Solutions for confluent and double-confluent Heun equations and some applications

Léa Jaccoud El-Jaick and Bartolomeu D. B. Figueiredo
Centro Brasileiro de Pesquisas Físicas (CBPF)
Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro, RJ, Brasil

ABSTRACT. This paper examines some solutions for confluent and double-confluent Heun equations and their applications to the Schrödinger equation with quasi-exactly solvable potentials. In the first place, we review two Leaver's solutions in series of regular and irregular confluent hypergeometric functions for the confluent equation [E. W. Leaver, *J. Math. Phys.* **27**, 1238 (1986)] and introduce an additional expansion in series of irregular confluent hypergeometric functions. Then, we find the conditions under which one of these solutions can be written as a linear combination of the others. In the second place, by means of limiting procedures we generate solutions for the double-confluent equation as well as for special limits of both the confluent and double-confluent equations. In the third place, solutions of the Heun equations are used to solve the one-dimensional Schrödinger equation for quasi-exactly solvable potentials. We consider a symmetric and an asymmetric double-Morse potentials which appear in the theory of quantum spin systems [O. B. Zaslavskii and V. V. Ulyanov, *Sov. Phys.* JETP **60**, 991 (1984)], a bottomless volcano-type potential which gives degenerate eigenstates [S. Kar and R. R. Parwani, *Europhys. Lett.*, **80**, 30004 (2007)], and a potential which leads to quasinormal modes, that is, to solutions presenting complex energies [H. T. Cho and C. L. Ho, *J. Phys. A: Math. Theor.* **40**, 1325 (2007)].

I- INTRODUCTION

We deal with solutions of Heun equations and their possible applications to the Schrödinger equation with quasi-exactly solvable (QES) potentials. We consider only the confluent (CHE) and the double-confluent (DCHE) Heun equations, and one limiting case of each of these. The solutions for the CHE come directly from the differential equation, while the solutions for the other equations are obtained from the solutions of the CHE by limiting processes. Initially, we briefly discuss each of these equations and their connections; some more details are found in previous works [15, 16, 17, 27]. Then, we outline the main features of their solutions and present the QES potentials that will be used as examples.

The CHE [11, 12, 36], also known as generalized spheroidal wave equation [44, 45], in the form used by Leaver [27] reads

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3-2\eta\omega(z-z_0)+\omega^2z(z-z_0)\right]U=0,\ \ (\omega\neq 0) \tag{1}$$

where B_i, η and ω are constants and $z=0$ and $z=z_0$ are regular singular points with indicial exponents $(0,1+B_1/z_0)$ and $(0,1-B_2-B_1/z_0)$, respectively. At the irregular point $z=\infty$ the behavior of the solutions, obtained from the normal Thomé solutions [27, 35], is given by

$$\lim_{z\to\infty}U(z)\sim e^{\pm i\omega z}z^{\mp i\eta-(B_2/2)}. \tag{2}$$

The singularity parameter z_0 may take any value and, when $z_0=0$, the CHE gives the following DCHE with five parameters [27]

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left(B_3-2\eta\omega z+\omega^2z^2\right)U=0,\ \ (B_1\neq 0,\ \omega\neq 0), \tag{3}$$

where now $z=0$ and $z=\infty$ are both irregular singularities ($B_1=0$ and/or $\omega=0$ are degenerate cases [16]). At $z=\infty$ the behavior is again given by Eq. (2), while at $z=0$ the normal Thomé solutions afford

$$\lim_{z\to 0}U(z)\sim 1,\ \text{or}\ \lim_{z\to 0}U(z)\sim e^{B_1/z}z^{2-B_2}. \tag{4}$$

The CHE and the DCHE admit a limit which changes the nature of the irregular singularity at $z=\infty$, keeping unaltered the other singular points. This limit is obtained by letting that [16, 17]

$$\omega\to 0,\quad \eta\to\infty,\ \text{such that}\quad 2\eta\omega=-q,\quad \text{(Whittaker-Ince limit)} \tag{5}$$

where q is a constant. It is called Whittaker-Ince limit because Whittaker and Ince have used a similar procedure to get the Mathieu equation (12) from the Whittaker-Hill equation (11) [19, 21]. The Whittaker-Ince limit of the CHE is

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3+q(z-z_0)\right]U=0,\qquad (q\neq 0) \tag{6}$$

(if $q=0$ this equation can be transformed into a hypergeometric equation), while the Whittaker-Ince limit of the DCHE is

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+(B_3+qz)\,U=0,\qquad (q\neq 0,\ B_1\neq 0) \tag{7}$$

(if $q=0$ and/or $B_1=0$ the equation degenerates into a confluent hypergeometric equation or simpler equations [16]). Eqs. (6) and (7) differ from the CHE and DCHE, respectively, by the behavior of their solutions at the irregular point $z=\infty$, which now is obtained from the subnormal Thomé solutions [35], namely,

$$\lim_{z\to\infty}U(z)\sim e^{\pm 2i\sqrt{qz}}z^{(1/4)-(B_2/2)}, \tag{8}$$

in contrast with the behavior of original equations (normal Thomé solutions). Eq. (7) also results when we take $z_0=0$ in Eq. (6).

The preceding equations and their connections are summarized in the following diagram which is a modified version of a diagram given in Ref. [17]. The upper boxes (9) display the CHE and the DCHE. The lower boxes (10) show the Whittaker-Ince limits corresponding to the CHE and DCHE, respectively.

$$\boxed{\begin{array}{l} z(z-z_0)\dfrac{d^2U}{dz^2}+(B_1+B_2z)\dfrac{dU}{dz}+ \\ \left[B_3-2\eta\omega(z-z_0)+\omega^2z(z-z_0)\right]U=0. \end{array}} \overset{z_0\to 0}{\Longrightarrow} \boxed{\begin{array}{l} z^2\dfrac{d^2U}{dz^2}+(B_1+B_2z)\dfrac{dU}{dz}+ \\ \left(B_3-2\eta\omega z+\omega^2z^2\right)U=0. \end{array}} \tag{9}$$

$$\Downarrow\quad (\omega\to 0\ \text{and}\ \eta\to\infty,\ \text{such that}\ 2\eta\omega=-q)\qquad \Downarrow$$

$$\boxed{\begin{array}{l} z(z-z_0)\dfrac{d^2U}{dz^2}+(B_1+B_2z)\dfrac{dU}{dz}+ \\ \left[B_3+q(z-z_0)\right]U=0. \end{array}} \overset{z_0\to 0}{\Longrightarrow} \boxed{\begin{array}{l} z^2\dfrac{d^2U}{dz^2}+\ (B_1+B_2z)\dfrac{dU}{dz}+ \\ (B_3+qz)\,U=0. \end{array}} \tag{10}$$

These connections among equations having different types of singularities become fully effective only when solutions of the CHE admit both the Leaver and the Whittaker-Ince limits. Counter-examples are provided by Hylleraas [20] and Jaffé's [22] solutions which admit none of these limits, as we can see by using Leaver's form for such solutions [27].

