e^{-\frac{\omega}{2}_1}\right)} \\ e^{i\frac{\omega_3}{2}} &= e^{i\frac{\omega_1+\omega_2}{2}}\left(\frac{1+\gamma_2\bar{\gamma}_1 e^{-i\frac{\omega}{2}_1}}{1+\gamma_2\gamma_1 e^{i\frac{\omega}{2}_1}}\right)^{\frac{1}{2}} \end{aligned} \tag{41}$$

From these composition laws one may obtain the irreducible transformation law of a (m,s)Wigner wave functions in terms of a one-component representation involving a Wigner phase $\varphi((\gamma,\omega),p)$.

But there are some quite interesting and physically potentially important positive energy representations for which the above covariantization does not work and the $H_R(\mathcal{O})$ do not have such a geometric description i.e. the modular localization is more "quantum" than geometric. These exceptional representations include $d=1+2$ spin$\neq$halfinteger anyons and the still somewhat mysterious $d\geq 1+3$ massless "infinite spin-tower" (called "continuous spin" by Wigner, unfortunately a somewhat misleading name). These are the cases which also resist Lagrangian quantization attempts. However the modular localization method reveal for the first time that those representations do not allow a compact (with pointlike as limiting case) localization in fact these cases are only consistent with a noncompact modular localization which extends to infinity. The associated multiparticle spaces do not have the structure of a Fock space and the localized operators describing creation and annihilation are too noncommutative for a Lagrangian quantization interpretation.

Before we look at those special cases let us note that the localization in wedges and in certain special intersection of two wedges is a general property of all positive energy representations of $\mathcal{P}_+$. The above proof of standardness of the $\mathfrak{s}$ operator only uses general properties of the boost and the r reflection which are evidently true in each positive energy representation of the extended Poincaré group $\widetilde{\mathcal{P}}_+$. A bit more tricky is the nontriviality of the following intersected spaces

Theorem 1 *(Guido and Longo [29]) Let W_1 and W be orthogonal wedges (in the sense of orthogonality of their spacelike edges) and define $W_2 = \Lambda_W(-2\pi t)W_1$. Then $H(W_1 \cap W_2) \equiv H_R(W_1 \cap W_2) + iH_R(W_1 \cap W_2)$ is dense in the positive energy representation space $\mathcal{P}_+$.*

The size of the intersection decreases with increasing t. It is conic with apex at the origin, but it does not look like a spacelike cone since it contains lightlike rays (for t$\rightarrow \infty$ its core is a lightlike string).

Proof. From the assumptions one obtains a geometric expression for $\mathfrak{s}_2\mathfrak{s}_1$

$$\mathfrak{s}_2\mathfrak{s}_1 = \Delta_W^{it}\Delta_{W_1}^{-\frac{1}{2}}\Delta_W^{it}\Delta_{W_1}^{\frac{1}{2}}$$

where we used the orthogonality assumption via $\jmath_{W_1}\Delta_W^{it}\ \jmath_{W_1} = \Delta_W^{-it}$. The claimed density is equivalent to the denseness of the subspace:

$$\{\psi|\ \mathfrak{s}_2\mathfrak{s}_1\psi = \psi\} \Leftrightarrow \left\{\psi|\ \Delta_{W_1}^{-\frac{1}{2}}\Delta_W^{it}\Delta_{W_1}^{\frac{1}{2}}\psi = \Delta_W^{-it}\psi\right\}$$

but according to a theorem in [29] this is a consequence of the denseness of the domain of $\Delta_{W_1}^{-\frac{1}{2}}\Delta_W^{it}\Delta_{W_1}^{\frac{1}{2}}$ which holds for every unitary representation of SL(2,R) which, as easily shown, is the group generated by the two orthogonal wedges. ■

Before this theorem will be applied to the localization of the exceptional Wigner representation it is instructive to recall the argument for the lack of compact localization in these cases.

Any localization beyond those of group theoretical origin requires the construction of at least partial intertwiners. Before we comment on this let us first show that in the cases of d=1+2 anyonic and d=1+3 infinite spin a compact localization is impossible (which also shows that there are no intertwiners in the previous sense). The typical causally closed simply connected compact region has the form of a double cone i.e. the intersection of the upper light cone with the lower one. Since in terms of wedges one needs infinitely many intersections, we will prove the even the larger region of the intersection of two wedges (which is infinite in transverse direction) has a trivial H_R.

In order to compute the action of $\mathfrak{s}$ we use the Wigner cocycle (25) for the t-x boost Λ_{W_0}

$$e^{is\Omega(\Lambda_{W_0},,p)} = \left(\frac{1-\gamma(p)\gamma_t + (\gamma_t - \gamma(p))\,\overline{\gamma(\Lambda_{W_0}(-t)p)}}{c.c.}\right)^s \tag{42}$$

$$= u(p)u(\Lambda_{W_0}(-t)p),\ u(p) \equiv (\frac{p_0 - p_1 + m + ip_2}{p_0 - p_1 + m - ip_2})^s$$

This formula results by specialization from the following formula for the action of the L-group on one-component massive Wigner wave functions [37][31]

$$(\mathfrak{u}\psi)(p,s) = e^{is\Omega(\tilde{R}(\Lambda,p))}\psi(\Lambda^{-1}p)$$

$$e^{is\Omega(\Lambda(\omega,\gamma),p)} = e^{is\frac{\omega}{2}}\left(\frac{1-\gamma(p)\bar{\gamma}e^{-i\frac{\omega}{2}} + (\gamma - \gamma(p)\bar{\gamma}e^{-i\frac{\omega}{2}})\bar{\gamma}(\Lambda(\gamma,\omega)^{-1}p))}{c.c.}\right)^s$$

and a similar phase factor for the massless case with a faithful little group representation.

In case of the d=1+3 massless spin-tower representation this is more tricky. One finds

$$(\mathfrak{u}(\Lambda, a)\psi)(p) = e^{iap} V_{\Xi,\pm}(\tilde{R}(\Lambda, p))\psi(\Lambda^{-1}p) \tag{43}$$

$$(V_{\Xi,\pm}(\Lambda_{z,\varphi})f)(\theta) = \begin{cases} \{\exp i(\Xi\,|z|\cos(\arg z - \vartheta))\}\, f(\vartheta - \varphi) \\ \left\{\exp i(\Xi\,|z|\cos(\arg z - \vartheta) + \frac{1}{2}\varphi)\right\} f(\vartheta - \varphi) \end{cases}$$

with the + sign corresponding to an integer valued spin tower. In this case the infinite component wave function $\psi(p)$ is a square integrable map from the momentum space mass shell to functions with values in the L_2 space on the circle (in which the noncompact $\tilde{E}(2)$ group is irreducibly represented by the last formula). Ξ is an invariant (Euclidean "mass") of the $\tilde{E}(2)$ representation. Scaling the Ξ to one and introducing a "spin basis" (discrete Fourier-basis) $e^{in\varphi}$, the $V_{\Xi,\pm}(\Lambda_\varphi)$ becomes diagonal and the translational part $V_{\Xi,\pm}(\Lambda_z)$ can be written in terms of Bessel functions

$$V_{\Xi,\pm}(\Lambda_z)_{n,m} = \left(\frac{z}{|z|}\right)^{n-m} J_{n-m}(\Xi\,|z|) \tag{44}$$

From this one can study the analyticity behavior needed for the modular localization.

The following theorem may is easily established

Theorem 2 *The d=1+2 representations with s≠halfinteger and the d=1+3 Wigner spin tower representations do not allow a compact double cone localization.*

For the spin tower this was already suggested by an ancient No-Go theorem of Yngvason [30] who showed that there is an incompatibility with the Wightman setting. We will prove in fact the slightly stronger statement that the space $H_R(W \cap W_a^{'})$ which describes the intersection of a wedge with its translated opposite (which has still a noncompact transversal extension) is trivial. This implies a fortiori the triviality of compact double cone intersections. The common origin of the weaker localization properties for the exceptional positive energy representations is the fact that the analytical continuation of the wave function to the opposite boundary of the strip (which combines together with the action of the charge-conjugating geometric involution to a would be $\mathfrak{s}$) has in addition a matrix part (a phase factor for d=1+2) which has to be cancelled by a compensating modification of the involution part

$$\mathfrak{j} = \mathfrak{t}\mathfrak{j}_{geo} \tag{45}$$

The $\mathfrak{t}$, which in the case of the spin-tower is a complicated operator in the representation space of the little group, is the preempted field theoretic twist operator T whose presence shows up in commutation relations of spacelike (noncompactly) localized operators (braid group statistics in case of d=1+2).

