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MORPHOLOGY OF THE COSMIC RADIATION BASED ON THE ONE FIRE-BALL MODEL

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ABSTRACT

The one-dimensional diffusion equations of the hadronic and electromagnetic components of the cosmic radiation in the atmosphere are integrated assuming a "one fire-ball model" of the hadron interactions with the air nuclei.

Some hints are also given for the numerical calculations necessary for the eventual comparison of the theoretical results with the experimental data.

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1.1. The Diffusion Equation for the Nuclear Active Component of the Cosmic Radiation

The development of the nuclear and the pion components of the cosmic radiation in the atmosphere can be described in a first approximation by the one-dimensional diffusion equations [1,2,3]

$$\frac{\partial F_N(x,E)}{\partial x} = -\frac{1}{\lambda_N} F_N(x,E) + \frac{1}{\lambda_N(1-K_N)} F_N(x,\frac{E}{1-K_N}) \qquad (1)$$

$$\frac{\partial F_\pi(x,E)}{\partial x} = -\frac{1}{\lambda_\pi} F_\pi(x,E) + \frac{1}{\lambda_\pi(1-K_\pi)} F_\pi(x,\frac{1}{1-K_\pi}) +$$

$$+ P_\pi^{NN}(x,E) + P_\pi^{\pi N}(x,E) \qquad (2)$$

where the functions F_N and F_π must satisfy the conditions

$$F_N(0,E) = G(E) \qquad (3)$$

$$F_\pi(0,E) = 0 \qquad (4)$$

G(E)dE is the primary nucleon differential spectrum in the top of the atmosphere. $F_\alpha(x,E)$ is the flux per (cm^2.s.ster.) of a generic hadron $\underline{\alpha}$ with energy in the range E, E+dE at the atmospheric depth $\underline{x}$ (g/cm^2).

We put $\alpha = N$ for nucleons and $\alpha = \pi$ for charged pions.

λ_α is the interaction length (g/cm^2) of the high energy hadron $\underline{\alpha}$ with an air nucleus; K_α is the inelasticity of the hadron α-interaction with an air nucleus.

$P_\pi^{\alpha N}(x,E)$ is the production rate of charged pions produced by the hadron α-interaction with energy in the range E, E+dE at the atmospheric depth $\underline{x}$ (g/cm^2).

In this approximation the production of Kaons is not taken into account; λ_α and K_α are supposed to be constant and independent of the hadron incident energy.

The $\pi \to \mu$ decay is not taken into account because we consider only pions of energy equal or greater than 1 TeV and consider only the diffusion along the vertical direction.

1.2. Solution of the Equation for Nucleons

The solution of the equation (1) that describes the diffusion of the nucleon component is given by[4,5]

$$F_N(x,E) = e^{-x/\lambda_N} \sum_{n=0}^{\infty} x^n/(\lambda_N^n n!)(1-K_N)^{-n}\, G(E/(1-K_N)^n) \qquad (5)$$

with the initial condition:

$$F_N(0,E) = G(E).$$

If the primary nucleon differential spectrum in the top of the atmosphere is approximated by a power function

$$F_N(0,E) = N_0 E^{-(\gamma+1)} \qquad (6)$$

The solution (5) reduces to

$$F_N(x,E) = F_N(0,E)\ e^{-x/La} \qquad (7)$$

where

$$L_a = \frac{\lambda_N}{1-(1-K_N)^{\gamma}} \qquad (8)$$

L_a is the absorption length of the nuclear component (N) in the atmosphere in (g/cm^2). The intensity of nucleons diminishes exponentially with the atmosphere dept x and, the exponent γ is independent of the atmosphere depth.

1.3. Solution of the Equation for Charged Pions

For the integration of the equation (2) we must know explicitly the production rates $P_{\pi}^{\alpha N}(x,E)$.

If we design by $\varphi_{\alpha}(E_0,E)dE$ the number of charged pions with energy range E, E+dE produced by the interaction of an hadron α of incident energy E_0 we have

$$P_{\pi}^{\alpha N}(x,E) = \frac{1}{\lambda_{\alpha}} \int_{(E_{0\alpha})_{MIN}}^{(E_{0\alpha})_{MAX}} \varphi_{\alpha}(E_0,E) F_{\alpha}(x,E_0)\, dE_0 \qquad (9)$$

The explicit form of the functions φ_{α} depends on the specific model we adopt to describe the interactions. Frequently the following conditions are verified

a) $\varphi_{\alpha}(E_0,E)dE$ is an homogeneous function[3] of E and E_0, that is

$$\varphi_{\alpha}(E_0,E)dE = f_{\alpha}\left(\frac{E}{E_0}\right) \frac{dE}{E_0} \qquad (10)$$

b) The minimum energy $(E_{0\alpha})_{MIN}$ is proportional to E, that is:

$(E_{0\alpha})_{MIN} = E/B_{\alpha}$ where B_{α} is some constant

c) The spectrum $F_{\alpha}(x,E)$ extends to $E = \infty$. If that conditions are satisfied, putting $\eta = E/E_{0\alpha}$ we have

$$P_{\pi}^{\alpha N}(x,E) = \frac{1}{\lambda_{\alpha}} \int_{0}^{B_{\alpha}} f_{\alpha}(\eta) F_{\alpha}(x,\frac{E}{\eta}) \frac{d\eta}{\eta} \tag{11}$$

The functions f_{α} depends on the specific model adopted to describe the multiple production of pions in the interaction. We suppose, for a moment, that these functions are known. The calculus of the $P_{\pi}^{\alpha N}(x,E)$ is very simplified, when the primary nucleon differential spectrum in the top of atmosphere is approximated by the power function (6).