Now we introduce the Whittaker-Hill and the Mathieu equations which are particular cases of both the CHE and the DCHE [11]. A trigonometric (hyperbolic) form of the Whittaker-Hill equation (WHE) is [2, 21]

$$\frac{d^2W}{du^2}+\kappa^2\left[\vartheta-\frac{1}{8}\xi^2-(p+1)\xi\cos(2\kappa u)+\frac{1}{8}\xi^2\cos(4\kappa u)\right]W=0,\quad \text{(WHE)}. \tag{11}$$

If u is a real variable, this equation represents the usual WHE when $\kappa = 1$ and the modified WHE when $\kappa = i$. On the other hand, the Mathieu equation has the form [32]

$$\frac{d^2w}{du^2} + \sigma^2\big[a - 2k^2\cos(2\sigma u)\big]w = 0, \qquad \text{(Mathieu equation)}, \tag{12}$$

where $\sigma = 1$ or $\sigma = i$ for the Mathieu or modified Mathieu equation, respectively. Some details about solutions for the WHE and Mathieu equation regarded as CHE or DCHE are given in Ref. [17]. The Mathieu equation is also a particular case of equation (7) as shown in section III. Incidentally, the original Whittaker-Ince limit [19, 21] is obtained when $\xi \to 0$, $p \to \infty$ so that $p\xi = 2k^2$, $\kappa = \sigma$ and $\vartheta = a$ in the WHE. This gives the Mathieu equation.

On the other hand, the solutions of the Heun equations assume one of following forms [36]

$$\sum_n a_n\ f_n(z) = \sum_{n=-\infty}^{\infty} a_n\ f_n(z), \qquad \sum_{n=0}^{\infty} a_n\ f_n(z), \qquad \sum_{n=0}^{N} a_n\ f_n(z), \tag{13}$$

where the series coefficients a_n satisfy three-term or higher order recurrence relations, $f_n(z)$ is a function of the independent variable and N is a non-negative integer. These are called, respectively, two-sided infinite series, one-sided infinite series and finite series. The finite series are also known as quasi-polynomial solutions, quasi-algebraic solutions or Heun polynomials. Notice the convention

$$\sum_n = \sum_{n=-\infty}^{\infty} \tag{14}$$

Expansions in two-sided infinite series are necessary to assure the series convergence when there is no free constant in the Heun equations. Thus, all the parameters of the CHE and DCHE which rule the time-dependence of Klein-Gordon and Dirac test-fields in some Friedmannian spacetimes [6, 15] are determined from conditions imposed on the spatial part of the wave functions [37, 38]. Similarly, in the scattering problem of ions by a finite dipole [27] or by polarizable targets [7, 16] all the parameters of the radial Schrödinger equation are known.

When some parameters of the Heun equations assume special values, one-sided infinite series truncate on the right giving expansions in finite series. These Heun polynomials are important to get solutions for QES problems [39, 40, 42, 43]. In effect, according to Kalnins, Miller and Pogosyan [23], a problem is exactly solvable if its solutions are given by (generalized) hypergeometric functions; a problem is QES if its solutions are given by finite-series whose coefficients necessarily satisfy three-term or higher order recurrence relations. This definition suggests a relation between Heun equations and QES problems. We will consider only the CHE and the DCHE, but in fact there are potentials for which the Schrödinger equation leads also to the general, biconfluent and triconfluent Heun equations, as explained in Appendix A.

Excepting possibly the Heun polynomials, in general the solutions for the Heun equations do not converge for the entire range of the independent variable. Then, it is necessary to consider two or more solutions converging over different domains and having the appropriate behaviors at the singular points. It is also necessary to take into account the transformation rules which generate new solutions from a known solution (these rules result from substitutions of variables which preserve the form of the Heun equations but modify their parameters).

We will start with a set of three solutions for the CHE, represented by two-sided infinite series which have coefficients that satisfy three-term recurrence relations. These solutions admit both the Leaver and the Whittaker-Ince limits and, so, we can generate sets of solutions for all the equations discussed above. Solutions obtained from this set by means of transformation rules are given in Appendices C and D.

More precisely, in Sec. II we take two Leaver's solutions in series of regular and irregular confluent hypergeometric functions for CHE and introduce another expansion in series of irregular confluent hypergeometric functions - see the solutions given in Eqs. (43a) and the recurrence relations (43b) and (43c). The expansion in series of regular functions converges for any z, whereas the two expansions in series of irregular functions converge for $|z|>|z_0|$. From the properties of the three-term recurrence relations and of the hypergeometric functions, we shall find conditions which permit to write one solution as a linear combination of the others in the region $|z|>|z_0|$. These conditions also assure that the series coefficients of the three solutions are proportional to each other.

The solutions for the limits of the CHE, given in the diagram, are obtained by the same procedure used in Ref. [17], where a set having only two solutions in terms of one-sided series of hypergeometric functions was considered. Thus, in Sec. III we find a set of three expansions in series of Bessel functions for Whittaker-Ince limit (6) of the CHE, the three solutions possessing exactly the same series coefficients. In Sec. IV we find that the solutions for the DCHE are again given by series of confluent hypergeometric and the solutions for Whittaker-Ince limit of the DCHE are given by series of Bessel functions once more.

The solutions of the CHE and DCHE are applied to the one-dimensional Schrödinger equation with QES potentials. For a particle with mass μ and energy E the time-independent Schrödinger equation is written as

$$\frac{d^2\psi}{du^2}+[\mathcal{E}-V(u)]\psi=0, \qquad u=ax, \qquad \mathcal{E}=\frac{2\mu E}{\hbar^2a^2}, \tag{15}$$

where a is a real constant, x is the spatial coordinate and $V(u)$ is a function proportional to the potential. This is the so-called normal form of a second order differential equation (Appendice A) and, for each potential, Eq. (15) must be converted to the canonical form of the CHE and DCHE, Eqs. (1) and (3), in order to use the mentioned solutions. The wave function ψ is required to be bounded for all values of the variable u and, in particular, at the singular points of the equation.