According to the second last theorem the localization in the noncompact intersection of two wedges in a selected relative position (where the second one results from applying an "orthogonal" boost to the first) is always possible for all positive energy representations in all spacetime dimensions. But only in d=1+2 this amounts to a spacelike cone localization (with a semiinfinite spacelike string as a core). In that case one knows that plektonic situations do not allow for a better localization. However there is a problem with the application of that theorem to anyons since it refers to the representation of the Poincare group in $d \geq 3$ spacetime but not to its covering $\tilde{\mathcal{P}}_+$ in $d = 3$ which would be needed for the case of anyons. Fortunately Mund has found a direct construction of spacelike cone C localized subspaces

$H_R(C)$ in terms of a partial intertwiner $u(p)$ and subspace of of doubled test functions Φ with supports in spacelike cones. If one starts from the standard x-t wedge and wants to localize in cones which contain the negative y-axis then Mund's localization formula and his partial u (to be distinguished from the previous $\mathfrak{u}$) are

$$u(p)E_m\Phi,\ u(p) = (\frac{p^0 - p^1}{m})^s(\frac{p_0 - p_1 + m + ip_2}{p_0 - p_1 + m - ip_2})^s \tag{46}$$

For spacelike cones along other axis the form of the partial intertwiner changes. Running through all C-localized test functions the formula describes a dense set of spacelike cone-localized Wigner wave function only for those spacelike cones which contain the negative y-axis after apex(C) has been shifted to the origin (which includes the standard x-t wedge as a limiting case). He then shows an interesting "spreading" mechanism namely that if one chooses a better localized function with compacr support in that region, the effect of the partial intertwiner " is to radially extend the support to spacelike infinity. The anyonic spin Wigner representation can be encoded into many infinite dimensional covariant representations [37] (also appendix), but this does not improve the localization since infinite dimensional covariant transformation matrices, unlike finite dimensional ones, are not entire functions of the group parameters.

For d=1+3 the intersection region has at its core a 2-dimensional spacelike half-plane. There is good reason to believe that this is really the optimally possible localization for the spin-tower representation. The argument is based on converting this representation into the factorizing form uE_mf where u is the infinite dimensional intertwiner from the covariant representation (appendix) to the Wigner representation. The best analytic behavior which the unitary representation theory of the L-group (necessarily infinite dimensional) can contribute to modular localization seems to be that of the above Guido-Longo theorem. Whereas for the standard representations the support of the classical test function multiplets determine the best localization region (because the finite dimensional representations of the Lorentz group are entire analytic functions), the exceptional representations spread any test function localization which tries to go beyond those which pass through the intertwiner. This goes hand in hand with a worsening of the spacelike commutativity properties in the associated operator algebras. Therefore in the case in which the modular localization cannot be encoded into the support property of a test function multiplet, we often use the word "quantum localization". These are the cases which cannot not be described as a quantized classical structure or in terms of Euclidean functional integrals.

As will be shown in the next section the QFT associated with such particles do not allow sub-wedge PFGs i.e. better than wedge-localized operators which applied to the vacuum create one-particle states free of vacuum polarization.

Whereas in standard Boson/Fermion systems (halfinteger spin representations) the vacuum polarization is caused by the interaction (this can be used to define the intrinsic meaning of interaction for such systems), the sub-wedge vacuum polarization phenomenon associated with the QFT of the exceptional Wigner representations is of a more kinematical kind; it occurs in those other cases already without interaction; the polarization clouds are simply there to sustain e.g. the anyonic spin&statistics connection.

3 From Wigner representations to the associated local quantum physics

In the following we will show that such net of operator algebras of free particles with halfinteger spin/helicity can be directly constructed from the net of modular localized subspaces in standard Wigner representations. For integral spin s one defines with the help of the Weyl functor $Weyl(\cdot)$ the local von Neumann algebras [17][18] generated from the Weyl operators as

$$\mathcal{A}(W) := alg\,\{Weyl(f)|f \in H_R(W)\} \tag{47}$$

a process which is sometimes misleadingly called "second quantization". These Weyl generators have the following formal appearance in terms of Wigner (momentum space) creation and annihilation operators and modular localized wave functions

$$H_R(W) \stackrel{\Gamma}{\rightarrow} Weyl: \; f \rightarrow Weyl(f) = e^{iA(f)} \tag{48}$$
$$A(f) = \sum_{s_3=-s}^{s} \int (a^*(p,s_3) f_{s_3}(p) + b^*(p,s_3) f_{s_3}^*(-p) + h.c.) \frac{d^3p}{2\omega}$$

It is helpful to interprete the operator $A(f)$ as an inner product

$$A(f) = \int \begin{pmatrix} a^*(p) & b^*(p) \end{pmatrix} \begin{pmatrix} f(p) \\ f^*(-p) \end{pmatrix} \frac{d^3p}{2\omega} + h.c \tag{49}$$

of an operator bra with a ket vector of a $2{\times}(2s{+}1)$ eigenfunction of $\mathfrak{s}$ representing a vector in $H_R(W)$. The formula refers only to objects in the Wigner theory; covariant fields or wave functions do not enter here. Unlike those covariant objects, the Weyl functor is uniquely related to the (m,s) Wigner representation. The special hermitian combination entering the exponent of the Weyl functor is sometimes called the I. Segal operator [27].

The local net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ may be obtained in two ways, either one first constructs the spaces $H_R(\mathcal{O})$ via (17) and then applies the Weyl functor, or one first constructs the net of wedge algebras (47) and then intersects the algebras according to

$$\mathcal{A}(\mathcal{O}) = \bigcap_{W\supset\mathcal{O}} A(W) \tag{50}$$

The proof of the net properties follows from the well-known theorem that the Weyl functor relates the orthocomplemented lattice of real subspaces of H_{Wig} (with the complement H_R' of H_R being defined in the symplectic sense of the imaginary part of the inner product in H_{Wig}) to von Neumann subalgebras $\mathcal{A}(H_R) \subset \mathcal{B}(H_{Fock})$

This functorial mapping Γ also maps the above pre-modular operators into those of the Tomita-Takesaki modular theory

$$J, \Delta, S = \Gamma(\,\mathfrak{j}, \delta, \mathfrak{s}) \tag{51}$$

Whereas the pre-modular operators of the spatial theory (denoted by small letters) act on the Wigner space, the modular operators J, Δ have an Ad action ($AdUA \equiv UAU^*$) on von Neumann algebras in

Fock space which makes them objects of the Tomita-Takesaki modular theory

$$\begin{aligned} SA\Omega &= A^*\Omega,\, S = J\Delta^{\frac{1}{2}} \\ Ad\Delta^{it}\mathcal{A} &= \mathcal{A} \\ AdJ\mathcal{A} &= \mathcal{A}' \end{aligned} \tag{52}$$

The operator S is that of Tomita i.e. the unbounded densely defined normal operator which maps the dense set $\{A\Omega |\, A \in \mathcal{A}(W)\}$ via $A\Omega \to A^*\Omega$ into itself and gives J and $\Delta^{\frac{1}{2}}$ by polar decomposition. The nontrivial miraculous properties of this decomposition are the existence of an automorphism $\sigma_\omega(t) = Ad\Delta^{it}$ which propagates operators within $\mathcal{A}$ and only depends on the state ω (and not on the implementing vector Ω) and a that of an antiunitary involution J which maps $\mathcal{A}$ onto its commutant $\mathcal{A}'$. The theorem of Tomita assures that these objects exist in general if Ω is a cyclic and separating vector with respect to $\mathcal{A}$.

