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ULTRA-RELATIVISTIC NEUTRINO GAS IN COSMOLOGY *

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ABSTRACT

We consider a neutrino field self-interacting through Heisenberg's fundamental equation. We show that it is possible to have a situation in which the neutrino field behaves as an ultra-relativistic gas.

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I - INTRODUCTION

It is widely accepted that there are three null-mass particles in Nature: the photon, the graviton and the neutrino. From these, the photon and the graviton are respectively the quanta of the quantized version of the long range electromagnetic and gravitational interactions. Neutrino does not have classical long range force analogue*.

There are many aspects of these three particles to be considered, but here we want to discuss their possible role on the structure of the cosmological metric.

Let us begin by the graviton. One should suspect that vacuum cosmological solutions of Einstein's equations are of relative interest once it is clear that they cannot describe our Cosmos with great accuracy. However, some of these solutions have been of great interest mainly due to the simple analysis that their symmetries permit and have been used as a prototype of more realistic models. Further, it was proved by Lifshitz et al.[6] that cosmological models may develop a vacuum stage near the singularity, at the origin of the Universe. Indeed, they have shown how matter, in any form, becomes less and less important for the dynamical behaviour of the metric as we came closer and closer to the singularity.

A general anisotropic solution founded by Kasner represents this final stage.

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* See reference 1 for a discussion of long range forces induced by excahnge of neutrinos.

For the case of photons, the highly isotropic $2.7^{o}k$ background radiation induces us to consider an electromagnetic gas in which its anisotropic components are smoothed out. Conservation of the energy-momentum tensor implies that the density of this gas varies inversely proportional to the fourth power of the radius of the Universe $a(t)$ in a Friedmann type metric given by $ds^2 = dt^2 - a^2(t)\, d\sigma^2$. $d\sigma^2$ is a three-geometry of a hypersurface orthogonal to the co-moving motion of the galaxies. By comparison with an Universe filled with a pressure-free gas (incoherent matter) we conclude that an era of photon could have been possible at early times. The present value of the matter density forbiddens photons to be the main responsible for the present curvature of the Universe.

Finally we arrive at the neutrino case. The possible importance of neutrino in Cosmology has been stressed by many authors (See Kuchowicz for an excelent review on Neutrino). However there are two great difficulties with neutrino cosmology. The first is the lack of an explicit solution of the combined system of Einstein and Dirac equations; that solution would make possible the construction of a cosmological metric with neutrino as the main source of the gravitational field. Second, we have no direct evidence of extra galactic neutrino and even for neutrino coming from the sun (see Davis).

Irrespectly of this we will discuss an Universe filled with neutrino radiation.

If we believe in the current theory of weak interactions, our Universe should be assymetrical with respect to left and right hand helicity. Indeed, the maximal parity violation of leptonic processes gives rise to the creation of left-handed neutrinos only. However, it is also our purpose in this paper to analyse an Universe filled with neutrinos of both polarizations. The reason for doing this rests on a recent proposal of Novello and Rotelli[5] of a cosmological dependence of weak interactions. These authors have shown that in a Friedmann type universe the dependence of weak processes on the cosmological metric appears only as a time-dependent weighting of the axial vector current relative to the vector current. So, produced neutrinos and anti-neutrinos are admixtures of both left and right hand polarized states.

In section II we present the equations for the neutrino field self-interacting through Heisenberg's fundamental equation. Section III deals with the gas components of the neutrino fluid as viewed by an observer travelling with a velocity vector v^μ and we summarise the status of the radiation (photon, neutrino) in our Universe. In section IV we study a cosmology based on the ultrarelativistic neutrino gas. Section V ends with the conclusions of the whole work.

