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ON THE ANALYTIC PROPERTIES OF PARTIAL AMPLITUDES

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ON THE ANALYTIC PROPERTIES OF PARTIAL AMPLITUDES.*

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ABSTRACT

The analytic properties of partial wave amplitudes in meson-nucleon scattering are investigated on the basis of the Mandelstam's representation and an integral representation is set up for them which explicitly exhibits those properties.

INTRODUCTION

A method for the separation of partial waves from non-forward relativistic dispersion relations has been put forward by Capps and Takeda[1]. The scattering amplitudes in the integrals are expanded in terms of Legendre polynomials which are analytically continued into the unphysical region. Each partial amplitude is then related to two integrals involving infinite series of partial amplitudes, which are conv

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ergent at small energies.

We have devised a different approach to this problem which consists in deducing the analytic properties of the partial amplitudes and establishing a representation for them based on those properties. In order to investigate the analytic properties of the partial amplitudes it is necessary to know the analytic structure of the scattering amplitudes as functions of two variables, the energy and momentum transfer. A full representation of these amplitudes has been proposed by Mandelstam[2] and is used here. The main result is that the partial amplitudes are analytic functions of energy throughout the complex plane except for cuts along the real axis and a cut along a circle with its centre on the real axis.

Integral representations are obtained by applying Cauchy's theorem to suitable combinations of pairs of partial amplitudes. The integrals along the real axis depend on the imaginary part of the function considered and one can apply the unitarity condition in the physical region. The meaning of the unphysical region as well as the integral along the circle is discussed.

I. The covariant amplitudes. Mandelstam's representation.

The matrix element of the S-matrix for meson-nucleon scattering may be written in the form[3].

$$\langle p' s, q' | S | p r, q \rangle = \langle p' s, q' | p r, q \rangle + \frac{1}{(2\pi)^6} \frac{1}{(4 q_0 q_0' p_0 p_0')^{1/2}} F_{sr} (p', q'; p, q) \quad (1)$$

where p, q are the initial momenta of the nucleon and the meson respectively p', q' the final momenta and r, s the spin states of the incoming and outgoing nucleon. The isotipic spin coordinates of the nucleons and the mesons have been omitted.

Due to conservation of total momentum be out of the four momenta there will only three independent vectors which determine the kinematics of the process. We choose the combinations:

$$P = \frac{1}{2}(p + p'), \quad Q = \frac{1}{2}(q + q'), \quad K = p - p' = q' - q \tag{2}$$

With these vectors one can form the invariants[4]:

$$\begin{aligned} &K^2 = -4\Delta^2 \qquad P^2 = M^2 + \Delta^2 \qquad Q^2 = m^2 + \Delta^2 \\ &P.\,Q = \xi \qquad\qquad P.\,K = Q.\,K = 0 \end{aligned} \tag{3}$$

so that there are only two independent invariants.

We introduce an invariant matrix $\mathcal{M}$ in space related to the Feynman amplitude Fs r (p', q'; p, q) by:

$$F\, s\, r\, (p', q'; p, q) = \bar{U}_s\, (p')\mathcal{M}U_r\, (p) \tag{4}$$

where $U_r(p)$ and $U_s\,(p')$ are Dirac spinors.

Assuming parity conservation and making use of the Dirac equation one can reduce $\mathcal{M}$ to the form[5]:

$$= U + \not{Q}\; V \tag{5}$$

where U and V are functions of the invariants ξ and Δ^2.

To fix our ideas consider the processes:

$$N + K \longrightarrow N' + K' \qquad \text{I}$$

$$N + \bar{K}' \longrightarrow N + \bar{K} \qquad \text{II}$$

$$N + \bar{N}' \longrightarrow K' + \bar{K} \qquad \text{III}$$

The Feynman amplitudes for these processes are obtained from the same Green's function taken in different and non-overlapping domains of the variables ξ and Δ^2 which of course have different physical meanings for each of them. The physical region for the respective processes are:

$$\text{(I)} \quad \xi > \xi_T \quad ; \quad \Delta^2 > 0$$

$$\text{(II)} \quad \xi < -\xi_T \quad ; \quad \Delta^2 > 0$$

$$\text{(III)} \quad \xi < \xi_T^2 \quad ; \quad \Delta^2 < -M^2$$

where

$$\xi_T = \sqrt{(M^2 + \Delta^2)(m^2 + \Delta^2)}$$

Mandelstam assumes that the Green's function is analytic in both variables except for poles on the real axis and cuts along certain hyperplanes.

