



CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Rio de Janeiro

Notas de Física

CBPF-NF-003/15

July 2015

**Invariant PDEs of Conformal Galilei Algebra as deformations:
cryptohermiticity and contractions**

N. Aizawa, Z. Kuznetsova and F. Toppan

Invariant PDEs of Conformal Galilei Algebra as deformations: cryptohermiticity and contractions

N. Aizawa*, Z. Kuznetsova[†] and F. Toppan[‡]

July 3, 2015

* *Department of Mathematics and Information Sciences,
Graduate School of Science, Osaka Prefecture University, Nakamozu Campus,
Sakai, Osaka 599-8531 Japan.*

[†] *UFABC, Av. dos Estados 5001, Bangu,
cep 09210-580, Santo André (SP), Brazil.*

[‡] *CBPF, Rua Dr. Xavier Sigaud 150, Urca,
cep 22290-180, Rio de Janeiro (RJ), Brazil.*

Abstract

In two papers, JPA: Math. Theor. 46, 405204 (2013) and JMP 56, 031701 (2015), second-order invariant PDEs of the $d = 1$ $\ell = \frac{1}{2} + \mathbb{N}_0$ centrally extended Conformal Galilei Algebras, were constructed (for continuous and, respectively, discrete spectrum). We investigate here the general class of second-order invariant PDEs, pointing out that they are deformations of decoupled systems. For $\ell = \frac{3}{2}$ the unique deformation parameter γ belongs to the fundamental domain $\gamma \in]0, +\infty[$. The invariant PDE with discrete spectrum induces a cryptohermitian operator possessing the same spectrum as two decoupled oscillators of given energy ω_1, ω_2 . The normalization $\omega_1 = 1$ implies, for ω_2 , the admissible critical values $\omega_2 = \pm\frac{1}{3}, \pm 3$ (the negative energy solutions correspond to a special case of Pais-Uhlenbeck oscillator).

Unitarily inequivalent operators, acting on the $\mathcal{L}^2(\mathbb{R}^2)$ Hilbert space, are obtained for the deformation parameter γ belonging to the fundamental domain. The undeformed $\gamma = 0$ case corresponds to a decoupled cryptohermitian operator with enhanced symmetry at the critical values $\omega_2 = \pm\frac{1}{3}, \pm 1, \pm 3$. Two inequivalent 12-generator symmetry algebras are found at $\omega_2 = \pm\frac{1}{3}, \pm 3$ and $\omega_2 = \pm 1$, respectively. The $\ell = \frac{3}{2}$ Conformal Galilei Algebra is not a subalgebra of the decoupled symmetry algebra. Its $\gamma \rightarrow 0$ contraction corresponds to a 8-generator subalgebra of the decoupled $\omega_2 = \pm\frac{1}{3}, \pm 3$ symmetry algebra.

The features of the $\ell \geq \frac{5}{2}$ invariant PDEs are briefly discussed.

CBPF-NF-003/15

*E-mail: aizawa@mi.s.osakafu-u.ac.jp

[†]E-mail: zhanna.kuznetsova@ufabc.edu.br

[‡]E-mail: toppan@cbpf.br

1 Introduction

In [1] and [2] second-order PDEs, invariant under the $\widehat{\mathfrak{cga}}_\ell$ ($\ell = \frac{1}{2} + \mathbb{N}_0$) centrally extended Conformal Galilei Algebra [3], were constructed. They were shown to possess a spectrum which is either continuous [1] or discrete (positive and bounded) [2]. In [1] the invariant PDEs were obtained via Verma module representation, while in [2] the so called on-shell condition was used (for the cases at hand the two approaches are proven to be equivalent).

In this paper we address several important issues that were not touched in these two previous works. We name a few: the identification of the general class of invariant PDEs (which turns out to depend on real parameters belonging to a fundamental domain), the existence of a contraction algebra, the reason for the cryptohermiticity (we use here the word adopted in [4]) of the discrete spectrum, the construction of the associated Hilbert spaces, the connection with Pais-Uhlenbeck oscillators with unbounded spectrum, etc.

Specifically, the following list of results is derived in the present paper (we limit here in the Introduction to discuss the first non-trivial case obtained for $\ell = \frac{3}{2}$, the $\ell > \frac{3}{2}$ cases are commented in Section 9): two special differential realizations of $\widehat{\mathfrak{cga}}_\ell$ produce, as invariant PDEs, Schrödinger-type equations with continuous (respectively, discrete) spectrum and no explicit dependence on the time coordinate. Both realizations depend on a parameter $\gamma \neq 0$. Unitarily inequivalent theories are recovered for γ belonging to the fundamental domain $\gamma \in]0, +\infty[$.

The $\gamma = 0$ PDEs are decoupled equations. The continuum spectrum case corresponds to the free Schrödinger equation in 1 + 1 dimensions, while the discrete spectrum case corresponds to a system of two decoupled *cryptohermitian oscillators* (namely, despite being non hermitian, possessing the same spectrum as two decoupled oscillators with the given frequencies). The parameter γ can therefore be regarded as a deformation parameter and as a coupling constant.

Without loss of generality we can fix $\omega_1 = 1$ to be the energy mode of the first oscillator in the coupled cryptohermitian PDE. Then, the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ invariance of the PDE is recovered if the energy mode of the second (crypto-oscillator) possesses the critical values $\omega_2 = \pm\frac{1}{3}, \pm 3$ ($\omega_2 = 3$ is the solution given in [2]). The negative values correspond to an unbound spectrum and, as explained later, are connected with special cases of the Pais-Uhlenbeck oscillators. At fixed $\omega_{1,2}$, the spectrum of the cryptohermitian operators does not depend on the value of γ .

The $\gamma \rightarrow 0$ limit of the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ algebra produces a contraction algebra which is a symmetry subalgebra of the decoupled systems. For the decoupled cryptohermitian oscillators (without loss of generality the analysis can be limited to the $\omega_1 = 1, \omega_2 \geq 1$ domain), the PDE possesses a 9-generator symmetry algebra at generic values, with enhanced symmetry at the critical values $\omega_2 = 1$ and $\omega_2 = 3$ (two different 12-generator symmetry algebras are obtained at these special points). The $\gamma \rightarrow 0$ contraction algebra is a 8-generator subalgebra of the $\omega_2 = 3$, 12-generator decoupled symmetry.

For all critical values $\omega_2 = \pm\frac{1}{3}, \pm 3$ and for all values of γ (including $\gamma = 0$), the cryptohermitian operators associated with the discrete spectrum act on the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$. The existence of similarity transformations prove that the spectrum is independent of γ . Unitary transformations change the phase of γ . Therefore, inequivalent cryptohermitian operators with the same spectrum are labeled by $\gamma \in [0, +\infty[$.

It is easily shown that the eigenvectors are not normalized in $\mathcal{L}^2(\mathbb{R}^2)$. A different Hilbert space, $\mathcal{L}^2(\widetilde{\mathbb{R}}^2)$, can be introduced. It is defined by preserving the canonical commutation relations while changing the conjugation properties of the creation/annihilation operators. Since the canonical commutation relations are unchanged, the spectrum of the operators acting on $\mathcal{L}^2(\widetilde{\mathbb{R}}^2)$ coincides with the spectrum of the operators acting on $\mathcal{L}^2(\mathbb{R}^2)$. In $\mathcal{L}^2(\widetilde{\mathbb{R}}^2)$ the $\gamma = 0$ operator

is hermitian and given by the sum of two decoupled harmonic oscillators. The $\gamma \neq 0$ operators are cryptohermitian and their eigenstates, which belong to $\mathcal{L}^2(\widetilde{\mathbb{R}}^2)$, are not orthogonal.