In this case $F_N(x,E)$ is given by (7)(8) and we have

$$P_{\pi}^{NN}(x,E) = \frac{1}{\lambda_N} \int_{0}^{B_N} f_N(\eta) F_N(0, \frac{E}{\eta}) \, e^{-x/L_a} \frac{d\eta}{\eta} \tag{12}$$

$$= \frac{c_N}{\lambda_N} F_N(x,E) = \frac{c_N}{\lambda_N} N_0 \, E^{-(\gamma+1)} \, e^{-x/L_a} =$$

$$= A(x) \, E^{-(\gamma+1)}$$

where

$$c_N = \int_{0}^{B_N} f_N(\eta) \eta^{\gamma} \, d\eta$$

The rate of production

$$P_{\pi}^{\pi N}(x,E) = \frac{1}{\lambda_{\pi}} \int_{0}^{B_{\pi}} f_{\pi}(\eta) F_{\pi}(x,\frac{E}{\eta}) \frac{d\eta}{\eta} \tag{13}$$

cannot be determined so straightfully because $F_{\pi}(x,E)$ is precisely the unknow function of the integro-differential equation (2). Fortunately, in this special cases this equation can be exactely integrated by the method of separation of variables.

The result is

$$F_\pi(x,E) = Y(x)\; E^{-(\gamma+1)} \qquad F_\pi(0,E) = 0 \tag{14}$$

with

$$Y(x) = \frac{c_N}{\lambda_N} N_0 \frac{e^{-x/\mathcal{L}_\pi} - e^{-x/L_a}}{\frac{1}{L_a} - \frac{1}{\mathcal{L}_\pi}} \tag{15}$$

where

$$\mathcal{L}_\pi = 1/\Lambda_\pi \tag{16}$$

$$\Lambda_\pi = \frac{1-(1-K_\pi)^\gamma}{\lambda_\pi} - \frac{c_\pi}{\lambda_\pi} = \frac{1}{L_\pi} - \frac{c_\pi}{\lambda_\pi} \tag{17}$$

and

$$c_\pi = \int_0^{B_\pi} f_\pi(\eta)\eta^\gamma \; d\eta \tag{18}$$

L_π is the absorption length of the π component in the atmosphere (g/m^2)

$$L_\pi = \frac{\lambda_\pi}{1-(1-K_\pi)^\gamma} \tag{19}$$

The exact solution (15) depends on two constants c_N and c_π that are to be determined with the introduction of an specific model for the interaction.

Taking into account the explicit form of $F_\pi(x,E)$ (14) and (15) the rate of production of charged pions can be easily obtained with the aid of (13). We have then

$$P_{\pi}^{\pi N}(x,E) = B(x)\ E^{-(\gamma+1)} \tag{20}$$

where

$$B(x) = \frac{c_N c_\pi}{\lambda_N \lambda_\pi} N_0 \frac{e^{-\frac{x}{\mathcal{L}_\pi}} - e^{-x/L_a}}{\frac{1}{L_a} - \frac{1}{\mathcal{L}_\pi}} \tag{21}$$

The solution $F_\pi(x,E)$ presents a singularity for $\frac{1}{L_a} = \frac{1}{\mathcal{L}_\pi}$. The intensity of charged pions is zero at the top of the atmosphere and varies with the atmospheric depth according to (14) (15).

1.4. The Ratio $F_\pi(x,E)/F_N(x,E)$

This ratio increases with the atmospheric depth x, but the rate of increasing depends on the relative value of L_a and $\mathcal{L}_\pi$. In fact, from (6) (7) (14) and (15) it results

$$\frac{F_\pi(x,E)}{F_N(x,E)} = \frac{c_N}{\lambda_N} \left(\frac{e^{yx}-1}{y}\right) \tag{22}$$

where

$$y = \frac{1}{L_a} - \frac{1}{\mathcal{L}_\pi}$$

if $y > 0$ F_π/F_N increases rapidly with x

if $y < 0$ F_π/F_N increases as $1 - e^{-|y|x}$ and tends to be saturated. In the limit case $y = 0$, we have $\lim_{y\to 0} F_\pi/F_N = \frac{c_N}{\lambda_N}\cdot x$ and the ratio increases linearly with x. What is really the case to be considered that is a question to be decided by the experimental determination of the parameters that figure in (22).

2

2.1. Isotropic Emission of Particles from a Moving Center

Up to here, we have delayed the more possibly the introduction of any specific model of multiple meson production, so that the theory can be applied to many ones.

The hypotheses we have introduced are of general character and seems to be consistent with the fact that all the essential characteristics of the strong interactions changes very slowly with the energy of the incident particle[7,8].

Now to make possible the introduction of some specific models we must consider the isotropic emission of particles from a moving center.

2.2. Integral and Differential Energy Spectra in the Laboratory System of Reference

Let be

$$\psi(E^*)dE^* \frac{d\Omega^*}{4\pi}$$

the energy distribution of the particles emitted isotropically from the moving center. The magnitudes labelled with an asteristic refer to the rest system S^* of the moving center. To obtain the energy integral spectrum in the laboratory sistem (L.S.) we use the Lorentz transformations

$$\left.\begin{aligned} E &= \Gamma(E^* + \beta\, p^* c \cos\theta^*) \\ p\cos\theta &= \Gamma(p^* \cos\theta^* + \beta E^*/c) \end{aligned}\right\} \qquad (23)$$

where E and p refer to the laboratory system and Γ is the Lorentz factor of the moving center relative to the (L.S.).

