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ABSTRACT

Fackerell-Crossman's solutions to a generalized spheroidal wave equation are extended for the case in which there is no free parameters in that equation. The results, together with Leaver's expansions in series of Coulomb wave functions, are used to find the time dependence of massive scalar test fields in nonflat Friedmann-Robertson-Walker models of universes with dust.

Key-words: Spheroidal wave equations; Klein-Gordon equation in Friedmann--Robertson-Walker universes

I. INTRODUCTION

In nonrelativistic quantum mechanics generalized spheroidal wave equations (GSWEs) occur when one tries to solve the two-center problem approximately, e.g., the Schrödinger equation for the ionized hydrogen molecule[1]. More recently, Leaver[2] showed that Teukolsky's equations are also special instances of the GSWEs and, at the same time, developed representations to their solutions supposing that there is no free parameter in the differential equation. In fact, the usual expansions in series for the solutions of the GSWE present coefficients that satisfy three-term recurrence relations. These relations lead to characteristic equations from which we can compute values for some undetermined parameter occurring in the differential equation; only for these values may the series converge. In order to satisfy those characteristic equations, when all the parameters in the differential equation are previously known, Leaver inserted an arbitrary phase parameter in the series expansions. This is an important point, but his expansions in Coulomb wave functions and in confluent hypergeometric functions are solutions appropriate solely for the so-called "radial" wave equation. Here we shall be concerned mainly with the problem of finding solutions to the "angular" wave equation when there is no free parameter in it. We proceed as outlined below.

In section II we modify one of Fackerell-Crossman's expansions in series of Jacobi polynomials[3] for the solutions of the angular GSWE by introducing in it a parameter ν in order to satisfy the characteristic equation. Another expansion is constructed from the preceding one by employing a symmetry of the differential equation. In section III we apply the expansions of section II and Leaver's expansions in series of Coulomb wave functions to find the time dependence of massive scalar test fields in nonflat Friedmann-Robertson-Walker (FRW) universes filled with dust. Additional comments are provided in section IV, whereas in Appendix A we write down the general expressions for expansions in series of Coulomb wave functions.

II. EXPANSIONS IN SERIES OF HYPERGEOMETRIC FUNCTIONS

Let

$$x(x-x_0)\frac{d^2U}{dx^2} + (B_1+B_2x)\frac{dU}{dx} + \left[B_3+\omega^2x(x-x_0)-2\omega\eta(x-x_0)\right]U=0 \qquad (1)$$

be the GSWE in Leaver's version, ω, η, x_0 and B_i being constants. We suppose that x and x_0 are real and that $0\le x\le x_0$. Besides we assume that all the parameters in (1) are known, but do not ascribe them any particular values. For this situation we modify one of Fackerell-Crossman's expansions, the modification consisting in rewriting the Jacobi polynomials in terms of hypergeometric functions and, then, substituting $n+\nu$ for n in the parameters of the hypergeometric functions. The parameter ν, introduced in order to guarantee the validity of a characteristic equation, requires that we extend the range of the summation index n from $0\le n<\infty$ to $-\infty<n<\infty$. Another solution is not obtained from the second Fackerell-Crossman expansion: it results from a symmetry of the differential equation. From these expansions we trivially obtain two independent limits for $\omega=0$.

In eq. (1) there are two regular singular points, x_0 and 0, while the infinity is an irregular singularity. The indicial equation for expansions of U(x) in power series about x_0 has the roots

$$r_1=0 \qquad \text{and} \qquad r_2=1-B_2-B_1/x_0.$$

According to Frobenius's method[4], one solution to (1) has the form

$$U_1= U_1(x-x_0), \tag{2}$$

where the right hand side is a series regular at $x=x_0$. Moreover, if

$$B_2+B_1/x_0\neq 0 \text{ or negative integer}, \tag{3}$$

a second solution linearly independent with respect to U_1 is given by

$$U_2=(x-x_0)^{r_2}\, g(x-x_0), \tag{4}$$

g being regular at $x=x_0$. Replacing U by U_2 in eq. (1), we see that g also satisfies that equation, provided that we make the substitutions

$$B_2 \to B_2'=2-B_2-2B_1/x_0 \qquad \text{and} \qquad B_3 \to B_3'=B_3+\frac{B_1}{x_0}\left(\frac{B_1}{x_0}+B_2-1\right) \tag{5}$$

Thence it follows that[5], if

$$U_1=U_1(B_1,B_2,B_3;\omega,\eta;x-x_0) \tag{6a}$$

is a solution to eq. (1), then

$$U_2=(x-x_0)^{1-B_2-B_1/x_0}\, U_1(B_1,B_2',B_3';\omega,\eta;x-x_0) \tag{6b}$$

is another solution.