An important thermal aspect of the Tomita-Takesaki modular theory is the validity of the Kubo-Martin-Schwinger (KMS) boundary condition [1]

$$\omega(\sigma_{t-i}(A)B) = \omega(B\sigma_t(A)),\;\; A, B \in \mathcal{A} \tag{53}$$

i.e. the existence of an analytic function $F(z) \equiv \omega(\sigma_z(A)B)$ holomorphic in the strip $-1 < Imz < 0$ and continuous on the boundary with $F(t-i) = \omega(B\sigma_t(A))$ or briefly (53). The fact that the modular theory applied to the wedge algebra has a geometric aspect (with J equal to the TCP operator times a spatial rotation and $\Delta^{it} = U(\Lambda_W(2\pi t)))$ is not limited to the interaction-free theory [1]. These formulas are identical to the standard thermal KMS property of a temperature state ω in the thermodynamic limit if one formally sets the inverse temperature $\beta = \frac{1}{kT}$ equal to $\beta = -1.$ This thermal aspect is related to the Unruh-Hawking effect of quantum matter enclosed behind event/causal horizons.

For halfinteger spin, the Weyl functor has to be replaced by the Clifford functor R. In the previous section we already noted that there exists a mismatch between the geometric and the spatial complement which led to the incorporation of an additional phase factor i into the definition of $\mathfrak{j}$.

A Clifford algebra is associated to a real Hilbert space H_R with generators

$$\begin{aligned} R &: \mathcal{S}(\mathbb{R}^4) \to B(H_R) \\ (f, g)_R &= \mathrm{Re}\,(f, g) \end{aligned} \tag{54}$$

where the real inner product is written as the real part of a complex one. One sets

$$R^2(f) = (f, f)_R \mathbb{I} \tag{55}$$

or

$$\{R(f), R(g)\} = 2(f, g)_R \mathbb{I} \tag{56}$$

where $\mathcal{S}(\mathbb{R}^4)$ is the Schwartz space of test functions over $\mathbb{R}^4$ and $B(H_R)$ is the space of bounded operators over H_R.

These $R(f)$'s generates $Cliff(H_R)$ as polynomials of R's. The norm is uniquely fixed by the algebraic relation, e.g.

$$||R(f)||^2 = ||R(f)^* R(f)|| - ||R^2(f)|| = ||f||_R \tag{57}$$

and similarly for all polynomials, i.e., on all $Cliff(H_R)$. The norm closure of the Clifford algebra is sometimes called $CAR(H_R)$ (canonical anti-commutation) C*-algebra. It is unique (always up to C*-isomorphisms) and has no ideals. This Clifford map may be used as the analog of the Weyl functor in the case of halfinteger *spin* $s = \frac{n}{2}, n$ odd.

It turns out to be more useful to work with a alternative version of CAR which is due to Araki: the selfdual CAR-algebra. In that description, the reality condition is implemented via a antiunitary involution Γ inside the larger complex Hilbert space H. Now

$$\begin{aligned} f &\longrightarrow B(f) \\ B(f)^* &= B(\Gamma f) \\ \{B^*(f), B(g)\} &= (f,g)\mathbb{I} \end{aligned} \tag{58}$$

is a complex linear map of H into generators a normed *-algebra whose closure is by definition the C*-algebra $CAR(K,\Gamma)$. The previous Clifford functor results from the selfadjoint objects $B(\Gamma f) = B(f)$ or $\Gamma f = f$. In physical terms Γ is the charge conjugation operation C which enters the definition of the $\mathfrak{s}$-operator. The functor maps this spatial modular object into an operator of the Clifford algebra; the analog of (49) is

$$f \in H_R(W) \rightarrow R(f) = \Psi \cdot f + h.c. \tag{59}$$

where, as explained in section 2.2, the Wigner wave function $f \in H_R(W)$ interpreted as a $4 \times (2s+1)$ component column vector and Ψ is a bra vector of Wigner creation and annihilation operators. As a consequence of the presence of a twist factor in the spatial involution $j = tj_{geo}$ one obtains a twist operator in the algebraic involution J

$$\begin{aligned} S &= J\Delta^{\frac{1}{2}},\ J = TJ_{geo} \\ T &= \frac{1-iU(2\pi)}{1-i} = \begin{cases} 1 \ on\ even \\ i\ on\ odd \end{cases} \\ SA\Omega &= A^*\Omega,\ A \in \mathcal{A}(W) = a\lg\{B(f)|\, f \in H_R(W)\} \end{aligned} \tag{60}$$

The presence of the twist operator (which is one on the even and i on the odd subspaces of H_{Fock}) accounts for the difference between the von Neumann commutant $\mathcal{A}(W)'$ and the geometric opposite $\mathcal{A}(W')$. The bosonic CCR (Weyl) and the fermionic CAR (Clifford) local operator algebras are the only ones which permit a functorial interpretation in terms of a "quantization" of classical function algebras. In the next section we will take notice of the fact that they are also the only QFTs which possess sub-wedge-localized PFGs.

In the case of d=1+2 anyonic spin representations the presence of a plektonic twist has the more radical consequences. Whereas the fermionic twist is still compatible with the existence of PFGs and free fields in Fock space, the twist associated with genuine braid group statistics causes the presence of vacuum polarization for any sub-wedge localization region. The same consequences hold for the spin tower representations.

Our special case at hand, in which the algebras and the modular objects are constructed functorially from the Wigner theory, suggest that the modular structure for wedge algebras may always have a geometrical significance associated with a fundamental physical interpretation in any QFT. This is indeed true, and within the Wightman framework this was established by Bisognano and Wichmann [1]. In the general case of an interacting theory in d=1+3 with compact localization (which according to the DHR theory is necessarily a theory of interacting Bosons/Fermions) the substitute for a missing functor between a spatial and an algebraic version of modular theory is the modular map between a real subspace of the full Hilbert space H and a local subalgebra of algebra of all operators $B(H)$. In a theory with asymptotic completeness i.e. with a Fock space incoming (outgoing) particle structure $H = H_{Fock}$ the scattering operator S_{scat} turns out to play the role of a relative modular invariant between the wedge algebra of the free incoming operators and that of the genuine interacting situation

$$J = J_0 S_{scat} \tag{61}$$

$$S = S_0 S_{scat} \tag{62}$$

This relation follows directly by rewriting the TCP transformation of the S-matrix and the use of the relation of J with the TCP operator. The computation of the real subspaces $H_R(W) \in H_{Fock}$ requires diagonalization of the S-matrix. The difficult step about which presently nothing is known is the passing from these subspaces to wedge-subalgebras whose selfadjoint part applied to the vacuum generate these subspaces. Although it is encouraging that the solution of the inverse problem $S_{scat} \rightarrow \{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ is unique [11], a general formalism which takes care of the existence part of the problem is not known apart from some special but very interesting cases which will be presented in the next section. Connes has developed a theory involving detailed properties of the natural modular cones $\mathcal{P}_{\mathcal{A}(W),\Omega}$ which are affiliated with a single standard pair $(\mathcal{A}(W), \Omega)$ (the net structure is not used) but it is not clear how to relate his facial conditions on these cones to properties of local quantum physics. As a matter of fact even in the case of standard Wigner representations it is not clear how one could obtain the modular algebraic structure if one would be limited to the Connes method [28] without the functorial relation. For these reasons the modular based approach which tries to use the twist/S-matrix factor in $J = J_0 T$ respectively $J = J_0 S_{scat}$ for the determination of the algebraic structure of $\mathcal{A}(W)$ and subsequently computes the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ by forming intersections is presently limited to theories which permit only vitual but no real particle creation. Besides the exeptional Wigner representation (anyons, spin towers) which lead to a twist and changed spacelike commutation relations, the only standard (bosonic, fermionic) interacting theories are the $S_{scat} = S_{el}$ models of the d=1+1 bootstrap-formfactor setting (factorizing models).

