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ON THE COMPLEX TWO-DIMENSIONAL INTERNAL SPACE IN GENERAL RELATIVITY

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ON THE COMPLEX TWO-DIMENSIONAL INTERNAL SPACE IN GENERAL RELATIVITY *

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ABSTRACT: A calculus of vectors in two-dimensional sympletic spaces is developed from the concept of existence of local basis systems. The similarities, as well as the differences, of this calculus with the tetrad formulation of four-dimensional curved spaces are discussed. The affinity and curvature of the sympletic space are derived and its relationship with the affinity and curvature of the usual spinor formalism are given. A system of hybrid geometrical objects displaying a tensor and a spinor index take over the role of the usual Hermitian matrices $\sigma_{\mu}^{K\dot{M}}(x)$.

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INTRODUCTION

The use of basis systems in the spinor calculus was suggested in connection with its applications to the theory of special relativity [1]. Presently we extend the applicability of this concept for spaces with curvature. In this way it is possible to construct in the two-dimensional complex spaces with skew symmetric metrics a formalism of local basis systems which display several similarities with the tetrad formulation on curved manifolds. The term complex two-legs is used for characterizing the geometrical objects which correspond to the tetrad in four-dimensional spaces.

However, the geometrical object which corresponds formally to the tetrad in the usual formalism is not the complex two-leg but instead a linear combination of these components, such combination involves a spinor and a tensor index. From the point of view of the four-dimensional space such object behaves as a set of four null complex vectors.

The notation which is used in this paper follows the usual conventions of the tetrad calculus, by denoting local degrees of freedom by means of the same letter as the "coordinate" index but inside a bracket. Presently both types of indices are spinor indices so that the above term "coordinate index" is purelly formal. All types of spinor indices are denoted by capital latin letters. Indices corresponding to the four-dimensional space are denoted by greek letters.

1. RECIPROCAL BASIS SYSTEM IN S_2

Let S_2 be a two dimensional sympletic space, that is, a linear vector space over the field of the complex numbers in which there exists a non-degenerate skew symmetric bilinear inner product. Explicitly, given $u, v \in S_2$ and α a complex

$$u \circ v = - v \circ u$$

$$(\alpha u) \circ v = \alpha u \circ v, \; u(\alpha v) = \alpha(u \circ v)$$

$$(u+v) \circ w = u \circ w + v \circ w$$

$$u \circ (v+w) = u \circ v + u \circ w$$

$$u \circ v = 0 \text{ for all } v \in S_2 \text{ implies } u = 0.$$

We introduce into S_2 a system of basis vectors $h_{(1)}$ and $h_{(2)}$ such that

$$h_{(1)} \circ h_{(2)} = 1 \tag{1}$$

In this paper we shall use explicitly the index notation since this will be important for our purposes, as will be clear during the treatment. The realization of the above relations in this notation is obtained by introducing a skew symmetric ϵ_{AB}, playing the same role as the symmetric metric tensor in the four-dimensional space.

$$u \circ v = \epsilon_{AB} \, u^A \, v^B = u_B \, v^B \; . \tag{2}$$

The quantity $h_{(1)}$ and $h_{(2)}$ have contravariant indices,

$$h_{(A)} = (\, h_{(A)}^{B})$$

but they may also be presented with covariant indices according to

$$h_{B(R)} = h^A_{(R)}\ \epsilon_{AB}\ .$$

The relation (1) reads as

$$h_{(1)} \circ h_{(2)} = \epsilon_{AB}\ h^A_{(1)}\ h^B_{(2)} = h_{B(1)}\ h^B_{(2)} = 1\ .$$

A reciprocal basis system may be introduced as the set of two vectors of S_2 which satisfy

$$h^{(A)} \circ h_{(B)} = -\ h_{(B)} \circ h^{(A)} = \delta^{(A)}_{(B)} \tag{3}$$

we have

$$h^{(1)}_A = -\ h_{A(2)}\ . \tag{4-1}$$

$$h^{(2)}_A = h_{A(1)}\ . \tag{4-2}$$

It is possible to construct an unimodular matrix with the components of the vectors $h_{(1)}$ and $h_{(2)}$,

$$M = \begin{pmatrix} h^1_{(1)} & h^1_{(2)} \\ h^2_{(1)} & h^2_{(2)} \end{pmatrix} \tag{5}$$