Now, to obtain the energy integral spectrum of the secondary particles in the (LS)(*) we observe that in order a secondary particle may have an energy $E \geq E_\pi$ in (LS) it must be emitted in S^* with an angle θ^* and energy E^* such that

$$\Gamma(E^* + \beta\, p^* c \cos\theta^*) \geq E_\pi \tag{24}$$

The total number of the particles satisfying this condition, is therefore

$$F(\geq E_\pi) = \iint\limits_{\Gamma(E^* + \beta\, p^* c \cos\theta^*) \geq E_\pi} \psi(E^*) dE^* \frac{d\Omega^*}{4\pi} \tag{25}$$

or

$$F(\geq E\ ) = \frac{1}{2} \int_{E^*_{min}}^{\infty} \psi(E^*) dE^* \left(1 - \frac{E_\pi/\Gamma - E^*}{\beta\, p^* c}\right) \tag{26}$$

where

$$E^*_{min} = \Gamma\ (E_\pi - \beta\, p_\pi c) \tag{27}$$

This value E^*_{min} is the minimum value of the energy E^* that must have an emitted particle in S^*, to have <u>exactly the energy</u> E_π in (LS). This value is obtained solving the equations (23) for E^* which gives.

$$E^* = \Gamma \left[E - \beta\, p_\pi c \cos\theta\right] \tag{28}$$

(*) This method was indicated to the author by Prof.Anna M.F.Endler. See also (8).

and observing that E^*_{min} is verified for cos θ = 1. The upper limit of the integral (26) corresponds to the assumption that the range of $\psi(E^*)$ extends to ∞. The differential energy spectrum in the lab. syst. (LS) will be, then

$$\vartheta(\Gamma,E_\pi) = -\frac{\partial F(>E_\pi)}{\partial E_\pi} = \tag{29}$$

$$= \frac{1}{2}\int_{E^*_{min}}^{\infty} \frac{\psi(E^*)dE^*}{\Gamma\beta\, p^* c} = \frac{1}{\Gamma}\, G(E^*_{min})$$

The term resulting of the differentiation of the integral (26) respect to its lower limit is zero, because E^*_{min} satisfies the condition

$$E_\pi = (E^*_{min} + \beta\, p^*_{min}\, c) \tag{30}$$

that is obtained putting $E = E_\pi$ (fixed) and $\cos\theta^* = 1$, in the first equation of (23).

<u>High Energy Interactions</u>. Let <u>m</u> be the mass of the emitted particles. Since we are specially interested in the study of high energy interactions we limit ourselves to the consideration or particles of energy ≥ 1 TeV. One can easily see that for $E_\pi \geq mc^2$ and $\Gamma^2 >> 1$, the minimum energy given by (27) can be approximated by

$$E^*_{min} \approx \frac{E_\pi}{2\Gamma} + \frac{m^2c^4}{2}\,\frac{\Gamma}{E_\pi} \tag{31}$$

In this case [6]

$$\vartheta(\Gamma,E_\pi)dE_\pi = \frac{dE_\pi}{\Gamma}\, G\left(\frac{E_\pi}{2\Gamma}+\frac{m^2c^4}{2}\,\frac{\Gamma}{E_\pi}\right)$$

$$= h\left(\frac{E_\pi}{\Gamma}\right)\, d\left(\frac{E_\pi}{\Gamma}\right) \tag{32}$$

is an homogeneous function of E_π and Γ.

2.3. The Fire-Ball Model

In this model, it is admitted that in a nuclear collision of very high energy (starting of the accelerator region of 10 GeV up to the cosmic rays region of several hundreds of Tev) one or more intermediate states of matter (fire balls) are formed and the multiple production of mesons results from the decay of these intermediate states.

In the last years, the emulsion chamber-experiments undertaken by the Brazilian Japanese Emulsion Chamber Groups made possible to measure the angles, the transverse momenta, and energy of the secondary particles and γ-rays, and to atribute to these intermediate states constant masses and constant temperatures narrowly distributed around some fixed values, independent of the other caracteristics of the interaction [9]. All the experimental data obtained not only by the Brazilian-Japanese E.C.G. but also those obtained by other Experimental Groups [1,2,3] in balòon experiments and in accelerator work can be consistently explained by assuming the production of "fire balls". Up to date the existence of two kinds of fire balls is well established:

a) a "small" fire-ball (mirim) of mass about 2 or 3 times the mass of the nucleon and

b) a large fire-ball (assū) of mass about 10 times the mass of the small one.

To develop further our calculations and having these facts in view we adopt a "simplified model" of fire-ball based on the following assumptions:

1 - If the incident hadron energy is sufficiently high, new intermediate states of matter (fire ball) are created in the collision.

2 - The fire-ball has constant mass and constant temperature, independent of the hadron incident energy.

3 - The fire-ball evaporates isotropically in its rest system of reference (S^*).

4 - The particles produced in the decay are only pions (first approximation).

5 - The average number as pions emitted (N) is constant and is independent of the hadron incident energy.

6 - The principle of independence of charge is verified.

7 - The fire-ball produced in the hadron-nucleon collisions are the same irrespective the incident hadron is a nucleon or a charged pion. To these assumption we add also the more restrictive one:

8 - In the rest system S^* of the fire-ball the pions emitted obey the distribution law of Bose for the momenta p^* [6] (*).

$$\phi(p^*)dp^* \frac{d\Omega^*}{4\pi} = \frac{kp^{*2}dp^*}{\exp(E^*/p_0c)-1} \frac{d\Omega^*}{4\pi} \qquad (33)$$

Here p_0c is related to the temperature of the intermediate state, measured in units of energy. k is a normalization factor.

(*) This argument was indicated to the author by Prof.C.M.G. Lattes.

