CBPF-NF-003/95

Maxwell's Equations in Spatially Homogeneous Cosmological Models

Renato Portugal

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

Abstract

Maxwell's equations with source are investigated using the background of Bianchi cosmological models. Exact solutions are given for all Bianchi types. For class A models, the electric current must be spacelike. For class B models, the current can be either spacelike or timelike.

Key-words: Cosmology, Maxwell's Equations; Bianchi Models.

1 Introduction

The discovery of a cosmological magnetic field of order 10^{-8} Gauss[1] increased the interest in the study of cosmological models that admit electromagnetic field. The presence of such field requires the use of a class of spacetimes more general than the Friedmann-Robertson-Walker ones, since isotropy is broken. The most natural extension is the study of spatially homogeneous models, which were classified as the Bianchi type models[2,3]. A great number of solutions of exact Einstein-Maxwell equations has been found and many physical results have been obtained using Bianchi type models[4].

A class exhaustively studied is the Bianchi magnetic models[4,5,6]. In this class, the electrical conductivity is supposed to be infinity. This implies a vanishing electrical field to avoid an infinite electrical current. These models can be said to be in a perfect magnetohydrodynamic regime[7], in contrast with those that have finite conductivity, which are said to be in a (generic) magnetohydrodynamic regime.

The study of cosmological models with finite and non-zero electrical conductivity was initiated, as far as the author knows, by Dunn and Tupper[8,9,10]. They studied spatially homogeneous cosmological models of Bianchi type I, II, III and IV_0 with perfect fluid plus electromagnetic field as material content. They found expanding exact solutions for Einstein-Maxwell equations with finite conductivity satisfying the relevant physical constraints, such as the dominant energy conditions, positivity of conductivity etc.

Dunn and Tupper[10] called attention to the importance of the study of cosmological models with finite conductivity, since the assumption of infinite conductivity lead to plasma models that have 'little contact with reality'[11] .

Portugal and Soares[12,13] found new contracting cosmological solutions for axisymmetric Bianchi type I, III and Kantowski-Sachs models. They showed that the Bertotti-Robinson-like models can evolve to contracting Kantowski-Sachs models in magnetohydrodynamic regime.

An important study is the phenomenological application of magnetohydrodynamic cosmological models to describe the Universe in the era prior to recombination, when the material content of the Universe was a hot radiating plasma. It is in general argued that the conductivity is too high in this period implying a vanishing electrical field[14]. Those calculations of the conductivity are valid under some assumptions, for example, when the electron Larmor frequence eB/m is small compared to the inverse of the mean collision time. In this case, the conductivity is given by

$$\sigma = \frac{n_e e^2 \tau}{m}$$

where n_e, e and m are the electron's density, charge and mass and τ is the collision time. When the plasma is in the presence of a magnetic field, the conductivity is given by[15]

$$\sigma = \frac{n_2 e^2 \tau}{m(1 + (eB/m)^2)}$$

where B is the modulus of magnetic field. For strong magnetic fields, it is no longer valid that $\sigma \approx \infty$.

Our interest here is the study of the Maxwell equations with source in spatially homogeneous models. The Bianchi cosmological models can be treated through a unified formalism using the theory of differential forms. Hughston and Jacobs[16] used this formalism to study sourceless Maxwell equations in Bianchi models. They solved these equations for vanishing Poynting vector and pure magnetic field. They also studied massive-vector-meson fields. Their work was further developed by Ftaclas and Cohen[17] and Lorenz[18]. The latter found new solutions of sourceless Maxwell equations with non-vanishing electrical field and non-vanishing Poynting vector. The former authors, on the other hand, analyzed the Maxwell equations with source. They showed that the models of class A must have spacelike current regardless of any consideration on the dynamics of the models. Only local arguments were used. They also showed that models with timelike current do not admit pure magnetic field, and models with vanishing Poynting vector are pure electric. In the present work we extend this analysis for a more general electric current.