since

$$|M| = h_{(1)} \circ h_{(2)} = 1$$

note that

$$h_{1(A)} = -h^2_{(A)},\ h_{2(A)} = h^1_{(A)} \tag{6}$$

as consequence of (1), (4) and (6) we get

$$h^A_{(1)} h^{(1)}_B + h^A_{(2)}\ h^{(2)}_B = \delta^A_B \tag{7}$$

the inverse matrix of (5) is

$$M^{-1} = \begin{pmatrix} h^2_{(2)} & -h^1_{(2)} \\ -h^2_{(1)} & h^1_{(1)} \end{pmatrix} = \begin{pmatrix} h^{(1)}_1 & h^{(1)}_2 \\ h^{(2)}_1 & h^{(2)}_2 \end{pmatrix} . \quad (8)$$

It should be noted that

$$h^A_{(1)} = \begin{pmatrix} h^1_{(1)} \\ h^2_{(1)} \end{pmatrix} = \begin{pmatrix} h_{2(1)} \\ -h_{1(1)} \end{pmatrix}, \quad h^{(1)}_A = \begin{pmatrix} h^{(1)}_1 \\ h^{(1)}_2 \end{pmatrix} = \begin{pmatrix} -h_{1(2)} \\ -h_2(2) \end{pmatrix} = \begin{pmatrix} h^2_{(2)} \\ -h^1_{(2)} \end{pmatrix}$$

$$h^A_{(2)} = \begin{pmatrix} h^1_{(2)} \\ h^2_{(2)} \end{pmatrix} = \begin{pmatrix} h_{2(2)} \\ -h_{1(2)} \end{pmatrix}, \quad h^{(2)}_A = \begin{pmatrix} h^{(2)}_1 \\ h^{(2)}_2 \end{pmatrix} = \begin{pmatrix} h_{1(1)} \\ h_{2(1)} \end{pmatrix} = \begin{pmatrix} -h^2_{(1)} \\ h^1_{(1)} \end{pmatrix}$$

Since the $h_{(A)}$ and $h^{(A)}$ are basis vactors we have for any vector of S_2,

$$u = u^{(A)} h_{(A)} = u_{(A)} h^{(A)} \quad (9)$$

where

$$u^{(A)} = h^{(A)} \circ u$$

$$u_{(A)} = u \circ h_{(A)} .$$

In index notation, these relations read, for u a contravariant vector,

$$u^A = u^{(B)} h^A_{(B)} = u_{(B)} h^{(B)A} \quad (10)$$

$$u^{(A)} = h^{(A)}_B u^B \quad (11)$$

$$u_{(A)} = h^B_{(A)} u_B \quad (12)$$

as it is clear from what was seen up to now, all indices of vectors are raised and lowered by the skew symmetric matrices ϵ^{AB}, ϵ_{AB} given by

$$\epsilon^{AB} = \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and all indices between brackets, denoting the different elements of the basis, are raised and lowered by means of the skew symmetric matrices $\epsilon^{(A)(B)}$ and $\epsilon_{(A)(B)}$, with matrix elements

$$\epsilon^{(A)(B)} = \epsilon_{(A)(B)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We may interpret the matrices $\epsilon^{(A)(B)}$ and $\epsilon_{(A)(B)}$ as the operators which transform the basis $h_{(B)}$ into the reciprocal basis $h^{(B)}$ and vice-versa.

$$h^{(B)}_{C} = \epsilon^{(B)(M)}\, h_{(M)C} \qquad (13)$$

$$h_{(B)C} = \epsilon_{(M)(B)}\, h^{(M)}_{C} \qquad (14)$$

These equations may be written in the free-index notation as

$$h^{(B)} = \epsilon^{(B)(M)}\, h_{(M)}$$

$$h_{(B)} = \epsilon_{(M)(B)}\, h^{(M)}$$

multiplying the first on the left by $h^{(N)}$ and the second on the left by $h_{(N)}$, we obtain

$$\epsilon^{(B)(N)} = h^{(N)} \cdot h^{(B)} \qquad (15)$$

$$\epsilon^{(B)(N)} = h_{(N)} \cdot h_{(B)} \qquad (16)$$

from which it follows the matrix elements of the equation written previously. Similarly, we obtain

$$\epsilon^{BM} = h^{M}_{(A)}\, h^{B(A)} = -\, h^{M(A)}\, h^{B}_{(A)} \qquad (17)$$