The poles arise from the bound states of the system and the location of the cuts is determined by the threshold energies for the allowed virtual transitions. It is then convenient to introduce the new variables W^2, $\bar{W}^2$, K^2 which are the square of the energy in the c.m.s. for the three processes respectively. They are related to ξ and Δ^2 by:

$$\frac{1}{2}(W^2 - M^2 - m^2) = \xi + \Delta^2 \qquad (6)$$

$$\frac{1}{2}(\bar{W}^2 - M^2 - m^2) = -\xi + \Delta^2 \qquad (7)$$

$$K^2 = -4\Delta^2 \tag{8}$$

Thence

$$W^2 + \overline{W}^2 + K^2 = 2(M^2 + m^2) \tag{9}$$

The momentum transfer between the mesons or nucleons in the first two processes is $-K^2$; in process III it is $-\overline{W}^2$ between the nucleon N and the meson K' and $-W^2$ between the nucleon N and the meson $\overline{K}$.

On the basis of Mandelstam assumptions one obtains for each of the covariant amplitudes a representation of the form:

$$A = \sum_y \frac{g_y^2}{\overline{W}^2 - M_y^2} + \frac{1}{\pi^2}\int_{(M+m)^2}^{\infty} d\,W'^2 \int_{(M_y+\;\;)^2}^{\infty} d\,\overline{W}'^2 \;\frac{A_{12}(W'^2,\ \overline{W}'^2)}{(W'^2 - W^2)(\overline{W}'^2 - \overline{W}^2)} +$$

$$+ \frac{1}{\pi^2}\int_{(M+m)^2}^{\infty} d\,W'^2 \int_{(2\mu)^2}^{\infty} d\,K'^2 \;\frac{A_{13}(W'^2,\ K'^2)}{(W'^2 - W^2)(K'^2 - K^2)}$$

$$+ \frac{1}{\pi^2}\int_{(M_y+\mu)^2}^{\infty} d\,\overline{W}'^2 \int_{(2\mu)^2}^{\infty} d\,K'^2 \;\frac{A_{23}(\overline{W}'^2,\ K'^2)}{(\overline{W}'^2 - \overline{W}^2)(K'^2 - K^2)} + \frac{1}{\pi}\int_{(M_y+\mu)^2}^{\infty} d\,\overline{W}'^2 \;\frac{A_2(\overline{W}'^2,\ \infty)}{\overline{W}'^2 - \overline{W}^2}$$

In this case the conservation laws and in particular the conservation of strangeness and parity allow for bound states only in process II, corresponding to the Λ and Σ particles. The energy thresholds for the three processes correspond to transitions into the virtual intermediate states $(K + N)$, $(Y + \pi)$ $(\pi + \pi)$ respectively. The bound state contributions exactly coincide with first Born approximation in the conventional perturbation theory.

From the reality of the absorptive parts of the amplitudes for the physical processes (I-III) it follows that A_{ij} are all real in the respective domains of integration.

At least one subtraction is necessary, in this case, for fixed $\bar{W}_0^2$. It has been performed in a suitable way giving rise to the last term in (10). In perturbation theory, this term, together with the first one corresponds to graphs of the form shown in Fig. 1-a.

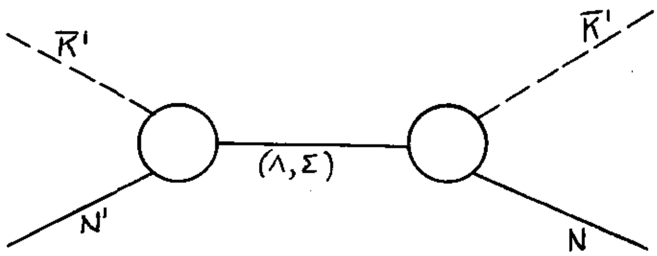

Fig. 1-a

Representation (10) as it stands does not allow for direct meson-meson interactions. If such interactions do exist one meson subtraction is necessary, now for fixed K^2. In fact in perturbation theory graphs of the form shown in Fig. 1-b.