We select three QES potentials. First we regard double-Morse potentials [26], more exactly the potentials used by Zaslavskii and Ulyanov in the study of quantum spin systems [46, 41], namely,

$$V(u)=\frac{B^2}{4}\left(\sinh u-\frac{C}{B}\right)^2-B\left(s+\frac{1}{2}\right)\cosh u, \qquad s=0,\frac{1}{2},1,\frac{3}{2},\cdots, \tag{16}$$

where C is a real constant such that $C\geq 0$. If $C=0$, the potential is symmetric $[V(u)=V(-u)]$ and the the Schrödinger equation reduces to a modified Whittaker-Hill equation which will be treated as a CHE in Sec. II.D. On the other side, if $C>0$ the potential is asymmetric and the Schrödinger equation becomes an instance of the DCHE, considered in Sec. IV.B. The second is a potential given by Cho and Ho [8], namely,

$$V(u)=-\frac{b^2}{4}\sinh^2 u-\frac{\left[(\ell+1)^2-\frac{1}{4}\right]}{\cosh^2 u}, \quad (\ell=0,1,2,\cdots) \tag{17}$$

where $b>0$ is a real constant. This is a bottomless potential in the sense that $V(u)\to-\infty$ when $u\to\pm\infty$. If $b^2<4(\ell+1)^2-1$, it is a volcano-type potential [24] given by an inverted double well; if $b^2\geq 4(\ell+1)^2-1$, the potential is similar to the one of an inverted oscillator. This potential is discussed in Sec. III.C where the Schrödinger is reduced to the Whittaker-Ince limit (6) of the CHE. The third example deals with a QES potential given again by Cho and Ho [9], namely,

$$V(u)=-\frac{b^2}{4}e^{2u}-(\ell+1)de^{-u}+\frac{d^2}{4}e^{-2u}, \qquad (\ell=0,1,2,\cdots) \tag{18}$$

where b and d are real constants. As $u\to\mp\infty$, $V(u)\to\pm\infty$ and so the potential is unbounded from below on the right, that is, when $u\to\infty$. This potential gives rise to quasinormal modes, that is, to solutions having discrete complex energies. In Sec. IV.B we find that these energies can be obtained from a DCHE.

The quasi-exact solvability results from the values attributed to the parameters s and ℓ, since these values imply finite-series solutions from which we can determine a limited number of energy levels. On the other hand, there are also infinite-series solutions which in principle could afford the remaining part of the spectrum, but these solutions have been discarded on the grounds that they are not finite (bounded) for all values of the independent variable [23].

We will find that for the potential (17) there are finite- and infinite-series solutions which are bounded for all values of u. For the potential (18) there are two possibilities: if $d > 0$, only finite series are bounded; if $d < 0$, only infinite series are bounded. For the double-Morse potentials (16) there are finite-series solutions which are bounded for all values of u, but there is no single infinite-series solution bounded for all values of u. However, for the latter problem there is one solution that is bounded at one singular point and another solution that is bounded at the other point. A third solution (which can be written as linear combination of the others) is bounded in the intermediate region.

Thus, in Sec. II we deal with the CHE, in Sec. III with the Whittaker-Ince limit (6) of the CHE, and in Sec. IV with the DCHE and its limit (7). Sec. V presents some conclusions. Appendix A discusses relations among Heun equations and quasi-exactly solvable problems, Appendix B gives some properties of the confluent hypergeometric functions, while Appendices C and D are devoted to the transformations of the CHE and its Whittaker-Ince limit, respectively.

II. THE CONFLUENT HEUN EQUATION

In this section we review the two Leaver solutions in series of regular and irregular confluent hypergeometric functions for the confluent Heun equation [27] and introduce the extra expansion in series of irregular confluent hypergeometric functions. The fact that the expansion in terms of regular functions converges for any z (the expansions in terms of irregular functions converge for $|z| > |z_0|$) distinguishes the present solutions from the Leaver expansions in series of Coulomb wave functions [27] since the latter converge only for $|z| > |z_0|$.

In Sec. II.A we recall some features of three-term recurrence relations for the series coefficients and supply properties of the confluent hypergeometric functions which enter the solutions for the CHE and DCHE. We also write down the Barber-Hassé solutions in power series since these are important to obtain finite-series solutions as well as to cover the cases in which the expansions in series of regular confluent hypergeometric functions are not valid. In addition, we derive a new result concerning the solutions of the Whittaker-Hill equation (11).

In Sec. II.B we analyze the set constituted by the expansions in series of confluent hypergeometric functions. This is called the fundamental set of solutions because, by means of the transformation rules given in Appendix C, it originates other sets of solutions for the CHE and, by way of limiting processes, it affords sets of solutions for the other equations given in the schema of the first section. We find the conditions under which one solution of the fundamental set is given as a linear combination of the others. After that, we truncate the two-sided series from below in order to get one-sided series solutions as well. In Sec. II.C we discuss the convergence of new expansion in series of irregular confluent hypergeometric functions and, in Sec. II.D we examine the solutions of the Schrödinger equation for the symmetric double-Morse potential.

II.A. General remarks and the Barber-Hassé expansions

If b_n denotes the series coefficients of one-sided series solutions, then the three-term recurrence relations are written in the form

$$\alpha_0 b_1 + \beta_0 b_0 = 0, \qquad \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \quad (n \geq 1) \tag{19}$$

where α_n, β_n and γ_n depend on the parameters of the differential equation. These relations form a infinite system of homogeneous linear equations which has nontrivial solutions for b_n only if the determinant of the respective infinite tridiagonal matrix vanishes. This demands some arbitrary parameter in the differential equation, that is, in the elements of the matrix. Equivalently, these recurrence relations imply a characteristic equation given by the infinite continued fraction [27]

$$\beta_0 = \frac{\alpha_0\gamma_1}{\beta_1-}\ \frac{\alpha_1\gamma_2}{\beta_2-}\ \frac{\alpha_2\gamma_3}{\beta_3-}\cdots, \tag{20}$$

which must be satisfied in order to assure the series convergence.