2.4. The Differential Energetic Spectrum of Pion in the Laboratory System for an Isotropic Distribution of Pions in the Rest System (S^*) of the Fire-Ball

If we denote by $\mathcal{M}_0$ the mass of the fire-ball and by E_o the incident hadron energy of the collision, we have

$$K_\alpha E_o = \Gamma \mathcal{M}_0 c^2 \tag{34}$$

where K_α is the inelasticity of the collision. Putting $\frac{1}{\Gamma} = \mathcal{M}c^2/K_\alpha E_o$ in the equation (31) we have

$$E^*_{min} = a_\alpha \frac{E_\pi}{E_o} + b_\alpha \frac{E_o}{E_\pi} \tag{35}$$

where

$$a_\alpha = \frac{\mathcal{M}c^2}{2K_\alpha} \qquad b_\alpha = \frac{K_\alpha W_o^2}{2\mathcal{M}c^2} \qquad W_o = m_\pi c^2 \tag{36}$$

Observing also that

$$\frac{1}{2\Gamma} = \frac{a_\alpha}{E_o} \tag{37}$$

the equation (24) becomes

$$\begin{aligned} \mathcal{V}(\Gamma, E_\pi)dE_\pi &= \frac{dE_\pi}{2\Gamma} \int_{E^*_{min}}^{\infty} \frac{\psi(E^*)dE^*}{\beta p^* c} = \\ &= a_\alpha \, g\left(\frac{E_\pi}{E_o}\right) \frac{dE_\pi}{E_o} \end{aligned} \tag{38}$$

Now, assuming the principle of <u>independence</u> of <u>charge</u> to be valid we obtain the differential energetic spectrum (in the laboratory system) of the <u>charged pions</u> emitted isotropically by the fire-ball in it's rest system (S^*).

$$\varphi_\alpha(E_o,E_\pi)dE_\pi = \frac{2}{3} a_\alpha \; g(\frac{E_\pi}{E_o}) \; \frac{dE_\pi}{E_o} \tag{39}$$

where

$$g_\alpha(\eta) = \int_{a_\alpha \cdot \eta + b_\alpha/\eta}^{\infty} \frac{\psi(E^*)dE^*}{\sqrt{E^{*2}-W_o^2}} \tag{40}$$

and

$$\eta = E_\pi/E_o$$

2.5. Rates of Production of Charged Pions

In order to fix the relation between $E_{o,min}$ and E_π in the lower limit of the integral that gives $P_\pi^{\alpha N}(x,E_\pi)$ (relation (9) we observe that the minimum value of E_o (for a fixed value of E_π) is verified when the major part the available energy of the interaction is transfered to a single pion of energy E_π and all the remaining pions take only a minor part of the energy(*). In the limit case we can put

$$K_\alpha \; E_{o,min} = E_\pi \; . \tag{41}$$

So that $E_{o,min} = E_\pi/K_\alpha$ ($K_\alpha \neq 0$). The relations (39) and (41) show that the conditions a) and b) of 1.3, (b) are satisfied, if we make

$$\left.\begin{aligned} f_\alpha(\frac{E}{E_o}) &= \frac{2}{3} a_\alpha \; g_\alpha(\frac{E}{E_o}) \\ \text{and} \qquad & \\ B_\alpha &= K_\alpha \end{aligned}\right\} \tag{42}$$

(*) For the small fire-ball only.

2.6. Rates of Production of Charged Pions for a Fire-Ball that Emits Pions Isotropically in it Rest System of Reference S^* and the Pions Obey the Distribution Law of Bose for the Momenta

The distribution of moments is given by (33). The differential spectrum of the energy in the rest system S^* of the fire-ball is therefore.

$$\psi(E^*)dE^* = \frac{Nk}{c^3} \frac{p^* E^* dE^*}{\exp(E^*/p_0 c)-1} \tag{43}$$

and $g_\alpha(\eta)$ is given explicitly by

$$g_\alpha(\eta) = \frac{Nk}{c^3} \int_{a_\alpha.\eta+b_\alpha/\eta}^{\infty} \frac{E^* dE^*}{\exp(E^*/p_0 c)-1} \tag{44}$$

If we put

$$t = E^*/p_0 c$$

we have

$$g_\alpha(\eta) = \frac{Nk}{c} p_0^2 \int_{\sigma_\alpha \eta+\delta_\alpha/\eta}^{\infty} \frac{tdt}{e^t-1} \tag{45}$$

where

$$\sigma_\alpha = \frac{a_\alpha}{p_0 c} = \frac{\mathcal{M} c^2}{2K_\alpha p_0 c} \qquad \delta_\alpha = \frac{K_\alpha W_0^2}{2\mathcal{M} c^2 (p_0 c)} \tag{46}$$

Now we introduce the Debye function[10]

$$\Phi(x) = \int_x^{\infty} \frac{tdt}{e^t-1} = \zeta(2) = \int_0^x \frac{tdt}{e^t-1} \tag{47}$$

$\zeta(2)$ is the value of the well known Riemann zeta function. $\zeta(x)$ for $x = 2$ ($\zeta(2) = \pi^2/6$).

Taking (47) into account we can write

$$g_\alpha(\eta) = \frac{Nkp_o^2}{c} \quad \Phi(\sigma_\alpha\eta + \delta_\alpha/\eta) \tag{48}$$

and

$$\mathcal{G}_\alpha(E_o,E_\pi)dE_\pi = f_\alpha(\frac{E_\pi}{E_o}) \frac{dE_\pi}{E_o} = f_\alpha(\eta)d\eta \tag{49}$$

where

$$f_\alpha(\eta) = \frac{2}{3} a_\alpha \frac{Nkp_o^2}{c} \Phi(\sigma_\alpha\eta + \delta_\alpha/\eta) \tag{50}$$

$$= \frac{2}{3} \frac{a_\alpha}{p_o c} N(kp_o{}^3)\Phi(\sigma_\alpha\eta + \delta_\alpha/\eta)$$

$$= \frac{2}{3} \sigma_\alpha N(kp_o{}^3)\Phi(\sigma_\alpha\eta + \delta_\alpha/\eta)$$

$$\eta = E_\pi/E_o$$

and

$$\sigma_\alpha = \mathcal{M}c^2/2K_\alpha(p_oc)$$

$$\delta_\alpha = KW_o{}^2/2\ c^2(p_oc)$$

2.7. Explicit Determination for the Constants c_α

Taking into account the explicit values of $f_\alpha(\eta)$ given in (50), we have