Within this framework we are now in position to discuss the subtle connection with Pais-Uhlenbeck oscillators. The Pais-Uhlenbeck model is a higher derivative system [5, 4, 6] which admits, via the Ostrogradskiĭ construction [7], a Hamiltonian formulation. The Ostrogradskiĭ Hamiltonian is canonically equivalent to a set of decoupled harmonic oscillators with alternating (positive and negative) energy modes. In a series of papers [8, 9, 10, 11, 12] the Pais-Uhlenbeck oscillators with energy modes given (up to a normalization factor) by the arithmetic progression $\omega_i = 2i - 1$ were linked to the Conformal Galilei Algebras $\widehat{\text{cga}}_\ell$ (with $\ell = n - \frac{1}{2}$). The present analysis proves that the connection is rather subtle. The PDE, invariant under the Conformal Galilei Algebra, is obtained for the coupled cryptohermitian operator with $\gamma \neq 0$ and unbounded spectrum. The derivation of the Pais-Uhlenbeck oscillator requires two non-trivial passages which (both) spoil the Conformal Galilei invariance: *i*) taking the $\gamma \rightarrow 0$ decoupling limit and *ii*) change the conjugation properties, by replacing the decoupled cryptohermitian operator with the hermitian decoupled harmonic oscillator.

Another result presented in the paper is the realization of a commutative diagram relating, via similarity transformations and a change of the time coordinate, the differential realizations for coupled and decoupled Schrödinger equations with continuous and discrete spectrum.

The scheme of the paper is as follows: in Section 2 we present the ($\gamma \neq 0$ -dependent) differential realization for the deformation of the free Schrödinger equation at $\ell = \frac{3}{2}$. The differential realization for the coupled cryptohermitian oscillator is presented in Section 3. The connection of the two differential realizations obtained by similarity transformations and change of the time coordinate is shown in Section 4. In Section 5 the most general solution of the $\widehat{\text{cga}}_{\frac{3}{2}}$ -invariant cryptohermitian oscillator is given. The symmetry of the decoupled cryptohermitian oscillator (with enhanced critical points at $\omega_2 = 1$ and $\omega_2 = 3$) is presented in Section 6. The $\ell = \frac{3}{2}$ contraction algebra in the $\gamma \rightarrow 0$ limit is given in Section 7. The Hilbert spaces for the cryptohermitian oscillators are discussed in Section 8. In Section 9 the extension to the $\ell > \frac{3}{2}$ cases and the relation to Pais-Uhlenbeck oscillators are commented. Generalizations of the present construction are discussed in the Conclusions.

2 Differential realization for the free system deformation

The $d = 1$ $\ell = \frac{3}{2}$ centrally extended Conformal Galilei algebra $\widehat{\text{cga}}_{\frac{3}{2}}$ admits, for an arbitrary parameter $\gamma \neq 0$, the following differential realization in terms of first-order differential operators

acting on functions of τ, x, y :

$$\begin{aligned}
\bar{z}_+ &= \partial_\tau, \\
\bar{z}_0 &= -2i\tau\partial_\tau - ix\partial_x - 3iy\partial_y - 2i, \\
\bar{z}_- &= -4\tau^2 - 4\left(\tau x - \frac{3}{\gamma}y\right)\partial_x - 12\tau y\partial_y - 8(\tau - ix^2), \\
\bar{w}_{+3} &= \partial_y, \\
\bar{w}_{+1} &= -2i\tau\partial_y + \frac{2i}{\gamma}\partial_x, \\
\bar{w}_{-1} &= -4\tau^2\partial_y + \frac{8}{\gamma}\tau\partial_x - \frac{8i}{\gamma}x, \\
\bar{w}_{-3} &= 8i\tau^3\partial_y - \frac{24i}{\gamma}\tau^2\partial_x - 48\left(\frac{1}{\gamma}\tau x + \frac{1}{\gamma^2}y\right), \\
\bar{c} &= 1.
\end{aligned} \tag{1}$$

The non-vanishing $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ commutators are

$$\begin{aligned}
[\bar{z}_0, \bar{z}_\pm] &= \pm 2i\bar{z}_\pm, \\
[\bar{z}_+, \bar{z}_-] &= -4i\bar{z}_0, \\
[\bar{z}_\pm, \bar{w}_k] &= (k \mp 3)i\bar{w}_{k\pm 2}, \\
[\bar{w}_{|k|}, \bar{w}_{-|k|}] &= (3 - 2k)\frac{16}{\gamma^2}\bar{c}.
\end{aligned} \tag{2}$$

Three second-order on-shell invariant differential operators $\bar{\Omega}_{\pm 1, 0}$ are encountered at degree $\pm 1, 0$ (measured by the degree operator $-\frac{i}{2}\bar{z}_0$), respectively:

$$\begin{aligned}
\bar{\Omega}_{+1} &= i\partial_\tau - i\gamma x\partial_y + \frac{1}{2}\partial_x^2 = i\bar{z}_+ - \bar{H}_+ = i\bar{z}_+ + \frac{\gamma^2}{16}(\{\bar{w}_{+3}, \bar{w}_{-1}\} - \{\bar{w}_{+1}, \bar{w}_{+1}\}), \\
\bar{\Omega}_0 &= -2i\tau\bar{\Omega}_{+1} = i\bar{z}_0 - \bar{H}_0 = i\bar{z}_0 + \frac{\gamma^2}{32}(\{\bar{w}_{+3}, \bar{w}_{-3}\} - \{\bar{w}_{+1}, \bar{w}_{-1}\}), \\
\bar{\Omega}_{-1} &= -4\tau^2\bar{\Omega}_{+1} = i\bar{z}_- - \bar{H}_- = i\bar{z}_- + \frac{\gamma^2}{16}(\{\bar{w}_{+1}, \bar{w}_{-3}\} - \{\bar{w}_{-1}, \bar{w}_{-1}\}).
\end{aligned} \tag{3}$$

The $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ on-shell invariant condition for $\bar{\Omega}_{\pm 1, 0}$ (see [13, 2] for a definition) is guaranteed by the fact that their only non-vanishing commutators with the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ generators are expressed as

$$\begin{aligned}
[\bar{z}_0, \bar{\Omega}_{+1}] &= 2i\bar{\Omega}_{+1}, \\
[\bar{z}_-, \bar{\Omega}_{+1}] &= 4i\bar{\Omega}_0 = 8\tau\bar{\Omega}_{+1}, \\
[\bar{z}_+, \bar{\Omega}_0] &= -2i\bar{\Omega}_{+1} = \tau^{-1}\bar{\Omega}_0, \\
[\bar{z}_-, \bar{\Omega}_0] &= 2i\bar{\Omega}_{-1} = 4\tau\bar{\Omega}_0, \\
[\bar{z}_+, \bar{\Omega}_{-1}] &= -4i\bar{\Omega}_0 = 2\tau^{-1}\bar{\Omega}_{-1}, \\
[\bar{z}_0, \bar{\Omega}_{-1}] &= -2i\bar{\Omega}_{-1}.
\end{aligned} \tag{4}$$