Accomplishing the following changes of variables in (1):

$$y=(x_0-x)/x_0 \quad \text{and} \quad U = e^{-i\omega x_0 y}\, Y, \tag{7}$$

we find

$$y(1-y)\frac{d^2Y}{dy^2} + \left[B_2 + \frac{B_1}{x_0} - \left(B_2 + 2i\omega x_0\right)y + 2i\omega x_0 y^2\right]\frac{dY}{dy} +$$

$$\left[-i\omega x_0\left(B_2 + \frac{B_1}{x_0}\right) - B_3 + i\omega x_0\left(B_2 + 2i\eta\right)y\right]Y = 0, \tag{8}$$

We note at once that in the limit $\omega\to 0$ (8) reduces to the hypergeometric equation

$$y(1-y)\,\frac{d^2Y}{dy^2} + \left[\gamma - (\alpha_+ + \alpha_- + 1)y\right]\frac{dY}{dy} - \alpha_+\alpha_-\, Y = 0 \tag{9a}$$

with

$$\gamma = B_2 + B_1/x_0, \quad \alpha_\pm = \frac{B_2-1}{2} \pm i\left[B_3 - \left(\frac{B_2-1}{2}\right)^2\right]^{1/2} \tag{9b}$$

Under the hypothesis (3), two linearly independent solutions to eq. (9) are[6]

$$Y_1 = U_1 = F(\alpha_+,\ \alpha_-,\ \gamma;\ y), \tag{10a}$$

$$Y_2 = U_2 = y^{1-B_2-B_1/x_0} F(\alpha'_+ \;, \alpha'_- \;, \gamma' \;; y), \tag{10b}$$

where $\alpha'_\pm$ and γ' are obtained from the right hand side of (9b) by means of (5) and F is the hypergeometric function. However, if $B_2+B_1/x_0=m=$ integer, (10a) stands for $m\geq1$ and (10b), for $m\leq0$. Thus (10) represents only one solution to each value of m; the other independent solution would be logarithmic and may be discarded if we demand regularity at $y=0$.

Coming back to eq. (8) and writing a solution to it as

$$Y_1 = e^{i\omega x_0 y} U_1 = \sum_{n=-\infty}^{n=\infty} b_n \, F(-n-\nu,\; n+\nu+B_2-1,\; B_2+B_1/x_0;\; y), \tag{11}$$

we find, after some straightforward calculation, the recurrence relations

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \tag{12a}$$

for b_n, where

$$\alpha_n = i\omega x_0 \frac{(n+\nu+1)(n+\nu-B_1/x_0)(n+\nu+B_2/2-i\eta)}{2(n+\nu+B_2/2)(n+\nu+B_2/2+1/2)} \,,$$

$$\beta_n = -B_3-(n+\nu)(n+\nu+B_2-1)-\eta\omega x_0 \frac{(B_2+B_1/x_0)(B_2-2)+2(n+\nu)(n+\nu+B_2-1)}{2(n+\nu+B_2/2-1)(n+\nu+B_2/2)}, \tag{12b}$$

$$\gamma_n = -i\omega x_0 \frac{(n+\nu+B_2-2)(n+\nu+B_2+B_1/x_0-1)(n+\nu+B_2/2+i\eta-1)}{2(n+\nu+B_2/2-3/2)(n+\nu+B_2/2-1)}.$$

Hence we obtain the infinite continued fractions

$$\alpha_n \frac{b_{n+1}}{b_n} = -\frac{\alpha_n\gamma_{n+1}}{\beta_{n+1}-}\frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-}\frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-}\cdots \tag{13a}$$

and

$$\alpha_n \frac{b_{n+1}}{b_n} = -\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1}-}\frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-}\frac{\alpha_{n-3}\gamma_{n-2}}{\beta_{n-3}-}\cdots \tag{13b}$$

Taking n=0 and equating the right hand sides of (13) we get the characteristic equation

$$\beta_0 = \frac{\alpha_{-1}\gamma_0}{\beta_{-1}-}\frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-}\frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-}\cdots + \frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\frac{\alpha_2\gamma_3}{\beta_3-}\cdots \tag{14}$$

Other solution to eq. (8), resulting from (6b), (7) and (11), is

$$Y_2 = e^{i\omega x_0 y} U_2 = y^{1-B_2-B_1/x_0} \sum_{n=-\infty}^{n=\infty} b'_n F(-n-\nu', n+\nu'+B'_2-1, B'_2+B_1/x_0; y) \tag{15a}$$

where the b'_n satisfy the relations

$$\alpha'_n\, b'_{n+1} + \beta'_n\, b'_n + \gamma'_n\, b'_{n-1} = 0 \tag{15b}$$

Here α'_n, β'_n and γ'_n are obtained from α_n, β_n and γ_n, respectively, by performing the substitutions given in (5) and changing ν by ν'. Once more, ν' is evaluated from a characteristic equation entirely analogous to (14). Furthermore, if B_2+B_1/x_0=m=integer, the remarks following eq. (10) are still valid in the present case: the solutions (11) hold for $m\geq 1$ and (15) for $m\leq 0$.