If $\gamma_n = 0$ for some $n = N+1$, where N is a positive integer, the one-sided series terminates at $n = N$ and, consequently, gives a finite-series solution with $0 \leq n \leq N$ [2]. In effect, if $\gamma_{N+1} = 0$ we can choose the parameters of the equation so that $b_{N+1} = 0$ in the relation

$$\alpha_{N+1} b_{N+2} + \beta_{N+1} b_{N+1} + 0 \times b_N = 0.$$

This implies that $b_{N+2} = 0$ and, from the recurrence relations (19), $b_n = 0$ for any $n \geq N+1$. Thus the recurrence relations can be written in the form

$$\begin{pmatrix} \beta_0 & \alpha_0 & 0 & \cdots & & & 0 \\ \gamma_1 & \beta_1 & \alpha_1 & & & & \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & & & \\ \vdots & & & & & & \\ & & & & \gamma_{N-1} & \beta_{N-1} & \alpha_{N-1} \\ 0 & \cdots & & & 0 & \gamma_N & \beta_N \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \\ b_N \end{pmatrix} = 0. \tag{21}$$

If the elements α_i, β_i and γ_i of the previous matrix are real and if

$$\alpha_i \gamma_{i+1} > 0, \qquad 0 \leq i \leq N-1, \tag{22}$$

then all the $N+1$ roots of its determinant are real and different [2]. This theorem is important to determine a part of the energy spectra in the case of quasi-exact potentials. On the other hand, if $\alpha_n = 0$ for some $n = N$, the series begins at $n = N+1$, but in this case one may set $n = m + N + 1$ and rename the series coefficients in order to obtain a series beginning at $m = 0$.

Now we suppose that there is a second solution with coefficients c_n satisfying

$$\tilde{\alpha}_0 c_1 + \beta_0 c_0 = 0, \qquad \tilde{\alpha}_n c_{n+1} + \beta_n c_n + \tilde{\gamma}_n c_{n-1} = 0 \quad (n \geq 1), \tag{23}$$

where β_n is the same as in Eq. (19). Then, if

$$\tilde{\alpha}_n \tilde{\gamma}_{n+1} = \alpha_n \gamma_{n+1}, \tag{24}$$

it follows from Eq. (20) that both solutions have the same characteristic equation if n takes the same values in both series: in these circumstances, b_n and c_n in general are proportional to each other. We emphasize that this proportionality requires the same range for n in Eqs. (19) and (23) because there are cases in which the relation (24) is formally satisfied, but one solution is given by a finite series while the other is given by an infinite series, as we will see in the paragraph after Eq. (49c). In these cases one series breaks off on the right and the other on the left.

We extend the previous remarks to doubly infinite (or two-sided) series. These expansions present a parameter ν which must be determined from a characteristic equation if there is no free parameter in the differential equation, or can be chosen at will if there is a free constant. The recurrence relations for the series coefficients b_n now take the form

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \quad (-\infty < n < \infty) \tag{25a}$$

where α_n, β_n, γ_n and b_n depend on the parameters of the differential equation as well as on ν. These recurrence relations lead to the characteristic equation [27]

$$\beta_0 = \frac{\alpha_{-1}\ \gamma_0}{\beta_{-1}-}\ \frac{\alpha_{-2}\ \gamma_{-1}}{\beta_{-2}-}\ \frac{\alpha_{-3}\ \gamma_{-2}}{\beta_{-3}-} \cdots + \frac{\alpha_0\ \gamma_1}{\beta_1-}\ \frac{\alpha_1\ \gamma_2}{\beta_2-}\ \frac{\alpha_2\ \gamma_3}{\beta_3-} \cdots . \tag{25b}$$

If there is a second doubly infinite series having the recurrence relation $\tilde{\alpha}_n c_{n+1} + \beta_n c_n + \tilde{\gamma}_n c_{n-1} = 0$ such that the condition (24) is fulfilled, then both solutions satisfy the same characteristic equation (25b). The two series are really doubly infinite if neither the coefficients of b_{n+1} and c_{n+1} nor the coefficients of b_{n-1} and c_{n-1} vanish, since such conditions assure that the summation extends from negative to positive infinity in both solutions. This requires that

$$\alpha_n,\ \tilde{\alpha}_n,\ \gamma_n \text{ and } \tilde{\gamma}_n \text{ do not vanish for any } n, \tag{26}$$

a requirement which imposes constraints on the parameters of differential equation and on the characteristic parameter ν, in the case of two-sided series. These conditions will be useful for studying the sets of two-sided solutions of the CHE and DCHE. In addition, if we choose ν such that $\alpha_{-1} = \tilde{\alpha}_{-1} = 0$, then the series are truncated on the left since the summation begins at $n = 0$. Thus, we have

$$\alpha_{-1} = \tilde{\alpha}_{-1} = 0 \quad \Rightarrow \quad \text{one-sided series with } n \geq 0. \tag{27}$$

Other restrictions on ν and on the parameters of the Heun equations come from the properties of the special functions used to construct the series solutions. Thus, let us consider expansions in series of regular and irregular confluent hypergeometric functions for the CHE, denoted by $\Phi(a, c; y)$ and $\Psi(a, c; y)$ respectively. These are solutions of the confluent hypergeometric equation [13]

$$y\frac{d^2\varphi}{dy^2} + (c - y)\frac{d\varphi}{dy} - a\ \varphi = 0, \tag{28}$$

where the parameters a and c will depend on summation index n, on the parameters of the Heun equations and also on the characteristic parameter ν in the case of two-sided infinite series. In fact, the following four solutions for Eq. (28)

$$\varphi_n^1(y) = \Phi(a, c; y), \qquad \varphi_n^2(y) = \Psi(a, c; y),$$

$$\varphi_n^3(y) = e^y\ y^{1-c}\Phi(1 - a, 2 - c; -y), \qquad \varphi_n^4(y) = e^y\ y^{1-c}\ \Psi(1 - a, 2 - c; -y) \tag{29}$$

are all of them defined and distinct only if c is not an integer [13]. Furthermore, if

$$a, \quad c \quad \text{and} \quad c - a \quad \text{are not integer}, \tag{30}$$

then any two of the solutions (29) form a fundamental system of solutions for confluent hypergeometric equation [13]. The formula

$$\Psi(a, c; y) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)}\ \Phi(a, c; y) + \frac{\Gamma(c - 1)}{\Gamma(a)}\ y^{1-c}\ \Phi(a - c + 1, 2 - c; y), \tag{31}$$

gives the analytic continuation of Ψ in terms of Φ. The expression of Φ in terms of Ψ is obtained from the previous one by using the relation $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$. One finds [13],

$$\Phi(a, c; y) = e^{i\pi a\varepsilon}\ \frac{\Gamma(c)}{\Gamma(c - a)}\ \Psi(a, c; y) + e^{i\pi\varepsilon(a-c)}\ \frac{\Gamma(c)}{\Gamma(a)}\ e^y\ \Psi(c - a, c; -y), \quad (\varepsilon = \pm 1) \tag{32}$$

where the plus or minus signs are to be taken throughout following the conventions

$$\varepsilon = 1,\ \text{if}\ -1 = e^{i\pi}; \quad \varepsilon = -1,\ \text{if}\ -1 = e^{-i\pi} \quad \Rightarrow \quad (-1)^c = e^{i\pi c}; \quad (-1)^c = e^{-i\pi c}.$$

The relation (31) allows writing $\Phi(a, c; y)$ as a combination of a regular and an irregular confluent hypergeometric functions; analogously, Eq. (32) gives $\Psi(a, c; y)$ in terms of regular and irregular functions.