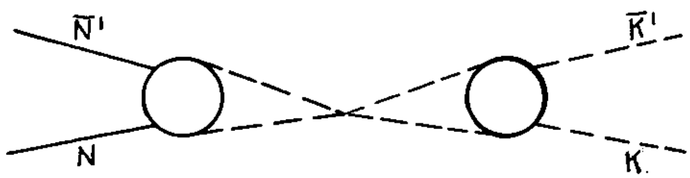

Fig. 1-b

depend only on the variable K^2 and contribute to the invariant amplitude U.

Therefore one must add to the right of (10) a term:

$$U\ (K^2,\ \infty) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} d\ K'^2\ \frac{U_3\ (K'^2, \infty)}{K'^2 - K^2} \tag{11}$$

II - The scattering matrix in the c.m.s.

The covariant transition matrix is related in the c.m.s. to the scattering matrix by:

$$\bar{u}_s(k')\mathcal{M}\,\mathcal{U}_r(k) = 4\pi \frac{W}{M}\ \mathcal{U}_s^t\ (o)\ f\ \mathcal{U}_r\ (o) \tag{12}$$

where $\mathcal{U}_r(o)$ are orthonormal eigenstates of γ_o $(\gamma_o\, \mathcal{U}_r(o) = \mathcal{U}_r(o))$ and

$$\mathcal{U}_r\ (k) = \frac{\not{k} + M}{\sqrt{2M\ (E+M)}}\mathcal{U}_r(o) \tag{13}$$

Here $k(E,\underline{k})$ and $k'(E,\ \underline{k}')$ are the four momenta of the incoming and outgoing nucleons in the c.m.s.

The matrix f may be written in the form:

$$f = f_1 + (\underline{\sigma}.\underline{\epsilon}')\ (\underline{\sigma}.\underline{\epsilon})\ f_2 \tag{14}$$

where $\underline{\epsilon}'$ and $\underline{\epsilon}$ are unit vectors in the direction $\underline{k}'$ and $\underline{k}$ respectively, and

$$f_1 = \sum (f_\ell^+\ \ P'_{\ell+1} - f_2^-\ \ P'_{\ell-1})\ ;\ \ f_2 = \sum (f_\ell^-\ - f_\ell^+)\ P'_\ell \tag{15}$$

The $f_\ell^\pm$ are amplitudes for transitions in given states of total angular momentum $j = \ell \pm \frac{1}{2}$ and parity $(-1)^\ell$. For K-nucleon scattering below the threshold for pion production the amplitudes $f_\ell^\pm$ have the form:

$$f_\ell^\pm = e^{i\,\delta_\ell^\pm}\ \sin\ \delta_\ell^\pm / k$$

where the phases $\delta_\ell^\pm$ are real functions of k. For $\bar{K}$-nucleon scattering as well as for K-nucleon scattering above that threshold the phases are complex.

From (5), (12) and (13) one obtains the following relations[6] between the covariant amplitudes U, V and the scattering amplitudes f_1, f_2:

$$4\pi f_1 = \frac{E+M}{2W}\left[U + (W-M)V\right] \tag{16}$$

$$4\pi f_2 = \frac{E-M}{2W}\left[-U + (W+M)V\right] \tag{17}$$

and hence

$$\frac{1}{4\pi} U = \frac{W+M}{E+M} f_1 - \frac{W-M}{E-M} f_2 \tag{18}$$

$$\frac{1}{4\pi} V = \frac{1}{E+M} f_1 + \frac{1}{E-M} f_2 \tag{19}$$

One can see that:

$$f_1(-W) = - f_2(W) \tag{20}$$

which is a consequence of invariance under Schwinger's space-time reflexion. The partial wave amplitudes may be projected out of f_1 and f_2. One obtains recurrence relations which on the assumption that $\lim_{\ell\to\infty} f_\ell^{\pm} = 0$ give[7]:

$$f_\ell^{\pm} = \frac{1}{2}\int_{-1}^{+1} (f_1 P_\ell + f_2 P_{\ell\pm1}) \tag{21}$$

From the symmetry (20) it follows that:

$$f_\ell^{+}(-W) = - f_{\ell+1}^{-}(W) \tag{22}$$

IV. The analytic properties of the partial amplitudes.

We investigate now the analytic properties of the partial am plitudes as defined by (21) as functions of the variable

$$\nu = \frac{1}{2}(W^2 - M^2 - m^2) = \xi + \Delta^2 \tag{23}$$

They are derived from the representation(10) for U and V, setting

$$\Delta^2 = \frac{1}{2} k^2 (1 - Z) \tag{24}$$

and varying Z from -1 to +1.