To study the convergence of the series (11), we must find

$$\lim_{n\to\infty} \frac{b_{n+1}\, F(-n-\nu-1,\ n+\nu+B_2,\ B_2+B_1/x_0;\ y)}{b_n\, F(-n-\nu,\ n+\nu+B_2-1\ ,\ B_2+B_1/x_0;\ y)} \tag{16a}$$

and

$$\lim_{n\to-\infty} \frac{b_n\, F(-n-\nu,\ n+\nu+B_2-1,\ B_2+B_1/x_0;\ y)}{b_{n+1}\, F(-n-\nu-1,\ n+\nu+B_2\ ,\ B_2+B_1/x_0;\ y)}\ . \tag{16b}$$

From relation (13a) and

$$\frac{b_n}{b_{n+1}} = -\ \frac{\alpha_n}{\beta_n -}\ \ \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} -}\ \ \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} -}\ \cdots ,$$

we find that

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} \cong \lim_{n\to-\infty} \frac{b_n}{b_{n+1}} \cong -\ \frac{i\omega x_0}{2n}\ . \tag{17}$$

On the other hand, from the asymptotic expansion of F(a,b,c;y) when the parameters a and b tend to infinity[6], we find that the ratio of the hypergeometric functions in (16) is finite. Thus, for $0\le x\le x_0$, the series converges whenever the characteristic eq. (14) is satisfied. The same is true for the series in (15).

The solutions (11) and (15) reduce to the solutions (10) when ω=0. In effect, for this limit the recurrence relations (12a) and (15b) read

$$\Big[\beta_n\ b_n\Big]_{\omega=0} = \Big[\ \beta'_n\ b'_n\ \Big]_{\omega=0} = 0 \tag{18}$$

or, in order to have nontrivial solutions,

$$B_3+(n+\nu)(n+\nu+B_2-1) = 0 \quad \text{and} \quad B'_3+(n+\nu')(n+\nu'+B'_2-1) = 0.$$

This implies that

$$n+\nu = \frac{1-B_2}{2} + i\left[B_3-\left(\frac{B_2-1}{2}\right)^2 \right]^{\frac{1}{2}}, \quad n+\nu' = \frac{1-B'_2}{2} + i\left[B'_3-\left(\frac{B'_2-1}{2}\right)^2 \right]^{\frac{1}{2}}. \tag{19}$$

Inserting these results into (11) and (15a) and suppressing the summation, we obtain (10a) and (10b), respectively. However, these are only formal limits since we have no *a priori* guarantee that those solutions converge. Actually, in general, we have to impose boundary conditions in order that solutions (10) be regular at $x=x_0$.

So far we have assumed that there is no free parameter in the GSWE (1) and have introduced ν and ν' into the solutions aiming to assure the characteristic equations. However, when x represents an angular variable, there is an arbitrary constant of separation and we must have $\nu=\nu'=0$. In this case, a careful analysis of the process by which we have obtained (12a) and (15b) shows that we can restrict n to $0\le n\le\infty$ on condition that $\gamma_{-1}=0$. On the other hand, the definition of the Jacobi polynomial[7],

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{n!(1+\alpha)_n} F(-n,\ n+\alpha+\beta+1,\ 1+\alpha;\ 1/2-z/2) \tag{20a}$$

where the Pochammer symbol is given by

$$(1+\alpha)_n = \Gamma(n+1+\alpha)/\Gamma(1+\alpha), \tag{20b}$$

allows us to write our solutions as

$$U_1 = e^{i\omega(x-x_0)} \sum_{n=0}^{n=\infty} A_n P_n^{(B_2+B_1/x_0-1,\ -B_1/x_0-1)}(2x/x_0-1), \qquad (21a)$$

$$U_2 = e^{i\omega(x-x_0)}\,(x-x_0)^{1-B_2-B_1/x_0} \times$$

$$\sum_{n=0}^{n=\infty} A'_n\, P_n^{(B'_2+B_1/x_0-1,\ -B_1/x_0-1)}(2x/x_0-1). \qquad (21b)$$