In the following we deal only with one expansion in series of regular and two expansions in series of irregular confluent hypergeometric functions as solutions for the CHE and DCHE. These solutions correspond to the above solutions $\varphi_n^1(y)$, $\varphi_n^2(y)$ and $\varphi_n^4(y)$ for the confluent hypergeometric equation. $\varphi_n^3(y)$ is discarded because it would lead to a solution whose domain of convergence excludes all the singular points of the CHE. Then, the formula (32) is the only necessary to link the solutions. Consequently, the conditions (30) are replaced by

$$a, \quad c \quad \text{and} \quad c-a \quad \text{are not zero or } \textbf{negative} \text{ integers.} \tag{33}$$

The present conditions combined with conditions (26) will allow us to use Eq. (32) in order to write one solution of the CHE or DCHE as a linear combination of the others.

Now we write down some of the Barber-Hassé solutions in series of $(z-z_0)$ for the CHE [4, 17, 27]. They converge for finite values of z. The first solution is

$$U_1^{\text{barber}}(z) = e^{i\omega z}\sum_{n=0}^{\infty} a_n^{(1)}(z-z_0)^n, \qquad (|z| = \text{finite}) \tag{34a}$$

where recurrence relations for the coefficients are given by $(a_{-1}^{(1)} = 0)$

$$z_0\left(n+B_2+\tfrac{B_1}{z_0}\right)(n+1)\,a_{n+1}^{(1)} + \Big[n\left(n+B_2-1+2i\omega z_0\right)+$$

$$B_3 + i\omega z_0\left(B_2+\tfrac{B_1}{z_0}\right)\Big]a_n^{(1)} + 2i\omega\left(n+i\eta+\tfrac{B_2}{2}-1\right)a_{n-1}^{(1)} = 0. \tag{34b}$$

This expansion will provide solutions to the CHE for the case in which the Leaver expansion in regular hypergeometric functions, given in the following, is not valid, and admits both the Leaver and the Whittaker-Ince limits. Furthermore, it provides finite-series solutions with $0 \leq n \leq N$ when $i\eta + B_2/2 = -N$.

In fact we may generate a group containing 16 sets of solutions for the CHE by applying to the previous solution the transformation rules given in Appendix C. We write the solutions obtained by using the rules T_1 and T_2 applied in the order given in Eq. (C6). The solution U_2^{barber}, which admits only the Whittaker-Ince limit, is

$$U_2^{\text{barber}}(z) = e^{i\omega z} z^{1+\frac{B_1}{z_0}}\sum_{n=0}^{\infty} a_n^{(2)}(z-z_0)^n, \tag{35a}$$

where the recurrence relations for $a_n^{(2)}$ are

$$z_0\left(n+B_2+\frac{B_1}{z_0}\right)(n+1)a_{n+1}^{(2)} + \left[n\left(n+1+2i\omega z_0+B_2+\frac{2B_1}{z_0}\right)+i\omega z_0\left(B_2+\frac{B_1}{z_0}\right)+\right.$$

$$\left.\left(1+\frac{B_1}{z_0}\right)\left(B_2+\frac{B_1}{z_0}\right)+B_3\right]a_n^{(2)} + 2i\omega\left(n+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)a_{n-1}^{(2)} = 0. \tag{35b}$$

The solution U_3^{barber} admits both the Leaver and Ince limits. It reads

$$U_3^{\text{barber}}(z) = e^{i\omega z} z^{1+\frac{B_1}{z_0}}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_{n=0}^{\infty} a_n^{(3)}(z-z_0)^n, \tag{36a}$$

where the $a_n^{(3)}$ satisfy the relations

$$z_0\left(n+2-B_2-\frac{B_1}{z_0}\right)(n+1)a_{n+1}^{(3)} + \Big[n\left(n+3+2i\omega z_0-B_2\right)+2-B_2+$$

$$B_3+i\omega z_0\left(2-B_2-\frac{B_1}{z_0}\right)\Big]a_n^{(3)} + 2i\omega\left(n+1+i\eta-\frac{B_2}{2}\right)a_{n-1}^{(3)} = 0. \tag{36b}$$

U_4^{barber}, which admits the Ince limit but not the Leaver limit, is

$$U_4^{\text{barber}}(z) = e^{i\omega z}(z-z_0)^{1-B_2-\frac{B_1}{z_0}} \sum_{n=0}^{\infty} a_n^{(4)}(z-z_0)^n, \tag{37a}$$

with the recurrence relations

$$z_0\left(n+2-B_2-\frac{B_1}{z_0}\right)(n+1)a_{n+1}^{(4)} + \left[n\left(n+1+2i\omega z_0-B_2-\frac{2B_1}{z_0}\right)+B_3+\right.$$

$$\left. i\omega z_0\left(2-B_2-\frac{B_1}{z_0}\right)+\frac{B_1}{z_0}\left(B_2+\frac{B_1}{z_0}-1\right)\right]a_n^{(4)} + 2i\omega\left(n+i\eta-\frac{B_1}{z_0}-\frac{B_2}{2}\right)a_{n-1}^{(4)} = 0. \tag{37b}$$

The solutions for the CHE give solutions for the Whittaker-Hill equation (WHE). If $U(z)$ symbolizes the solutions for the CHE, the solutions $W(u)$ for the WHE (11) are obtained by writing [17]

$$W(u) = U(z), \qquad z = \cos^2(\kappa u), \qquad (\kappa = 1, i) \tag{38a}$$

where the parameters of $U(z)$ are given in terms of the parameters of the WHE by

$$z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = \frac{1}{4}[(p+1)\xi - \vartheta], \quad i\omega = \frac{\xi}{2}, \quad i\eta = \frac{p+1}{2}. \tag{38b}$$

For the WHE, the Baber-Hassé solutions solutions become even or odd solutions with respect to the change of the sign of u.

