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ABSTRACT

A time-independent spherically symmetric solution of a general relativistic nonlinear field equations is obtained. It is shown that the nonlinear negative energy scalar field has a localized solution with a positive mass. It can be regarded as a 3-dimensional extension of the usual kink solution on the Schwarzschild geometry, connecting the two vacuum states from one assymptotically flat space to the other through the Rosen--Einstein bridge.

1. INTRODUCTION

Recent development of nonlinear field model of elementary particles brought an interesting view point on the origin of their mass spectrum and structure. A stable and static solution of nonlinear field equation has been shown to have a particle character and is called "kink" or "soliton"[1,2].

The simplest version of such theories is the so-called $\lambda\phi^4$ model. For the one-dimensional space case, its kink solution has several interesting properties. One of them is that it is topologically separated from the vacuum state so that it is considered to represent a fermion in this space[2]. Such an idea to define a fermion as a state topologically distinguishable from the vacuum is very interesting and useful to, for example, explain the conservation of baryon numbers.

However, unfortunately a simple extension of this model for a 3-dimensional case encounters a difficulty. The pseudo--virial theorem[3,4] does not permit a static, nonsingular, spherically symmetric solution if the potential term of scalar field is defined positive definite.

On the other hand, the formulation of the problem in the view of general relativity brings a new feature in the theory[4,5]. The point is that the effect of general relativity alters the curvature of the space-time as well as its topological structure, hence the pseudovirial theorem is also affected.

In this paper, we show that the general relativistic treatment permits the kink-like solution of the simple $\lambda\phi^4$ source-free Lagrangean. This is possible only if we introduce a Schwarzschild type geometry of the space-time structure.

In § 2 we briefly review why the non-general relativistic $\lambda\phi^4$ theory does not have a 3-dimensional kink solution. We then show in §3 how the effect of general relativity alters the situation. In §4 we show some numerical examples of solutions and discuss the consequences.

2. FIELD EQUATIONS

We write the Lagrangean density as

$$\kappa \mathcal{L} = (-g)^{1/2} \left[\frac{1}{2} R + \varepsilon\{S_{,\alpha}S_{,\beta}\, g^{\alpha\beta} - V(S^2)\}\right] \quad , \kappa = 8\pi G/c^4 \ , \qquad (1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, R the scalar curvature, S a scalar field, and the notation $S_{,\alpha}$ denotes the derivative of S with respect to the coordinate x^α . V is a potential only depending on S^2, and ε is the signature of the field S, and takes the value +1 (usual field) or -1 (ghost field).

For a static and spherically symmetric case, we may choose the line element as

$$ds^2 = e^{2\eta}(dx^o)^2 - e^{2\alpha}(dr)^2 - r^2 \ d\Omega^2 \quad , \qquad (2)$$

where η and α are functions of radial coordinate r.

Together with the definition of line element (2) , Einstein's equation reduces to the following equations[4,5]:

$$\eta_1 = \varepsilon\ r\ S_1^2 - \alpha_1 \quad , \qquad (3)$$

$$2r\alpha_1 = \varepsilon\ r^2\ S_1^2 + 1 - (1 - \varepsilon r^2 V)\ e^{2\alpha} \quad , \qquad (4)$$

$$S_{11} + (\eta_1 - \alpha_1 + 2/r)S_1 - Se^{2\alpha}\ dV/dS^2 = 0 \quad , \qquad (5)$$

where the subscript 1 means the derivative with respect to r. In the weak gravitation limit, the equivalence of energy source and gravitation source gives the pseudo-virial theorem[4] written as

$$\langle S_1{}^2\rangle + 3\ \langle V\rangle = 0 \quad , \qquad (6)$$

where $\langle A\rangle$ means the total space integration of A:

$$\langle A\rangle = 4\pi \int_0^\infty r^2\ dr\ A \quad .$$

Thus it is clear to see that the above pseudo-virial theorem does not permit a nonsingular static and spherically symmetric solution in the weak gravitation limit if V is positive definite. The situation is found to be the same even in the nonlinear limit of gravitation provided that the metrics are non singular everywhere[4].