$$\epsilon_{BM} = h_{M(A)} h^{(A)}_{B} = -h^{(A)}_{M}\, h_{B(A)} \qquad (18)$$

which give the matrix elements written before. It should be noted that the equation (7) which was used in the proof of (17), may be written in two forms which differ by a sign,

$$h^{A}_{(K)}\, h^{(K)}_{B} = -h^{A(K)} h_{(K)B} = \delta^{A}_{B} .$$

The equations (15) and (16), and similarly (17) and (18), can be presented in the form, using once more the fact that the ϵ_{AB}, ϵ^{AB} lower and raise vector indices,

$$\epsilon^{(B)(M)} = h^{(M)}_{A}\, h^{(B)}_{R}\, \epsilon^{AR} \qquad (19\text{-}1)$$

$$\epsilon_{(B)(M)} = h^{A}_{(M)}\, h^{R}_{(B)}\, \epsilon_{AR} \qquad (19\text{-}2)$$

$$\epsilon^{BM} = h^{M}_{(A)}\, h^{B}_{(R)}\, \epsilon^{(A)(R)} \qquad (19\text{-}3)$$

$$\epsilon_{BM} = h^{(A)}_{M}\, h^{(R)}_{B}\, \epsilon_{(A)(R)} \qquad (19\text{-}4)$$

As it is clear, the equations (10), (11), (12) and (19) present the same behaviour as the equations which define a set of tetrad vectors in the four-dimensional space. The only difference is that here we deal with an anti-symmetric ϵ playing the role of a "metric tensor", and the $h_{(A)}$ and $h^{(A)}$ are complex vectors with two components. We may call them by complex two-legs instead of tetrads.

2. AFFINITIES AND CURVATURE IN S_2

Following the usual method of tetrad calculus we may interpret the equations (10) and (11) as defining two vector spaces spanned respectivelly by the basis vectors $h_{(A)}$ and h_A with components

$$h_{(A)} = (h_{(1)}, h_{(2)})$$

$$h_{(1)} = \begin{pmatrix} h^1_{(1)} \\ h^2_{(1)} \end{pmatrix} \quad h_{(2)} = \begin{pmatrix} h^1_{(2)} \\ h^2_{(2)} \end{pmatrix}$$

$$h_A = (h_1, h_2)$$

$$h_1 = \begin{pmatrix} h_1^{(1)} \\ h_1^{(2)} \end{pmatrix} \quad h_2 = \begin{pmatrix} h_2^{(1)} \\ h_2^{(2)} \end{pmatrix}$$

The elements of these vector spaces are the vectors $u = (u^A)$ and $\overset{*}{u} = (u^{(A)})$. In order to distinguish one from the other we used the symbol [2] for the later one. We have, in free-index notation,

$$u = u^{(A)} \, h_{(A)}$$

$$\overset{*}{u} = u^A \, h_A$$

Therefore, we may define the covariant derivatives [2] of both u^A and $u^{(A)}$ according the usual method,

$$u^A_{;\mu} = u^A_{,\mu} + \Gamma_\mu{}^A{}_B \, u^B \tag{20}$$

$$u^{(A)}_{;\mu} = u^{(A)}_{,\mu} + \Lambda_\mu^{(A)}{}_{(B)} \, u^{(B)} \tag{21}$$

From the equation (3) we obtain

$$h^{(A)}_{B;\mu} = - h^{(C)}_B \, h^D_{(C);\mu} \, h^{(A)}_D, \tag{22}$$

thus, the vanishing of $h^A_{(B);}$ implies in the vanishing of $h^{(A)}_{B;\mu}$.
We have

$$h^A_{(B);\mu} = h^A_{(B),\mu} + \Gamma_\mu{}^A{}_R \, h^R_{(B)} - \Lambda_\mu^{(R)}{}_{(B)} \, h^A_{(R)} , \tag{23}$$

imposing the condition $h^{A}_{(B);\mu} = 0$, we can solve (23) for the Λ_μ. We find

$$\Lambda_{\mu\,(B)}^{(R)} = h^{M}_{(B)}\, h^{(R)}_{A}\, \Gamma_{\mu\ \ M}^{\ A} + h^{(R)}_{A}\, h^{A}_{(B),\mu} \ . \qquad (24)$$