Now we derive a result concerning the solutions of the WHE, to be used in Sec. II.D. For this, first we apply the transformation rule T_4 given in Eq. (C4) to the solution U_1^{barber} and find the solution U_5^{barber} for the CHE, namely,

$$U_5^{\text{barber}}(z) = e^{i\omega z} \sum_{n=0}^{\infty} (-1)^n a_n^{(5)} z^n, \tag{39a}$$

where the coefficients are given by ($a_{-1}^{(5)} = 0$)

$$z_0\left(n-\frac{B_1}{z_0}\right)\left(n+1\right)a_{n+1}^{(5)} + \left[n\left(n+B_2-1-2i\omega z_0\right)+\right.$$

$$\left. B_3 + i\omega B_1 + 2\eta\omega z_0\right]a_n^{(5)} - 2i\omega\left(n+i\eta+\frac{B_2}{2}-1\right)a_{n-1}^{(5)} = 0. \tag{39b}$$

Hence, we find that

$$\alpha_i^{(1)}\gamma_{i+1}^{(1)} = 2i\omega z_0\left(n+B_2+\frac{B_1}{z_0}\right)(n+1)\left(n+i\eta+\frac{B_2}{2}\right),$$

$$\alpha_i^{(5)}\gamma_{i+1}^{(5)} = -2i\omega z_0\left(n-\frac{B_1}{z_0}\right)(n+1)\left(n+i\eta+\frac{B_2}{2}\right),$$

which, for the WHE, reduce to

$$\alpha_i^{(1)}\gamma_{i+1}^{(1)} = -\alpha_i^{(5)}\gamma_{i+1}^{(5)} = 2i\omega(n+1)\left(n+\frac{1}{2}\right)\left(n+i\eta+\frac{1}{2}\right). \tag{40}$$

We can show that for the WHE

$$\alpha_i^{(j)}\gamma_{i+1}^{(j)} = -\alpha_i^{(j+4)}\gamma_{i+1}^{(j+4)}, \qquad (j = 1,2,3,4) \tag{41}$$

where the coefficients on the the left-hand side are obtained from the recurrence relations of the solution U_j^{barber}, and the coefficients on the the right-hand side are obtained from the recurrence relations of the solution U_{j+4}^{barber} obtained from the former solution by the rule T_4. Consequently, if there is one finite-series solution satisfying the condition (22) which assures real roots for the characteristic equation, then there is also a solution for which that condition is not fulfilled.

II.B. The fundamental set of solutions

First we examine the two-sided infinite series solutions and then we obtain the one-sided series. It is worth advancing that the solutions in terms of regular confluent hypergeometric functions $\Phi(a,c;y)$ are not valid when the first parameter a is zero or a negative integer. This is true if a does not depend on the summation index n, since the Kummer relations (B1) give another hypergeometric function in which the first parameter depends on n.

Defining the function $\tilde{\Phi}(a,b;y)$ by [27]

$$\tilde{\Phi}(a,c;y)=\frac{\Gamma(c-a)}{\Gamma(c)}\;\Phi(a,c;y)=\frac{\Gamma(c-a)}{\Gamma(c)}\left[1+\frac{a}{1!c}y+\frac{a(a+1)}{2!c(c+1)}y^2+\cdots\right], \tag{42}$$

the fundamental set reads

$$U_1(z)=e^{-i\omega z}\sum_n(-1)^n b_n^{(1)}\;\tilde{\Phi}\left(\frac{B_2}{2}-i\eta,n+\nu+B_2;2i\omega z\right),$$

$$U_1^{\infty}(z)=e^{-i\omega z}\sum_n(-1)^n b_n^{(1)}\Psi\left(\frac{B_2}{2}-i\eta,n+\nu+B_2;2i\omega z\right),$$

$$\bar{U}_1^{\infty}(z)=e^{i\omega z}\sum_n c_n^{(1)}(-2i\omega z)^{1-n-\nu-B_2}\;\Psi\left(1+i\eta-\frac{B_2}{2},2-n-\nu-B_2;-2i\omega z\right), \tag{43a}$$

where the recurrence relations for $b_n^{(1)}$ and $c_n^{(1)}$ are

$$\begin{aligned}&(n+\nu+1)\left(n+\nu+i\eta+\tfrac{B_2}{2}\right)b_{n+1}^{(1)}+\Big[(n+\nu)(n+\nu+B_2-1+2i\omega z_0)+\\&B_3+i\omega z_0\left(B_2+\tfrac{B_1}{z_0}\right)\Big]b_n^{(1)}+2i\omega z_0\left(n+\nu+B_2+\tfrac{B_1}{z_0}-1\right)b_{n-1}^{(1)}=0.\end{aligned} \tag{43b}$$

and

$$\begin{aligned}&(n+\nu+1)c_{n+1}^{(1)}+\Big[(n+\nu)(n+\nu+B_2-1+2i\omega z_0)+B_3+i\omega z_0\left(B_2+\tfrac{B_1}{z_0}\right)\Big]c_n^{(1)}+\\&2i\omega z_0\left(n+\nu+B_2+\tfrac{B_1}{z_0}-1\right)\left(n+\nu+i\eta+\tfrac{B_2}{2}-1\right)c_{n-1}^{(1)}=0.\end{aligned} \tag{43c}$$

Apart from a multiplicative constant, the coefficients $b_n^{(1)}$ and $c_n^{(1)}$ are connected by

$$c_n^{(1)}=\Gamma\left(n+\nu+i\eta+\frac{B_2}{2}\right)b_n^{(1)}, \tag{44}$$

provided that the argument of the gamma function is not zero or negative integer.

The solutions $U_1(z)$ and $U_1^{\infty}(z)$ have been taken from Eqs. (166) and (167) of Leaver's paper [27] with ν replaced by $\nu+B_2$. On the other hand, by using a Kummer relation given in Eq. (B1), $\bar{U}_1^{\infty}(z)$ is rewritten as

$$\bar{U}_1^{\infty}(z)=e^{i\omega z}\sum_n c_n^{(1)}\Psi\left(n+\nu+i\eta+\frac{B_2}{2},n+\nu+B_2;-2i\omega z\right). \tag{45}$$

Then, it becomes clear that this solution can be obtained by substituting $n+\nu$ for n in the one-sided solution given in Eq. (33a) of Ref. [17] and by allowing that the summation runs from negative to positive infinity. The solution U_1 converges for any z [27], while both U_1^{∞} and $\bar{U}_1^{\infty}$ converge for $|z|>|z_0|$. From the fact that $\Phi(a,c;0)=1$ and from Eq. (B2), it follows that

$$\lim_{z\to 0} U_1(z)\sim 1,\qquad \lim_{z\to\infty} U_1^{\infty}(z)\sim e^{-i\omega z}z^{i\eta-\frac{B_2}{2}},\qquad \lim_{z\to\infty} \bar{U}_1^{\infty}(z)\sim e^{i\omega z}z^{-i\eta-\nu-\frac{B_2}{2}}. \tag{46}$$

Thus, two different behaviors at $z=\infty$ are included in the solutions belonging to the same set.