Let us first consider the Born approximation $f_\ell^{\pm(1)}$. In the K-nucleon scattering the poles of U and V give rise to branch lines of $f_{\ell(\nu)}^{\pm(1)}$ in the intervals of the real axis along which

$$\nu_y + \nu - k^2(1 - Z) = 0$$

that is $(-\infty, -\frac{1}{2}(M^2 + m^2))$ and $(-\nu_y' \ , -\nu_y)$ where:

$$\nu_y = \frac{1}{2}(M_y^2 - M^2 - m^2)$$

$$-\nu_y = \frac{1}{2}\left[\frac{(M^2 - m^2)^2}{M_y^2} - M^2 - m^2\right]$$

By contrast in $\bar{K}$-nucleon scattering the Born approximation vanishes for all amplitudes except the S and $P_{1/2}$ states in which there remain poles at $\nu = \nu_y$.

Let us now consider the singularities of $f_\ell^{\pm(2)} = f_\ell^{\pm} - f_\ell^{\pm(1)}$. They lie on the lines of the complex -plane defined by:

$$W'^2 - W^2 = 0 \tag{25}$$

$$\bar{W}'^2 - \bar{W}^2 = 0 \tag{26}$$

$$K'^2 - K^2 = 0 \tag{27}$$

where W'^2, $\bar{W}'^2$ and K'^2 are parameters assuming values within the intervals of integration in (10).

For K-nucleon scattering $\bar{W}^2$ and K^2 are expressed in terms of ν and z by means of (7), (8), (23), and (27).

Equation (25) defines a cut on the real axis in the interval $(m\,M, \infty)$. For a fixed value of $\bar{W}'^2$ equation(26) defines cuts in the intervals $(-\infty, -\frac{1}{2}(M^2+m^2))$ and $(-\bar{\nu}', -\bar{\nu})$ where:

$$\bar{\nu} = \frac{1}{2}(\bar{W}'^2 - M^2 - m^2)$$
$$-\bar{\nu}' = \frac{1}{2}\left[\frac{(M^2 - m^2)^2}{\bar{W}'^2} - M^2 - m^2\right]$$

As $\bar{W}'^2$ varies from $(M_y + \mu)^2$ to ∞ the last cut covers the interval$(-\infty, -\bar{\nu}_o)$ where

$$\bar{\nu}_o = \frac{1}{2}\left[(M_y + \mu)^2 - M^2 - m^2\right] \qquad (28)$$

Equation (27) is equivalent to

$$k^2 = -K'^2 / 2\,(1 - z)$$

where the right hand side varies from $-\infty$ to $-\mu^2$. As k^2 varies from $-\infty$ to $-M^2$, ν describes two branch lines on the real axis along $(-\infty, -M^2)$ and $(-\frac{1}{2}(M^2 + m^2), -M^2)$; as k^2 varies from $-M^2$ to $-m^2$, ν goes from $-M^2$ to $-m^2$ along two branches (above and below the real axis)of a circle in the complex plane, with centre at $\nu = -\frac{1}{2}(M^2 + m^2)$ and radius $\frac{1}{2}(M^2 - m^2)$; as k^2 varies from $-m^2$ to $-\mu^2$, ν describes two branch lines on the real axis along $(-m^2, -\nu_1)$, $(-m^2, \nu_2)$ where $-\nu_1$ and ν_2 are the roots of $k^2 + \mu^2 = 0$ (See fig. 2).

One can use these analytic properties to obtain a representation for the partial amplitudes in the form of dispersion relations. It is convenient to introduce the combinations:

$$\varphi_\ell^+ = \frac{1}{W}\left(f_\ell^+ + f_{\ell+1}^-\right) = \frac{1}{2W}\int_{-1}^{+1}(f_1 + f_2)(P_\ell + P_{\ell+1})\, d\, z$$

$$\varphi_\ell^- = \left(f_\ell^+ - f_{\ell+1}^-\right) = \frac{1}{2}\int_{-1}^{+1}(f_1 - f_2)\,(P_\ell - P_{\ell+1})\, d\, z \qquad (29)$$

which are symetric functions of W.