The three operators $\bar{\Omega}_{\pm 1, 0}$ close the $\mathfrak{sl}(2)$ algebra, with $\bar{\Omega}_0$ the Cartan element:

$$\begin{aligned}
[\bar{\Omega}_0, \bar{\Omega}_{\pm 1}] &= \mp 2\bar{\Omega}_{\pm 1}, \\
[\bar{\Omega}_{+1}, \bar{\Omega}_{-1}] &= 4\bar{\Omega}_0.
\end{aligned} \tag{5}$$

The degree 1 invariant equation

$$\bar{\Omega}_1 \Psi(\tau, x, y) = 0 \quad \equiv \quad i\partial_\tau \Psi = -\frac{1}{2}\partial_x^2 \Psi + i\gamma x \partial_y \Psi \quad (6)$$

is a Schrödinger equation with no explicit time dependence (τ is the time coordinate). The parameter γ is a coupling constant. This equation can be regarded as a γ -deformation of the free Schrödinger equation in 1 + 1 dimensions.

3 Differential realization for the crypto-oscillator deformation

The following differential realization of the $\widehat{\text{cga}}_{\frac{3}{2}}$ algebra (2) (in terms of first-order differential operators acting on functions of t, x, y)

$$\begin{aligned} z_0 &= \partial_t, \\ z_+ &= e^{2it}(\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\ z_- &= e^{-2it}(\partial_t - ix\partial_x - 3iy\partial_y + \frac{12}{\gamma}y\partial_x + 7ix^2 + \frac{12}{\gamma}xy - 2i), \\ w_{+3} &= e^{3it}\partial_y, \\ w_{+1} &= e^{it}(\partial_y + \frac{2i}{\gamma}\partial_x + \frac{2i}{\gamma}x), \\ w_{-1} &= e^{-it}(\partial_y + \frac{4i}{\gamma}\partial_x - \frac{4i}{\gamma}x), \\ w_{-3} &= e^{-3it}(\partial_y + \frac{6i}{\gamma}\partial_x - \frac{18i}{\gamma}x - \frac{48}{\gamma^2}y), \\ c &= 1 \end{aligned} \quad (7)$$

produces the second-order on-shell invariant operators $\Omega_{\pm 1,0}$, given by

$$\begin{aligned} \Omega_{+1} &= e^{2it}\Omega_0 = iz_+ - H_+ = iz_+ + \frac{\gamma^2}{16}(\{w_{+3}, w_{-1}\} - \{w_{+1}, w_{+1}\}), \\ \Omega_0 &= i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 - 3y\partial_y - i\gamma x \partial_y - \frac{3}{2} = iz_0 - H_0 = iz_0 + \frac{\gamma^2}{32}(\{w_{+3}, w_{-3}\} - \{w_{+1}, w_{-1}\}), \\ \Omega_{-1} &= e^{-2it}\Omega_0 = iz_- - H_- = iz_- + \frac{\gamma^2}{16}(\{w_{+1}, w_{-3}\} - \{w_{-1}, w_{-1}\}). \end{aligned} \quad (8)$$

The on-shell invariant condition is guaranteed by the fact that their only non-vanishing commutators with the $\widehat{\text{cga}}_{\frac{3}{2}}$ generators are expressed as

$$\begin{aligned} [z_0, \Omega_{+1}] &= 2i\Omega_{+1}, \\ [z_-, \Omega_{+1}] &= 4i\Omega_0 = 4ie^{-2it}\Omega_{+1}, \\ [z_+, \Omega_0] &= -2i\Omega_{+1} = -2ie^{2it}\Omega_0, \\ [z_-, \Omega_0] &= 2i\Omega_{-1} = 2ie^{-2it}\Omega_0, \\ [z_+, \Omega_{-1}] &= -4i\Omega_0 = -4ie^{2it}\bar{\Omega}_{-1}, \\ [z_0, \Omega_{-1}] &= -2i\Omega_{-1}. \end{aligned} \quad (9)$$

The $\Omega_{\pm 1,0}$ operators close the $\mathfrak{sl}(2)$ algebra

$$\begin{aligned} [\Omega_0, \Omega_{\pm 1}] &= \mp 2\Omega_{\pm 1}, \\ [\Omega_{+1}, \Omega_{-1}] &= 4\Omega_0. \end{aligned} \quad (10)$$

The degree 0 invariant equation

$$\Omega_0\Psi(t, x, y) = 0 \quad \equiv \quad i\partial_t\Psi = \left(-\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + 3y\partial_y + i\gamma x\partial_y + \frac{3}{2}\right)\Psi \quad (11)$$

is a Schrödinger-type equation with no explicit dependence on the time coordinate t and with a non-hermitian operator (which is proven to be cryptohermitian) in the right hand side. The parameter γ is a coupling constant. The equation (11) can be regarded as a γ -deformation of a decoupled “cryptohermitian oscillator” discussed in the following.

4 Connection of the two differential realizations

The two differential realizations of the $\widehat{\mathfrak{cga}}_{\frac{3}{2}}$ algebra introduced in Section 2 and 3 are characterized by inducing a Schrödinger equation with no explicit dependence on the time coordinate from, respectively, degree 1 and degree 0 invariant operators.

The two differential realizations are connected via a similarity transformation coupled with a redefinition of the time coordinate.

Let us denote as g an operator entering (7) or (8) and as \bar{g} its corresponding operator entering (1) or (3). For convenience we introduce the operator X_+ by setting, for z_+ in (7),

$$z_{\pm} = e^{\pm 2it}(\partial_t + X_{\pm}), \quad X_+ = ix\partial_x + 3iy\partial_y + ix^2 + 2i. \quad (12)$$

The connection is explicitly realized by the similarity transformation

$$\begin{aligned} g \mapsto \bar{g} &= e^S g e^{-S}, & (e^S &= e^{S_2} e^{S_1}), \\ S_1 &= tX_+, \\ S_2 &= \frac{1}{2}x^2, \end{aligned} \quad (13)$$

supplemented by the redefinition of the time coordinate

$$t \mapsto \tau = \frac{i}{2}e^{-2it}. \quad (14)$$

The first similarity transformation (induced by S_1) allows to map

$$z_+ \mapsto \hat{z}_+ = e^{S_1} z_+ e^{-S_1} = e^{2it} \partial_t = \partial_{\tau}, \quad (15)$$

so that

$$\Omega_{+1} \mapsto \widehat{\Omega}_{+1} = e^{S_1} \Omega_{+1} e^{-S_1} = ie^{2it} \partial_t - \widehat{H}_{+1}, \quad (16)$$

with

$$\widehat{H}_{+1} = e^{2it} (iX_+ + e^{tX_+} H_0 e^{-tX_+}). \quad (17)$$

Due to the commutators

$$[X_+, H_0] = 2iK_+, \quad [X_+, K_+] = -2iK_+, \quad (18)$$

where

$$K_+ = \frac{1}{2}(\partial_x + x)^2 - i\gamma x\partial_y, \quad (19)$$

we obtain

$$\widehat{H}_{+1} = e^{2it} (iX_+ + H_0 + K_+) - K_+. \quad (20)$$

The remarkable identity

$$iX_+ + H_0 + K_+ = 0 \quad (21)$$

implies that \widehat{H}_{+1} does not depend on the time coordinate (either t or τ).