$$c_\alpha = \int_o^{K_\alpha} f_\alpha(\eta)\eta^\gamma\ d\eta = \tag{51}$$

$$= \frac{2}{3}\,\sigma_\alpha N(kp_o{}^3) \int_0^{K_\alpha} \Phi(\sigma_\alpha \eta + \delta_\alpha/\eta)\eta^\gamma \, d\eta$$

$$= \frac{2}{3}\, N\sigma_o(kp_o{}^3)\, K_\alpha^\gamma \int_0^1 \Phi(\sigma_o \eta + \delta_o/\eta)\eta^\gamma d\eta$$

where

$$\sigma_o = \sigma_\alpha K_\alpha = \frac{\mathcal{M}c^2}{2(p_o c)} \qquad (52)$$

$$\delta_o = \frac{\delta_\alpha}{K_\alpha} = \frac{W_o^2}{2\mathcal{M}c^2(p_o c)}$$

and, finally

$$c_\alpha = \frac{2}{3}\, N\sigma_o(kp_o{}^3)K_\alpha^\gamma \,.\, \langle \phi\eta^\gamma \rangle \qquad (53)$$

3.1. <u>The Electromagnetic Component $F_{\gamma,e}(x,E)$</u>

The production rate $P_\gamma(x,E)$ of gamma rays produced from the decay of neutral pi-mesons and the production rate $P_{\pi^0}(x,E)$ of neutral pi-mesons are related by [11].

$$P_\gamma(x,E_\gamma) = \int_{\overline{E}_\pi}^{\infty} 2P_{\pi^0}(x,E_\pi^0)\, \frac{dE_{\pi^0}}{p_{\pi^0}} \qquad (54)$$

where

$$\overline{E}_\pi = E_\gamma + m_\pi^2 c^4/4E_\gamma$$

If we restrict ourselves to the consideration of high energy gamma rays (E_γ > 1 Tev) we have approximately

$$P_\gamma(x,E_\gamma) = \int_{E_\gamma}^{\infty} \frac{dE'}{E'}\, P_\pi(x,E') \qquad (55)$$

where $P_\pi(x,E')$ is the rate of production of charged pions.

Now, for the special case of a primary spectrum of the form (6) that we are considering we have by (12) and (20)

$$P_\pi(x,E) = P_\pi^{NN}(x,E) + P_\pi^{\pi N}(x,E) = \left[A(x)+B(x)\right] E^{-(\gamma+1)} \tag{56}$$

$$= C(x)\; E^{-(\gamma+1)}$$

Therefore we have

$$P_\gamma(x,E_\gamma) = \frac{C(x)}{\gamma+1} E_\gamma^{-(\gamma+1)} \tag{57}$$

Now we observe that the number of electrons and gamma rays with energy between E, E+dE, at the atmospheric depth x, (measured in radiation lengths) of an electromagnetic cascade initiated by a gamma of energy E_0 at X = 0 is given by (approximation A).

$$(\pi+\gamma)(E_0,E,X) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{E_0^{-S}}{E^{S+1}} W(S,X)\,dS \tag{58}$$

where

$$W(u,X) = N_1(u)\; e^{\lambda_1(u)X} + N_2(u)\; e^{\lambda_2(u)X} \tag{59}$$

If we denote by X_0 the radiation length in air measured in (g/cm^2) we have

$$W(u,x) = N_1(u)\; e^{\lambda_1(u)x/X_0} + N_2(u)\; e^{\lambda_2(u)x/X_0} \tag{60}$$

where $N_i(u)$ (i = 1,2) are the well known functions of the cascade theory [11,12,13].

Thus we can write for the differential spectrum $F_{\gamma e}(x,E)dE$ of the electromagnetic component

$$F_{\gamma e}(x,E)dE = dE \int_0^x dt \int_E^\infty dE_\gamma \; P_\gamma(t,E_\gamma)(\pi+\gamma)(E_\gamma,E,x-t) \qquad (61)$$

and taking (57), (58) and (60) into account we have

$$F_{\gamma e}(x,E) = D(x,\gamma)\; E^{-(\gamma+1)} \qquad (62)$$

where

$$D(x,\gamma) = \frac{1}{\gamma+1} \int_0^x C(t)W(\gamma,x-t)dt \qquad (63)$$

From (62) the corresponding integral spectrum is readly obtained and we have:

$$F_{\gamma e}(>E,x) = D(x,\gamma)\; \frac{E^{-\gamma}}{\gamma} \; . \qquad (64)$$

3.2. Explicit Form for $D(x,\gamma)$

Performing the integral (62) we have

$$D(x,\gamma) = \frac{N_o C_N}{\gamma+1} \Big[(a_1+b_1)\; e^{\lambda_1^* x/\lambda_N} + (a_2+b_2)\; e^{\lambda_2^* x/\lambda_N} + \\ - (a_1+a_2)\; e^{-\mu(\gamma+1)x} - (b_1+b_2)\; e^{-\Lambda_\pi x} \Big] \qquad (65)$$

where

$$a_i = \frac{R\; N_i(\gamma)}{\mu(\gamma+1)\lambda_N+\lambda_i^*} \qquad i = (1,2) \qquad (66)$$

$$b_i = \frac{S\; N_i(\gamma)}{\lambda_N\; \Lambda_\pi + \lambda_i^*} \qquad (67)$$

$$S = \frac{c_\pi}{\lambda_\pi} \frac{1}{\mu(\gamma+1) - \Lambda_\pi} \qquad (68)$$

$$R = 1 - S \tag{69}$$

$$\mu(\gamma+1) = \frac{1}{L_a} \tag{70}$$

$$\Lambda_\pi = \frac{1}{L_\pi} - \frac{c_\pi}{\lambda_\pi} = \frac{1}{\mathcal{L}_\pi} \tag{71}$$

$$\lambda_i^*(\gamma) = \lambda_i(\gamma) \frac{\lambda_N}{X_o} \qquad i = (1,2) \tag{72}$$