Since we are dealing with three solutions for a second order linear differential equation, now we establish the conditions to get one of these as a linear combination of the others in a domain where the three solutions are valid. From the recurrence relations (43b) and (43c) we find that the three series are really doubly infinite if

$$\nu,\quad \nu+B_2+\frac{B_1}{z_0}\quad \text{and}\quad \nu+i\eta+\frac{B_2}{2}\quad \text{are not integers,} \tag{47a}$$

since under these conditions neither the coefficients of $b_n^{(1)}$ and $c_n^{(1)}$ nor the coefficients of $b_{n-1}^{(1)}$ and $c_{n-1}^{(1)}$ vanish, that is to say, the series do not truncate on the left or on the right. The last condition also assures that the series coefficients are linked by Eq. (44) and in turn this implies that there is a unique characteristic equation. If, in addition to conditions (47a) are valid, that is,

$$n+\nu+B_2,\quad \frac{B_2}{2}-i\eta\quad \text{and}\quad n+\nu+i\eta+\frac{B_2}{2}\quad \text{are not zero or negative integers,}$$

then Eq. (32) may be used to prove that any of the three solutions is a linear combination of the others. Since n extends from $-\infty$ to ∞ (two-sided series), the above conditions are equivalent to

$$\nu+B_2 \text{ and } \nu+i\eta+\frac{B_2}{2} \text{ are not integers;}\quad \frac{B_2}{2}-i\eta \text{ is not zero or negative integer,} \tag{47b}$$

which repeat one of the conditions (47a).

In summary, to express one solution in terms of the others, the three solutions must be given by two-sided series and the formula (32) for analytic continuation of the hypergeometric functions must hold. These are the general conditions which may be applied to any set of solutions generated from the first set through the transformation rules of the CHE. In fact, they are equivalent to the conditions (26) and (33).

If U_1 is a superposition of U_1^{∞} and $\bar{U}_1^{\infty}$ in the common domain of convergence ($|z|>|z_0|$), then the behavior of U_1 when $z\to\infty$ must be given by a combination of the behaviors of U_1^{∞} and $\bar{U}_1^{\infty}$. However, for certain problems as in Sec. II.D, one of the expansions in irregular functions may be inadequate when $z\to\infty$ and, consequently, U_1 becomes inappropriate as well. This conclusion also results if we consider only the solution U_1 and the behavior of $\Phi(a,c;y)$ when $y\to\infty$ [13].

Only the restriction on the values of $(B_2/2)-i\eta$ cannot be satisfied by a convenient choice of ν. This restriction also arises if we consider the solution $U_1(z)$ by itself, disregarding its connection with the other solutions. In fact, if $(B_2/2)-i\eta=-m$ $(m=0,1,2,\cdots)$, the hypergeometric function $\Phi(a,c;y)$ which appears in U_1 becomes a polynomial of degree m with respect to its argument [1] and, then, the summation from negative to positive infinity is meaningless. The solution

$$U_1^{\mathrm{p}}(z)=e^{-i\omega z}\sum_{n=0}^{\infty} d_n^{(1)}(z-z_0)^n,\qquad (|z|=\text{finite}) \tag{48a}$$

where recurrence relations for the coefficients are ($d_{-1}^{(1)}=0$)

$$z_0\left(n+B_2+\frac{B_1}{z_0}\right)(n+1)d_{n+1}^{(1)}+\Big[n\left(n+B_2-1-2i\omega z_0\right)+$$

$$B_3-i\omega z_0\left(B_2+\frac{B_1}{z_0}\right)\Big]d_n^{(1)}-2i\omega\left(n-i\eta+\frac{B_2}{2}-1\right)d_{n-1}^{(1)}=0, \tag{48b}$$

takes the place of U_1 when $(B_2/2) - i\eta = -m$. Notice that U_1^{p} was obtained from the Barber-Hassé expansion (34a) by substituting $(-\omega, -\eta)$ for (ω, η). Furthermore, even if $(B_2/2) - i\eta = -m$, by means of the transformation rules we can find two-sided series expansions in terms of regular confluent hypergeometric functions.

Now we consider the one-sided series solutions. From the recurrence relations (43b) and (43c) we see that for truncating the three solutions on the left at $n = 0$ the only choice of ν common to the three solutions is $\nu = 0$ – see Eq. (27). We rewrite these one-sided series solutions as

$$U_1(z) = e^{-i\omega z} \sum_{n=0}^{\infty} \frac{(-1)^n c_n^{(1)}}{\Gamma(n + B_2)} \,\Phi\left(\frac{B_2}{2} - i\eta, n + B_2; 2i\omega z\right),$$

$$U_1^{\infty}(z) = e^{-i\omega z} \sum_{n=0}^{\infty} b_n^{(1)} \,\Psi\left(\frac{B_2}{2} - i\eta, n + B_2; 2i\omega z\right),$$

$$\bar{U}_1^{\infty}(z) = e^{i\omega z} \sum_{n=0}^{\infty} c_n^{(1)} \,\Psi\left(n + i\eta + \frac{B_2}{2}, n + B_2; -2i\omega z\right), \tag{49a}$$

where the recurrence relations are $\left(b_{-1}^{(1)} = c_{-1}^{(1)} = 0\right)$:

$$(n+1)\left(n + i\eta + \frac{B_2}{2}\right) b_{n+1}^{(1)} + \beta_n^{(1)} b_n^{(1)} + 2i\omega z_0 \left(n + B_2 + \frac{B_1}{z_0} - 1\right) b_{n-1}^{(1)} = 0, \tag{49b}$$

$$(n+1) c_{n+1}^{(1)} + \beta_n^{(1)} c_n^{(1)} + 2i\omega z_0 \left(n + B_2 + \frac{B_1}{z_0} - 1\right)\left(n + i\eta + \frac{B_2}{2} - 1\right) c_{n-1}^{(1)} = 0, \tag{49c}$$

in which

$$\beta_n^{(1)} = n(n + B_2 - 1 + 2i\omega z_0) + B_3 + i\omega z_0 \left(B_2 + \frac{B_1}{z_0}\right).$$

According to the previous subsection, if $i\eta + (B_2/2) = -l$ $(l = 0, 1, 2, \cdots)$, the series in U_1 and $\bar{U}_1^{\infty}$ break off on the right and these solutions reduce to Heun polynomials $(0 \le n \le l)$, while the solution U_1^{∞} truncates on the left $(n \ge l + 1)$. However, there is no need of considering these Heun polynomials since the Barber-Hassé solution (34a) also suplies finite-series solutions with the same characteristic equation.