The second similarity transformation (induced by S_2) allows to express

$$\widehat{\Omega}_{+1} \mapsto \overline{\Omega}_{+1} = e^{S_2} \widehat{\Omega}_{+1} e^{-S_2} = i\partial_\tau + \frac{1}{2}\partial_x^2 - i\gamma x\partial_y \quad (22)$$

in the form which reduces, in the $\gamma \rightarrow 0$ limit, to the standard free Schrödinger equation in 1 + 1 dimensions.

One should observe that the similarity transformation preserves the symmetry properties of the equations, mapping first-order invariant operators into first-order invariant operators.

The following commutative diagram is obtained:

$$\begin{array}{ccc} \text{coupled } (\gamma \neq 0) : & \mathbf{Free}_\gamma^{0,\pm 1}(\tau) & \xleftrightarrow{\mathbf{S}} \mathbf{Osc}_\gamma^{0,\pm 1}(t) \\ & \downarrow \mathbf{r} & \downarrow \mathbf{r} \\ \text{decoupled } (\gamma = 0) : & \mathbf{Free}^{0,\pm 1}(\tau) & \xleftrightarrow{\mathbf{S}} \mathbf{Osc}^{0,\pm 1}(t) \end{array} \quad (23)$$

The left (right) part of the diagram denotes the equations obtained from the differential realizations of Section **2 (3)**. The horizontal arrows indicate the similarity transformation together with the change of the time coordinate, τ and t respectively.

The three invariant PDEs (at degree 0, ± 1) are mapped into each other.

In the left part, the Schrödinger-type invariant PDE corresponds to *deg 1* and possesses a continuous spectrum.

In the right part the Schrödinger-type invariant PDE corresponds to *deg 0* and possesses a real, discrete spectrum which coincides with the spectrum of two decoupled harmonic oscillators.

The vertical arrows denote the mapping to the decoupled systems. This mapping can be reached in two ways:

i) the singular similarity transformation

$$g \mapsto R_1 g R_1^{-1}, \quad \text{with } R_1 = e^{\alpha y \partial_y} \quad (24)$$

(such that $\gamma \rightarrow e^{-\alpha} \gamma$) in the $\alpha \rightarrow \infty$ limit. Despite the singularity of the limit, the invariant equations of the upper part of the diagram admits as non-singular limit the decoupled equations of the lower part of the diagram. This similarity transformation preserves the symmetry of the equations, mapping first-order invariant operators into first-order invariant operators;

ii) the non-singular similarity transformation

$$g \mapsto R_2 g R_2^{-1}, \quad R_2 = e^{\left(\frac{3i}{8}\gamma x + \frac{i}{8}\gamma \partial_x - \frac{1}{96}\gamma^2 \partial_y\right) \partial_y}. \quad (25)$$

This non-singular transformation does not preserve the symmetry of the equation because some of the transformed generators are no longer first-order differential operators. Nevertheless, it proves that the deformed cryptohermitian operators possess the same spectrum of eigenvalues as the $\gamma = 0$ decoupled cryptohermitian operators of the same frequency.

The four Schrödinger-type equations associated with the commutative diagram, starting from the upper right corner and proceeding clockwise, are: (I) the deformed cryptohermitian oscillator (11), (II) the decoupled cryptohermitian oscillator, (III) the free Schrödinger equation in $1 + 1$ dimensions and, finally, (IV) the deformed free Schrödinger equation (6).

The three inequivalent (with constant, linear and quadratic potential, see [13, 14, 15, 16]) time-independent Schrödinger equations in $1 + 1$ dimensions invariant under the Schrödinger algebra are recovered as restrictions of the $\ell = \frac{3}{2}$ invariant PDEs. Indeed, if we introduce the x, y separation of variables, the equation of the harmonic oscillator and the free Schrödinger equation are recovered by setting $\partial_y \equiv 0$ from, respectively, equations (I) and (IV). The linear Schrödinger equation is recovered from equation (IV) after setting $\Psi(\tau, x, y) = \psi(\tau, x)\phi(y)$, with the restriction $\partial_y\phi(y) = k\phi(y)$.

5 The general $\ell = \frac{3}{2}$ cryptohermitian oscillator

The $\widehat{\text{cga}}_{\frac{3}{2}}$ conformal Galilei invariance requires the coupling parameter $\gamma \neq 0$. Since a unitary transformation changes its phase, we can assume without loss of generality that γ belongs to the fundamental domain $\gamma \in]0, +\infty[$.

For γ real, the invariant PDEs in the left part of the commutative diagram (23) are hermitian. This is not the case for the invariant PDEs in the right part of the diagram. Under hermitian conjugation, the deformed cryptohermitian oscillator equation is transformed into its conjugate

$$\Omega_0^\dagger(\gamma)\Psi(t, x, y) = 0 \quad \equiv \quad (i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 + 3y\partial_y - i\gamma x\partial_y + \frac{3}{2})\Psi(t, x, y). \quad (26)$$

All operators

$$K = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + \omega y\partial_y - i\gamma x\partial_y + C, \quad (27)$$

for any arbitrary constant C and any $\gamma \neq 0$, induce a Schrödinger-type invariant equation with $\ell = \frac{3}{2}$ Conformal Galilei Symmetry, if ω is restricted to the values

$$\omega = \pm\frac{1}{3}, \pm 3. \quad (28)$$

The $\omega \leftrightarrow -\omega$ change of sign is explained by the hermitian conjugation. Understanding the $\omega \leftrightarrow \frac{1}{\omega}$ transformation is subtler. One should note at first that in the $\gamma = 0$ decoupled case the role of the space coordinates x, y can be exchanged by performing the canonical transformation

$$y \leftrightarrow \frac{1}{\sqrt{2}}(x - \partial_x), \quad \partial_y \leftrightarrow \frac{1}{\sqrt{2}}(x + \partial_x).$$

Next, the coupling term is introduced in terms of the non-singular similarity transformation. As it turns out, this procedure guarantees the conformal Galilei invariance of the resulting PDE.

An explicit check of the symmetries of this class of PDEs proves that, in order to have the on-shell invariant equations $[z_\pm, \Omega_0] = f_\pm \cdot \Omega_0$, with f_\pm arbitrary functions of the coordinates and symmetry generators of the form $z_\pm = e^{\pm i\lambda t}(\partial_t + \check{X}_\pm)$, (\check{X}_\pm time-independent operators and $\lambda \neq 0$), the following necessary and sufficient condition has to be satisfied: the two equations

$$\begin{aligned} \lambda(\omega^2 + 1 - \frac{5}{2}\lambda^2) &= 0, \\ -3\lambda^2 + 3\lambda^4 + 2\lambda\omega + 4\lambda^3\omega - \lambda^2\omega^2 - 2\lambda\omega^3 &= 0, \end{aligned} \quad (29)$$

must be simultaneously satisfied. The only non-vanishing solutions for λ are encountered at $\omega = \pm 3$ and $\omega = \pm\frac{1}{3}$. Therefore, the $\omega = \pm\frac{1}{3}, \pm 3$ critical values are special points of enhanced symmetry.