Recalling that $\varphi_\ell^\pm(\nu)$ behaves like $k^{2\ell} \sim (\nu^2 - m^2 M^2)^\ell$ as $k^2 \to 0$, and applying Cauchy's theorem to $\varphi_\ell^\pm(\nu')/(\nu' - \nu)(\nu'^2 - m^2 M^2)^\ell$ for the contour shown in Fig. 2 one obtains:

$$\varphi_\ell^\pm(\nu) = \varphi_\ell^{\pm(1)}(\nu) + \frac{1}{\pi}\int_{mM}^{\infty} \mathrm{Im.}\, \varphi_\ell^\pm(\nu')\left(\frac{\nu^2 - m^2 M^2}{\nu'^2 - m^2 M^2}\right)\frac{d\nu'}{\nu' - \nu} + \frac{1}{\pi}\int_{-\infty}^{\nu_2} \mathrm{Im.}\, \varphi_\ell^{\pm(2)}(\nu').$$

$$\left(\frac{\nu^2 - m^2 M^2}{\nu'^2 - m^2 M^2}\right)^\ell \frac{d\nu'}{\nu' - \nu} + \frac{1}{\pi}\int_S \varphi_\ell^\pm(\nu')\left(\frac{\nu^2 - m^2 M^2}{\nu'^2 - m^2 M^2}\right)^\ell \frac{d\nu'}{\nu' - \nu} \qquad (30)$$

where $\varphi_\ell^{(1)}$ is the Born approximation, corresponding to the first term in (10) and $\varphi_\ell^{(2)} = \varphi_\ell - \varphi_\ell^{(1)}$.

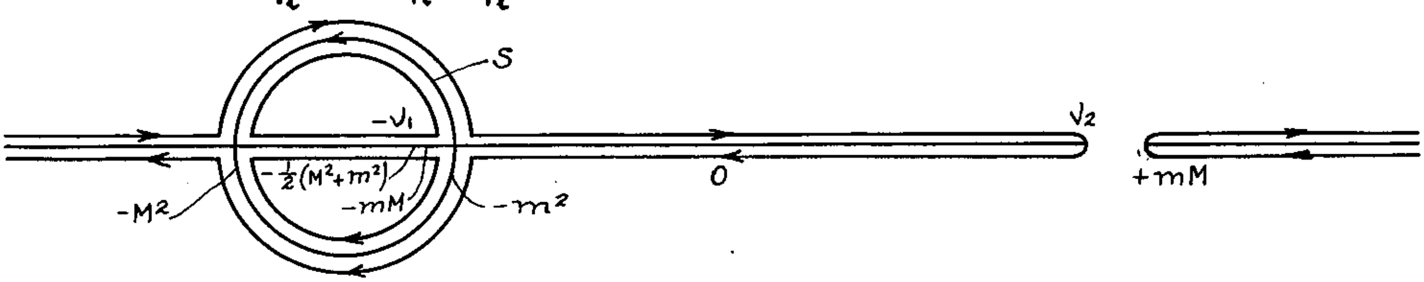

Fig. 2 - Complex ν-plane, showing the singularities of the partial am

plitudes in the K-nucleon scattering.

From the representation (10) one obtains, for in the interval $(-\infty, \nu_2)$:

$$\mathrm{Im.} \int_{-1}^{1} A(\nu_1 z)\, P_l\,(z)\, d\, z = \mathrm{Re}\left(I_{l2}(\nu) + I_{l3}(\nu)\right) \qquad \nu < -\tfrac{1}{2}(M^2 + m^2)$$

$$= I_{l2}(\nu) \qquad -\tfrac{1}{2}(M^2 + m^2) < \nu < -\nu_1$$

$$= I_{l2}(\nu) + I_{l3}(\nu) \qquad -\nu_1 < \nu < -\overline{\nu}_o \tag{31}$$

$$= I_{l3}(\nu) \qquad -\overline{\nu}_o < \nu < \nu_2$$

where

$$I_{l2}(\nu) = -\frac{1}{\pi} \int_{z_o}^{1} A_2\,(W^2, K^2)\, P_l\,(z)\, d\, z \tag{32}$$

$$I_{l3}(\nu) = \pm \frac{1}{\pi} \int_{-1}^{1-2\mu^2/k^2} A_3\,(K^2, W^2)\, P_l\,(z)\, d\, z \tag{33}$$

The lower limit of integration in (32) is $z_o = -1$ when $-\frac{1}{2}(M^2 + m^2)$ where $-\overline{\nu}_o'$ is the solution of

$$\overline{\nu}_o + \nu - 2\,k^2 = 0$$

and

$$z_o = 1 - \frac{\overline{\nu}_o + \nu}{k^2}$$

for ν outside that interval.