II - NEUTRINO FIELD

The equation of the neutrino field is obtained from the Lagrangian

$$\mathcal{L} = i\sqrt{-g} \quad \overline{\psi}(x)\, \gamma^\mu(x) \quad D_\mu\, \psi(x) \tag{1}$$

where g is the determinant of $g_{\mu\nu}$; $\gamma^\mu(x)$ are the generalized Dirac metrices related to the metric tensor $g^{\mu\nu}(x)$ through the anticommutation relations

$$\{ \gamma^\alpha(x), \gamma^\beta(x) \} = 2\, g^{\alpha\beta}(x)$$

$D_\mu \psi$ is the covariant derivative given by

$$D_\mu \psi = \partial_\mu \psi - \tau_\mu \psi$$

$\tau_\mu(x)$ is the so-called internal connection. If we assume that the covariant derivative of the γ's are null*, then we obtain

$$\gamma^\mu(x) \, D_\mu \psi(x) = 0 \qquad (2)$$

The energy-momentum tensor $T_{\mu\nu}$ is obtained through the relation

$$\delta \int \sqrt{-g} \, \mathcal{L} \, d^4x = \int \sqrt{-g} \, T_{\mu\nu} \, \delta g^{\mu\nu} \, d^4x \qquad (3)$$

and we find

$$- iT_{\mu\nu} = \bar{\psi} \gamma_\mu D_\nu \psi + \bar{\psi} \gamma_\nu D_\mu \psi - (D_\mu \bar{\psi}) \gamma_\nu \psi - (D_\nu \bar{\psi}) \gamma_\mu \psi \qquad (4)$$

The difficulty in selving the coupled Einstein and Dirac's system of equations rests basically on the intrincated structure of the covariante derivative. Recently, Inomata[3] has called attention to a very powerful simplification that is obtained by imposing a restriction on the solutions of

* Usually it is not made clear enough that this is an additional hypothesis in the theory. See (2) for a complete discussion on the subject.

equation (2) through an additional constraint given by

$$(5) \qquad D_\mu \psi = (\bar{\psi} \gamma_{\lambda\mu} \psi) \gamma^\lambda \psi - (\bar{\psi} \gamma_{\lambda\mu} \gamma_5 \psi) \gamma^\lambda \gamma_5 \psi$$

where

$$\gamma_{\nu\mu} \equiv \gamma_\nu \gamma_\mu - \gamma_\mu \gamma_\nu .$$

The origin of this expression rests on Heisenberg's fundamental theory of matter. Inomata has used an energy-momentum tensor (4) submitted to condition (5) in order to be able to elaborate a geometrization of the neutrino field through Rainich's scheme (already unified model). It is not our purpose here to consider the geometrical aspects of the above model. Our interest on it is limited to the simplification that it can introduce in a discussion of neutrino radiation as source of the cosmological metric.

III - THE NEUTRINO GAS

It is a well known result that the energy-momentum tensor can be split with respect to a velocity field V^μ by

$$(6) \qquad T_{\mu\nu} = \rho V_\mu V_\nu - p h_{\mu\nu} + q_{(\mu} V_{\nu)} + \Pi_{\mu\nu}$$

where

where

$$q_\mu V^\mu = 0$$

$$\Pi_{\mu\nu} V^\mu = 0$$

(7) $$\Pi_{\mu\nu} g^{\mu\nu} = 0$$

$$h_{\mu\nu} = g_{\mu\nu} - V_\mu V_\nu$$

$$X_{(\mu} Y_{\alpha)} = X_\mu Y_\alpha + X_\alpha Y_\mu$$

The physical meaning of the quantities appearing in (6) is clear: ρ is the energy density measured from an observer that moves with velocity V^α ; q_α is the heat flux; $\Pi_{\mu\nu}$ is the anisotropic pressure and p is the isotropic pressure.

Let us performe this decomposition in the case of neutrino presented above.

We define

(8) $$\Sigma_{\lambda\mu} = \overline{\psi} \gamma_{\lambda\mu} \psi$$

$$\Omega_{\lambda\mu} = \overline{\psi} \gamma_{\lambda\mu} \gamma_5 \psi$$

Then, the tensor $T_{\mu\nu}$ given by (4) is written as

(9) $$- i\, T_{\mu\nu} = - \Sigma_{\lambda\mu}\, \Sigma^{\lambda}{}_{\nu} + \Omega_{\lambda\mu}\, \Omega^{\lambda}{}_{\nu}$$