Here we again used the Pochammer symbols to write

$$A_n = n!\,(B_2+B_1/x_0)_n\, b_n \text{ and } A'_n = n!\,(B'_2+B_1/x_0)_n\, b'_n\,. \qquad (21c)$$

The expansion (21a) is a little more general than the solution (19) of Fackerell and Crossman, in the sense that we have not specified the parameters appearing in (1). However, the expansion (21b) — derived from (6b) and (21a) — does not include the second Fackerell-Crossman solution; in despite of this, it affords (10b) when ω=0 on the contrary of that second Fackerell-Crossman expression. A generalization of the second Fackerell and Crossman expansion could be obtained by taking

$$0\ 0\ ?\qquad y=(x_{9\bar{p}}-x)/x_0 \text{ and } \quad U = e^{i\omega x_0 y}\,Y,$$

instead of (7); then we would find

$$y(1-y)\frac{d^2Y}{dy^2} + \left[B_2 + \frac{B_1}{x_0} - \left(B_2 - 2i\omega x_0\right)y - 2i\omega x_0 y^2\right]\frac{dY}{dy} +$$

$$\left[i\omega x_0\left(B_2 + \frac{B_1}{x_0}\right) - B_3 - i\omega x_0\left(B_2 - 2i\eta\right)y\right]Y = 0.$$

This means that, if $U(B_1,B_2,B_3;\eta,\omega;x-x_0)$ is a solution to eq. (1), then $U(B_1,B_2,B_3;-\eta,-\omega;x-x_0)$ is another one[5].

III. APPLICATION: SCALAR TEST FIELDS IN FRW UNIVERSES WITH DUST

In this section we deal with the equation for the time dependence of massive scalar test fields coupled conformal and minimally to the gravity of nonflat FRW universes filled with dust. By changes of variables we reduce it to a GSWE and, since there is no free parameters in it, we expand its solutions in series of hypergeometric or Coulomb functions (with $\nu\neq0$) depending on whether the spatial curvature is positive or negative, respectively.

Thus, consider the FRW line element in the conformally static form

$$ds^2=[A(\tau)]^2\left[d\tau^2-d\chi^2-\frac{\sin^2(\sqrt{\varepsilon}\chi)}{\varepsilon}\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] , \tag{22}$$

where $\varepsilon=\pm1$ is the spatial curvature. Then the time dependence of a scalar test field with mass M satisfies the equation[8] ($\hbar=c=1$)

$$\frac{d^2T}{d\tau^2} +\left[k^2+M^2A^2+(6\xi-1)\left(\frac{1}{A}\frac{d^2A}{d\tau^2}+ \varepsilon\right)\right]T=0, \tag{23}$$

in which ξ is zero or 1/6 for minimal or conformal coupling, respectively. k is a constant of separation that is determined by imposing boundary conditions on the spatial dependence of the scalar field; its values are

$$k=1,2,\dots, \quad \text{if } \varepsilon=-1 \text{ and } 0<k=\text{real}<\infty \text{ , if } \varepsilon=1. \tag{24}$$

For an universe filled with dust, that is, for the scale factor

$$A=a_0[1-\cos(\sqrt{\varepsilon}\tau)]/\varepsilon, \tag{25}$$

the equation (23) becomes

$$\frac{d^2T}{d\tau^2}+\left\{k^2+(Ma_0)^2\left[1-\cos(\sqrt{\varepsilon}\ \tau)\right]^2+\frac{\varepsilon(6\xi-1)}{1-\cos(\sqrt{\varepsilon}\tau)}\right\}T=0. \tag{26}$$

Defining a parameter d by

$$d=1 \ (\text{if } \xi=0) \text{ and } d=0 \ (\text{if } \xi=1/6) \tag{27a}$$

and performing the substitutions

$$x=1+\cos(\sqrt{\varepsilon}\tau), \qquad T=(x-2)^d\ U(x) \tag{27b}$$

in the eq. (26), it results that U must satisfy the equation

$$x(x-2)\frac{d^2U}{dx}+\left[-1+(1+2d)x\right]\frac{dU}{dx}+\left[d-\varepsilon k^2-\varepsilon M^2a_0^2x(x-2)+2M^2a_0^2(x-2)\right]U=0 \tag{28a}$$

which is a GSWE in the form given in eq. (1) and with the constants ω, η, x_0

and B_1 given by

$$\omega = \eta = Ma_0\sqrt{-\varepsilon}, \quad x_0=2, \quad B_1=-1, \quad B_2=2d+1 \quad \text{and} \quad B_3=d-\varepsilon k^2. \qquad (28b)$$

As the constant of separation k is given by (24) and d by (27a), all these parameters are known and , consequently, we have to use representations to the solutions with an arbitrary parameter in order to satisfy the characteristic equations. On the other hand, for M=0 we have $\omega=0$. Then from (10) and (27) we find the solutions

$$T_1=y^d F(d+k\sqrt{\varepsilon}, d-k\sqrt{\varepsilon}, 2d+1/2;y), \qquad (29a)$$

$$T_2=y^{1/2-d} F(1/2-d+k\sqrt{\varepsilon}, 1/2-d-k\sqrt{\varepsilon}, 3/2-2d;y), \qquad (29b)$$

where $y=[1-\cos(\sqrt{\varepsilon}\tau)]/2$. For conformal coupling (d=0) the hypergeometric functions in (29) may be written in terms of elementary functions and the solutions reduce to linear combinations of exponentials, as expected.