6 Symmetry of the decoupled cryptohermitian oscillator

By applying the same considerations as in Section 5, it is sufficient to analyze the symmetry of the decoupled ($\gamma = 0$) cryptohermitian operator

$$\Omega = i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}x^2 + \omega y\partial_y \quad (30)$$

in the range $\omega \in [1, \infty[$.

For a generic ω the following invariant operators can be encountered at degree $0, \pm\frac{1}{2}, \pm\frac{\omega}{2}, \pm 1$:

$$\begin{aligned} \bar{z}_\pm &= e^{\pm 2it}(\partial_t \pm ix\partial_x + i\omega y\partial_y + ix^2 \pm \frac{i}{2}), \\ \bar{z}_0 &= \partial_t + i\omega y\partial_y, \\ \bar{d} &= -\frac{i}{2}\partial_t, \\ \bar{c} &= 1, \\ \bar{w}_\omega &= e^{i\omega t}\partial_y, \\ \bar{w}_1 &= e^{it}(\partial_x + x), \\ \bar{w}_{-1} &= e^{-it}(\partial_x - x), \\ \bar{w}_{-\omega} &= e^{-i\omega t}y. \end{aligned} \quad (31)$$

\bar{d} is the degree operator. Explicitly, the degree is

$$\pm 1 : \bar{z}_\pm; \quad 0 : \bar{z}_0, \bar{d}, \bar{c}; \quad \pm\frac{\omega}{2} : \bar{w}_{\pm\omega}; \quad \pm\frac{1}{2} : \bar{w}_{\pm 1}. \quad (32)$$

This 9-generator symmetry algebra closes the $\mathfrak{u}(1) \oplus (\mathfrak{sch}(1) \oplus \mathfrak{h}(1))$ algebra, with non-vanishing commutation relations given by

$$\begin{aligned} [\bar{d}, \bar{z}_\pm] &= \pm \bar{z}_\pm, \\ [\bar{d}, \bar{w}_k] &= \frac{k}{2}\bar{w}_k, \\ [\bar{z}_0, \bar{z}_\pm] &= \pm 2i\bar{z}_\pm, \\ [\bar{z}_+, \bar{z}_-] &= -4i\bar{z}_0, \\ [\bar{z}_0, \bar{w}_{\pm 1}] &= \pm i\bar{w}_{\pm 1}, \\ [\bar{z}_\pm, \bar{w}_{\mp 1}] &= \mp 2i\bar{w}_{\pm 1}, \\ [\bar{w}_1, \bar{w}_{-1}] &= -2\bar{c}, \\ [w_\omega, w_{-\omega}] &= \bar{c}. \end{aligned} \quad (33)$$

\bar{d} is the generator of the $\mathfrak{u}(1)$ subalgebra, while $\bar{z}_0, \bar{z}_\pm, \bar{w}_{\pm 1}, \bar{c}$ generate the Schrödinger algebra $\mathfrak{sch}(1)$ and $w_{\pm\omega}, \bar{c}$ generate the Heisenberg algebra $\mathfrak{h}(1)$.

The critical values $\omega = 1$ and $\omega = 3$ are points of enhanced symmetry for the decoupled system.

6.1 The enhanced symmetry for the decoupled $\omega = 1$ system

At the critical value $\omega = 1$ three extra generators are found at degree 0 and -1 :

$$\begin{aligned} q_1 &= y(\partial_x + x), \\ q_2 &= e^{-2it}y^2, \\ q_3 &= e^{-2it}y(\partial_x - x). \end{aligned} \quad (34)$$

They have to be added to the previous set of (generic) symmetry generators

$$\begin{aligned}
\bar{z}_{\pm} &= e^{\pm 2it}(\partial_t \pm ix\partial_x + iy\partial_y + ix^2 \pm \frac{i}{2}), \\
\bar{z}_0 &= \partial_t + iy\partial_y, \\
\bar{d} &= -\frac{i}{2}\partial_t, \\
\bar{c} &= 1, \\
\bar{w}_{1b} &= e^{it}\partial_y, \\
\bar{w}_{1a} &= e^{it}(\partial_x + x), \\
\bar{w}_{-1a} &= e^{-it}(\partial_x - x), \\
\bar{w}_{-1b} &= e^{-it}y.
\end{aligned} \tag{35}$$

The extra non-vanishing commutation relations involving the q_i 's generators are

$$\begin{aligned}
[\bar{z}_0, q_1] &= iq_1, \\
[\bar{d}, q_2] &= -q_2, \\
[\bar{d}, q_3] &= -q_3, \\
[\bar{z}_+, q_3] &= -2iq_3, \\
[\bar{z}_-, q_1] &= 2iq_3, \\
[\bar{w}_{1b}, q_1] &= \bar{w}_{1a}, \\
[\bar{w}_{-1a}, q_1] &= 2\bar{w}_{-1b}, \\
[\bar{w}_{1b}, q_2] &= 2\bar{w}_{-1b}, \\
[\bar{w}_{1b}, q_3] &= \bar{w}_{-1a}, \\
[\bar{w}_{1a}, q_3] &= -2\bar{w}_{-1b}, \\
[q_1, q_3] &= -2q_2.
\end{aligned} \tag{36}$$

The symmetry algebra closes as a non semi-simple, 12-generator, Lie algebra.

6.2 The enhanced symmetry for the decoupled $\omega = 3$ system

At the $\omega = 3$ critical value the three extra generators \bar{r}_{-j} , $j = 1, 2, 3$, of degree $-j$, are encountered. We have, explicitly,

$$\begin{aligned}
\bar{r}_{-1} &= e^{-2it}y(\partial_x + x), \\
\bar{r}_{-2} &= e^{-4it}y(\partial_x - x), \\
\bar{r}_{-3} &= e^{-6it}y^2.
\end{aligned} \tag{37}$$

At $\omega = 3$ the symmetry algebra is a 12-generator algebra which differs from the 12-generator symmetry algebra of the $\omega = 1$ decoupled system.

The extra non-vanishing commutation relations involving the \bar{r}_{-j} generators are given by

$$\begin{aligned}
[\bar{d}, \bar{r}_{-j}] &= -j\bar{r}_{-j}, \\
[\bar{z}_0, \bar{r}_{-1}] &= i\bar{r}_{-1}, \\
[\bar{z}_0, \bar{r}_{-2}] &= -i\bar{r}_{-2}, \\
[\bar{z}_-, \bar{r}_{-1}] &= 2i\bar{r}_{-2}, \\
[\bar{z}_+, \bar{r}_{-2}] &= -2i\bar{r}_{-1}, \\
[\bar{w}_{+3}, \bar{r}_{-1}] &= \bar{w}_{+1}, \\
[\bar{w}_{-1}, \bar{r}_{-1}] &= 2\bar{w}_{-3}, \\
[\bar{w}_{+3}, \bar{r}_{-2}] &= \bar{w}_{-1}, \\
[\bar{w}_{+1}, \bar{r}_{-2}] &= -2\bar{w}_{-3}, \\
[\bar{w}_{+3}, \bar{r}_{-3}] &= 2\bar{w}_{-3}, \\
[\bar{r}_{-1}, \bar{r}_{-2}] &= -2\bar{r}_{-3}.
\end{aligned} \tag{38}$$