3.3. Two Approximations a) and b)

There are two special cases in which the expression of $D(x,\gamma)$ is very simplified and may be useful for a preliminary choi ce of the parameters at a comparison of the theoretical results with the experimental data.

a) - The pion production of the second generation is disregarded $P_\pi^{\pi N}(x,t) = 0$

$$S = 0 \qquad\qquad R = 1 \tag{73}$$

$$a_i = \frac{N_i(\gamma)}{\frac{\lambda_N}{L_a} + \lambda_i(\gamma)\frac{\lambda_N}{X_o}} = \frac{L_a}{\lambda} \frac{N_i(\gamma)}{1 + \lambda_i(\gamma)\, L_a/X_o} \tag{74}$$

$$b_i = 0 \qquad\qquad i = (1,2) \tag{75}$$

$$D(x,\gamma) = \frac{N_o c_N}{\gamma + 1}\left[a_1\, e^{\lambda_1(\gamma)x/X_o} + a_2\, e^{\lambda_2(\gamma)x/X_o} - (a_1+a_2)\, e^{-x/L_a}\right] \tag{76}$$

$$= \frac{N_o c_N}{\gamma + 1} \sum_{i=1,2} a_i \left(e^{\lambda_i(\gamma) x / X_o} - e^{-x/L_a} \right) \tag{77}$$

$$= \frac{N_o c_N}{\gamma + 1} \sum_{i=1,2} \frac{L_a}{\lambda_N} N_i(\gamma) \frac{e^{\lambda_i(\gamma) x / X_o} - e^{-x/L_a}}{1 + \lambda_i(\gamma) L_a / X_o} \tag{78}$$

$$D(x,\gamma) = \frac{1}{\gamma+1} \frac{N_o c_N}{\Lambda_N} L_a P(\gamma, x, L_a) \tag{79}$$

from which a well known formula of Hayakawa,[11,14,15] can be derived. For this purpose we introduce the integral production spectrum of charged π mesons

$$P_\pi(>E,x) = \int_E^\infty P_\pi(x,E) dE = \int_E^\infty C(x) E^{-(\gamma+1)} dE \tag{80}$$

$$= C(x) \frac{E^{-\gamma}}{\gamma}$$

From (80) (56) and (12) we have

$$P_\pi(>E,0) = C(0) \frac{E^{-\gamma}}{\gamma} = A(0) \frac{E^{-\gamma}}{\gamma} = \frac{c_N N_o E^{-\gamma}}{\lambda_N \gamma} \tag{81}$$

The integral spectrum of $F_{\gamma e}$ (64) will be then

$$F_{\gamma e}(>E,\gamma) = D(x,\gamma) \frac{E^{-\gamma}}{\gamma} =$$

$$= \frac{1}{\gamma+1} P_\pi(>E,0) L_a P(\gamma,x,L_a) \tag{82}$$

$$= \frac{G_\pi(E)}{\gamma + 1} P(\gamma,x,L_a) \quad (*)$$

(*) See formula A-8, ref. [14].

where

$$G_\pi(E) = L_a \, P_\pi(>E,0) \tag{83}$$

b) <u>The absorption lengths of pions and nucleons are supposed to be equal</u>

We have then

$$L_a = L_\pi = \frac{1-(1-K_N)^\gamma}{\lambda_N} = \frac{1-(1-K\ )^\gamma}{\lambda_\pi} \tag{84}$$

The consideration of this special case is suggested by the experimental fact that in the region where we can atribute an absorption length to the nuclear active component the observed absorption length do not change or changes very slowly with the atmospheric depth (x), although the composition π/N varies considerably with $\underline{x}$.

The equality $L_a = L_\pi$ give the same absorption length to both π and N components and so the absorption length of the nuclear active component will be independent of the relation π/N, and will be equal to the common value of $L_a = L_\pi$. To obtain the explicit form of $D(x,\gamma)$ in this case, we suppose that the condition (84) is satisfied. If $c_\pi \neq 0$ we have

$$\mu(\gamma+1) - \Lambda_\pi = \frac{1}{L_a} - \left(\frac{1}{L_\pi} - \frac{c_\pi}{\lambda_\pi}\right) = \frac{c_\pi}{\lambda_\pi}$$

$$S = 1 \qquad R = 0 \qquad a_i = 0$$

$$b_i = \frac{N_i(\gamma)}{\frac{\lambda_N}{\mathcal{L}_\pi} + \frac{\lambda_i(\gamma)\lambda_N}{X_o}} = \frac{\mathcal{L}_\pi}{\lambda_N} \quad \frac{N_i(\gamma)}{1 + \lambda_i(\gamma)\frac{\mathcal{L}_\pi}{X_o}}$$

and

$$D(x,\gamma) = \frac{N_0 c_N}{\gamma+1} \frac{\mathcal{L}_\pi}{\lambda_N} \sum_{i=1,2} N_i(\gamma) \frac{e^{\lambda_i(\gamma)x/X_0} - e^{-x/\mathcal{L}_\pi}}{1 + \lambda_i(\gamma) \frac{\mathcal{L}_\pi}{X_0}} \qquad (85)$$

We see that $D(x,\gamma)$ is given by the same formula (79) with the substitution

$$\mathcal{L}_a \rightarrow \mathcal{L}_\pi$$

where

$$\frac{1}{\mathcal{L}_\pi} = \frac{1}{L_\pi} - \frac{c_\pi}{\lambda_\pi} = \frac{1}{L_a} - \frac{c_\pi}{\lambda_\pi}$$

The integral spectrum $F_{\gamma e}$ will be therefore

$$F_{\gamma e}(>E,x) = \frac{1}{\gamma+1} P_\pi(>E,0). \mathcal{L}_\pi \; P(\gamma,x,\mathcal{L}_\pi) \qquad (86)$$