In section 2 we establish the Maxwell equations for the Bianchi models following Hughston & Jacobs[16] and Lorenz[18]. In section 3 we discuss the form of the electric current and in section 4 we present the exact solutions.

2 The Maxwell Equations

The Maxwell equations can be written as

$$
\begin{aligned}
dF &= 0 \\
d\overset{*}{F} &= \overset{*}{J}
\end{aligned} \tag{1}
$$

where F is the electromagnetic field 2-form, $\overset{*}{F}$ is its dual and $\overset{*}{J}$ is the dual electrical current 3-form. We use an orthonormal synchronous basis of 1-forms $\sigma^\alpha (\alpha = 0, 1, 2, 3)$, defined by[3,18]

$$
\begin{aligned}
\sigma^0 &= \omega^0 = dt \\
\sigma^i &= R_i \omega^i \qquad (i = 1, 2, 3)
\end{aligned} \tag{2}
$$

where ω^i satisfy

$$
d\omega^i = -\frac{1}{2}\, C^i{}_{jk} \omega^j \wedge \omega^k \tag{3}
$$

where $C^i{}_{jk}$ are the structure constants of the 3-dimensional isometry groups. There are alternative conventions for the choice of $C^i{}_{jk}$. We follow the conventions of Luminet[19] and Eardley[20]. The non-vanishing values of the structure constants are shown in table I.

CLASS A $(C^\ell{}_{i\ell} = 0)$

Bianchi I : $C^i{}_{jk} = 0$

Bianchi II : $C^1{}_{23} = 1$

Bianchi VI$_0$: $C^1{}_{23} = C^2{}_{13} = 1$

Bianchi VII$_0$: $C^1{}_{23} = C^2{}_{31} = 1$

Bianchi VIII : $C^1{}_{23} = C^2{}_{31} = C^3{}_{21} = 1$

Bianchi IX := $C^1{}_{23} = C^2{}_{31} = C^3{}_{12} = 1$

CLASS B $(C^{\ell}{}_{i\ell} \neq 0)$

Bianchi III : Bianchi VI_{-1}

Bianchi IV : $C^1{}_{31} = C^1{}_{23} = C^1{}_{32} = 1$

Bianchi V : $C^1{}_{31} = C^2{}_{32} = 1$

Bianchi VI_h : $C^1{}_{23} = C^2{}_{13} = 1$, $C^1{}_{31} = C^2{}_{32} = \sqrt{-h}$, $h < 0$

Bianchi VII_h : $C^1{}_{23} = C^2{}_{31} = 1$, $C^1{}_{31} = C^2{}_{32} = \sqrt{h}$, $h > 0$

TABLE I: Structure constants of Bianchi models: $C^i{}_{jk} = -C^i{}_{kj}$

The metric of the spacetimes is given by

$$ds^2 = -dt^2 + (R_1\omega^1)^2 + (R_2\omega^2)^2 + (R_3\omega^3) \tag{4}$$

We choose $u^a = (1,0,0,0)$, therefore the components of the electric and magnetic fields are

$$\begin{aligned} E^i &= F^{oi} \\ B^i &= \frac{1}{2}\,\varepsilon^{ijk}F_{jk} \end{aligned} \tag{5}$$

where ε^{ijk} is the totally antisymmetric permutation symbol. Due to the homogeneity of the spatial section, we assume that $\vec{E}$ and $\vec{B}$ are function of time only.

We can project F and J in the basis ω^A, to obtain

$$\begin{aligned} F &= E_iR_i\omega^i \wedge \omega^0 + \frac{1}{2}\,\varepsilon_{ijk}B^2R_jR_k\;\omega^j \wedge \omega^k \\ \overset{*}{F} &= -B_iR_i\omega^i \wedge \omega^0 + \frac{1}{2}\,\varepsilon_{ijk}E^iR_jR_k\;\omega^j \wedge \omega^k \\ \overset{*}{J} &= \frac{1}{3!}\,\varepsilon_{ijk}J^0R_iR_jR_k\omega^i \wedge \omega^j \wedge \omega^k - \frac{1}{2}\,\varepsilon_{ijk}J^iR_jR_k\;\omega^0 \wedge \omega^j \wedge \omega^k \end{aligned} \tag{6}$$

The index of the term R_i is not subjected to the summation convention, it just follows a neighbouring index of the same name.