III.1. EXPANSIONS IN SERIES OF HYPERGEOMETRIC FUNCTIONS ($\varepsilon=1$)

Now let us consider (26) when $\varepsilon=1$. We have

$$\omega = \eta = iMa_0, \quad x_0=2, \quad B_1=-1, \quad B_2=2d+1, \quad B_3=d-k^2, \quad B'_2=2-2d,$$
$$B'_3=(d-1/2)^2-k^2 \quad \text{and} \quad y=\frac{1-\cos\tau}{2}=[\sin(\tau/2)]^2. \qquad (30)$$

Thus, (11), (15) and (27) yield

$$T_1 = e^{Ma_0(1-\cos\tau)} (\sin(\tau/2))^{2d} \times$$

$$\sum_{n=-\infty}^{n=\infty} b_n \, F[-n-\nu,\ n+\nu+2d,\ 2d+1/2;\ (\sin(\tau/2))^2], \tag{31a}$$

$$T_2 = e^{Ma_0(1-\cos\tau)} (\sin(\tau/2))^{1-2d} \times$$

$$\sum_{n=-\infty}^{n=\infty} b'_n \, F[-n-\nu',\ n+\nu'+1-2d, 3/2-2d;\ (\sin(\tau/2))^2]. \tag{31b}$$

The recurrence relations for b_n and b'_n are obtained by substituting (30) into (12) and (15); we find:

$$\alpha_n = -Ma_0 \frac{(n+\nu+1)(n+\nu+1/2)(n+\nu+d+1/2+Ma_0)}{(n+\nu+d\ +1/2)(n+\nu+d+1)},$$

$$\beta_n = k^2-d-(n+\nu)(n+\nu+2d)+M^2a_0^2 \frac{(2d+1/2)(2d-1)+2(n+\nu)(n+\nu+2d)}{(n+\nu+d+1/2)(n+\nu+d+1)}, \tag{32a}$$

$$\gamma_n = Ma_0 \frac{(n+\nu+2d-1)(n+\nu+2d-1/2)(n+\nu+d-1/2-Ma_0)}{(n+\nu+d-1)(n+\nu+d-1/2)};$$

$$\alpha'_n = -Ma_0 \frac{(n+\nu'+1)(n+\nu'+1/2)(n+\nu'-d+1+Ma_0)}{(n+\nu'-d\ +1)(n+\nu'-d+3/2)},$$

$$\beta'_n = k^2-(d-1/2)^2-(n+\nu')(n+\nu'-2d+1)+$$

$$-2M^2a_0^2\,\frac{d(3/2-2d)-(n+\nu')(n+\nu'-2d+1)}{(n+\nu'-d)(n+\nu'-d+1)}, \tag{32b}$$

$$\gamma'_n = Ma_0\,\frac{(n+\nu'-2d)(n+\nu'-2d+1/2)(n+\nu'-d-Ma_0)}{(n+\nu'-d-1/2)(n+\nu'-d)}\ .$$

The limits for vanishing masses are given by (29) . On the other hand, for conformal coupling (d=0), the hypergeometric functions in (31) can be written as[6]

$$F(-n-\nu,n+\nu,1/2;\sin^2(\tau/2))=\cos[(n+\nu)\tau], \tag{33a}$$

$$F(-n-\nu',n+\nu'+1,3/2;\sin^2(\tau/2))=\frac{\sin[(n+\nu'+1/2)\tau]}{(2n+2\nu'+1)\sin(\tau/2)}, \tag{33b}$$

and then we have

$$T_1 = e^{Ma_0(1-\cos\tau)} \sum_{n=-\infty}^{n=\infty} b_n \cos[(n+\nu)\tau], \tag{34a}$$

$$T_2 = e^{Ma_0(1-\cos\tau)} \sum_{n=-\infty}^{n=\infty} \frac{b'_n}{n+\nu'+1/2} \sin[(n+\nu'+1/2)\tau]. \tag{34b}$$

These solutions are both finite at $\tau=0$, that is, at the singularity of the spacetime defined by (22) and (25). For minimal coupling (d=1), on the contrary, the solution T_2 given in (33b) is divergent at $\tau=0$.Nevertheless, the full wave function appears multiplied by $A^{-1/2}$ and, consequently, diverges at $\tau=0$ in either cases.