7 The contraction algebra

For $\gamma \neq 0$ and in the $\gamma \rightarrow 0$ limit, a contraction algebra is recovered from (7) by suitably rescaling the generators. The contraction requires the rescaling $g \mapsto \tilde{g} = \gamma^s g$ (g is any generator entering (7)), with

$$\begin{aligned}
s = 0 & : z_0, z_+, w_3, c, \\
s = 1 & : z_-, w_1, w_{-1}, \\
s = 2 & : w_{-3}.
\end{aligned} \tag{39}$$

The contracted 8-generator algebra expressed by $\tilde{z}_\pm, \tilde{z}_0, \tilde{c}, \tilde{w}_k$ ($k = \pm 1, \pm 3$) is a subalgebra of the full 12-generator symmetry algebra. The identification goes as follows

$$\begin{aligned}
\tilde{z}_+ &= e^{\tilde{S}} \bar{z}_+ e^{-\tilde{S}} = e^{2it} (\partial_t + ix\partial_x + 3iy\partial_y + ix^2 + 2i), \\
\tilde{z}_0 &= e^{\tilde{S}} (2i\bar{d} - \frac{3i}{2}\bar{c}) e^{-\tilde{S}} = \partial_t, \\
\tilde{z}_- &= e^{\tilde{S}} (12i\bar{r}_{-1}) e^{-\tilde{S}} = 12ie^{-2it} y (\partial_x + x), \\
\tilde{w}_{+3} &= e^{\tilde{S}} \bar{w}_{+3} e^{-\tilde{S}} = e^{3it} \partial_y, \\
\tilde{w}_{+1} &= e^{\tilde{S}} (-2i\bar{w}_{+1}) e^{-\tilde{S}} = 2ie^{it} (\partial_x + x), \\
\tilde{w}_{-1} &= e^{\tilde{S}} (-4i\bar{w}_{-1}) e^{-\tilde{S}} = 4ie^{-it} (\partial_x - x), \\
\tilde{w}_{-3} &= e^{\tilde{S}} (48\bar{w}_{-3}) e^{-\tilde{S}} = -48e^{-3it} y, \\
\tilde{c} &= e^{\tilde{S}} \bar{c} e^{-\tilde{S}} = 1,
\end{aligned} \tag{40}$$

with the similarity transformation given by $\tilde{S} = -\frac{3}{2}it$.

The contraction algebra corresponds to the two-dimensional Euclidean algebra acting on two set of creation/annihilation operators. We have the $\mathfrak{e}(2) \oplus \mathfrak{h}(2)$ algebra, with non-vanishing

commutators given by

$$\begin{aligned}
[\tilde{z}_0, \tilde{z}_\pm] &= \pm 2i\tilde{z}_\pm, \\
[\tilde{z}_0, \tilde{w}_k] &= k\tilde{w}_k, \\
[\tilde{z}_+, \tilde{w}_{-1}] &= -4i\tilde{w}_{+1}, \\
[\tilde{z}_-, \tilde{w}_{+3}] &= 6i\tilde{w}_{+1}, \\
[\tilde{z}_-, \tilde{w}_{-1}] &= 2i\tilde{w}_{-3}, \\
[\tilde{w}_{|k|}, \tilde{w}_{-|k|}] &= (3 - 2k)16\tilde{c}.
\end{aligned} \tag{41}$$

8 Cryptohermitian operators and Hilbert space

The non-hermitian operator derived from (11) is

$$H_0(\gamma) = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + 3y\partial_y + i\gamma x\partial_y + \frac{3}{2}. \tag{42}$$

It acts on the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$, with \mathbb{R}^2 expressed by the coordinates x, y . The $H_0(\gamma)$ eigenvalues $E_{n,m}$ are real, discrete, positive and bounded. For $n, m \in \mathbb{N}_0$ we have

$$E_{n,m} = n + 3m + \frac{3}{2}. \tag{43}$$

Due to the reality of its eigenvalues $H_0(\gamma)$ belongs to the class of cryptohermitian operators as defined by Smilga in [4]. Its eigenvectors, however, are not normalized in $\mathcal{L}^2(\mathbb{R}^2)$. We have indeed, for the $\psi_{n,m}(x, y)$ eigenvector,

$$\psi_{n,m}(x, y) = y^m \varphi_n(x - i\gamma m), \tag{44}$$

with $\varphi_n(x)$ the n -th eigenvalue of the $\omega_1 = 1$ harmonic oscillator.

Associated to $H_0(\gamma)$ one can introduce a different operator with the same canonical commutation relations (therefore, the same spectrum, up to a vacuum energy constant), but different hermitian conjugation properties. One is naturally induced to define $K(\bar{\gamma})$ as

$$K(\bar{\gamma}) = a^\dagger a + 3b^\dagger b + \frac{1}{2} + \bar{\gamma}(a + a^\dagger)b, \tag{45}$$

through the positions

$$a = \frac{1}{\sqrt{2}}(x + \partial_x), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x), \quad b = \frac{1}{\sqrt{2}}(z + \partial_z), \quad b^\dagger = \frac{1}{\sqrt{2}}(z - \partial_z), \quad \bar{\gamma} = \frac{i}{\sqrt{2}}\gamma. \tag{46}$$

The two independent creation/annihilation operators (a, a^\dagger and b, b^\dagger) have non vanishing commutators: $[a, a^\dagger] = [b, b^\dagger] = 1$.

The operator $K(\bar{\gamma})$ acts on a new $\mathcal{L}^2(\tilde{\mathbb{R}}^2)$ Hilbert space, where $\tilde{\mathbb{R}}^2$ is now expressed by the coordinates x, z .

Unlike $H_0(\gamma)$, the operator $K(\bar{\gamma})$ is mapped in the $\bar{\gamma} \rightarrow 0$ limit on the *hermitian* Hamiltonian of two decoupled harmonic oscillators. The $K(\bar{\gamma})$ eigenvectors belong to $\mathcal{L}^2(\tilde{\mathbb{R}}^2)$. They are given by the (unnormalized) states $|n, m\rangle = (a^\dagger)^n (b^\dagger)^m |vac\rangle$, where $|vac\rangle \equiv |0, 0\rangle$ is the Fock vacuum defined by the conditions $a|vac\rangle = b|vac\rangle = 0$.

One can read from the commutators

$$[K(\bar{\gamma}), A_\lambda] = \lambda A_\lambda \tag{47}$$

which excited modes are created.