Equations (1) yield, for $i \neq j \neq k$:

$$
\begin{aligned}
&B^i R_j R_k C^\ell{}_{i\ell} = 0 \\
&E^i R_j R_k C^\ell{}_{i\ell} = -J^0 R_1 R_2 R_3 \\
&\frac{\partial}{\partial t}\left(B^i R_j R_k\right) - \frac{1}{2}\,\varepsilon^{i\ell m} E_p R_p C^p{}_{\ell m} = 0 \\
&\frac{\partial}{\partial t}\left(E^i R_j R_k\right) + \frac{1}{2}\,\varepsilon^{i\ell m} E_p R_p C^p{}_{\ell m} = -J^i R_j R_k
\end{aligned} \tag{7}
$$

From these equations we conclude that[17]:

1. The electric charge density vanishes for models of class A.

2. Pure magnetic solutions are forbidden when $J_0 \neq 0$.

3. The magnetic field must vanish when $J_0 \neq 0$ and $\vec{B} = f(t)\vec{E}$ (vanishing Poynting vector).

3 The Electric Current

In general, the electric current is supposed to obey the Ohm's law

$$J^\alpha = \rho u^\alpha + \sigma E^\alpha \tag{8}$$

where ρ is the charge density measured by the comoving observer and σ is the conductivity. In the collision time approximation, σ is given by

$$\sigma = \frac{ne^2\tau}{m}$$

where n, e and m are the electron's density charge and mass and τ is the collision time.

In the presence of a magnetic field, there are new terms due to the Hall effect, and the current obeys Ohm's generalized law[15]

$$J^\alpha = \rho u^\alpha + \frac{\sigma}{1+\lambda^2 B^2}\left[\eta^{\alpha\beta} + \lambda\eta^{\alpha\beta\mu\nu} u_\mu B_\nu - \lambda^2 B^\alpha B^\beta\right] E_\beta \tag{9}$$

where $\lambda = e\tau/m$.

There are two important situations where expression (9) simplifies to expression (8). First, when the Poynting vector vanishes, i.e., $\varepsilon_{ijk}E^j B^k = 0$. Second, when the Larmor frequency eB/m is small compared to the collision frequency. In this case the terms in (9) involving the magnetic field are small compared to the term $\sigma\eta^{\alpha\beta}E_\beta$.

4 Solutions

Let us solve Maxwell's equations for the electric current satisfying eq. (8). The form of the solutions will depend on the choice of the structure constants. In our case, they are given by table I. We solve the equations in a general form and afterwords we discuss the restriction for each Bianchi model. We analyze the classes A and B apart.

For models of class A, the structure constants vanish when there are repeated indices, then eq. (7) simplify to

$$\begin{aligned} &\frac{\partial}{\partial t}\left(B^i R_j R_k\right) - \varepsilon^{ijk} E^i R_i C^i{}_{jk} = 0 \qquad (i \neq j \neq k, \text{ no sum}) \\ &\frac{\partial}{\partial t}\left(E^i R_j R_k\right) + \varepsilon^{ijk} B^i R_i C^i{}_{jk} = -\sigma E^i R_j R_k \end{aligned} \tag{10}$$

and $\rho = 0$.

For $\vec{B} = 0$, we have

$$E^i = \frac{e_i}{R_j R_k} \exp \int -\sigma dt \tag{11}$$

with the restriction $E^i C^i{}_{jk} = 0$. This restriction implies that for Bianchi II we must put the constant $e_1 = 0$. For Bianchi VII$_0$ and VI$_0$ we must put $e_1 = e_2 = 0$. For Bianchi VIII and IX we must put all constants e_i equal zero, then $\vec{B} = 0$ implies $\vec{E} = 0$.