III.2 EXPANSIONS IN SERIES OF COULOMB WAVE FUNCTIONS ($\varepsilon=-1$)

Taking $\varepsilon=-1$ in (27-28) we obtain

$$x=1+\cosh\tau, \qquad T=(x-2)^{d}\, U(x), \tag{35a}$$

$$\omega=\eta=Ma_0, \quad x_0=2, \quad B_1=-1, \quad B_2=2d+1 \quad \text{and } B_3=d+k^2 \tag{35b}$$

and we see that $2\le x=\text{real}<\infty$. In terms of Leaver's expansions in series of Coulomb wave functions (Appendix A) a general solution to (26-28) consists of linear combinations of $T^{(+)}$ and $T^{(-)}$, where

$$T^{(\pm)}=(x-2)^{d}U^{(\pm)}(x)=(x-2)^{d}x^{-d-1/2}\sum_{n=-\infty}^{n=\infty} b_n\, \mathcal{U}^{(\pm)}_{n+\nu}(Ma_0,Ma_0x), \tag{36a}$$

with

$$\mathcal{U}^{(\pm)}_{n+\nu}(Ma_0,Ma_0x) = (-1)^n\left[\frac{\Gamma(n+\nu+1\pm iMa_0)}{\Gamma(n+\nu+1\mp iMa_0)}\right]^{1/2}\exp\left[\pi Ma_0/2 \mp i\pi(\nu+1/2)\right]\times$$

$$(2Ma_0x)^{n+\nu+1}\exp(\pm iMa_0x_0)\quad U(n+\nu+1\pm iMa_0,2n+2\nu+2,\mp 2iMa_0x), \tag{36b}$$

being $U(a,b;z)$ the irregular confluent hypergeometric function[6]. The coefficients b_n in (36a) satisfy the relations (A3) with

$$\alpha_n = -2Ma_0\left[(n+\nu+1)^2+(Ma_0)^2\right]^{1/2} \frac{(n+\nu-d+1)(n+\nu-d+3/2)}{(n+\nu+1)(2n+2\nu+3)},$$

$$\beta_n=(n+\nu+1/2-ik)(n+\nu+1/2+ik) + 2(Ma_0)^2\left[1+ \frac{d(d-1/2)}{(n+\nu)(n+\nu+1)}\right], \tag{37}$$

$$\gamma_n=-2Ma_0\left[(n+\nu)^2 + (Ma\ \ )^2\right]^{1/2}\frac{(n+\nu+d-1/2)(n+\nu+d)}{(2n+2\nu-1)(n+\nu)} .$$

With these redefintions, the characteristic equation (14) remains valid. From the asymptotic behavior of U(a,b;z) (see Ref.4) and from (35-36) we obtain that for $x\to\infty$

$$T^{(\pm)}(x) \propto x^{-1/2}(2Ma_0x)^{\mp iMa_0}\exp(-iMa_0x)\to 0 . \tag{38}$$

On the other hand, the full time dependence is obtained multiplying (36a) by $A^{-1/2}$. Thus, we see that for d=0 it diverges at x=2 regardless of the convergence of the series in (36a). For d=1, it must also diverges since so happens with its limit when $M\to 0$. For $\varepsilon=1$ the solutions (35) were obtained trivially as limits of (31). Now we want to show how and in what sense we can obtain the above solutions for $\varepsilon=-1$ as a limit of the expansions in series of Coulomb wave functions given in (36). In this case, the nontriviality of the limits comes from the behavior of the irregular confluent hypergeometric functions when $\omega\to 0$. In effect, for $\omega\to 0$, Leaver found that we must have

$$\nu = \nu_0 = \frac{1}{2}\left\{-1\pm\left[1+B_2(B_2-2)-4B_3\right]^{1/2}\right\} = -\frac{1}{2} \pm ik . \tag{39}$$