In this paper, we investigate the potential

$$V(S^2) = \frac{1}{2} \left(\frac{\mu}{f}\right)^2 (1 - f^2 S^2)^2 \tag{7}$$

where f and μ are constants. Then the classical vacuum state of the field is given by $S = \pm f^{-1}$. Eq. (7) is nothing but the usual $\lambda\phi^4$ Lagrangean term except the additional constant $\frac{1}{2}\mu^2/f^2$, which is necessary to eliminate the gravitational source at the classical vacuum state of S.

For the sake of convenience we introduce new variables,

$$x = \mu r \ , \tag{8}$$

$$y = fS \ , \tag{9}$$

$$u = f^2 \mu \left[1 - e^{-2\alpha}\right] r \ , \tag{10}$$

$$v = (f/\mu)^2 V \ . \tag{11}$$

From Eqs. (3), (4) and (5) we get

$$y'' + 2y'/x = - \left[y(1-y^2) + y' \ f^{-2}(u/x^2 - \varepsilon x v)\right] e^{2\alpha} \ , \tag{12}$$

$$u' = \varepsilon x^2 (y'^2 \ e^{-2\alpha} + v) \ , \tag{13}$$

where a prime denotes d/dx, and

$$e^{-2\alpha} = 1 - f^{-2}\, u/x \quad , \quad v = \frac{1}{2}\,(1 - y^2)^2 \quad . \tag{14}$$

Note that in the limit $f \to \infty$ with finite u, Eq. (12) tends to the non-general relativistic $\lambda\phi^4$ model, and hence there is no nonsingular solution which satisfies the boundary condition. To alter the situation, the second term in the parenthesis of the right-hand-side of Eq. (12) should be predominant some where. However it was found[4] that the smallness of f alone (strong gravity) is not sufficient to have a consistent solution as long as metrics are nonsingular. In the following section we show that the introduction of Schwarzschild type geometry in the spacetime structure permits static nonsingular solutions for y.

3. SCHWARZSCHILD GEOMETRY

As stated before, the set of equations (12), (13) and (14) does not have a nonsingular solution which satisfies the boundary condition i.e. $|y|$ tends to unity for large x, as long as the metric potential α is finite everywhere.

Now let us drop this condition so that $e^{2\alpha}$ may have a singularity at $x = x_0$. By a simple order analysis of singularity, we find that a consistent solution is possible only if $e^{2\alpha} \propto (x-x_0)^{-1}$. This nothing but the Schwarzschild type singularity. It is well known that such a singularity in the metric does not imply any physical singularity of the spacetime structure. It can also be seen that the behaviour of y and u near $x = x_0$ are given as $y \propto (x-x_0)^{1/2}$ and $u = \text{const.}$, respectively .

Then we write

$$y = \sqrt{\rho}\ Z(\rho) \quad , \tag{15}$$

$$e^{-2\alpha} = \rho\ E(\rho) \quad , \tag{16}$$

where $\rho \equiv x - x_0$, and Z and E are analytic functions of ρ near $\rho = 0$. In order to maintain the order of singularity, we should have $E(0) \neq 0$.

Inserting Eqs. (15) and (16) into Eqs. (12), (13) and (14), we get

$$Z'' = \frac{1}{\rho^2}\left[\frac{1}{4} - \frac{1}{2f^2\ E}\left(\frac{u}{x^2} - \varepsilon\ xv\right)\right] Z$$
$$- \frac{1}{\rho}\left[Z' + \frac{Z}{x} + \frac{1}{E}\left\{Z + \frac{1}{f^2}\left(\frac{u}{x^2} - \varepsilon\ xv\right)Z'\right\}\right]$$
$$+ \left(\frac{Z^3}{E} - \frac{2}{x}\ Z'\right) \quad , \tag{17}$$

$$E = \left[1 - \frac{1}{f^2}\frac{u}{x}\right] / \rho \ , \tag{18}$$

$$u' = \varepsilon\ x^2 \left\{ \frac{1}{4}\left(Z + 2\rho Z'\right)^2 E + v \right\} , \tag{19}$$