The + sign in (33) is taken in the region outside the circle where $\frac{d\,k^2}{d\,\nu} > 0$, and the - sign is taken inside it, where $\frac{d\,k^2}{d\,\nu} < 0$. All these limits and intervals can be visualized better by an inspection of

Fig. 3.

The functions $A_2(\bar{W}^2, K^2)$ and $A_3(K^2, W^2)$ are the absorptive amplitudes for processes II and III as defined in reference. The following discussion is based on the properties of these functions as described by Mandelstam.

The functions $A_2(\bar{W}^2, K^2)$ and $A_3(K^2, W^2)$ are real in the respective intervals $-\frac{1}{2}(m^2 + M^2) < \nu < -\bar{\nu}_0$ and $-\nu_1 < \nu < \nu_2$, where they coincide with the imaginary part of the amplitudes for processes II and III For $\nu < -\frac{1}{2}(M^2 + m^2)$ the branch lines arising from (26) and (27) overlap and these functions may become complex. The overlapping along the inverval $(-\bar{\nu}_0, -\nu_1)$ is actually fictitious since it would arise from processes like those shown in Fig. 4.

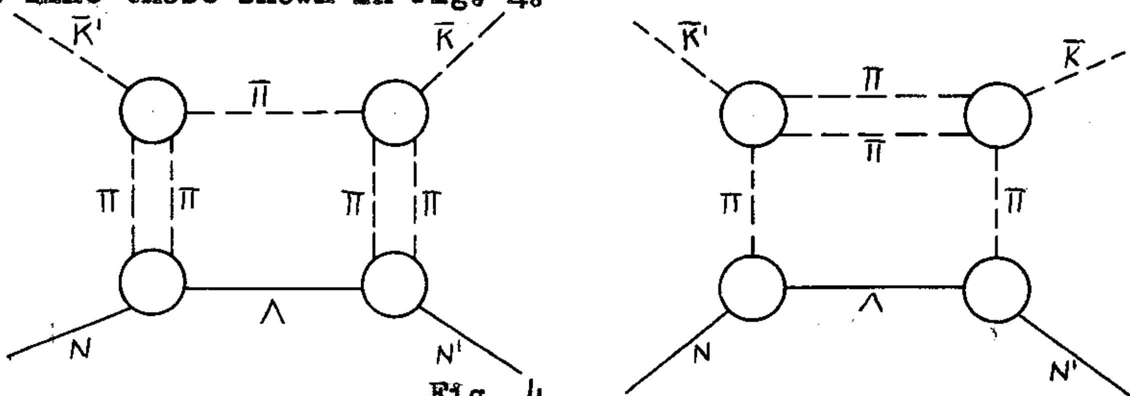

Fig. 4.

which are forbidden on account of conservation of strangeness, and isotopic spin. The branch lines of graphs with K-mesons in the intermediate states, already start outside that interval.

When ν is in the interval $(-\infty, \nu_2)$, Im. $\varphi_l^{\pm(2)}(\nu)$ is given by the right hand side of (31), provided the following modifications are introduced in (32) and (33):

(i) Replace P_ℓ by $\frac{1}{2}(P_\ell \pm P_{\ell+1})$

(ii). Replace A_1 by:

$$A_1^+ = \frac{H}{W^2}(MU_1 - \nu V_1)$$
$$A_1^- = \frac{H}{W}(EU_1 - M\omega V_1) \qquad (34a)$$

with $E + \omega = W$; $U_{2,3}$ and $V_{2,3}$ are the absorptive covariant amplitudes for processes (II,III). At this point we assume charge independence and introduce the column matrices in isospin space:

$$U = \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \quad , \quad V = \begin{pmatrix} V^0 \\ V^1 \end{pmatrix}$$

whose elements correspond to transitions in states of isotopic spin $T=0$ and $T = 1$ respectively. Isotopic spin has been taken into account in (34), where

$$H = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

is a matrix in isospin space[8].