Let us choose $\nu=-1/2+ik$. Then, in (37) we can perform the approximations

$$\alpha_n=-Ma_0(n+ik+1/2-d)(n+ik+1-d)+(n+ik+1),$$

$$\beta_n=n(n+2ik), \tag{40}$$

$$\gamma_n=-Ma_0(n+d+ik-1)(n+d+ik-1/2)+(n+ik-1).$$

Observe that now we cannot put $\alpha_n=\gamma_n=0$ as we did in eq. (18). In effect, for $M\to 0$ the argument of $U(a,b;z)$ in (36b) also goes to zero and we obtain[6]

$$\lim_{M\to 0} U(a,b;\mp 2iMa_0x)=\frac{\Gamma(2n+2ik)}{\Gamma(n+1/2+ik)}\,(\mp 2iMa_0x)^{-2n-2ik}, \quad (n\geq 1) \tag{41a}$$

$$\lim_{M\to 0} U(a,b;\mp 2iMa_0x)=\frac{\Gamma(-2n-2ik)}{\Gamma(1/2-n-ik)}\,, \qquad (n\leq -1) \tag{41b}$$

$$\lim_{M\to 0} U(a,b;\mp 2iMa_0x)=\frac{\Gamma(-2ik)}{\Gamma(1/2-ik)}+\frac{\Gamma(2ik)}{\Gamma(1/2+2ik)}(\mp 2iMa_0x)^{-2ik}, \quad (n=0) \tag{41c}$$

We see that the expressions on the right hand side present the factor M, except for n=0. In addition, since (41c) is just (41a)+(41b) with n=0, we can write

$$\lim_{M\to 0} U^{(\pm)}(x)=x^{-d-1/2}\lim_{M\to 0}\left[\sum_{n=-\infty}^{n=0}\ldots+\sum_{n=0}^{n=\infty}\ldots\right]$$

and use (41a,b) also for n=0. Thus using also (36) we find

$$\lim_{M\to 0} U^{(\pm)}(x)=x^{-d}\left[(2Ma_0x)^{-ik}\sum_{n=0}^{n=\infty}c_n^{(+)}x^{-n}+e^{\pm\pi k}(2Ma_0x)^{ik}\sum_{n=-\infty}^{n=0}c_n^{(-)}x^{n}\right] \qquad (42)$$

where we left off an irrelevant common constant and defined $c_n^{(\pm)}$ as

$$c_n^{(\pm)}=\frac{\Gamma(\pm 2n\pm 2ik)}{\Gamma(1/2\pm n\pm ik)}(2Ma_0)^{\mp n}\ b_n\ . \qquad (43)$$

Once these new constants are related with b_n, we can infer their recurrence relations from the limit of the recurrence relations to b_n. Taking also into account eqs.(40), we find two-term relations as $M\to 0$, namely,

$$c_n^{(\pm)}=2^{-1\pm 2ik\pm n}\Gamma(\pm ik)\ \frac{(d\pm ik)_{\pm n}(d\pm ik+1/2)_{\pm n}}{\sqrt{\pi}\ (\pm n)!\,(1\pm 2ik)_{\pm n}}\ b_0\ . \qquad (44)$$

Hence (42) and the definition of the hypergeometric function furnish

$$\lim_{M\to 0} U^{(\pm)}=b_0\ \frac{x^{-d}}{2\sqrt{\pi}}\left[\ H^{(-)}(x)\ +\ e^{\pm\pi k}\ H^{(+)}(x)\right] \qquad (45a)$$

where

$$H^{(\pm)}(x)=\Gamma(\pm ik)\left(Ma_0x/2\right)^{\pm ik}F(d+1/2\pm ik,d\pm ik,1\pm 2ik;2/x). \qquad (45b)$$

In (45) it is impossible to absorb the indefinite limits $M^{\pm ik}$ in b_0. However, the general solution to eq. (1) is given by linear combinations of $U^{(+)}$ and $U^{(-)}$, say,

$$U(x)=K_1U^{(+)}(x)+K_2U^{(-)}(x) \tag{46a}$$

and, consequently, we find that

$$\lim_{N\to 0} U(x) \propto x^{-d}\left[\bar{K}_1H^{(-)}(x)+\bar{K}_2\ H^{(+)}(x)\right], \tag{46b}$$

being $\bar{K}_1$ and $\bar{K}_2$ given by

$$\bar{K}_1=K_1+K_2 \quad \text{and} \quad \bar{K}_2=e^{\pi k}\ K_1+e^{-\pi k}\ K_2\ .$$

As $\bar{K}_1$ and $\bar{K}_2$ are linearly independent they may be chosen so as to eliminate the indefinite factors that appear in $H^{(-)}$ and $H^{(+)}$. At last, to recover the solutions given in (29), we must use (27b) and apply a linear transformation on the hypergeometric functions of (45b), for example[6],

$$F(a,b,c;z)=L_1z^{-a}\ F(a,a+1-c,a+b+1-c;1-1/z)+$$

$$L_2z^{a-c}(1-z)^{c-a-b}\ F(c-a,1-a,c+1-a-b;1-1/z),$$

where

$$L_1=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad L_2=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ .$$

Finally, note that we selected only one of the roots given in (39). The reason for this is that the other root gives the same result cited above.