For any $\bar{\gamma} \neq 0$, the solutions of the (47) equation are obtained for $\lambda = \pm 3, \pm \frac{1}{3}$. The corresponding modes are

$$\begin{aligned} A_{-3} &= b, \\ A_{-1} &= a + \frac{1}{2}\bar{\gamma}b, \\ A_{+1} &= a^\dagger - \frac{1}{4}\bar{\gamma}b, \\ A_{+3} &= b^\dagger - \frac{1}{2}\bar{\gamma}a^\dagger - \frac{1}{4}\bar{\gamma}a + \frac{1}{24}\bar{\gamma}^2b. \end{aligned} \quad (48)$$

In this basis the non-vanishing commutators are

$$[A_{-i}, A_j] = \delta_{ij}, \quad (i, j = 1, 3). \quad (49)$$

The non-hermitian operator $K(\bar{\gamma})$ commutes with the “non-hermitian analog of the Number operator”, $N(\bar{\gamma})$. In terms of the A_k modes, the operators are given by

$$\begin{aligned} K(\bar{\gamma}) &= 3A_3A_{-3} + A_1A_{-1} + \frac{1}{2}, \\ N(\bar{\gamma}) &= A_3A_{-3} + A_1A_{-1}, \\ [K(\bar{\gamma}), N(\bar{\gamma})] &= 0. \end{aligned} \quad (50)$$

The Fock vacuum $|vac\rangle$ satisfies

$$a|vac\rangle = b|vac\rangle = 0, \quad A_{-1}|vac\rangle = A_{-3}|vac\rangle = 0. \quad (51)$$

The Hilbert space $\mathcal{L}^2(\tilde{\mathbb{R}}^2)$ can be spanned by both sets of (unnormalized) states,

$$\begin{aligned} |n, m\rangle &= (a^\dagger)^n (b^\dagger)^m |vac\rangle, \\ |\bar{n}, \bar{m}\rangle &= A_1^n A_3^m |vac\rangle. \end{aligned} \quad (52)$$

We can therefore write

$$|vac\rangle = |0, 0\rangle = |\bar{0}, \bar{0}\rangle. \quad (53)$$

The spectrum of $K(\bar{\gamma})$, $N(\bar{\gamma})$ coincides with the spectrum of the Hamiltonian and Number operator of a decoupled harmonic oscillator. $|\bar{n}, \bar{m}\rangle$ is an eigenvector for $K(\bar{\gamma})$, $N(\bar{\gamma})$ with respective eigenvalues $n + 3m + \frac{1}{2}$ and $n + m$. In increasing order of $K(\bar{\gamma})$ eigenvalues, the first (unnormalized) common eigenvectors of $K(\bar{\gamma})$, $N(\bar{\gamma})$ are

$$\begin{aligned} \left(\frac{1}{2}, 0\right) : & \quad |\bar{0}, \bar{0}\rangle = |0, 0\rangle = |vac\rangle, \\ \left(\frac{3}{2}, 1\right) : & \quad |\bar{1}, \bar{0}\rangle = |1, 0\rangle, \\ \left(\frac{5}{2}, 2\right) : & \quad |\bar{2}, \bar{0}\rangle = |2, 0\rangle, \\ \left(\frac{7}{2}, 1\right) : & \quad |\bar{0}, \bar{1}\rangle = |0, 1\rangle - \frac{1}{2}\bar{\gamma}|1, 0\rangle, \\ \left(\frac{7}{2}, 3\right) : & \quad |\bar{3}, \bar{0}\rangle = |3, 0\rangle, \\ \left(\frac{9}{2}, 2\right) : & \quad |\bar{1}, \bar{1}\rangle = |1, 1\rangle - \frac{1}{2}\bar{\gamma}|2, 0\rangle - \frac{1}{4}\bar{\gamma}|0, 0\rangle, \\ \left(\frac{9}{2}, 4\right) : & \quad |\bar{4}, \bar{0}\rangle = |4, 0\rangle. \end{aligned} \quad (54)$$

Since the operators are non-hermitian, their eigenvectors are non-orthogonal. This implies measurable physical consequences. Let us suppose that we are able to prepare the system in a given common eigenvector of $K(\bar{\gamma})$, $N(\bar{\gamma})$, let's say the state $|\bar{1}, \bar{1}\rangle$. Following the standard rule of Quantum Mechanics we can compute the probability for this state to collapse, after a measurement operation, to the vacuum state. A simple computation shows that the probability is $p = |\langle N < \bar{1}, \bar{1} | \bar{0}, \bar{0} \rangle_N|^2$ ($|\bar{0}, \bar{0}\rangle_N$, $|\bar{1}, \bar{1}\rangle_N$ are the normalized states). In the case at hand we have

$$p = \frac{|\bar{\gamma}|^2}{16 + 9|\bar{\gamma}|^2}. \quad (55)$$

This probability is restricted in the range $0 \leq p < \frac{1}{9} < 1$. The deformation coupling parameter $\bar{\gamma}$, via its squared modulus, has testable consequences.

9 Comment on Pais-Uhlenbeck oscillators and the $\ell \geq \frac{5}{2}$ cases

The same spectrum of eigenvalues is obtained for

- i)* the coupled ($\gamma \neq 0$) cryptohermitian operator (42),
- ii)* the decoupled ($\gamma = 0$) cryptohermitian operator and (up to a vacuum energy shift)
- iii)* the hermitian Hamiltonian (given by (45) for $\bar{\gamma} = 0$) of two decoupled oscillators.

The construction of Section 8 can be repeated by starting with the hermitian conjugate of the (42) operator. In this case the spectrum of the three resulting operators is unbounded. It is given, up to the vacuum energy shift, by $E_{n,m} = n - 3m$. The Hilbert space of the decoupled harmonic oscillators with energy modes $1, -3$ continues to be $\mathcal{L}^2(\tilde{\mathbb{R}}^2)$, obtained by applying the creation operators a^\dagger, b^\dagger to the Fock state $|0, 0\rangle$ ($a|0, 0\rangle = b|0, 0\rangle = 0$). Due to the unboundedness of the spectrum, $|0, 0\rangle$ can no longer be interpreted as the vacuum state.

The system with unbounded spectrum is related to the Pais-Uhlenbeck oscillators. We recall [5, 4] that the Pais-Uhlenbeck model is a higher derivative system. It admits, via the Ostrogradskii construction [7] (see [17] for a review), a Hamiltonian formulation. The resulting Ostrogradskii Hamiltonian is canonically equivalent to a set of decoupled harmonic oscillators with alternating (positive and negative) energy modes. The n -oscillator Pais-Uhlenbeck system is canonically expressed as

$$H_n = \sum_{i=1}^n (-1)^{i+1} \omega_i a_i^\dagger a_i, \quad (56)$$

where $\omega_i \in \mathbb{R}$ and the constraint $\omega_i < \omega_{i+1}$ is satisfied.

The harmonic oscillator with energy modes $1, -3$ is a special case of the 2-oscillator Pais-Uhlenbeck model. In a series of papers [8, 9, 10, 11, 12] the Pais-Uhlenbeck oscillators with energy modes given (up to a normalization factor) by the arithmetic progression $\omega_i = 2i - 1$ were linked to the Conformal Galilei Algebras $\widehat{\mathfrak{cga}}_\ell$ (with $\ell = n - \frac{1}{2}$).

The present analysis proves that this association is rather subtle. The PDE, invariant under the Conformal Galilei Algebra, is obtained for the coupled cryptohermitian operator with $\gamma \neq 0$. The derivation of the Pais-Uhlenbeck oscillators requires two non-trivial passages:

- 1) to perform the $\gamma \rightarrow 0$ decoupling limit. The decoupled PDE no longer possesses the Conformal Galilei Algebra as invariance. Its symmetry algebra has been discussed in Section 6. The contraction of the Conformal Galilei Algebra is a symmetry subalgebra;
- 2) to change the conjugation properties, by replacing the decoupled cryptohermitian operator with the hermitian decoupled harmonic oscillator.