For those readers who are familiar with Weinberg's method of passing from Wigner representation to covariant pointlike free fields, it may be helpful to add a remark which shows the connection to the modular approach. For writing covariant free fields in the (m,s) Fock space

$$\psi^{[A,\dot{B}]}(x) = \frac{1}{(2\pi)^{3/2}} \int \{e^{-ipx} \sum_{s_3} u(p_1,s_3)a(p_1,s_3) + \qquad (63)$$

$$+ e^{ipx} \sum_{s_s} v(p_1,s_3)b^*(p_1,s_3)\} \frac{d^3p}{2\omega}$$

where $a^{\#}, b^{\#}$ are creation/annihilation opertors of Wigner (m,s) particles and $\psi^{[A,\dot{B}]}$ are covariant dotted/undotted fields in the SL(2,C) spinor formalism, it is only necessary to find intertwiners

$$u(p)D^{(s)}(\tilde{R}(\tilde{\Lambda},p)) = D^{[A,\dot{B}]}(\tilde{\Lambda})u(\tilde{\Lambda}^{-1}p) \qquad (64)$$

between the Wigner $D^{(s)}(\tilde{R}(\tilde{\Lambda},p))$ and the covariant $D^{[A,\dot{B}]}(\tilde{\Lambda})$ and these exist for all $A, \dot{B}$ which relative to the given s obey

$$\mid A - \dot{B} \mid \leq s \leq A + \dot{B} \qquad (65)$$

For each of these infinitely many values $(A, \dot{B})$ there exists a rectangular

$(2A+1)(2\dot{B}+1) \times (2s+1)$ intertwining matrix $u(p)$. Its explicit construction using Clebsch-Gordan methods can be found in Weinberg's book [15]. Analogously there exist antiparticle (opposite charge) intertwiners $v(p)$: $D^{(s)*}(R(\Lambda,p) \longrightarrow D^{[A,\dot{B}]}(\Lambda)$. All of these mathematically different fields in the same Fock space describe the same physical reality; they are just the linear part of a huge local equivalence class and they do not exhaust the full "Borchers class" which consists of all Wick-ordered polynomials of the $\psi^{[A,\dot{B}]}$. They generate the same net of local operator algebras and in turn furnish the singular coordinatizations. Free fields for which the full content of formula (63) can be described by the totality of all solutions of an Euler-Lagrange equation exist for each (m,s) but are very rare (example Rarita-Schwinger for s=$\frac{3}{2}$). It is a misconception that they are needed for physical reason. The causal perturbation theory can be done in any of those field coordinates and that one needs Euler-Lagrange fields in the setting of Euclidean functional integrals is an indication that differential geometric requirements and quantum physical ones do not always go into the same direction.

On the other hand our modular method for the construction of localized spaces and algebras use only the minimal intertwiners which are described by square $(2s+1) \times (2s+1)$ matrices. Without their use there would be no purely analytic characterization of the domain of the modular Tomita S-operator.

4 Vacuum polarization and breakdown of functorial relations

The functorial relation of the previous section between Wigner subspaces and operator algebras are strictly limited to the standard halfinteger spin representations for which generating pointlike free fields exist. The noncompactly localizable exceptional Wigner representations (anyonic spin, faithful spin-tower representations of the massless little group) as well as interacting theories involving standard (halfinteger spin/helicity) particles do not permit a direct functorial relations between wave function spaces and operator algebras.

In order to understand the physical mechanism which prevents a functorial relation it is instructive to look directly to the operators algebras. Given an operator algebra $\mathcal{A}(\mathcal{O})$ localized in a causally closed region $\mathcal{O}$ with a nontrivial causal complement $\mathcal{O}'$ (so that $(\mathcal{A}(\mathcal{O}),\Omega)$ is standard pair) we may ask whether this algebra admits a "polarization-free-generator" (PFG) namely an affiliated possibly unbounded closed operator G such that Ω is in the domain of G, G^* and $G\Omega$ and $G^*\Omega$ are vectors in E_mH with E_m projector on the one-particle space.

It turns out that if one admits very crude localizations as that in wedges then one can reconcile the standardness of the pair $(\mathcal{A}(W),\Omega)$ (i.e. physically the unique $A\Omega \leftrightarrow A \in \mathcal{A}(W)$ relationship) with the absense of polarization clouds caused by localization. For convenience of the reader we recall the abstract theorem from modular theory whose adaptation to the local quantum physical situation at hand will supply the existence of wedge-affiliated PFGs.

An interesting situation emerges if these operators which always generate a dense one-particle subspace also generate an algebra of unbounded operators which is affiliated to a corresponding von Neumann algebra $\mathcal{A}(\mathcal{O})$. For causally complete sub-wedge regions $\mathcal{O}$ such a situation inevitably leads to interaction-free theories i.e. the local algebras generated by ordinary free fields are the only $\mathcal{A}(\mathcal{O})$-affiliated PFGs. Such a situation is achieved by domain restrictions on the (generally unbounded) PFGs. Without any further domain restriction on these (generally unbounded) operators it would be difficult to imagine a constructive use of PFGs.

Before studying PFGs it is helpful to remind the reader of the following theorem of general modular theory.

Theorem 3 *Let S be the modular operator of a general standard pair $(\mathcal{A},\Omega)$ and let Φ be a vector in the domain of S. There exists a unique closed operator F affiliated with F (notation $F\eta\mathcal{A}$) which together with F^* has the reference state Ω in its domain and satisfies*

$$F\Omega = \Phi,\ F^*\Omega = S\Phi \tag{66}$$

A proof of this and the following theorem can be found in [32].

For the special field theoretic case $(\mathcal{A}(W),\Omega)$, the domain of S which agrees with that of $\Delta^{\frac{1}{2}} = e^{\pi K}, K =$ boost generator has evidently a dense intersection $\mathcal{D}^{(1)} = H^{(1)} \cap \mathcal{D}_{\Delta^{\frac{1}{2}}}$ with the one-particle space $H^{(1)} = E_mH$. Hence the operator F for $\Phi^{(1)} \in \mathcal{D}^{(1)}$ is a PFG G as previously defined. However the abstract theorem contains no information on whether the domain properties admit a repeated use of PFGs similar to smeared fields in the Wightman setting, nor does it provide any clew about the position of a *domG* relative to scattering states. Without such a physically motivated input, wedge-supported PFGs would not be useful. An interesting situation is encountered if one requires the G to be tempered. Intuitively speaking this means that $G(x) = U(x)GU(x)^*$ has a Fourier transform as needed if one wants to use PFGs in scattering theory. If one in addition assumes that the wedge algebras to which the PFGs are affiliated are of the standard Bose/Fermi type i.e. $\mathcal{A}(W') = \mathcal{A}(W)'$ or the twisted Fermi commutant $\mathcal{A}(W)^{tw}$, one finds

Theorem 4 *PFGs for the wedge localization always region exist, but the assumption that they are tempered leads to a purely elastic scattering matrix $S_{scat} = S_{el}$, whereas in d>1+1 is only consistent with $S_{scat} = 1$.*

Together with the recently obtained statement about the uniqueness of the inverse problem in the modular setting of AQFT [11] one finally arrives at the interaction-free nature in the technical sense that the PFGs can be described in terms of free Bose/Fermi fields.

The nonexistence of PFGs in interacting theories for causally completed localization regions smaller than wedges (i.e. intersections of two or more wedges) can be proven directly i.e. without invoking scattering theory

Theorem 5 *PFGs localized in smaller than wedge regions are (smeared) free fields. The presence of interactions requires the presence of vacuum polarization in all state vectors created by applying operators affiliated with causally closed smaller wedge regions.*

The proof of this theorem is an extension of the ancient theorem [26] that pointlike covariant fields which permit a frequency decomposition (with the negative frequency part annihilating the vacuum) and commute/anticommute for spacelike distances are necessarily free fields in the standard sense. The frequency decomposition structure follows from the PFG assumption and the fact that in a given wedge one can find PFGs whose localization is spacelike disjoint is sufficient for the analytic part of the argument to still go through, i.e. the pointlike nature in the old proof is not necessary to show that the (anti)commutator of two spacelike disjoint localized PFGs is a c-number (which only deviates from the Pauli-Jordan commutator by its lack of covariance). The most interesting aspect of this theorem is the inexorable relation between interactions and the presence of vacuum polarization which for the first time leads to a completely intrinsic definition of interactions which is not based on the use of Lagrangians and particular field coordinates. This poses the interesting question how the shape of localization region (e.g. size of double cone) and the type of interaction is related with the form of the vacuum polarization clouds which necessarily accompany a one-particle state. We will have some comments in the next section.

