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ON THE FINITE DIMENSIONALITY OF EVERY IRREDUCIBLE UNITARY REPRESENTATION OF A COMPACT GROUP

by

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ON THE FINITE DIMENSIONALITY OF EVERY IRREDUCIBLE UNITARY REPRESENTATION OF A COMPACT GROUP*

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We shall prove that every irreducible unitary representation of a compact group is finite dimensional.

Our argument is a variation of known proofs and it hardly could be based on an idea different from those already current.

It makes no use of compact or Hilbert-Schmidt operators and seems simpler than the proofs in [1], [2], [3], [4].

Its crucial point is that the prospectively finite dimension of the representation Hilbert space is expressible by a known integral formula.

Let $\mathcal{H} \neq 0$ be a Hilbert space and $x \rightarrow U_x$ be a group homomorphism of a compact group G into the group U $(\mathcal{H})$ of all uni

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tary operators in $\mathcal{H}$, such that the scalar product $\langle \xi, U_x \eta \rangle$ is a continuous function of $x \in G$ for all $\xi, \eta \in \mathcal{H}$.

Suppose that this representation is irreducible, namely that there is no closed vector subspace of $\mathcal{H}$ invariant under all U_x expect the trivial ones 0 and $\mathcal{H}$. Then there results that $\mathcal{H}$ is finite dimensional.

In fact, let $\xi, \eta, \xi', \eta' \in \mathcal{H}$. Denoting complex conjugation by a star, since

$$| \int \langle \xi, U_x \eta \rangle \cdot \langle \xi', U_x \eta' \rangle^* \, dx \, | \leqslant \|\xi\| \cdot \|\eta\| \cdot \|\xi'\| \cdot \|\eta'\| ,$$

there is an operator T on $\mathcal{H}$ depending on η , η' such that

$$\int \langle \xi, U_x \eta \rangle \cdot \langle \xi', U_x \eta' \rangle^* \, dx = \langle T \xi, \xi' \rangle .$$

T commutes with every U_t since

$$\langle TU_t \xi, \xi' \rangle = \int \langle \xi, U_{t^{-1}x} \eta \rangle \cdot \langle \xi', U_x \eta' \rangle^* \, dx =$$

$$= \int \langle \xi, U_x \eta \rangle \cdot \langle \xi', U_{tx} \eta' \rangle^* \, dx = \langle T \xi, U_t^* \xi' \rangle = \langle U_t T \xi, \xi' \rangle ,$$

from which $TU_t = U_t T$ follows. The irreducibility of the representation then implies that T is scalar operator, that is $T = \lambda(\eta, \eta') I$ and we get

$$\int \langle \xi, U_x \eta \rangle \cdot \langle \xi', U_x \eta' \rangle^* \, dx = \lambda(\eta, \eta') \langle \xi, \xi' \rangle .$$

By interchanging the roles of the couples (ξ, ξ') and (η, η') and using the rule $\int f(x^{-1})dx = \int f(x)dx$, we get

$$\lambda(\eta, \eta') \langle \xi, \xi' \rangle = \lambda(\xi, \xi')^* \langle \eta, \eta' \rangle^* .$$

Hence

$$\lambda(\eta, \eta') = c\langle \eta, \eta' \rangle^* ,$$

where c is a constant, and

(1) $\int \langle \xi, U_x \eta \rangle \cdot \langle \xi', U_x \eta' \rangle^* \, dx = c \langle \xi, \xi' \rangle \cdot \langle \eta, \eta' \rangle^*$.

If we let ξ, η, ξ', η' all become equal to a unit vector α we get

$$c = \int |\langle \alpha, U_x \alpha \rangle|^2 \, dx .$$

Hence $c > 0$, since the positive continuous function whose integral is c has strictly positive value at the identity of G.

Now let $e_1, \ldots, e_n$ be orthonormal in $\mathcal{H}$. Let η, η' become equal to e_i and ξ, ξ' become equal to α in (1). By adding the resulting equalities and using Bessel's inequality

(2)
$$\sum_{i=1}^{n} \langle \alpha, U_x e_i \rangle \cdot \langle \alpha, U_x e_i \rangle^* \leqslant \|\alpha\|^2 = 1$$

since $U_x e_1, \ldots, U_x e_n$ are orthonormal, we get $nc \leqslant 1$, that is $n \leqslant 1/c$. This completes the proof that the dimension of $\mathcal{H}$ is finite.

We remark that, if n is supposed to be the finite dimension of $\mathcal{H}$, then (2) holds as an equality and so we get $nc = 1$, that is $c = 1/n$. Then (1) becomes a known formula (see [5], Chap. V) which we took as motivation for the above proof.

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