The formula (24) closely resembles the relations which exist between the affinity Γ_μ and the Christoffel symbols [3]. Indeed, the condition which conducts to those relationships, namely, $\sigma_{\mu;\nu}=0$ are formally similar to ours conditions $h^{A}_{(B);\mu} = 0$. In passage, we note that the condition that the $h^{A}_{(B)}$ are constant under covariant differentiation implies through (22) that the internal metric components may be set constant under the operation of covariant differentiation.

$$\epsilon^{AB}_{;} = \epsilon_{AB;\mu} = 0 \ , \qquad (25\text{-}1)$$

$$\epsilon^{(A)(B)}_{;} = \epsilon_{(A)(B);\mu} = 0 \ . \qquad (25\text{-}2)$$

This in turn implies that the two matrices $\Gamma_{\mu AB}$ and $\Lambda_{\mu(A)(B)}$ are symmetric. A direct inspection on (24) shows that this symmetry property is satisfied for Λ_μ, if Γ_μ is symmetric. The symmetry of Γ_μ can be obtained from its explicit representation in terms of the matrices σ_λ. [3]

Still from (23) we may write,

$$h^{A}_{(B),\mu} = -\Gamma_{\mu\ \ R}^{\ A}\, h^{R}_{(B)} + \Lambda_{\mu\ \ (B)}^{(R)}\, h^{A}_{(R)} \ . \qquad (26)$$

Since the left hand side of this equation is a gradient, the four-dimensional rotational of the right hand side vanishes. This furnishes us with an integrability condition for the existence of solutions of (26). A direct calculation gives

$$S^{(K)}_{\mu\nu\ (B)} = h^{(K)}_{\ \ A}\, h^{R}_{(B)}\, P^{A}_{\mu\nu\ R} \tag{27}$$

where

$$P^{A}_{\mu\nu\ R} = -\frac{\partial \Gamma_{\mu\ R}^{\ A}}{\partial x^{\nu}} - \frac{\partial \Gamma_{\nu\ R}^{\ A}}{\partial x^{\mu}} + \Gamma_{\nu\ S}^{\ A}\, \Gamma_{\mu\ R}^{\ S} - \Gamma_{\mu\ S}^{\ A}\, \Gamma_{\nu\ R}^{\ S} \tag{28}$$

$$S^{(A)}_{\mu\nu\ (R)} = \frac{\partial \Lambda^{(A)}_{\ \ (R)}}{\partial x^{\nu}} - \frac{\partial \Lambda_{\nu}^{(A)}{}_{(R)}}{\partial x^{\mu}} + \Lambda_{\mu}^{(A)}{}_{(S)} \Lambda_{\mu}^{(S)}{}_{(R)} - \Lambda_{\mu}^{(A)}{}_{(S)} \Lambda_{\nu}^{(S)}{}_{(R)} \tag{29}$$

But these geometrical objects are just the internal curvatures defined by

$$u^{A}_{;\mu\nu} - u^{A}_{;\nu\mu} = P^{A}_{\mu\nu B}\, u^{B}, \quad u^{(A)}_{\ \ ;\mu\nu} - u^{(A)}_{\ \ ;\nu\mu} = S^{(A)}_{\mu\nu\ (B)}\, u^{(B)},$$

and we have the result that $S^{(A)}_{\mu\nu\ (B)}$ is the projection of $P^{A}_{\mu\nu\ B}$ over the space of the vectors $u^{(A)}$, according to the usual relations of the tetrad calculus (by tetrad calculus we mean the calculus of point dependent basis systems in four-dimensions). As it is also clear, we can obtain the equation (27) by using (24), (28) and (29), but the way that was followed for the obtention of (27) looks more elegant to us.

We finish this section by recalling that $\Lambda_{\mu}^{(A)}{}_{(B)}$ and $S^{(A)}_{\mu\nu\ (B)}$ may be written in terms of the four-dimensional objects $\left\{ {\mu \atop \nu\alpha} \right\}$ and $R_{\mu\nu\rho\sigma}$ by using the formulas which connect $\Gamma_{\mu\ B}^{\ A}$ and $P^{A}_{\mu\nu\ B}$ with $\left\{ {\mu \atop \nu\alpha} \right\}$ and $R_{\mu\nu\rho\sigma}$.