As Mund has recently shown, this theorem has an interesting extension to d=1+2 QFT with braid group (anyon) statistics.

Theorem 6 *([33]) There are no PFGs affiliated to field algebras localized in spacelike cones with anyonic commutation relations i.e. sub-wedge localized fields obeying braid group commutation relations applied to the vacuum are always accompanied by vacuum polarization clouds. Even in the absence of any genuine interactions this vacuum polarization is necessary to sustain the braid group statistics and maintain the spin-statistics relation.*

This poses the interesting question whether quantum mechanics is compatible with a nonrelativistic limit of braid group statistics. The nonexistence of vacuum polarization-free locally (sub-wedge) generated one particle states suggests that as long as one maintains the spin-statistics connection throughout the nonrelativistic limit procedure, the result will preserve the vacuum polarization contributions and hence one will end up with nonrelativistic field theory instead of quantum mechanics[9].

Using the concept of PFGs one can also formulate this limitation of quantum mechanics in a more provocative way by saying that (using the generally accepted fact that QFT is more fundamental than QM) QM owes its physical relevance to the fact that the permutation group (Boson/Fermion) statistics permits sub-wedge localized PFGs (free fields which create one particle states without vacuum polarization admixture) whereas the more general braidgroup statistics does not.

Another problem which even in the Wigner setting of noninteracting particles is interesting and has not yet been fully understood is the pre-modular theory for disconnected or topologically nontrivial

[9]The Leinaas-Myrheim geometrical arguments [34] do not take into account the true spin-statistics connection.

regions e.g. in the simplest case for disjoint double intervals of the massless $s = \frac{1}{2}$ chiral model on the circle. Such situations give rise to nongeometric (fuzzy) "quantum symmetries" of purely modular origin without a classical counterpart.

5 Construction of models via modular localization

Since up to date more work had been done on the modular construction of d=1+1 factorizing models, we will first illustrate our strategy in that case and then make some comments of how we expect our approach to work in the case of higher dimensional d=1+2 anyons and $d \geq 1+3$ spin towers.

The construction consists basically of two steps, first one classifies the possible algebraic structures of tempered wedge-localized PFGs and then one computes the vacuum polarization clouds of the operators belonging to the double cone intersections.

Let us confine ourself to the simplest model which we may associate with a massive selfconjugate scalar particle. If there would be no interactions the appropriate theorem of the previous section would only leave the free field which is a PFG for any localization

$$\begin{aligned} A(x) &= \frac{1}{\sqrt{2\pi}} \int \left(e^{-ip(\theta)x} a(\theta) + e^{ip(\theta)x} a^*(\theta) \right) d\theta \\ A(f) &= \int A(x) \hat{f}(x) d^2x = \frac{1}{\sqrt{2\pi}} \int_C a(\theta) f(\theta) d\theta, \ supp \hat{f} \in W \\ p(\theta) &= m(\cosh\theta, \sinh\theta) \end{aligned} \tag{67}$$

where in order to put into evidence that the mass shell only carries one parameter, we have used the rapidity parametrization in which the plane wave factor is an entire function in the complex extension of θ with $p(\theta - i\pi) = -p(\theta)$. The last formula for the smeared field with the localization in the right wedge has been written to introduce a useful notation; the integral extends over the upper and lower conture $C : \theta$ and $\theta - i\pi, -\infty < \theta < \infty$ where the Fourier transform $f(\theta)$ is analytic and integrable in the strip which C encloses as a result of its x-space test function support property. Knowing that tempered PFGs only permit elastic scattering (see previous section), we make the "nonlocal" Ansatz

$$\begin{aligned} G(x) &= \frac{1}{\sqrt{2\pi}} \int \left(e^{-ipx} Z(\theta) + e^{ipx} Z^*(\theta) \right) d\theta \\ G(\tilde{f}) &= \frac{1}{\sqrt{2\pi}} \int_C Z(\theta) f(\theta) d\theta \end{aligned} \tag{68}$$

where the Zs are defined on the incoming n-particle vectors by the following formula for the action of $Z^*(\theta)$ for the rapidity-ordering $\theta_i > \theta > \theta_{i+1}, \ \theta_1 > \theta_2 > ... > \theta_n$

$$\begin{aligned} & Z^*(\theta) a^*(\theta_1)...a^*(\theta_i)...a^*(\theta_n)\Omega = \\ & S(\theta-\theta_1)...S(\theta-\theta_i) a^*(\theta_1)...a^*(\theta_i) a^*(\theta)...a^*(\theta_n)\Omega \\ & + contr.\, from\, bound\, states \end{aligned} \tag{69}$$

In the absence of bound states (which we assume in the following) this amounts to the commutation relations[10]

[10]In the presence of bound states such commutation relations only hold after applying suitable projection operators.

$$Z^*(\theta)Z^*(\theta') = S(\theta - \theta')Z^*(\theta')Z^*(\theta),\, \theta < \theta' \tag{70}$$
$$Z(\theta)Z^*(\theta') = S(\theta' - \theta)Z^*(\theta')Z(\theta) + \delta(\theta - \theta')$$

where the structure functions S must be unitary in order that the Z-algebra be a *-algebra. It is easy to show that the domains of the Zs are identical to free field domains. We still have to show that our "nonlocal" Gs are wedge localized. According to modular theory for this we have to show the validity of the KMS condition. It is very gratifying that the KMS condition for the requirement that the $G(\tilde{f})$ $supp\tilde{f} \subset W$ are affiliated with the algebra $\mathcal{A}(W)$ is equivalent with the crossing property of the S.

Proposition 7 *The PFG's with the above algebraic structure for the Z's are wedge-localized if and only if the structure coefficients $S(\theta)$ in (70) are meromorphic functions which fulfill crossing symmetry in the physical θ-strip i.e. the requirement of wedge localization converts the Z-algebra into a Zamolodchikov-Faddeev algebra.*

Improving the support of the wedge-localized test function in $G(\hat{f})$ by choosing the support of $\hat{f}$ in a double cone well inside the wedge does not improve $locG(\hat{f})$, it is still spread over the entire wedge. This is similar to the spreading property of (46) and certainly very different from the behavior of smeared pointlike fields.

By forming an intersection of two oppositely oriented wedge algebras one can compute the double cone algebra or rather (since the control of operator domains has not yet been accomplished) the spaces of double-cone localized bilinear forms (form factors of would be operators).

The most general operator A in $\mathcal{A}(W)$ is a LSZ-type power series in the Wick-ordered Zs

$$A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_1, ...\theta_n) : Z(\theta_1)...Z(\theta_n) : d\theta_1...d\theta_n \tag{71}$$
$$A \in \mathcal{A}_{bil}(W) \tag{72}$$

with strip-analytic coefficient functions a_n which are related to the matrix elements of A between incoming ket and outgoing bra multiparticle state vectors (formfactors). The integration path C consists of the real axis, associated with annihilation operators and the line $Im\theta = -i\pi$, corresponding to creators. Writing such power series without paying attention to domains of operators means that we are only dealing with these objects (as in the LSZ formalism) as bilinear forms (72) or formfactors whose operator status still has to be settled.

Now we come to the second step of our algebraic construction, the computation of double cone algebras. The space of bilinear forms which have their localization in double cones are characterized by their relative commutance (this formulation has to be changed for Fermions or more general objects) with shifted generators $A^{(a)}(f) \equiv U(a)A(f)U^*(a)$

$$\left[A, A^{(a)}(f)\right] = 0,\, \forall f \ suppf \subset W \tag{73}$$
$$A \subset \mathcal{A}_{bil}(C_a)$$

where the subscript indicates that we are dealing with spaces of bilinear forms (formfactors of would-be operators localized in C_a) and not yet with unbounded operators and their affiliated von Neumann algebras. This relative commutant relation [35] on the level of bilinear forms is nothing but the famous "kinematical pole relations" which relate the even a_n to the residuum of a certain pole in the a_{n+2} meromorphic functions. The structure of these equations is the same as that for the formfactors of pointlike fields; but whereas the latter lead (after splitting off common factors [14] which are independent of the chosen field in the same superselection sector) to polynomial expressions with a hard to control asymptotic behavior, the a_n of the double cone localized bilinear forms are solutions which have better asymptotic behavior controlled by the Paley-Wiener-Schwartz theorem. We will not discuss here the problem of how this improvement can be used in order to convert the bilinear forms into genuine operators. Although we think that this is largely a technical problem which does not require new concepts, the operator control of the second step is of course important in order to convince our constructivist friends that modular methods really do provide a rich family of nontrivial d=1+1 models. We hope to be able to say more in future work.