The function $\phi_\ell^\pm(\nu)$ in the last integral of (30) is also given by $I^\pm_{\ell 3}(\nu)$ where A_3 has been replaced by $A_3^\pm$. If the integration around the circle S follows the external lines in Fig. 2 the + sign must be taken above and the - sign below the real axis.

The method described in this section applies equally to process II and pion-nucleon scattering. But then the functions $\varphi_0^\pm$ (S and $P_{1/2}$ waves) have a pole at $W^2=0$ and an arbitrary parameter, the S-wave scattering length, is conveniently introduced by means of a subtraction at $\nu = mM$, where the $P_{1/2}$ scattering amplitude vanishes. The fortuitous absence of this pole in K-nucleon scattering is due to conservation of strangeness.

V. Final remarks

The objective of setting up dispersion relations for partial amplitudes was to establish an integral representation of functions of a single variable $f_l^{\pm}(\nu)$, for which the unitarity condition takes on in the physical region, the simple form:

$$\Im m.\ f_l\ (\nu) = \sum_n \rho_n \left| f_l^n(\nu)\right|^2 \tag{35}$$

where the index n refers to all allowed channels, compatible with conservation of energy and ρ_n is a phase space factor; in the elastic channel $\rho_n = k$.

In our integral representation there exist however regions of frequencies where the unitarity condition has no simple form. If we attempt to apply the unitarity condition for all ν' in order to obtain an integral equation we find that the equations for partial amplitudes of different angular momentum and parity and corresponding to processes I, II, III, in different states of isotopic spin are all coupled. Moreover in the region $\nu' - \frac{1}{2}(M^2+m^2)$ and along the circle S the unitarity condition can only be obtained by means of analytical continuation.

In the interval $-\frac{1}{2}(M^2+m^2)<\nu<-mM$ the variables $\bar{\bar{W}}^2$ and K^2 in (32) are in the physical region for process II. One can re-express U_2 and V_2 in terms of the imaginary parts of the scattering amplitudes and expand in partial waves. This expansion is also valid in the unphysical region $-mM < \nu < -\bar{\nu}_0$, that is $(M_Y + \mu)^2 < \bar{W}^2 < (M+m)^2$. Only the virtual states $(Y + \pi)$ and $(\Lambda + 2\pi)$ contribute to the amplitudes in this region and if $N + \bar{K}' \longrightarrow \Lambda + 2\pi$ is neglected one can show[9] that the unitarity condition takes the form (35) except for a factor $(-1)^l$.

We remark that if all graphs in perturbation theory involving closed baryon loops and four-meson primary interactions are neglected, then the branch line around the circle S disappears as well as the contribution of A_3 along the interval $(-\nu_1, \nu_2)$.

Another possible approach is to use the method of Haber Schaim and replace the integrals in the unphysical regions by an effective range approximation. One might then try to obtain a solution for the inverse of the amplitudes.

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2. S. Mandelstam. Determination of the Pion-Nucleon Scattering Amplitude from Dispersion relations and Unitarity I (To be published).

3. For an expression of the Feynman amplitudes in terms of Heiserberg operators see F. E. Low, Phys. Rev., 97, 1392 (1955), M. L. Goldberger Phys. Rev., 97, 508 (1955) and H. Lehmann, K. Symanzik, and W. Zimmermann Nuovo Cimento, 1, 205 (1955).

4. We use the metric (1, -1, -1, -1) and units $\hbar = c = 1$.

5. A. Salam, Nuovo Cimento, 3, 427 (1956).

6. G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. 106, 337 (1957).

7. Note that this assumption is necessary for the validity of the partial wave expansion.

8. D. Amati and B. Vitale, Nuovo Cimento, 7, 190 (1958).

9. One has to consider the amplitudes defined for fictitious meson momenta such that $q^2 = q'^2 = \lambda = -\Delta$. The proof then depends on the assumption that for real $W^2 > (My + \mu)^2$ and Δ^2 within certain limits the absorptive part of the amplitudes may be analytically continued as a function of λ to $\lambda = m^2$.