3. CONNECTION WITH THE METRIC IN FOUR-DIMENSIONAL SPACE

We may use the two vectors on S_2 defined by

$$J^A = h^A_{(1)}, \quad L^A = h^A_{(2)} \tag{30}$$

according to the condition (1) they satisfy

$$J \cdot L = J_A L^A = 1 \tag{31}$$

with the vectors J and L we form the mixed quantities

$$K^A_\mu = \sigma_\mu^{A\dot{B}} J_{\dot{B}} = \sigma_\mu^{A\dot{B}} \bar{J}_B \tag{32}$$

$$M^A_\mu = \sigma_\mu^{A\dot{B}} L_{\dot{B}} = \sigma_\mu^{A\dot{B}} \bar{L}_B \tag{33}$$

where a bar over J_B and L_B means the conjugate complex of those quantities. The two separate relations (32) and (33) may be written as a column as

$$\phi^A_{(\dot{R})\mu} = \begin{pmatrix} \phi^A_{(\dot{1})\mu} \\ \phi^A_{(\dot{2})\mu} \end{pmatrix} = \begin{pmatrix} K^A \\ M^A_\mu \end{pmatrix} = \sigma_\mu^{A\dot{B}} \bar{h}_{B(R)}$$

from (32) and (33) we get,

$$K^A_{(\mu} K^B_{\nu)} = \sigma_{(\mu}^{A\dot{R}} \sigma^{B\dot{S}}_{\nu)} J_{\dot{R}} J_{\dot{S}}, \quad M^A_{(\mu} M^B_{\nu)} = \sigma_{(\mu}^{A\dot{R}} \sigma^{B\dot{S}}_{\nu)} L_{\dot{R}} L_{\dot{S}} \tag{34}$$

$$K^A_{(\mu} M^B_{\nu)} = \sigma_{(\mu}^{A\dot{R}} \sigma^{B\dot{S}}_{\nu)} J_{\dot{R}} L_{\dot{S}} \tag{35}$$

Using the equations

$$\sigma_{\mu B}{}^{\dot{R}} \sigma_\nu{}^{B\dot{S}} + \sigma_{\nu B}{}^{\dot{R}} \sigma_\mu{}^{B\dot{S}} = 2 g_{\mu\nu} \epsilon^{\dot{R}\dot{S}} \tag{36}$$

We get for the above expressions (34) and (35),

$$K_\mu \cdot K_\nu = -K_\nu \cdot K_\mu$$

$$M_\mu \cdot M_\nu = -M_\nu \cdot M_\mu$$

$$K_\mu \cdot M_\nu + K_\nu \cdot M_\mu = 2g_{\mu\nu}$$

These equations show that $(K^A_\mu) = K_\mu$ and $(M^A_\mu) = M_\mu$ are a set of eight vectors of S_2. The symmetrized scalar product of the vector K_μ by the vector M_ν gives $2g_{\mu\nu}$ as result. Multiplying (34) and (35) by $g^{\mu\nu}$ we obtain

$$K^\mu_A \, K^B_\mu = M^\mu_A \, M^B_\mu = 0 \tag{37}$$

$$K_{A\mu} \, M_{B\nu} \, g^{\mu\nu} = 2 \, \epsilon_{AB} \; . \tag{38}$$

The equations (37) show that the M^μ_A and the K^μ_A are a set of four-vectors of the four-dimensional space with a null norm. Each one of those vectors is a complex vector, so that we have in all four complex null vectors.

Finally, we may write

$$K^A_\mu = H^{(\alpha)}_\mu \, \overset{\circ}{\sigma}{}_\alpha^{A\dot{S}} \, \bar{J}_s \tag{39}$$

$$M^A_\mu = H^{(\alpha)}_\mu \, \overset{\circ}{\sigma}{}_\alpha^{A\dot{S}} \, \bar{L}_s \tag{40}$$

where the $\overset{\circ}{\sigma}_\alpha$ are the Pauli matrices for α equal to 1, 2, 3 and the two-by-two identity matrix for α equal to 4.

The $H^{(\alpha)}_\mu$ are the tetrad components in the four-dimensional space, satisfying

$$H^{(\alpha)}_\mu \, H^{(\lambda)}_\nu \, \overset{\circ}{g}_{\alpha\lambda} = g_{\mu\nu} , \tag{41}$$

where $\overset{\circ}{g}_{\alpha\lambda}$ is the metric of special relativity.