3.4. The Integral Energy Spectrum (in the L.S.) of the γ-Rays Emitted by a "Fire-Ball" at the Level of the Interaction

The differential energy spectrum (in the L sistem) of the secondary pions emitted by the "fire-ball" in a generic hadron-nucleon interaction is: (50)

$$\varphi_\alpha(E_0,E_\pi)dE_\pi = \sigma_\alpha N k p_0^3 \; \Phi(\sigma_\alpha \eta + \delta\alpha/\eta)d\eta \qquad (87)$$

with

$$\eta = E_\pi/E_0$$

If we admit the principle of independence of charge, the production rate of γ-rays will be (55)

$$P_\gamma(E_\gamma) = 2 \int_{\overline{E}_\pi}^{\infty} \frac{\frac{1}{3} \varphi_\alpha(E_0, E_\pi)}{(E_\pi^2 - m_\pi^2 c^4)^{1/2}} \, dE_\pi \qquad (88)$$

with

$$\overline{E}_\pi = E_\gamma + \frac{m_\pi^2 c^4}{4 E_\gamma}$$

Therefore

$$P_\gamma(E_\gamma) = \frac{2}{3} \sigma_\alpha \, Nkp_0^3 \int_{\overline{E}_\pi/E_0}^{\infty} \frac{\Phi(\sigma_\alpha y + \delta_\alpha/y)dy}{\left|y^2 - (m_\pi c^2/E_0)^2\right|^{1/2}} \qquad (89)$$

with

$$y = \frac{E_\pi}{E_0}$$

The function P_γ given by (89) is not an homogeneous function of E_γ and E_0, because the quotient $\overline{E}_\pi/E_0$

$$\overline{E}_\pi/E_0 = \frac{E_\gamma}{E_0} + \left(\frac{m_\pi c^2}{2E_0}\right)^2 \frac{1}{E_\gamma/E_0}$$

depends on E_0 too, but if

$$\frac{m_\pi^2 c^4}{4E_\gamma} \leq 1,$$

$P_\gamma(E_\gamma)$ will be a function of E_γ/E_0, that is

$$P_\gamma(E_\gamma) = \frac{2}{3} \sigma_\alpha Nkp_0^3 \int_{E_\gamma/E_0}^{\infty} \Phi(\sigma_\alpha y + \delta_\alpha /y)\frac{dy}{y} \qquad (90)$$

The integral spectrum of the γ-rays in the (L.S.) will be then

$$F_{\gamma e}(>E_\gamma) = \frac{2}{3} \sigma_\alpha Nkp_0^3 \int_{E_\gamma}^{\infty} \frac{dW}{E_0} \int_{W/E_0}^{\infty} \Phi(\sigma_\alpha y + \delta_\alpha/y) \, \frac{dy}{y} \qquad (91)$$

$$= \frac{2}{3}\,\sigma_\alpha N k p_0^3 \int_{E_\gamma/E_0}^{\infty} du \int_u^{\infty} \Phi(\sigma_\alpha y + \delta_\alpha/y)\,\frac{dy}{y}$$

This repeated integral can be reduced to a single one (see Appendix II)

$$F_{\gamma e}(>E_\gamma) = \frac{2}{3}\,\sigma_\alpha N k p_0^3 \int_\alpha^{\infty} \Phi(\sigma_\alpha t + \delta_\alpha/t)(1 - \frac{\alpha}{t})dt \qquad (92)$$

(with $\alpha = E_\gamma/E_0$) wich is more convenient for numerical computation.

We see that $F_{\gamma e}(>E_\gamma)$ is an homogeneous function of E_γ and E_0, that is

$$F_{\gamma e}(>E_\gamma) = F_{\gamma e}\left(\frac{E_\gamma}{E_0}\right) \quad .$$

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APPENDIX I

1. To obtain the values of the normalization constants k and the average energy $<\varepsilon^*>$ of the pions in the fire-ball rest-system (S^*) we considered the integrals

$$I = k \int_0^\infty \frac{p^{*2} dp^*}{\exp(E^*/p_o c)-1} = k p_o^3 \int_0^\infty \frac{u^2 du}{\exp\{u^2+\alpha^2\}^{1/2}-1}$$

and

$$<\varepsilon^*> = k \int_0^\infty \frac{p^{*2} E^* dp^*}{\exp(E^*/p_o c)-1} = k p_o^4 \int_0^\infty \frac{u^2 (u^2+\alpha^2)^{1/2} du}{\exp\{u^2+\alpha^2\}^{1/2}-1}$$

where $\alpha = \frac{m_\pi c}{p_o}$ and $u = p^*/p_o$ and computed numerically the integrals that figure in the right side of the equations using Laguerre integration formula

$$\int_0^\infty g(x)dx = \sum_{i=1}^n \omega_i \, e^{x_i} \, g(x_i)$$

with n = 6, abscissas x_i and weight factors ω_i from table 25.9, pg 923 of reference [10].

The same proceeding was used for the computation of $<p^*>$

$$<p^*> = k p_o^4 \int_0^\infty \frac{u^3 du}{\exp\{u^2+\alpha^2\}^{1/2}-1}$$

so doing we obtained

a) for p_o = 80 MeV/c:

$$<p^*> = 265.8 \text{ MeV/c and } <\varepsilon^*> = 307.9 \text{ MeV}.$$

Assuming the value N = 9 for the multiplicity, the resulting mass for the fire-ball is

$$\mathcal{M}_0 = 9 \times 0.308 \text{ GeV} \sim \underline{2.8} \text{ GeV}$$

b) for p_0 = 90 MeV/c:

$$<p^*> = 292.6 \text{ MeV/c and } <\varepsilon^*> = 331.9 \text{MeV}$$

The resulting mass of the fire-ball, with the same value of N = 9, is

$$\mathcal{M}_0 \sim 3.0 \text{ GeV}$$

To control the degree of approximation obtained we computed the value of kp_0^3 (for p_0 = 90 MeV/c) using the formulae of Dobrotin [3]