IV . FINAL COMMENTS

Our main results are in section II, where we found generalizations of Fackerell-Crossman's solutions to a GSWE. Our expansions in series of hypergeometric functions are convergent for $0 \le x \le x_0$ (x being a real variable and x_0 a real constant) and they allow us to satisfy the characteristic equations even when there is no free parameter in the GSWE. In addition, they formally give two independent limits for $\omega \to 0$, contrarily to Fackerell-Crossman's expansions. On the other hand, if there is an arbitrary parameter in equation (1), we must take $\nu=\nu'=0$ and restrict the summation index to the range $0 \le n < \infty$; then we can rewrite the hypergeometric functions in terms of Jacobi polynomials, as in Fackerell and Crossman. We remark that Leaver's expansions in series of Coulomb wave functions and in series of confluent hypergeometric functions are valid only when there is no free parameter in the differential equation, in contrast with the solutions of section II.

It is worthwhile to emphasize that the results of section II, as well as Leaver's expansions, hold only when the independent variable is real. This is important because, when we try to solve the Dirac equation for massive test fermions in FRW universes with radiation, we find GSWEs for its time dependence. However, if the radiation effective pressure is negative, as in Ref. 9, the dependent variable in eq. (1) will be complex.

In section III, results of sec. II, along with Leaver's expansions in series of Coulomb wave functions, were used to find the time dependence to massive Klein-Gordon test fields in nonflat FRW universes filled with dust .

For conformal coupling, the expansions in series of hypergeometric functions (ϵ=1) reduced to expansions in sine and cosine series. We also found the time dependence for massless fields as limits of the massive cases and saw that for minimal coupling this dependence is given by hypergeometric functions, not by exponential ones.

Finally, the following complementary problems deserve further consideration: (i) solutions to the characteristic equations of section II; (ii) time dependence of massive Dirac test fields in nonflat FRW universes with radiation. In the latter case, it is necessary to find solutions to a GSWE with a complex independent variable, as mentioned before.

APPENDIX : LEAVER'S EXPANSIONS IN SERIES OF COULOMB WAVE FUNCTIONS

In this representation we can write the general solution to eq.(1) as a linear combination of $U^{(+)}$ and $U^{(-)}$, where

$$U^{(\pm)}(x) = x^{-B_2/2} \sum_{n=-\infty}^{n=\infty} b_n \, \mathcal{U}^{(\pm)}_{n+\nu}(\eta, \omega x) \tag{A1}$$

with

$$\mathcal{U}^{(\pm)}_{n+\nu}(\eta, \omega x) = G_{n+\nu}(\eta; \omega x) \pm iF_{n+\nu}(\eta; \omega x)$$

$$= (-1)^n e^{\pi\eta/2} e^{\mp i\pi(\nu+1/2)} \left[\frac{\Gamma(n+\nu+1\pm i\eta)}{\Gamma(n+\nu+1\mp i\eta)}\right]^{1/2} (2\omega x)^{n+\nu+1} e^{\pm i\omega x} \times$$

$$U(n+\nu+1\pm i\eta, 2n+2\nu+2, \mp 2i\omega x). \tag{A2}$$

Here $F_{n+\nu}$ and $G_{n+\nu}$ are the regular and irregular Coulomb wave functions, respectively. The coefficients b_n are given by

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \tag{A3}$$

where

$$\alpha_n = -\omega \frac{[(n+\nu+1)^2+\eta^2]^{1/2}}{(n+\nu+1)(2n+2\nu+3)} \times$$

$$[(n+\nu+1)(n+\nu+2)x_0-(n+\nu+2)(B_1+B_2x_0)+\tfrac{1}{2}B_2(\tfrac{1}{2}B_2x_0+x_0+B_1)], \tag{A4}$$

$$\beta_n = (n+\nu)(n+\nu+1)+B_3 - \tfrac{1}{2}B_2(\tfrac{1}{2}B_2-1) + \frac{\omega\eta}{(n+\nu)(n+\nu+1)} \times$$

$$[(n+\nu)(n+\nu+1)x_0-(B_1+B_2x_0)+\tfrac{1}{2}B_2(\tfrac{1}{2}B_2x_0+x_0+B_1)], \tag{A5}$$

$$\gamma_n = -\omega \frac{[(n+\nu)^2+\eta^2]^{1/2}}{(n+\nu)(2n+2\nu-1)} \times$$

$$[(n+\nu)(n+\nu-1)x_0+(n+\nu-1)(B_1+B_2x_0)+\frac{1}{2}B_2(\frac{1}{2}B_2x_0+x_0+B_1)]. \tag{A6}$$

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