From the definition (32), or similarly from (35), we may reobtain directly the relationship between the curvature tensor $R^\lambda{}_{\mu\nu\alpha}$ and the $P^A_{\mu\nu B}$ as the integrability condition for the existence of solutions of

$$K^A_{\mu;\nu} = 0 \tag{42}$$

(this condition follows from $\sigma_{\mu;\nu}^{K\dot{M}} = 0$ together with our conditions $h^A_{(B);\nu} = 0$). Indeed, from (42) we get

$$K^A_{\mu;\nu} = \left\{ {\lambda \atop \mu\nu} \right\} K^A_\lambda - \Gamma_\nu{}^A{}_R \, K^R_\mu ,$$

and the condition that $K^A_{\mu;\nu\beta} = K^A_{\mu;\beta\nu}$ gives as result

$$R^\lambda_{\mu\nu\beta} K^A_\lambda + P^A_{\beta\nu R} K^R_\mu = 0 .$$

Using again (32), we find after some calculations

$$P^A_{\beta\nu\ B} = \frac{1}{4} R^\lambda{}_{\mu\beta\nu} \sigma_\lambda{}^{A\dot{S}} \sigma^\mu_{B\dot{S}} , \tag{43}$$

which is the well known relationship between these two curvatures [4].

4. THE RELATIONS WITH THE TETRAD FORMULATION IN THE FOUR-DIMENSIONAL SPACE

So far we have stablished a calculus of local basis systems in S_2, with the same general properties of the tetrad calculus in four-dimensional space. However, such similarities are only possible up to some extent. In the usual tetrad calculus the rôle of the metric is substituted by the tetrad.* In our present formulation, what substitutes the metric is not the tetrad (or more properly the complex-two legs) but a combination of these quantities as is shown in the equations (32), (33) and (38). That is, the role of the metric is taken over by a set of hybrid quantities displaying a vector and a spinor index. Such quantities are complex four-vectors with a null norm, and are simultaneously vectors on the sympletic space. If we write,

$$K_\mu^A = V_\mu^A + i\, J_\mu^A \quad ,$$
$$M_\mu^A = F_\mu^A + i\, W_\mu^A \quad ,$$

we obtain from (37)

$$V_A^\mu\, V_\mu^B - J_A^\mu\, J_\mu^B = 0$$
$$V_A^\mu\, J_\mu^B + J_A^\mu\, V_\mu^B = 0$$
$$F_A^\mu\, F_\mu^B - W_A^\mu\, W_\mu^B = 0$$
$$W_A^\mu\, W_\mu^B + W_A^\mu\, W_\mu^B = 0$$

which are relations limiting the total number of independent components in the K_μ^A and M_μ^A .

It is also interesting to note that we may construct a new set of null four-vectors by taking the scalar product of the K_μ and the M_μ with the two-legs,

$$a_\mu = J\circ K_\mu = \sigma_\mu^{A\dot{B}}\, J_A\, J_{\dot{B}} \ ,$$
$$b_\mu = L\circ K_\mu = \sigma_\mu^{A\dot{B}}\, L_A\, J_{\dot{B}} \ ,$$
$$c_\mu = L\circ M_\mu = \sigma_\mu^{A\dot{B}}\, L_A\, L_{\dot{B}} \ ,$$

all those four-vectors are null four-vectors since the matrices multiplying $\sigma_\mu^{A\dot{B}}$ in the above equations are singular Hermitian matrices.

We now stablish the relation existing between the present formalism and the usual tetrad formalism. This relationship will be stablished by means of formulas relating the curvatures on both formalisms.

Starting from the relation (43), and using the formula from the tetrad calculus which relate the Riemann tensor to the curvature in terms of the tetrads, the quantity $T_{\mu\nu}{}^{(\alpha)}{}_{(\beta)}$ defined by

$$A^{(\alpha)}_{;\mu\nu} - A^{(\alpha)}_{;\nu\mu} = T^{(\alpha)}_{\mu\nu\,(\beta)}\, A^{(\beta)} .$$

Which has the form

$$R^{\rho}{}_{\sigma\mu\nu} = H^{\rho}_{(\alpha)}\, H^{(\beta)}_{\sigma}\, T^{(\alpha)}_{\mu\nu\,(\beta)} . \tag{44}$$

We get,

$$P^{A}_{\mu\nu\,R} = \frac{1}{4}\, \overset{\circ}{\sigma}{}^{\lambda A\dot{S}}\, \overset{\circ}{\sigma}_{\alpha\dot{S}R}\, T^{(\alpha)}_{\mu\nu\,(\lambda)} , \tag{45}$$

from (27) we have

$$P^{A}_{\mu\nu\,R} = h^{A}_{(C)}\, h^{(M)}_{R}\, S^{(C)}_{\mu\nu\,(M)} . \tag{46}$$