The extension to the general factorizing d=1+1 models should be obvious. One introduces multi-component Zs with matrix-valued structure functions S. The contour deformation from the original integral to the "crossed" contour which is necessary to establish the KMS conditions in the presence of boundstate poles in the physical θ-strip compensates those pole contributions against the boundstate contributions in the state vector Ansatz (69) [35]. The fact that the structure matrix $S(\theta-\theta')$ is the 2-particle matrix element of the elastic S-matrix of the constructed algebraic net of double cone algebras is not used in this construction. Of the two aspects of an S-matrix in local quantum physics namely the large time LSZ (or Haag-Ruelle) scattering aspect and that of the S-matrix as a relative modular invariant of the wedge algebra we only utilized the latter.

As a side remark we add that the $Z^{\#}$ operators are conceptually somewhere between the free incoming and the interacting Heisenberg operators in the following sense: whereas any particle state in the theory contributes to the structure of the Fock space and has its own incoming creation/annihilation operator, the $Z^{\#}$ operators are (despite the rather rough wedge localization properties of their spacetime related PFGs G) similar to charge-carrying local Heisenberg operators in the sense that all other operators belonging to particles whose charge is obtained by fusing that of Z and Z^* are functions of Z [36]. The particle-field duality which holds for free fields becomes already incalidated by the interacting wedge-localized PFG G before one gets to the double-cone-localized operators.

Let us finally make some qualitative remarks about a possible adaptation of the above two-step processs to the higher dimensional exceptional Wigner cases. Since their are many wedges, one uses a θ-ordering with respect to the standard wedge as in [32]. Then the nongeometrical nature of the twist modification $\mathfrak{t}$ of the spatial $\mathfrak{j}$ operator in the Wigner representation leads to a field-theoretic twist operator T which is the analog of the S_{el} operator in the previous discussion. This T is responsible for the modification similar to (70), but this time with piecewise constant structure constants in the Z-analogs which still refer to the standard wedge (R-operators acting on the tower indices in case of spin towers). With other words the wedge formalism with respect to the standard wedge is like a tensor product formalism i.e. the n-"particle" states are analog to n-fold tensor products in a Fock space. The mismatch between the algebraic commutant and the geometric opposite of the wedge algebra is responsible for a drastic modification of the Bisognano-Wichmann theorem and leads to braid commutation relations between wedge and opposite wedge operators. The next step namely the formation of the intersection is analog to the previous case except that instead of a lightlike translation we now have to take the orthogonal wedge intersection as in section 2.2. The intersection naturally has to be taken with respect to the twisted relative commutant. It is expected to build up a rich vacuum polarization structure for the d=1+2 massive anyons as well as for the spin towers.

The impossibility of a compact localization in the case of the exceptional Wigner representation places them out of reach by Lagrangian quantization methods. The charge-carrying PFG operators corresponding to the wedge-localized subspaces as well as their best localized intersections are more "noncommutative" than those for standard QFT and the worsening of the best possible localization is inexorably interwoven with the increasing spacelike noncommutativity. This kind of noncommutativity should however be kept apart from the noncommutativity of spacetime itself whose consistency with the Wigner representation theory will be briefly mentioned in the subsequent last section.

6 Outlook

In the past the power of Wigner's representation theory has been somewhat underestimated. As a completely intrinsic relativistic quantum theory which stands on its own feet (i.e. it does not depend on any classical quantization parallelism and thus gives quantum theory its deserved dominating position) it was used in order to back up the Lagrangian quantization procedure [15], but thanks to its modular localization structure it is capable to do much more and shed new light also on problems which remained outside Lagrangian quantization and perturbation theory. This includes problems where, contrary to free fields, no PFG operator (one which creates a pure one-particle state without a vacuum polarization admixture) for sub-wedge regions exist, but where wedge-localized algebras still have tempered generators as d=1+1 factorizing models d=1+2 "free" anyons and "free" Wigner spin towers. It should however be mentioned that the braid group statistics particles refered to as anyons associated to d=1+2 continuous spin Wigner representations in this particular way (i.e. by extending the one-particle twist to multiparticle states with abelian phase composition) do not exhaust all possibilities of plektonic statistcs.

Since conformal theories in any dimensions (even beyond chiral theories) are "almost free" (in the sense that the only structure which distinguishes them from free massless theories is the spectrum of anomalous dimension which is related to an algebraic braid-like structure in timelike direction [38]), we believe that they also can be classified and constructed by modular methods.

This leaves the question of how to deal with interacting massive theories which have in addition to vacuum polarization real (on shell) particle creation. For such models PFG generators of wedge algebras are (as a result of their non-temperedness) too singular objects. One either must hope to find different (non-PFG) generators, or use other modular methods [20] related to holographically defined modular inclusions or modular intersections. For example holographic lightfront methods are based on the observation that the full content of a d-dimensional QFT can be encoded into d-1 copies of one abstract chiral theory whose relative placement in the Hilbert space of the d-dimensional theory carries the information. What remains to be done is to characterize the kind of chiral theory and its relative positions in a constructively manageable way.

Another insufficiently understood problem is the physical significance of the infinitely many modular symmetry groups which (beyond the Poincaré or conformal symmetry groups which leave the vacuum invariant) act in a fuzzy way within the localization regions and in their causal complements [39]. An educated guess would be that they are related to the nature of the vacuum polarization clouds which local operators in that region generate from the vacuum.

Finally the present viewpoint of QFT is also very well suited to address a problem which, after lying dorment for a very long time, in recent years returned to the focus of interests, namely the question whether besides the macro-causal relativistic quantum mechanics mentioned in the introduction and the micro-causal local quantum physics there are other relativistic non-micro causal quantum theories[11]. In particular one would be interested in relativistic theories which permit the physical notion of time-dependent scattering (i.e. obey cluster factorization properties) and which unlike the the relativistic mechanics preserve some of the vacuum polarization properties, especially those which are necessary to keep the TCP theorem intact (to which the existence of antiparticles is inexorably tied) and address the question of localization (in string theory the issue of localization remains a mystery).

All post-renormalization attempts to obtain ultraviolet improved theories by allowing nonlocal interactions, starting from the Kristensen-Moeller-Bloch [41][43] replacement of pointlike Lagrangian interactions by formfactors and the Lee-Wick complex pole modification [42] of Feynman rules, up to some of the recent proposals to implement nonlocality via noncommutative spacetime failed on different counts. The old attempts retained Lorentz-invariance and unitarity but failed on the starting motivation namely "finiteness" [43]. Of course even without this motivation it would have been very interesting to know if there are any physically viable nonlocal relatistic theories at all. By this we mean the survival of the physically indispensible macro-causality[12] without which the formalism has no physical interpretation. For the relativistic particle theory mentioned in the introduction this macro-causality was insured via the cluster-separability properties of the S-matrix. The almost 50 years of history on this issue has taught us time and again that the naive idea that a mild modification of pointlike Lagrangian interactions will still retain macro-causality turns out to be wrong under closer scrutiny. In fact the general message is that the notion of a mild violation of micro-causality (i.e. maintaining macro-causality) within the standard framework is a questionable concept [45] (akin to being a little bit pregnant). One surprising No-Go theorem states that if one replaces spacelike commutativity by a faster than exponential asymptotic decrease, one falls right back onto local commutativity [46].