For $\vec{B} \neq 0$ and $\varepsilon_{i\ell m} E^\ell B^m = 0$ (vanishing Poynting vector), we impose $B^i = B^k = E^j = E^i = 0$. For the values of i such that $C^i{}_{jk} = 0$, the solutions are

$$\begin{aligned} B^i &= \frac{b_i}{R_j R_k} \qquad i \neq j \neq k \\ E^i &= \frac{e_i}{R_j R_k} \exp \int -\sigma dt \end{aligned} \tag{12}$$

where b_i and e_i are constants. For the value of i such that $C^i{}_{jk} \neq 0$, we impose the relation

$$\sigma = \frac{2kR_i}{R_j R_k} \tag{13}$$

where k is a positive constant. The solutions of eqs. (10) depend on k

i) $0 \leq k < 1$

$$E^i = \frac{a}{R_j R_k} \sin\left(\sqrt{1-k^2}(\eta-\eta_0)\right) \exp(-k\eta) \tag{14}$$

$$B^i = -\frac{a\varepsilon^{ijk}}{C^i{}_{jk} R_j R_k} \left(k \sin(\sqrt{1-k^2}(\eta-\eta_0)) + \sqrt{1-k^2}\cos(\sqrt{1-k^2}(\eta-\eta_0))\right) \exp(-k\eta)$$

ii) $k = 1$

$$E^i = \frac{a+b\eta}{R_j R_k} \exp(-\eta) \tag{15}$$

$$B^i = -\frac{a+b+b\eta}{\varepsilon^{ijk} C^i{}_{jk} R_j R_k} \exp(-\eta)$$

iii) $k > 1$

$$E^i = \frac{a}{R_j R_k} \sinh(\sqrt{1-k^2}(\eta-\eta_0)) \exp(-k\eta) \tag{16}$$

$$B^i = -\frac{a\varepsilon^{ijk}}{C^i{}_{jk} R_j R_k} \left(k \sinh(\sqrt{1-k^2}(\eta-\eta_0)) + \sqrt{1-k^2}\ \cosh(\eta-\eta_0)\right) \exp(-k\eta)$$

where $d\eta = R_i dt / R_j R_k$.

By inspection of Table I, we see that

Bianchi I: For $i = 1, 2, 3$ solutions are given by eqs. (12)

Bianchi II: For $i = 2, 3$ solutions are given by eqs. (12) and for $i = 1$, by eqs. (14-16)

Bianchi VI_0 and VII_0: For $i = 3$, solutions are given by eqs. (12) and for $i = 1, 2$ by eqs. (14-16)

Bianchi VIII and IX: For $i = 1, 2, 3$, solutions are given by eqs. (14-16).

Still in Class A, let us consider now the case of non-vanishing Poynting vector. For Bianchi I, the solutions are given by eqs. (12). For Bianchi II, the solutions are given by eqs. (12) for $i = 2, 3$ and by eqs. (14-16) for $i = 1$ and with $\sigma = 2kR_1/R_2R_3$. For Bianchi VI_0 and VII_0, the solutions are given by eqs. (12) for $i = 3$ and by eqs. (14-16) for $i = 1, 2$, with the restriction $R_1 = R_2$ and $\sigma = 2k/k_3$. For Bianchi VIII and IX, the solutions are given by eqs. (14-16) but with one of the following restrictions: First, $E^i = 0$, $R_j = R_k$ and $\sigma = 2k/R_i$ where $i \neq j \neq k$. Second, $R_1 = R_2 = R_3$ and $\sigma = 2k/R$.

Now, let us analyze models of class B. The Maxwell equations (7) simplify to

$$\begin{aligned} & B^3 = 0 \\ & \rho = -\frac{C^\ell{}_{3\ell} E^3}{R_3} \\ & \frac{\partial}{\partial t}\left(B^i R_j R_k\right) - \frac{1}{2}\,\varepsilon^{i\ell m} E_p R_p C^p{}_{\ell m} = 0 \\ & \frac{\partial}{\partial t}\left(E^i R_j R_k\right) + \frac{1}{2}\,\varepsilon^{i\ell m} B_p R_p C^p{}_{\ell m} = -\sigma E^i R_j R_k \end{aligned} \tag{17}$$

where $i \neq j \neq k$.