$$f(p^*)dp^* = \frac{Z^3 p^{*2} dp^*}{m_\pi^3 F(z)} \{\exp Z \sqrt{(p^*/m_\pi)^2+1} -1\}^{-1}$$

$$F(z) = Z^2 \sum_{m=o}^{\infty} k_2 \frac{Z(m+1)}{m+1} \qquad Z = \frac{m_\pi}{T}$$

$$k = \frac{Z^3}{m_\pi^3 F(z)} = \frac{1}{T^3 F(z)}$$

since p_0 = T (Dobrotin) we have $kp_0^3 = \frac{1}{F(z)}$

Computing the first six terms of the series F(z) we obtained

$$kp_0^3 = \frac{1}{1.3503}$$

The calculus performed with the formula of Laguerre gives

$$kp_o^3 = \frac{1}{1.3501}$$

so we have $kp_o^3 = \frac{1}{1.350}$ for p_o = 90 MeV/c.

When these calculations are to be repeated many times for other values of p_o it is prefarable to use a computer for obtaining the indefinite integrals

$$I_1(x) = \int_o^x \frac{u^2 du}{\exp\{u^2+\alpha^2\}^{1/2}-1} \qquad \alpha = \frac{m_\pi c}{p_o}$$

and

$$I_2(x) = \int_o^x \frac{u^2(u^2+\alpha^2)^{1/2} du}{\exp\{u^2+\alpha^2\}^{1/2}-1}$$

This procedure has the advantage of giving at the same time the pion distribution and the pion energy integral distributions in the "fire-ball" system of reference S^*.

In considering values of x sufficiently great for the increasing of $I_1(x)$ and $I_2(x)$ to be negligible we obtained, I_1 = 1,35ı I_2 = 4,978 $<\varepsilon^*> = p_o\ I_2/I_1$ = 331.9 MeV and $\mathcal{M}_o$ = 2.66 GeV, for the selected value of p_o = 90 MeV/c. Note that the value I_1 give the same value for kp_o^3 as that obtained before with different methods of computation.

2. Knowing the values of p_o, kp_o^3 and $\mathcal{M}_o$ we can compute the values of c_α

$$c_\alpha = \frac{2}{3} N\sigma_o(kp_o)^3\ K_\alpha^\gamma <\phi\eta^\gamma>$$

The integral

$$<\phi\eta^{\gamma}> = \int_0^1 \phi(\sigma_0\eta + \delta_0/\eta)\eta^{\gamma}\, d\eta$$

may be easily computed using the well known series representation for $\Phi(x)$ [10].

$$\Phi(x) = \sum_{k=1}^{\infty} e^{-kx} \left(\frac{x}{k} + \frac{1}{k^2}\right) \qquad (x \geq 0 \quad n \geq 1)$$

As an example, using a computer IBM-1620-11 we obtained for the <u>small</u> fire-ball the values of $<\phi\eta^{\gamma}>$ given in the following table

<u>Table I</u>

γ					
2.0	2.05	2.1	2.15	2.2	2.3
0.2180×10^{-2}	0.2025×10^{-2}	0.1882×10^{-2}	0.1750×10^{-2}	0.1629×10^{-2}	0.1415×10^{-2}

The values used for $\mathcal{M}_0$ and p_0 were $\mathcal{M}_0 = 2.66$ GeV and $p_0 = 90$ MeV/c.

For the selected value of $\gamma = 2.1$ we have $c_{\alpha} = 0.110\ k_{\alpha}^{2.1}$ for the "small" fire-ball.

3. The same procedure can be used in computing the integral

$$\int_{\alpha}^{\infty} \Phi(\sigma_{\alpha} t + \delta_{\alpha}/t)(1 - \frac{\alpha}{t})dt \qquad \alpha = E_{\gamma}/E_0$$

that appears in equation (92).

APPENDIX II

Consider the repeated integral

$$I = \int_{\alpha}^{\infty} du \int_{u}^{\infty} \phi(\sigma_{\alpha} y + \delta_{\alpha}/y) dy \qquad \text{where } \alpha = E_{\gamma}/E_{0}.$$

To reduce this integral to a single one we proceed as follows.

First we make the transformation of variables $u = 1/\zeta$ and $y = \frac{1}{\eta}$ which give

$$\text{for } y = u \rightarrow \eta = \frac{1}{u} = \zeta$$

$$\text{for } y = \infty \rightarrow \eta = 0$$

Thus we have

$$I = \int_{0}^{1/\alpha} \frac{d\zeta}{\zeta^2} \int_{0}^{\zeta} \Phi\left(\frac{\sigma_{\alpha}}{\eta} + \delta_{\alpha}\eta\right) \frac{d\eta}{\eta}$$

Now we can apply the well known Dirichlet's formula for inverting the order of integrations

$$\int_{0}^{a} dx \int_{0}^{x} f(x,y) dy = \int_{0}^{a} dy \int_{y}^{a} f(x,y) dx$$

so doing we have

$$I = \int_{0}^{1/\alpha} \phi\left(\frac{\sigma_{\alpha}}{\eta} + \delta_{\alpha}\eta\right) \frac{d\eta}{\eta} \int_{\eta}^{1/\alpha} \frac{d\zeta}{\zeta^2} =$$

$$= \int_{0}^{1/\alpha} \phi\left(\frac{\sigma_{\alpha}}{\eta} + \delta_{\alpha}\eta\right) \frac{1}{\eta} \left(\frac{1}{\eta} - \alpha\right) d\eta$$

Now putting $t = 1/\eta$ we have finally

$$I = \int_{\alpha}^{\infty} \phi(\sigma_{\alpha} t + \delta_{\alpha}/t)(1-\alpha/t)dt$$

as was mentioned in (3.4).