These negative results suggest that in order to find a consistent way to get away from local commutativity one needs a much more radical Ansatz which modifies the very spacetime structure. In more recent times Doplicher Fredenhagen and Roberts [47] discovered a Bohr-Rosenfeld like argument which uses a quasiclassical interpretation of the Einstein field equation (coupled with a requirement of absence of measurement-caused black hole "photon traps") and leads to uncertainty relations of spacetime. Although the initiating idea was very conservative, the authors were nevertheless led to quite drastic conceptual changes since the localization indexing of field theoretic observables is now done in terms of noncommutative spacetime in which points correspond to pure states on a quantum mechanical spacetime substrate on which the Poincaré group acts. They found a model which saturate their commutation relations but still maintains the Poincaré symmetry. In more recent times it was realized [48], that when one recasts such models into the setting of Yang-Feldman perturbation theory with a kind of nonlocal interaction, the Lorentz-invariance and unitarity of their new general framework can even be upheld in

[11] A recent paper by Lieb and Loss [40] contains an interesting attempt to combine relativistic QM with local quantum field theory. To make this model fully cluster separable (macro-causal) one probably has to combine the localization properties of relativistic quantum mechanics with those of modular localization for the photon field.

[12] In case of formfactor modifications of pointlike interaction vertices this was shown in [43] and in case of the Feynman rule modifications by complex poles in [44].

perturbation theory. This is interesting because in many papers which appeared after the DFR work in which the main message DHR was not heeded [49][51] an inevitable violation of L-invariance and of the optical theorem was claimed. Most of these incorrect conclusions have their origin that the authors did not rethink the formalism but just copies old Feynman formalism without being aware that i.e. the $i\varepsilon$ prescription is not anymore the same as the spacetime time-ordering. Interestingly enough the formalism of Yang-Feldman perturbation theory which works directly with the field equations and seems to be more secure against comitting tacit conceptual mistakes is precisely the technique used in the first post-renormalization investigations of nonlocal interactions [41].

So there seems to be at least some hope that those specific nonlocalities caused by those models whose lowest nontrivial perturbative order is discussed in [48] may be exempt from the historical lessons. It is encouraging that the Gaussian decrease of the noncommutative analog of pointlike localization [47] goes beyond the scope of the aforementioned No-Go theorem [46]. The DFR noncommutative theory would be a theory to which the Wigner approach is applicable and the Fock space structure is maintained but with different localization concepts. It would be very interesting indeed if besides the two mentioned relativistic theories build on different localization concept treated in this article there could exist a theory of Wigner particles interacting on noncommutative spacetime in a possibly macro-causal way and uphold the significant gains concerning the TCP structure and antipartices which are so inexorably linked to vacuum polarization. Such a quest on a fundamental level should not be confused with the phenomenological use of the language of noncommutative geometry for certain conventional Schroedinger systems involving constant magnetic fields [51] since in those cases the localization concepts of the Schroedinger theory are in no way affected by the observation that one may write the system in terms of different dynamical variables.

In the context of potential particle-physics applications of noncommutative spacetime it is worthwhile to remember that the full local (anti)commutativity is not used in e.g. the derivation of the TCP theorem. In the present modular terminology of this paper the TCP property is in fact known to be equivalent to wedge localization (which in turn is related to "weak locality" [26]). It seems that the question of whether a modular wedge localization is possible in the context of the correctly formulated noncommutative L-invariant and unitary models [47][48] may well have a positive answer [50]. This point is certainly worthwhile to return to in future work.

It is very regrettable that such conceptually subtle points[13] seem to go unnoticed in the new globalized way of doing particle physics [51]. It seems that the ability of recognizing conceptually relevant points, which has been the hallmark of part of 20 century physics, has been lost in the semantic efforts of attaching physical-sounding words to mathematical inventions.

It is well-known to quantum field theorist with some historical awareness that the role of causality and localization was almost never appreciated/understood by most mathematicians. This has a long tradition. A good illustration is the impressive scientific curriculum of Irvine Segal, one of the outstanding pioneers of the algebraic approach. If in those papers localization concepts would have been treated with the same depth and care as global mathematical aspects of AQFT, quantum field theory probably would have undergone a more rapid development and we would have been spared the many differential geometric

[13] The claim in [51] that "noncommutativity of the space-time coordinates generally conflicts with Lorentz invariance" contradicts the results of the 1995 seminal paper [47] and a fortiori the forthcoming explicit perturbative model calculations in [48].

traps and pitfalls, including the banalization of Euclidean methods.

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7 Appendices

Here we have collected some mathematical details for the convenience of the reader.

7.1 Appendix A: The abstract spatial modular theory

Suppose we have a "standard" spatial modular situation i.e. a closed real subspace H_R of a complex Hilbert space H such that $H_R \cap iH_R = \{0\}$ and the complex space $H_D \equiv H_R + iH_R$ is dense in H. Let e_R and e_I be the projectors onto H_R and iH_R and define operators

$$t_\pm \equiv \frac{1}{2}(e_R \pm e_I) \tag{74}$$

Because of the reality restriction the two operators have very different conjugation properties, t_+ turns out to be positive $0 < t_+ < \mathbf{1}$, but t_- is antilinear. These properties follow by inspection through the use of the projection- and reality-properties. There are also some easily derived quadratic relations between involving the projectors and $t\pm$

$$\begin{aligned} e_{R,I}t_+ &= t_+(1 - e_{I,R}) \\ t_+t_- &= t_-(1 - t_+) \\ t_-^2 &= t_+(1 - t_+) \end{aligned} \tag{75}$$

Theorem 8 *([16]) In the previous setting there exist modular objects*[14] J, Δ *and* $S = \mathrm{j}\Delta^{\frac{1}{2}}$ *which reproduce* H_R *as the +1 eigenvalue real subspace of* S*. They are related to the previous operators by*

$$\begin{aligned} t_- &= J\,|t_-| \\ \Delta^{it} &= (1 - t_+)^{it}\, t_+^{-it} \end{aligned}$$

The proof consists in showing the commutation relation $J\Delta^{it} = \Delta^{it}\ J$ ($\curvearrowright J\Delta = \Delta^{-1}\ J$ since J is antiunitary) which establishes the dense involutive nature $S^2 \subset 1$ of S by using the previous identities. It is not difficult to show that 0 is not in the point spectrum of Δ^{it}.

Corollary 9 *If* H_R *is standard, then* iH_R*,* $H_R^\perp$ *and* $iH_R^\perp$ *are standard. Here the orthogonality* $\perp$ *refers to the real inner product* $Re(\psi, \varphi)$*. Furthermore the* J *acts on* H_R *as*

$$JH_R = iH_R^\perp$$

[14] In the physical application the Hilbert space can be representation space of the Poincaré group which carries an irreducible positive energy representation or the bigger Fock space of (free or incoming) multi-particle states. In order to have a uniform notation we use (different from section 2) big letters for the modular objects and the transformations, i.e. $S, J, \Delta, U(a, \Lambda)$.

We leave the simple proofs to the reader (or look up the previous reference [16]). The orthogonality concept is often expressed in the physics literature by $iH_R^{\perp} = H_R^{symp\perp}$ referring to symplectic orthogonality in the sense of $Im(\psi, \varphi)$. There is also a more direct analytic characterization of Δ and J

Theorem 10 *(spatial KMS condition) The functions f(t)=$\Delta^{it}\psi$, $\psi \in H_R$ permits an holomorphic continuation f(z) holomorphic in the strip -$\frac{1}{2}\pi$ $<$ $\mathrm{Im}\, z$ $<$ 0, continuous and bounded on the real axis and fulfilling $f(t - \frac{1}{2}i) = Jf(t)$ which relates the two boundaries. The two commuting operators Δ^{it} and j are uniquely determined by these analytic properties i.e. H_R does not admit different modular objects.*

Another important concept in the spatial modular theory is "modular inclusion"

Definition 11 *(analogous to Wiesbrock) A inclusion of a standard real subspace K_R into a standard space $K_R \subset H_R$ is called "modular" if the modular unitary $\Delta_{H_R}^{it}$ of H_R compresses K_R for one sign of t*

$$\Delta_{H_R}^{it} K_R \subset K_R \quad t < 0$$

If necessary one adds a -sign i.e. if the modular inclusion happens for t>0 one calls it a $-$modular inclusion.