For general half-integer ℓ , the invariant PDEs which possess the Conformal Galilei algebra $\widehat{\mathfrak{cga}}_\ell$ (for a definition, see [3]) as a symmetry algebra, depend on $\ell + \frac{3}{2}$ coordinates. The invariant PDEs are deformations of decoupled equations, depending on $\ell - \frac{1}{2}$ deformation parameters $\gamma_j \neq 0$ ($j = 1, \dots, \ell - \frac{1}{2}$). The decoupled systems are recovered in the limit, for any $j, \gamma_j \rightarrow 0$.

The invariant PDEs with continuous spectrum are

$$i\partial_\tau \Psi(\tau, \vec{x}) = \left(-\frac{1}{2}\partial_{x_1}^2 + i \sum_{j=1}^{\ell-\frac{1}{2}} \gamma_j x_j \partial_{x_{j+1}} \right) \Psi(\tau, \vec{x}). \quad (57)$$

The invariant PDEs with discrete spectrum are

$$i\partial_t \Psi(t, \vec{x}) = \left(-\frac{1}{2}\partial_{x_1}^2 + \frac{1}{2}x_1^2 + \sum_{i=2}^{\ell+\frac{1}{2}} \omega_i x_i \partial_{x_i} + i \sum_{j=1}^{\ell-\frac{1}{2}} \gamma_j x_j \partial_{x_{j+1}} \right) \Psi(t, \vec{x}). \quad (58)$$

The energy modes ω_i are normalized so that $|\omega_i| = 2i - 1$. The solution $\omega_i = \epsilon_i |\omega_i|$ with all positive signs ($\forall i, \epsilon_i = +1$) corresponds to the bounded discrete spectrum discussed in [2]. By taking the hermitian conjugation, the solution with flipped signs, $\epsilon_i = -1$ for all i , also leads to a $\widehat{\mathfrak{cga}}_\ell$ -invariant PDE.

An explicit computation of the on-shell condition for $\ell = \frac{5}{2}$ (the procedure straightforwardly follows the one presented in Section 5), proves that the $\widehat{\mathfrak{cga}}_{\frac{5}{2}}$ invariance is guaranteed by both choices of signs, $\epsilon_2 = \pm 1$ and $\epsilon_3 = \pm 1$. As explained above, the alternating choice ($\epsilon_2 = -1, \epsilon_3 = +1$) is related to a special case of three Pais Uhlenbeck oscillators (with 1, -3, 5 energy modes). An open problem is finding a general proof, valid for all half-integer ℓ , that every choice of $\epsilon_i = \pm 1$ signs lead to the $\widehat{\mathfrak{cga}}_\ell$ symmetry algebra of the PDE equation (58).

10 Conclusions

In this paper we analyzed in detail the properties of the second-order PDEs, invariant under the $d = 1$ centrally extended Conformal Galilei Algebra $\widehat{\mathfrak{cga}}_\ell$, with half-integer ℓ . Two classes of PDEs are derived with, respectively, continuous and discrete spectrum. In the latter case the spectrum is either bounded or unbounded. The PDEs depend on $\ell - \frac{1}{2}$ non-vanishing parameters γ_j which can be regarded as coupling constants. In the $\gamma_j \rightarrow 0$ limit the PDEs are decoupled equations and the conformal Galilei algebra is mapped into a contraction algebra.

The extension of this construction to the $\ell = \frac{1}{2} + \mathbb{N}_0$ centrally extended Conformal Galilei Algebras with $d > 1$ (see [3]) is immediate. The invariant PDEs with discrete spectrum correspond to cryptohermitian operators whose spectrum is given by d copies of the energy modes created in the $d = 1$ case.

An interesting non-trivial extension of the methods here presented can be applied to determine the second-order invariant PDEs for the class of $d = 2$ centrally extended Conformal Galilei Algebras with integer ℓ . For these theories invariant PDEs with continuous spectrum were constructed in [1]. So far, on the other hand, the invariant PDEs with discrete spectrum have not been discussed in the literature. They will be presented in a forthcoming paper.

Acknowledgements

We thank T. Yumibayashi for helpful discussions on Pais-Uhlenbeck oscillators. Z.K. and F.T. are grateful to the Osaka Prefecture University, where this work was elaborated, for hospitality. F.T. received support from CNPq (PQ Grant No. 306333/2013-9). N. A. is supported by the grants-in-aid from JSPS (Contract No. 26400209).

References

- [1] N. Aizawa, Y. Kimura, and J. Segar, *J. Phys. A: Math. Theor.* **46**, 405204 (2013).
- [2] N. Aizawa, Z. Kuznetsova, and F. Toppan, *J. Math. Phys.* **56**, 031701 (2015).
- [3] J. Negro, M. Del Olmo, and A. Rodriguez-Marco, *J. Math. Phys.* **38**, 3786 (1997).
- [4] A. Smilga, *SIGMA* **5**, 017 (2009).
- [5] A. Pais and G. Uhlenbeck, *Phys. Rev.* **79**, 145 (1950).
- [6] S. V. Ketov, G. Michiaki, and T. Yumibayashi, Quantizing with a higher time derivative, in *Advances in Quantum Field Theory*, edited by S. Ketov, InTech Publishers, p. 49 (2012).
- [7] M. Ostrogradskiĭ, *Mem. Ac. St. Petersburg* **VI**, 385 (1850).
- [8] A. Galajinsky and I. Masterov, *Phys. Lett. B* **723**, 190 (2013).
- [9] K. Andrzejewski, A. Galajinsky, J. Gonera, and I. Masterov, *Nucl. Phys. B* **885**, 150 (2014).
- [10] K. Andrzejewski, *Phys. Lett. B* **738**, 405 (2014).
- [11] K. Andrzejewski, *Nucl. Phys. B* **889**, 333 (2014).
- [12] A. Galajinsky and I. Masterov, *Nucl. Phys. B* **896**, 244 (2015).
- [13] F. Toppan, *J. Phys. Conf. Ser.* **597**, 012071 (2015).
- [14] U. Niederer, *Helv. Phys. Acta* **45**, 802 (1972).
- [15] U. Niederer, *Helv. Phys. Acta* **46**, 191 (1973).
- [16] U. Niederer, *Helv. Phys. Acta* **47**, 167 (1974).
- [17] R. Woodard, arXiv:1506.02210 [hep-th] (2015).

NOTAS DE FÍSICA é uma pré-publicação de trabalho original em Física.
Pedidos de cópias desta publicação devem ser enviados aos autores ou ao:

Centro Brasileiro de Pesquisas Físicas
Área de Publicações
Rua Dr. Xavier Sigaud, 150 – 4^o andar
22290-180 – Rio de Janeiro, RJ
Brasil
E-mail: socorro@cbpf.br/valeria@cbpf.br
<http://portal.cbpf.br/publicacoes-do-cbpf>

NOTAS DE FÍSICA is a preprint of original unpublished works in Physics.
Requests for copies of these reports should be addressed to:

Centro Brasileiro de Pesquisas Físicas
Área de Publicações
Rua Dr. Xavier Sigaud, 150 – 4^o andar
22290-180 – Rio de Janeiro, RJ
Brazil
E-mail: socorro@cbpf.br/valeria@cbpf.br
<http://portal.cbpf.br/publicacoes-do-cbpf>