A straightforward calculation shows that it is possible to decompose $\Sigma_{\lambda\mu}$ $(\Omega_{\lambda\mu})$ in an electric Σ_{λ} (Ω_{λ}) and a magnetic part ε_{λ} (χ_{λ}) such that

(10a)
$$\Sigma_{\lambda} = \Sigma_{\lambda\beta}\, V^{\beta}$$
$$\varepsilon_{\lambda} = \frac{1}{2}\, \eta_{\lambda\beta}{}^{\mu\nu}\, V^{\beta}\, \Sigma_{\mu\nu}$$

and

(10b)
$$\Omega_{\lambda} = \Omega_{\lambda\beta}\, V^{\beta}$$
$$\chi_{\lambda} = \frac{1}{2}\, \eta_{\lambda\beta}{}^{\mu\nu}\, V^{\beta}\, \Omega_{\mu\nu}$$

where $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\ \varepsilon_{\alpha\beta\gamma\delta} \cdot \varepsilon_{\alpha\beta\gamma\delta}$ is the Levi - Civita symbol. Indeed we have

(11a) $$\Sigma_{\mu\nu} = - V_{\nu}\, \Sigma_{\nu} + V_{\nu}\, \Sigma_{\mu} - \eta_{\nu\mu}{}^{\rho\sigma}\, V_{\rho}\, \varepsilon_{\sigma}$$

and

(11b) $$\Omega_{\mu\nu} = - V_\mu \Omega_\nu + V_\nu \Omega_\mu - \eta_{\nu\mu}{}^{\rho\sigma} V_\rho \chi_\sigma$$

Then, putting these results back into equation (9) we find

(12) $$- i T_{\mu\nu} = V_\mu V_\nu \{\Omega^2 - \Sigma^2\} - \Sigma_\mu \Sigma_\nu + \Omega_\nu \Omega_\mu - \varepsilon_\nu \varepsilon_\mu + \chi_\mu \chi_\nu - \eta_{\mu\nu}{}^{\lambda\alpha\beta} V_\lambda V_\alpha \{\Omega_\lambda \chi_\beta - \Sigma_\lambda \varepsilon_\beta\} + h_{\mu\nu} \{\varepsilon^2 - \chi^2\}$$

Let us analyse this energy-momentum tensor as the source of the curvature of the Universe.

There may be a cosmic Heisenberg neutrino gas at all wavelengths, similar to the existence of the cosmic background photon radiation. What could we say about the behaviour of the different parts of the energy momentum tensor after the completion of the summation over all components present in the Universe? At this point one should inquire the experimentalists. However the present state of art of experimental neutrino astronomy is not yet capable to help us. So, one is obliged to find some other radiation process from which to extract any information that may guide us into the subject.

Photon radiation, the 2.7°k background radiation, is a natural candidate. What do the experiments tell us?

The observed high degree of isotropy present in this radiation shows that by summing over all components the anisotropic part must vanish.

An inspection of the value of the anisotropic pressure for the electromagnetic field enables us to write that the electric (E_μ) and the magnetic (H_μ) parts when summed over all components have the value

$$E_\mu E_\nu = k\, h_{\mu\nu} \tag{13}$$

$$H_\mu H_\nu = \ell\, h_{\mu\nu}$$

$$q_\alpha = 0$$

where k and l are functions of the global time.

Indeed, (13) is sufficient to annihilate the anisotropic pressure given by

$$\Pi_{\mu\nu}^{(EM)} = \frac{1}{3}(E^2 + H^2)\, h_{\mu\nu} - E_\mu E_\nu - H_\mu H_\nu \tag{14}$$

We find for the density ρ_{EM} the value

$$\rho_{(EM)} = \frac{1}{2}(E^2 + H^2) = \frac{3}{2}(k + \ell) \tag{15}$$

and for the isotropic pressure

$$p_{(EM)} = \frac{1}{2}(k + \ell) \tag{16}$$

This shows the correct equation of stat $p = \frac{\rho}{3}$.