Theorem 12 *The modular group of a modular inclusion i.e. $\Delta_{K_R}^{it}$ together with $\Delta_{H_R}^{it}$ generate a unitary representation of the two-parametric affine group of the line.*

The proof consists in observing that the positive operator $\Delta_{K_R} - \Delta_{H_R} \geq 0$ is essentially selfadjoint. Hence we can define the unitary group

$$U(a) = e^{i\frac{1}{2\pi}a\overline{(\Delta_{K_R} - \Delta_{H_R})}} \tag{76}$$

The following commutation relation

$$\begin{aligned} \Delta_{H_R}^{it} U(a) \Delta_{H_R}^{-it} &= U(e^{\pm 2\pi t} a) \\ J_{H_R} U(a) J_{H_R} &= U(-a) \end{aligned} \tag{77}$$

and several other relations between $\Delta_{H_R}^{it}, \Delta_{K_R}^{it}$, J_{H_R}, $J_{K_R}, U(a)$. The above relations are the Dilation-Translation relations of the 1-dim. affine group. It would be interesting to generalize this to the modular intersection relation in which case one expects to generate the SL(2,R) group.

The actual situation in physics is opposite: from group representation theory of certain noncompact groups $\pi(G)$ one obtains candidates for Δ^{it} and J from which one passes to S and H_R. In the case of the Poincaré or conformal group the boosts or proper conformal transformations in positive energy representations lead to the above situation. The representations do not have to be irreducible; the representation space of a full QFT is also in the application range of the spatial modular theory. If the positive energy representation space is the Fockspace over a one-particle Wigner space, the existence of the CCR (Weyl) or CAR functor maps the spatial modular theory into operator-algebraic modular theory of Tomita and Takesaki. In general such a step is not possible. Connes has given conditions on the spatial theory which lead to the operator-algebraic theory. They involve the facial structure of positive cones associated with the space H_R. Up to now it has not been possible to use them for constructions in QFT. The existing ideas of combining the spatial theory of particles with the Haag-Kastler framework of spacetime localized operator algebras uses the following 2 facts

- The wedge algebra $\mathcal{A}(W)$ has known modular objects

$$\Delta^{it} = U(\Lambda_W(-2\pi t)) \tag{78}$$
$$J = S_{scat} J_0$$

Whereas the wedge affiliated L-boost (in fact all $P_+^\uparrow$ transformations) is the same as that of the interacting or free incoming/outgoing theory, the interaction shows up in those reflections which involve time inversion as J. In the latter case the scattering operator S_{scat} intervenes in the relation between the incoming (interaction-free) J_0 and its Heisenberg counterpart J. In the case of interaction free theories the J_0 contains in addition to the geometric reflection (basically the TCP) a "twist" operator which is particularly simple in the case of Fermions.

- The wedge algebra $\mathcal{A}(W)$ has PFG-generators. In certain cases these generators have nice (tempered) properties which makes them useful in explicit constructions. Two such cases (beyond the standard free fields) are the interacting d=1+1 factorizing models and the free anyonic and Wigner spin-tower representations in both cases the PFG property is lost (vacuum polarization is present) for sub-wedge algebras. In the last two Wigner cases the presence of the twist requires this, only the fermionic twist in the case of $S_{scat} = 1$ is consistent with having PFGs for all localizations.

7.2 Appendix B: Infinite dimensional covariant representations

In terms of the little group generators relative to the fixed vector $\frac{1}{2}(1,0,0,1)$ the Pauli-Lubanski operators has the form

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\sigma\tau} J^{\nu\sigma} P^\tau = \frac{1}{2}(M_3, \Pi_1, \Pi_2, M_3)$$

where M_3 is the 3-component of the angular momentum and Π_i are the two components of the Euclidean translations which together make up the infinitesimal generators of $\tilde{E}(2)$. An representation of the little group can be given in any of the Gelfand at al. irreducible representation spaces of the homogeneous Lorentz group. These consist of homogeneous functions of two complex variables $\zeta = (\zeta_1, \zeta_2)$ which are square integrable with respect to the following measure

$$d\mu(\zeta) = \frac{1}{4\pi}\left(\frac{i}{2}\right)^2 d^2\zeta d^2\bar{\zeta}\delta(\frac{1}{2}\zeta q \zeta^* - 1),\ q{=}\sigma^\mu q_\mu,\ q^2 = 0, q_0 > 0 \tag{79}$$
$$(f,g) = \int d\mu(\zeta)\bar{f}(\zeta)\, g(\zeta),\ \ f(\rho e^{i\alpha}\zeta) = \rho^{2(c-1)} e^{2il_0\alpha} f(\zeta),\ \lambda_0 = 0, \pm\frac{1}{2}, \pm 1, .., c = i\nu,$$

The inner product is independent of the choice of the lightlike vector q if $c = i\nu$ because the integrand has total homogeneous degree -4 and on functions $F(\rho\zeta) = \rho^{-4}F(\zeta)$ with this degree the integral is q-independent. This family of unitary irreducible representations $\chi = [\lambda_0, c = i\nu]$ for $-\infty < \nu < \infty$ of SL(2,C) is called the *principal series* representation. Another such family, the *supplementary series* $\chi = [\lambda_0, c]\,,\ -1 < c < 1$ contains an additional integral operator $K(\zeta,\eta)$

$$(f,g) = \int d\mu(\zeta)\bar{f}(\zeta) \int K(\zeta,\eta) g(\eta) \tag{80}$$
$$K(\zeta,\eta) = N^{-1}\,(\eta\varepsilon\zeta)^{-l_0-c-1}\,\overline{(\eta\varepsilon\zeta)}^{-l_0-c-1}$$

We now define basisvectors in the above representation spaces which carry a representation of the little group

$$\left(\Pi_1^2+\Pi_2^2\right) f_\lambda^{\chi,\rho}(\zeta)=\rho^2 f_\lambda^{\chi,\rho}(\zeta),\ M_3 f_\lambda^{\chi,\rho}(\zeta)=-\lambda f_\lambda^{\chi,\rho}(\zeta) \tag{81}$$
$$\left(U(\tilde{E}) f_\lambda^{\chi,\rho}\right)(\zeta)=\sum_{\lambda'} f_{\lambda'}^{\chi,\rho}(\zeta\tilde{E}) d_{\lambda',\lambda}(\tilde{E})$$
$$f_\lambda^{\chi,\rho}(\zeta)=|\zeta_2|^{2c-2} e^{-i\lambda\phi} J_{l_0-\lambda}(2\rho\,|z|) e^{il_0\alpha},\ \phi$$

In a similar way, the d=1+2 anyonic representations may be rewritten in terms of infinite dimensional covariant representations. It has been shown [37] that the following family of covariant unitary representations of $\tilde{\mathcal{P}}_3^{\uparrow}$ are useful in the covariant description of the (m,s) Wigner representation

$$(U(a,(\gamma,\omega))\psi)(p,z)=e^{ipa}\tau_{h,\sigma}((\gamma,\omega);z)\psi(\Lambda(\gamma,\omega)^{-1}p,(\gamma,\omega)^{-1}z)$$
$$\tau_{h,\sigma}((\gamma,\omega);z)=e^{-i\omega h}\left(\frac{1+z\bar{\gamma}}{1+z^{-1}\gamma}\right)^h(1+z\bar{\gamma})^{-1-2\sigma}(1+|\gamma|)^{\frac{1}{2}+\sigma}$$
$$(\gamma,\omega)\cdot z=e^{-i\omega}\frac{z-\gamma e^{i\omega}}{1-z\bar{\gamma}e^{-i\omega}}$$

Here the τ are Bargmann's principle series representations of $\widetilde{SL(2,R)}$ acting on the covering of the circle with the circular coordinate being z, $|z|=1$. The last formula is the action of the Moebius group on the circle. The wave functions $\psi(p,z)$ in this formula are from $L^2(p\in H_m^{\uparrow}, z=e^{i\varphi}; \frac{dp}{2p_0}, d\varphi)$ and in the range $-\frac{1}{2}<h\leq\frac{1}{2}$, $\sigma\in iR$ the action is unitary. It has been shown that this covariant representation can be decomposed into a direct sum of Wigner representations $(m,s=k-h)$. $k\in\mathbb{Z}$.

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