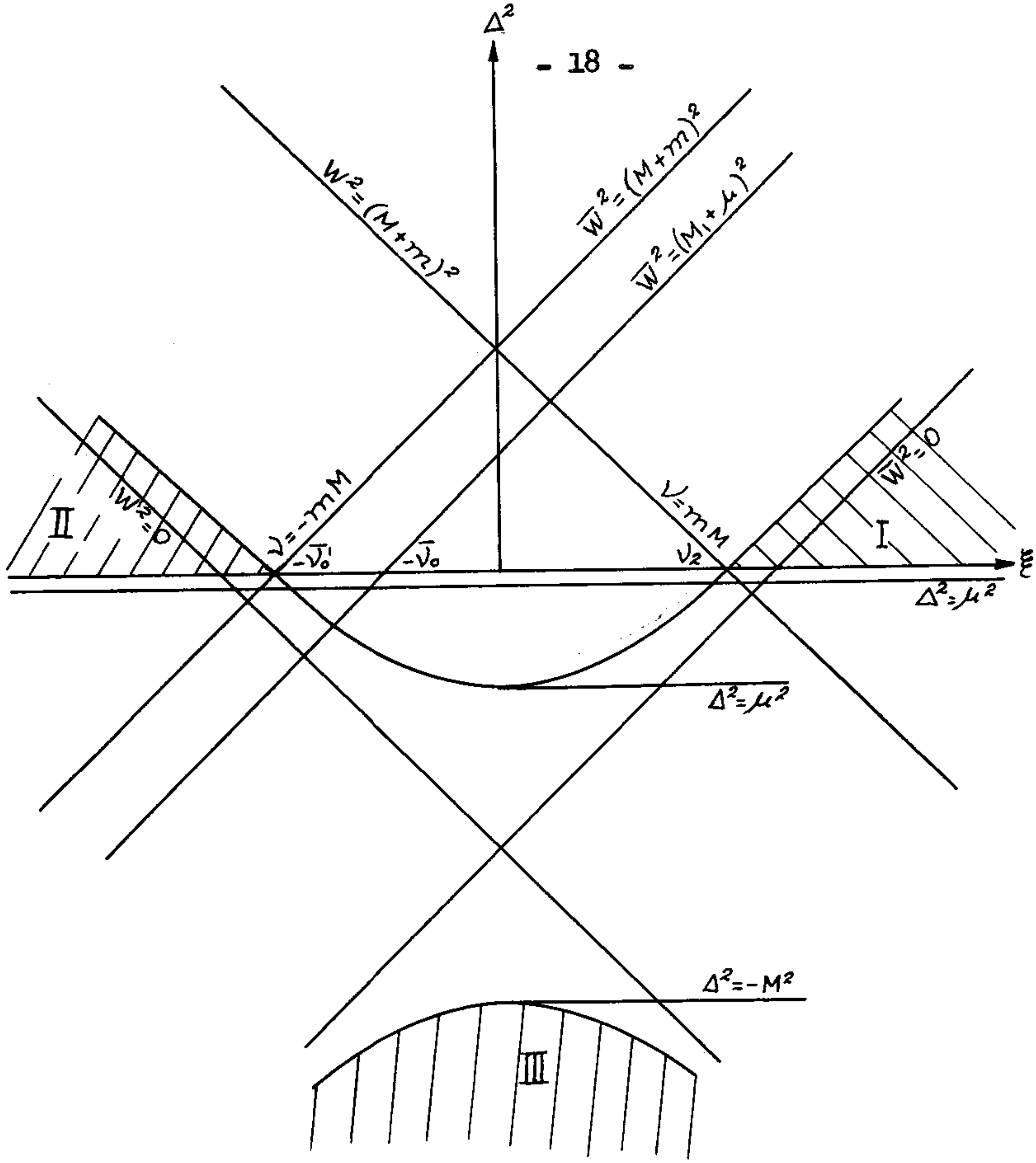

Fig. 3. Diagram of the (ξ , Δ^2) plane. The hyperbolis represents the curve ξ = ξ_T^2 or equivalently $k^2 = \Delta^2$.

The physical regions for processes I, II, III are shaded. We have marked values of ν on the ξ -axis recalling that $\nu = \xi$ for $\Delta^2 = 0$.