For $\vec{B} = 0$, we have

$$\begin{aligned} E^i &= \frac{e_i}{R_j R_k}\, \exp \int -\sigma dt \\ \rho &= -\frac{C^\ell{}_{3\ell} e_3}{R_1 R_2 R_3}\, \exp \int -\sigma dt \end{aligned} \tag{18}$$

with the restrictions $E_\ell R_\ell C^\ell{}_{23} = 0$ and $E_\ell R_\ell C^\ell{}_{31} = 0$. For Bianchi III these restrictions are automatically satisfied. For Bianchi IV, V, VI$_h$ ($h \neq -1$) e VII$_h$ these restrictions imply $E^1 = E^2 = 0$.

For vanishing Poynting vector, E^3 and ρ satisfy eqs. (18). To obtain the other components we impose $R_1 = R_2 = R$. and $\sigma = 2k/S$, where k is a constant and $S = R_3$. We give the solutions for each Bianchi type

Bianchi III:

$$\begin{aligned} B^1 &= \frac{1}{RS}\left(a + b\exp(-2k\eta)\right) \\ B^2 &= \frac{1}{RS}\left(a - 2kc + b\exp(-2k\eta)\right) \\ E^1 &= \frac{1}{RS}\left(c + (d + 2kb)\exp(-2k\eta)\right) \\ E^2 &= \frac{1}{RS}\left(c + d\exp(-2k\eta)\right) \end{aligned}$$

where a, b, c, d are constants and $d\eta = dt/S$.

Bianchi IV:

$$B^1 = \frac{1}{RS}\left(\sum_{i=1}^{4}\ -a_i\lambda_i(\lambda_i + 2k)\exp(\lambda_i\eta)\right)$$
$$B^2 = \frac{1}{RS}\left(\sum_{i=1}^{4}\ a_i\exp(\lambda_i\eta)\right)$$
$$E^1 = \frac{1}{RS}\left(\sum_{i=1}^{4} -a_i\lambda_i\exp(\lambda_i\eta)\right)$$
$$E^2 = \frac{1}{RS}\left(\sum_{i=1}^{4}\ -a_i\lambda_i^2(\lambda_i + 2k)\exp(\lambda_i\eta)\right)$$

where a_i are constants and λ_i are the roots of

$$\lambda^4 + 4k\lambda^3 + (4k^2 - 1)\lambda^2 - 2k\lambda + 1 = 0$$

Bianchi V:

$$B^1 = \frac{1}{RS}(a\alpha_+\exp(\alpha_-\eta) + b\exp(\alpha_+\eta))$$
$$B^2 = \frac{1}{RS}(c\exp(\alpha_+\eta) + d\exp(\alpha_-\eta))$$
$$E^1 = \frac{1}{RS}(-c\alpha_+\exp(\alpha_+\eta) - d\alpha_-\exp(\alpha_+\eta))$$
$$E^2 = \frac{1}{RS}(b\alpha_+\exp(\alpha_+\eta) - a\exp(\alpha_+\eta))$$

where $\alpha_\pm = -k \pm \sqrt{k^2 + 1}$ and a, b, c, d are constants.