The above result induces us to try analogue relations for the Heisemberg neutrino gas:

$$\Sigma_\mu \; \Sigma_\nu = a \, h_{\mu\nu}$$

$$\Omega_\mu \; \Omega_\nu = b \, h_{\mu\nu}$$

(17) $$\varepsilon_\mu \; \varepsilon_\nu = c \, h_{\mu\nu}$$

$$\chi_\mu \; \chi_\nu = d \, h_{\mu\nu}$$

$$q_\alpha = 0$$

Then, after a straightforward calculation and by means of equation (9) and using the decomposition (6) we find for the density ρ_ν and the pressure p_ν the values

(18) $$\rho_\nu = 3(b - a)$$

$$p_\nu = a - b - 2(c - d)$$

Furthermore, the anisotropic pressure vanishes indenticaly.

If we further have

(19) $$c - d = m(a - b)$$

we obtain en equation of state given by

(20) $$p = \frac{\rho}{3}(2m - 1)$$

The above relations show that different distributions of the components present in the neutrino gas induce distinct behaviours of the whole fluid. This situation is not possible to be attained with the photon gas - - and the reason rests on the absence of self-interacting mechanism on it.

Two particular cases are of special importance. The first one, for $m = 1$ - that gives the radiation equation $p = \frac{\rho}{3}$; and the second, for $m = 2$ - that gives the behaviour of an ultra-relativistic gas, $p = \rho$.

In the next section we will analyse some details of having case 2 in our Universe.

IV - COSMOLOGY BASED ON THE ULTRA-RELATIVISTIC GAS -

The main difficulty in finding the metric of an Universe filled solely with neutrino rests on the nondiagonality of its energy-momentum tensor. This difficulty is by-passed here and we try to find a Cosmology for the ultra-relativistic gas in which left and right handed neutrino are present*.

We write

(21) $$ds^2 = dt^2 - a^2(t)\{dx^2 + dy^2 + dz^2\}$$

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* Expression (5) is non-null only if we treat neutrino as a four-component object.

Einstein's equations take the form

$$(22)\quad \left(\frac{\dot a}{a}\right)^2 = \frac{\rho}{3}$$

$$2\,\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 = -\,\rho$$

(a dot means derivative with respect to t).

A solution of equation (22) may be easily found under the form

(23) $$a(t) \sim t^{1/3}$$

and for the density we obtain

(24) $$\rho = \frac{1}{3}\, t^{-2}$$

In this case the deceleration parameter q defined by $q = -\dfrac{a\ddot a}{(\dot a)^2}$ takes the value 2.

V - CONCLUSION -

The above model of a self-interacting neutrino gas, based on Heisenberg's fundamental theory, gives an example of an ultra-relativistic gas.

Furthermore, from equations (23) and (24) we can evaluate the present density of this hypothetical gas in our Universe. Taking the age of the Universe as 7×10^9 years we find that the neutrino density attains the value 10^{-29} g/cm^3 .

So, one arrives at the result that this ultra-relativistic gas could dominate the whole history of the Universe until our days. However, we have to wait until experimental neutrino astronomy can decide about it. Or try to justify, by theoretical means the reasons to believe in the existence of such Heisenberg neutrino gas in our Universe. This situation is very similar to the search for the mechanism of isotropization of the photon background radiation.

So, if this is the case and an invisible neutrino sea is the main responsible for the curvature of our Universe then Heisenberg's fundamental equation should have a role in Cosmology.

APPENDIX

In this appendix we want to make some comments concerning the inevitability of a singularity origin for the Universe. In its modern version, this situation is represented by the existence of the theorems of Penrose et al.

In all of these theorems there are some hypothesis concerning inequalities satisfied by the density and pressure of matter. They are called the energy conditions:

i) $\rho + p \geq 0$ (weak energy condition)

ii) $\rho + 3p \geq 0$ (energy condition)

In the case treated in the present paper of a Heisemberg neutrino gas, condition i) is satisfied if $\rho > 0$ and $m > -1$ (see equation (20)). Condition ii) will be satisfied if $\rho > 0$ and $m > 0$. A simple inspection shows that we can violate one of these inequalities without violating the other.

So, the conditions of the singularity theorems are not fulfilled by the neutrino gas treated here.

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