These last two equations allow us to express the curvature $S^{(C)}_{\mu\nu\,(M)}$ in terms of $T^{(\alpha)}_{\mu\nu\,(\beta)}$, which is the relation we want to obtain.

$$S^{(C)}_{\mu\nu\,(M)} = \frac{1}{4}\, h^{(C)}_{A}\, h^{B}_{(M)}\, \overset{\circ}{\sigma}{}^{\beta A\dot{S}}\, \overset{\circ}{\sigma}_{\alpha\dot{S}B}\, T^{(\alpha)}_{\mu\nu\,(\beta)} . \tag{47}$$

As it can be shown (see the appendix),

$$h^{(B)}_{A}\, \overset{\circ}{\sigma}_{\lambda}{}^{A\dot{S}} = - M^{(B)}_{(\lambda)}\, \bar{J}^{S} + K^{(B)}_{(\lambda)}\, \bar{L}^{S} \tag{48}$$

so that

$$S^{(C)}_{\mu\nu\,(M)} = \frac{1}{4} \left(K^{(C)(\beta)}\, M_{(M)(\alpha)} - M^{(C)(\beta)}\, K_{(M)(\alpha)} \right) T^{(\alpha)}_{\mu\nu\,(\beta)}$$

with

$$M^{(C)\,(\beta)} = \overset{\circ}{g}{}^{\beta\sigma}\, M^{(C)}_{\sigma} ,$$

$$M_{(M)(\alpha)} = M^{(B)}{}_{(\alpha)}\, \epsilon_{(B)(M)} ,$$

similar formulas holds for the $K^{(C)(\beta)}$ and $K_{(M)(\alpha)}$.

APPENDIX

From the equation (32) we obtain, by taking the projection on the two legs,

$$K_{\mu}^{(B)} = h^{(B)}_{A}\, K^{A}_{\mu} = h^{(B)}_{A}\, \sigma^{A\dot{R}}\, J_{\dot{R}}$$

which gives

$$K_{\mu}^{(B)} = h^{(B)}_{A}\, \overset{\circ}{\sigma}_{\alpha}^{\,A\dot{R}}\, H_{\mu}^{(\alpha)}\, \bar{h}_{R(1)}$$

thus,

$$K_{(\lambda)}^{(B)} = H^{\mu}_{(\lambda)}\, K_{\mu}^{(B)} = h_{A}^{(B)}\, \overset{\circ}{\sigma}_{\lambda}^{\,A\dot{R}}\, \bar{h}_{R(1)} \quad \text{(A-1)}$$

Similar calculations starting up with (33) give

$$M_{(\lambda)}^{(B)} = h^{(B)}_{A}\, \overset{\circ}{\sigma}_{\lambda}^{\,A\dot{R}}\, \bar{h}_{R(2)} \quad \text{(A-2)}$$

multiplying (A-1) by $\bar{h}^{S(1)}$ on the right hand side, and (A-2) by $\bar{h}^{S(2)}$ also on the right, and adding up both relations, we obtain

$$K_{(\lambda)}^{(B)}\, \bar{h}^{S(1)} + M_{(\lambda)}^{(B)}\, \bar{h}^{S(2)} = h^{(B)}_{A}\, \overset{\circ}{\sigma}^{\,A\dot{R}}\, \overline{h_{R(M)}\, h^{S(M)}}$$

since

$$h_{R(M)}\, h^{S(M)} = -\, \delta^{S}_{R}$$

we get

$$h^{(B)}_{A}\, \overset{\circ}{\sigma}_{\lambda}^{\,AR} = -\left(K_{(\lambda)}^{(B)}\, \bar{h}^{R(1)} + M_{(\lambda)}^{(B)}\, \bar{h}^{R(2)} \right) \quad \text{(A-3)}$$

but

$$h^{R(1)} = \epsilon^{(1)(K)}\, h^{R}_{(K)} = -\, h^{R}_{(2)} = -\, L^{R}$$

$$h^{R(2)} = \epsilon^{(2)(K)}\, h^{R}_{(K)} = h^{R}_{(1)} = J^{R}$$

therefore (A-3) takes the form of the equation (48) of the text.

* * *

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