with

$$v = \frac{1}{2}\left(1 - \rho Z^2\right)^2 \quad . \tag{20}$$

The boundary conditions for Z, E and u at $\rho = 0$ $(x=x_0)$ are determined by the following equations:

$$\left[\frac{1}{4} - \frac{1}{2f^2\ E}\left(\frac{u}{x^2} - \varepsilon\ xv\right)\right]_{\rho=0} = 0 \quad , \tag{21}$$

$$Z(0)\left[\frac{1}{4}-\frac{1}{2f^2\,E}\left(\frac{u}{x^2}-\varepsilon\,xv\right)\right]'_{\rho=o}-$$

$$-\left[Z'+\frac{Z}{x}+\frac{1}{E}\{Z+\frac{1}{f^2}\left(\frac{u}{x^2}-\varepsilon\,xv\right)Z'\}\right]_{\rho=o}=0\quad,\tag{22}$$

$$\left[1-\frac{1}{f^2}\frac{u}{x}\right]_{\rho=o}=0\quad.\tag{23}$$

A straight forward but tedious algebra gives

$$u(0)=f^2x_o\quad,\tag{24}$$

$$E(0)=\frac{1}{x_o}\left(2-\varepsilon\,x_o^2\,f^{-2}\right)\quad,\tag{25}$$

$$x_of^{-2}\,Z^2(0)=-2\varepsilon\;.\tag{26}$$

From Eq. (26) we conclude that the positive signature case ($\varepsilon=+1$) has no solution. For $\varepsilon=-1$, we get

$$Z^2(0)=\frac{2}{x_o}\,f^2\quad,\tag{27}$$

$$E(0)=\frac{1}{x_o}\left(2+x_o^2\,f^{-2}\right)\quad.\tag{28}$$

The first derivative of Z at $\rho=0$, $Z'(0)$ is also calculated from Eq. (22) as a function of x_o and f.

Thus for a given value of f, solutions are completely determined specifying only x_o. On the other hand the boundary condition $y\to 1$ for $x\to\infty$ sets an eigenvalue problem for x_o. Note that if this boundary condution is satisfied, the metric $e^{2\alpha}$ automatically presents the Schwarzschild asymptotic behavior $e^{2\alpha}\to(1-2m/x)^{-1}$ for $x\to\infty$, where m is a constant related to the mass of the system. Thus, our potential Eq. (7) completely specifies the mass without introducing any constant of integration.

4. SOLUTIONS AND DISCUSSION

Eqs. (17), (18) and (19) together with the boundary conditions at $x = x_0$, can be solved numerically. For a given value of f, the value of x_0 for which y satisfies the boundary condition at infinity is uniquely determined. In Fig. 1, three solutions of y for different f values are shown. In Fig. 2 we show the typical dependence of u and $e^{-2\alpha}$ on x for the case of f = 1.

The asymptotic value of u, $u(\infty)$ is related the mass of the system M by

$$M = \frac{f^{-2}}{2\mu} \frac{c^2}{G} u(\infty) \tag{29}$$

where $u(\infty)$ is a function of f. In Fig. 3, we plotted the quantity $\left[f^{-2} u(\infty)\right]^{1/2}$ versus f. We note that this quantity tends to zero linearly so that u behaves as $u \sim (f-f_0)^2$ near $f = f_0 \simeq 0.645$. For $f < f_0$, it seems that there is no solution, although we failed to confirm this because of the computational difficulty. In this figure we also plotted x_0/f^2 as a function of f. For large f, it is found that x_0 behaves as $x_0 \simeq 1.30\, f^2$.

The metric potential η can be obtained from the equation

$$\eta' = \frac{1}{2} f^{-2} \{e^{2\alpha} \left[\frac{u}{x^2} + x\, v\right] - x\, y'^2\} . \tag{30}$$

In virtue of Eqs. (21) and (27), we verify that η does not have singularity at $x = x_0$. Taking the boundary condition $\eta(\infty) = 0$ we get

$$\eta = -\frac{1}{2} f^{-2} \int_x^\infty \{e^{2\alpha}\left[\frac{u}{x^2} + x\ v\right] - xy'^2\}dx \quad . \tag{31}$$

The time-component of the metric, $e^{2\eta}$ is shown also in Fig. 2.