Bianchi VI$_h$, $\quad h < 0$

$$B^1 = \frac{1}{RS}\left(c\exp(\alpha_+\eta) + \frac{d - b\alpha_+}{\sqrt{-h}}\exp(\alpha_-\eta)\right)$$
$$B^2 = \frac{1}{RS}((a\alpha_- + \sqrt{-h}c)\exp(\alpha_+\eta) + d\exp(\alpha_-\eta))$$
$$E^1 = \frac{1}{RS}\left((\sqrt{-h}a - c\alpha_+)\exp(\alpha_+\eta) + \frac{b - d\alpha_-}{\sqrt{-h}}\exp(\alpha_-\eta)\right)$$
$$E^2 = \frac{1}{RS}(a\exp(\alpha_+\eta) + b\exp(\alpha_+\eta))$$

where $\alpha_\pm = -k \pm \sqrt{k^2 - 1 - h}$ and a, b, c, d constants

Bianchi VII_h, $\quad h > 0$

$$B^1 = \frac{1}{RS}\left(\sum_{i=1}^{4} \frac{a_i(h - 1 - 2k\lambda_i - \lambda_i^2)}{2\sqrt{h}}\ \exp(\lambda_i \eta)\right)$$

$$B^2 = \frac{1}{RS}\left(\sum_{i=1}^{4} a_i \exp(\lambda_i \eta)\right)$$

$$E^1 = \frac{1}{RS}\left(\sum_{i=1}^{4} \frac{a_i\lambda_i(\lambda_i^2 + 2k\lambda_i + 1 - 3h)}{(1+h)\sqrt{h}}\ \exp(\lambda_i \eta)\right)$$

$$E^2 = \frac{1}{RS}\left(\sum_{i=1}^{4} \frac{-a_i\lambda_i(\lambda_i^2 + 2\lambda_i + 3 - h)}{1+h}\ \exp(\lambda_i \eta)\right)$$

where a_i are constants and λ_i are the roots of

$$\lambda^4 + 4k\lambda^3 + 2(2k^2 - h + 1)\lambda^2 + 4k(1-h)\lambda + (h+1)^2 = 0$$

In these solutions $d\eta = dt/S$. For Bianchi IV, VI_h and VII_h we must take the real part of the solutions.

Acknowledgments

The author would like to thank J. Tiomno abd I.D. Soares for helpfull discussions during this work, and M.O. Calvão for revising the final version.

References

[1] Sofue, Y., Takano, T., and Fujimoto, M. (1980), Astron. Astrophys. **91**, 335.

[2] Taub, A.H., (1951), Ann. Math. **53**, 472.

[3] Ryan, M. and Shepley, L.C. 1975, Homogeneous Relativistic Cosmologies, Princeton University Press, Princeton, New Jersey.

[4] Kramer, D., Stephani, M., MacCallum, M. and Hertl, (1980), Exact Solutions of Einsteins Field Equations, VEB Deustscher Verlag der Wissenschaften Berlin;

[5] Ozsváth, J. 1977, GRG, **8**, 737;

[6] Jacobs, K.G. 1968, Ap. J., **155**, 379.

[7] Lichnerowicz, A. 1967, Relativistic Hydrodynamics and Magnetohydrodynamics, W.A. Benjamin, New York.

[8] Dunn, K.A. and Tupper, B.O.J. (1976) Ap. J., **204**, 322.

[9] Tupper, B.O.J. (1977) Ap. **216**, 192.

[10] Dunn, K.A. and Tupper, B.O.J. (1980), Ap. J. **235**, 307.

[11] Alfvén, H. 1972, Cosmic Plasma Physics, ed. K. Schindler, New York: Plenum Press.

[12] Portugal, R. and Soares, I.D. (1987), Ap. J. **316**, 483.

[13] Portugal, R. and Soares, I.D. (1991), Ap.J. **380**, 330.

[14] Fennelly, A.J. 1980, Phys. Rev. Letters, **44**, 955.

[15] Bekenstein, J.D. and Oron, E. (1978), Phys. Rev. D., **18**, 1809.

[16] Hughston, L.P. and Jacobs K.C. (1970) Ap. J. **160**, 147.

[17] Ftaclas, C. and Cohen, J.M. (1979), Ap. J. **227**, 13.

[18] Lorenz, D., Astrophys. Space Sci. (1982) **83**, 63.

[19] Luminet, J.P. 1978, Gen. Rel. Grav., **9**, 673.

[20] Eardley, D.M. 1974, Commun. Math. Phys., **27**, 287.