Because of the nonsingular behaviour of the metric $e^{2\eta}$ at $x = x_0$, the structure of our spacetime is different from that of the Schwarzschild solution. The line element near $r = r_0 \equiv x_0/\mu$ has the form

$$ds^2 \approx A(dx^0)^2 - \left[(1 - r_0/r)^{-1}\ dr^2 + r^2\ d\Omega^2\right] \tag{32}$$

rather than

$$ds^2 \approx (1 - r_0/r)(dx^0)^2 - \left[(1 - r_0/r)^{-1}\ dr^2 + r^2\ d\Omega^2\right] \tag{33}$$

where A is a constant ($0 < A < 1$). For $r >> r_0$, the line element has the assymptotic form:

$$ds^2 \approx (1 - 2m/r)(dx^0)^2 - \left[(1 - 2m/r)^{-1}\ dr^2 + r^2\ d\Omega^2\right] \tag{34}$$

In spite of the above difference, it is easy to see that our space geometry still exhibits a similar topological structure to the Rosen-Einstein bridge[6,7,8] on the spacelike hypersurface x^0 = const., i.e, two assymptotically flat spaces connected by a throat[7,8] of radius r_0. Thus the square-root of the variable ρ in Eq. (15) is equivalent to the singularity-free coordinate u of Rosen and Einstein which was used to describe the topology of the Schwarzschild geometry. The two assymptotically flat spaces correspond to regions $u > 0$ and $u < 0$, respectively. On the other hand, we observe that our solution $y = \sqrt{\rho}\, Z(\rho)$ is a part of an entire function $y^2 = \rho\, Z^2(\rho)$. The counterpart $y = -\sqrt{\rho}\, Z(\rho)$ is also a solution of the field equations. Taking the branch $y = +\sqrt{\rho}\, Z(\rho)$ in one space (say, $u > 0$) and $y = -\sqrt{\rho}\, Z(\rho)$ in the other ($u < 0$), we get an analytic function y defined on this Schwarzschild-like geometry. Now we conclude that the entire function $y^2 = \rho\, Z^2(\rho)$ is a natural extension of the usual one-dimensional kink solution in our space geometry, connecting the two vacuum states from one flat space to the other through the Rosen-Einstein bridge.

It should be emphasized that the apparent singularity in $e^{2\alpha}$ at $x = x_0$ does not imply any singular behaviour of the space geometry but it is due to the particular topological nature of our space, which is completely nonsingular everywhere[8]. This is the reason why the radius of the Rosen-Einstein bridge is uniquely determined without introducing any arbitrary integral constant when the two parameters μ and f in the original Lagrangean density are specified. The parameter μ^{-1} describes the dimension of the system and f decides the mass of the system except for the scale factor μ^{-1} in Eq. (29). If μ^{-1} is not extremely small ($\mu^{-1} > 10^{-53}$ cm), then the value of f to give the order of elementary particle masses ($\simeq 10^{-24}$ g) is practically f_0. In an appropriate limit of $\mu^{-1} \to 0$ and $f \to f_0$, our model contains a point particle with an arbitrary mass M.

One of objections to our model would arise from the fact that only the negative value of the field signature ε is permissible. Such a field carries a negative energy density in a flat space and, when quantized, it behaves as a ghost.

However in our model the ghost field S generates a curvature of the spacetime and the total energy of the system recovers the positive value. Furthermore, the scalar field tends rapidly to its vacuum state, having no physical effect outside the particle. Thus from the pure classical view point, the ghost scalar field S does not cause any serious difficulties. It seems that the field S is not observable as a usual particle field but it composes fermions and guarantees their stability. Such a situation is quite analogous to that of Weyl's gauge field studied by Utiyama[9].

In spite of the difficulty which arises from the simple quantization of the ghost field in a flat space, we may have to wait to decide whether such a ghost field is really unacceptable or not, untill a satisfactory quantum field theory in a curved space (or quantized general theory of relativity) is established.

An extension of our model to a nonlinear spinor field will be required in order to study the properties of realistic fermions.

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FIGURE CAPTIONS

Fig. 1 - Solutions of y for f = 0.77, 1 and 1.25 plotted versus x.

Fig. 2 - Quantities $\left[u/f^2\right]^{1/2}$ and x_o/f^2 plotted versus f.

Fig. 3 - The function u and metrics $e^{-2\alpha}$ and $e^{2\eta}$ plotted as functions of x for the case of f = 1. The asymptotic value of u, $u(\infty)$ defines the mass of the system.

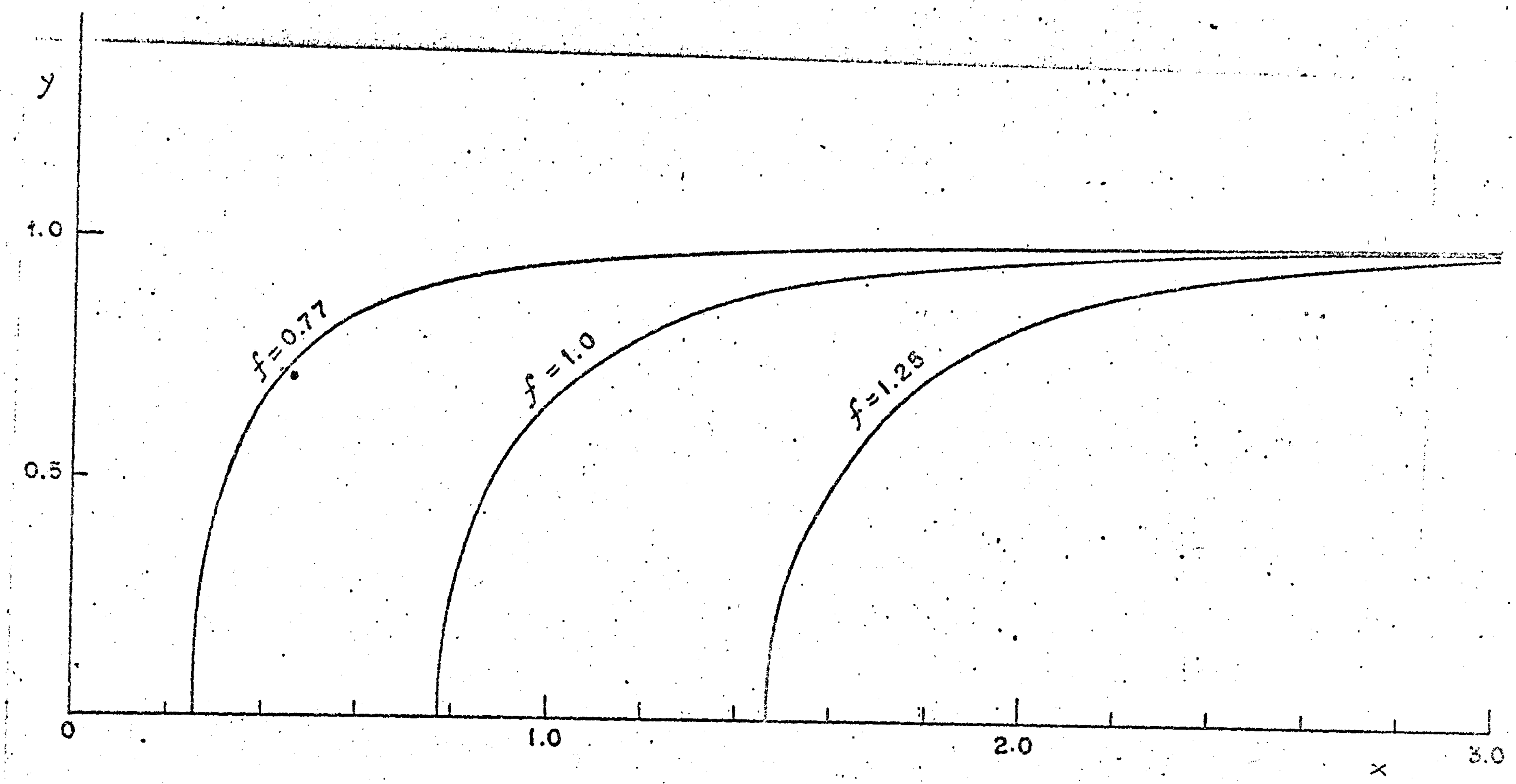

Fig 1. T. Kodama

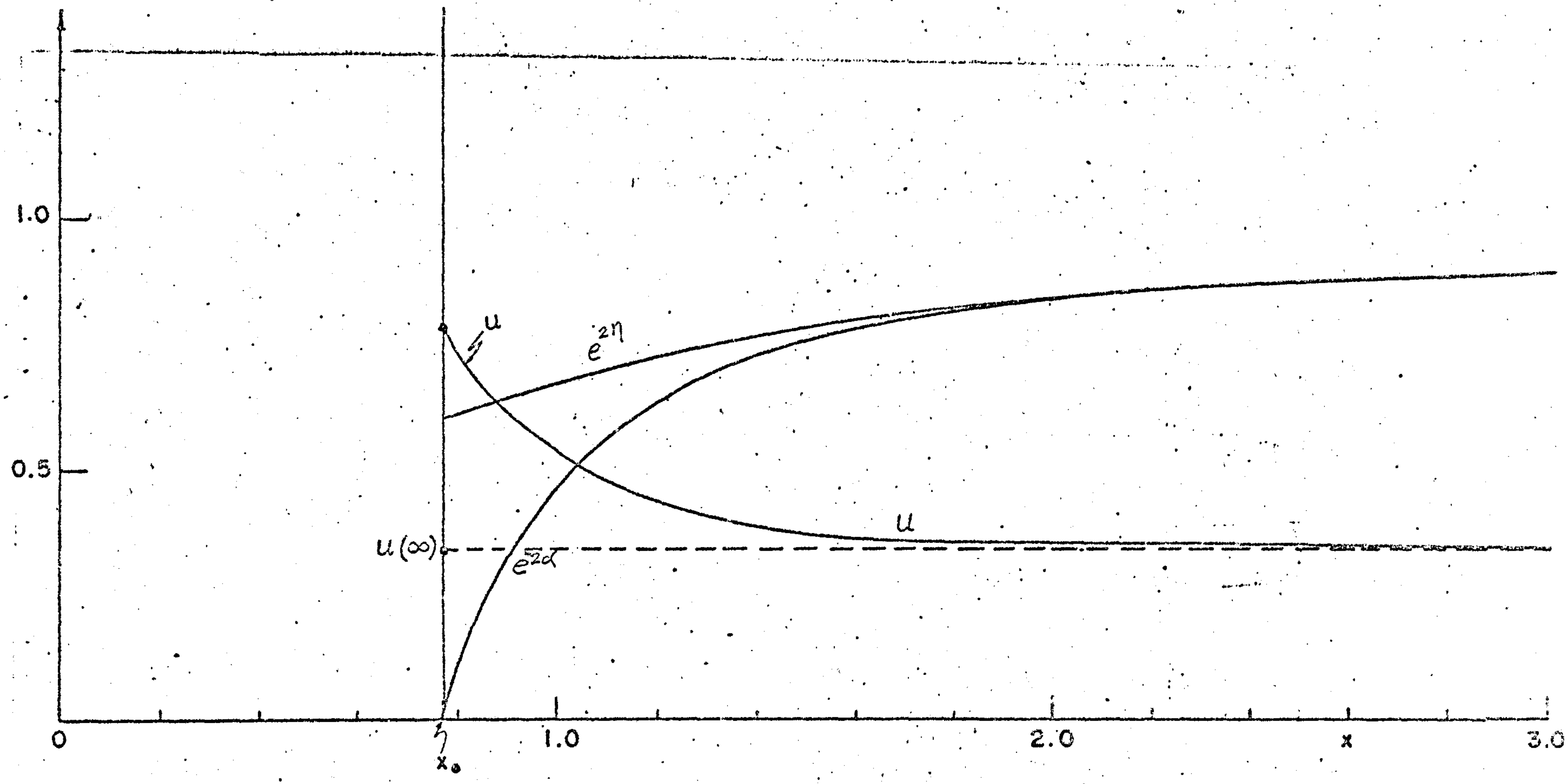

Fig 2. T Kodama

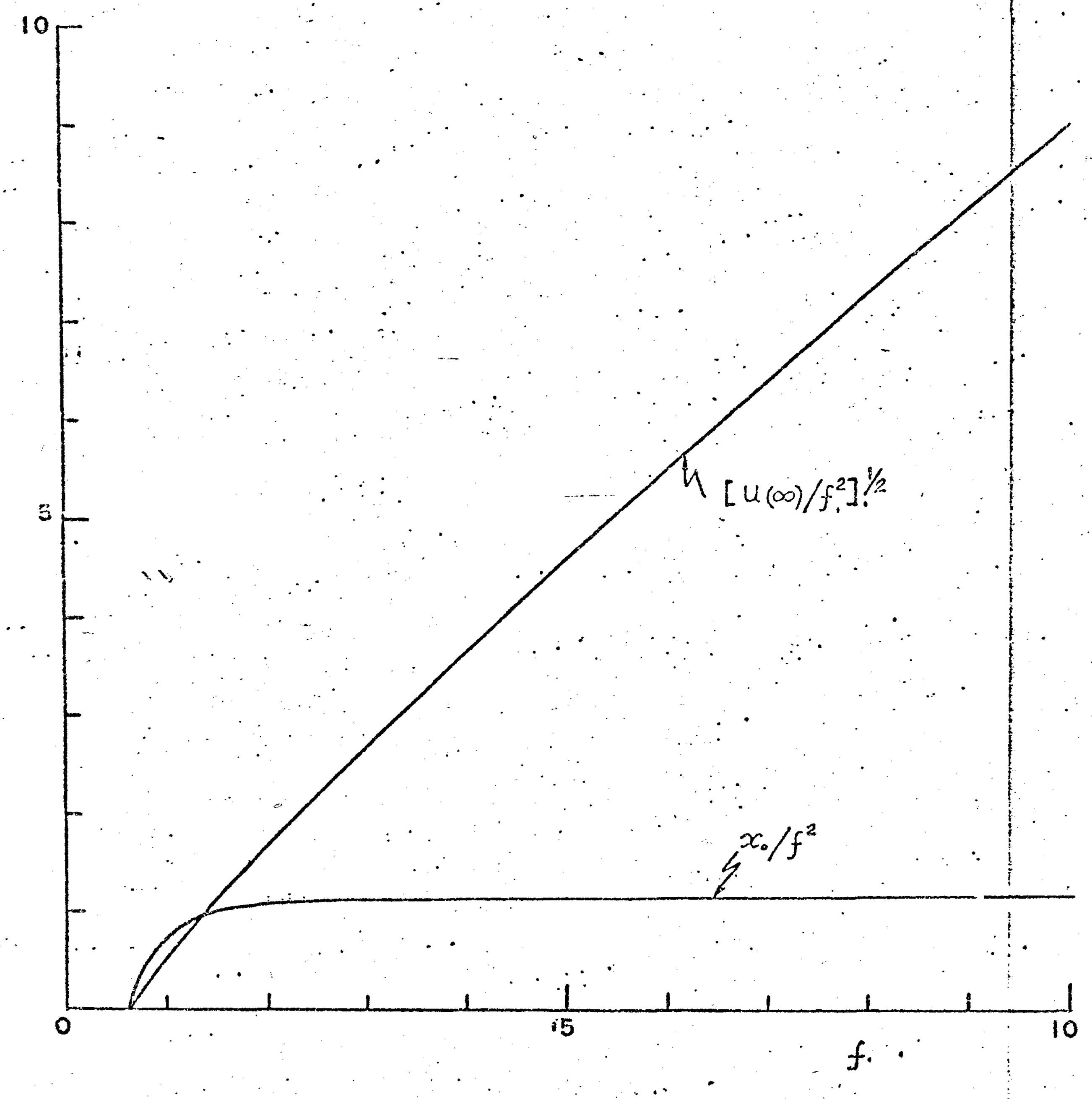

Fig 3 T. Kodama