The preceding remarks suggest that the one-sided solutions become useful only when the three solutions are given by infinite series and each solution of a fixed set can be expressed as a superposition of the others. For the solutions (49a) the series are infinite, with n running from 0 to infinity, if

$$B_2 + \frac{B_1}{z_0} \quad \text{and} \quad i\eta + \frac{B_2}{2} \quad \text{are not zero or negative integers,} \tag{50a}$$

as we see from the recurrence relations (49b) and (49c). The solutions can be connected by means of Eq. (32) if

$$B_2, \quad \frac{B_2}{2} - i\eta \quad \text{and} \quad i\eta + \frac{B_2}{2} \quad \text{are not zero or negative integers,} \tag{50b}$$

as we see from conditions (33). Then, the relation $c_n^{(1)} = \Gamma\left(n + i\eta + \frac{B_2}{2}\right) b_n^{(1)}$ is well defined and Heun polynomials are excluded from (49a). The one-sided infinite series solutions, together with the Baber-Hassé solutions, are used in Sec. II.D.

II.C. Convergence of the third solution

For one-sided infinite series the convergence of $\bar{U}_1^{\infty}$ have already been established in Ref. [17]. Next we show that the two-sided infinite series converges in both directions, that is, when $n \to \infty$ and when $n \to -\infty$. First we write the solution as

$$\bar{U}_1^{(\infty)}(z) = e^{i\omega z} \sum_{n=-\infty}^{\infty} c_n^{(1)} y^{1-B_2-n-\nu} \Psi_n(y), \quad \Psi_n(y) = \Psi\left(1 + i\eta - \frac{B_2}{2}, 2 - n - \nu - B_2; y\right), \tag{51}$$

where $y = -2i\omega z$. To determine the convergence of the series, we have to find the ratios

$$\lim_{n\to\infty} \frac{c_{n+1}^{(1)}\ \Psi_{n+1}(y)}{c_n^{(1)}\ y\ \Psi_n(y)}, \qquad \lim_{n\to-\infty} \frac{c_{n-1}^{(1)}\ y\ \Psi_{n-1}(y)}{c_n^{(1)}\ \Psi_n(y)}. \tag{52}$$

For this, in the first place we divide the recurrence relations (43c) by $nc_n^{(1)}$ and retain only the leading terms, that is,

$$\left(n + \frac{\nu+1}{n}\right) \frac{c_{n+1}^{(1)}}{c_n^{(1)}} + \left[n + 2\nu + B_2 - 1 + 2i\omega z_0 + O\left(\frac{1}{n}\right)\right]$$

$$+2i\omega z_0 \left[n + 2\nu + \frac{3}{2}B_2 + \frac{B_1}{z_0} + i\eta - 2 + O\left(\frac{1}{n}\right)\right] \frac{c_{n-1}^{(1)}}{c_n^{(1)}} = 0.$$

The minimal solutions for this equation are

$$\lim_{n\to+\infty} \frac{c_{n+1}^{(1)}}{c_n^{(1)}} = -2i\omega z_0 \left[1 + \frac{1}{n}\left(i\eta - 1 + \frac{B_1}{z_0} + \frac{B_2}{2}\right)\right], \tag{53a}$$

$$\lim_{n\to-\infty} \frac{c_{n-1}^{(1)}}{c_n^{(1)}} = -\frac{1}{n}\left[1 - \frac{B_2 + \nu - 3}{n}\right]. \tag{53b}$$

On the other hand, by applying Eq. (B6) to the function $\Psi_n(y)$, we get

$$-\left(n + \nu + i\eta + \frac{B_2}{2}\right)\Psi_{n+1}(y) + \big(n + \nu + B_2 - 1 - y\big)\Psi_n(y) + y\Psi_{n-1}(y) = 0.$$

Dividing this equation by $n\Psi_n$, we obtain

$$-\left[1 + \frac{1}{n}\left(\nu + i\eta + \frac{B_2}{2}\right)\right]\frac{\Psi_{n+1}}{\Psi_n} + \left[1 + \frac{\nu + B_2 - 1 - y}{n}\right] + \frac{y}{n}\frac{\Psi_{n-1}}{\Psi_n} = 0.$$

Then we can verify that

$$\lim_{n\to+\infty} \frac{\Psi_{n+1}}{\Psi_n} = 1 - \frac{1}{n}\left(1 + i\eta - \frac{B_2}{2}\right), \qquad \lim_{n\to-\infty} \frac{\Psi_{n-1}}{\Psi_n} = -\frac{1}{y}\left(n + \nu + B_2 - 1\right). \tag{54}$$

In fact there are other possibilities, but the preceding are the only ones compatible with Eq. (B7). Hence, since $y = -2i\omega z$, we find

$$\lim_{n\to\infty} \frac{c_{n+1}^{(1)}\ \Psi_{n+1}(y)}{c_n^{(1)}\ y\ \Psi_n(y)} = \frac{z_0}{z}\left[1 + \frac{1}{n}\left(\frac{B_1}{z_0} - 2\right)\right], \tag{55a}$$

$$\lim_{n\to-\infty} \frac{c_{n-1}^{(1)}\ y\ \Psi_{n-1}(y)}{c_n^{(1)}\ \Psi_n(y)} = 1 + \frac{2}{n}. \tag{55b}$$

Therefore, by the ratio test the series in Eq. (51) converges in the region $|z| > |z_0$.

II.D. Schrödinger equation for the symmetric double-Morse potential

For $C=0$, the potential (16) given by Ulyanov and Zaslavskii is

$$V(u)=\frac{B^2}{4}\sinh^2 u-B\left(s+\frac{1}{2}\right)\cosh u,\quad s=0,\frac{1}{2},1,\frac{3}{2},2,\cdots$$

where B is a positive constant and $u\in(-\infty,\infty)$. Then, the Schrödinger equation reads

$$\frac{d^2\psi}{du^2}+\left[\mathcal{E}-\frac{B^2}{4}\sinh^2 u+B\left(s+\frac{1}{2}\right)\cosh u\right]\psi=0, \tag{56}$$

By transforming this equation into a CHE, we will find even and odd solutions for each value of s. First, we use the four Barber-Hassé solutions to find bounded states given by Heun polynomials associated to a finite number of real energy levels. Second, we consider the solutions given by one-sided infinite series in terms of confluent hypergeometric functions and find that only one expansion in irregular functions vanishes at infinity ($u=\pm\infty$). This constitutes an example in which the Leaver expansion in regular confluent hypergeometric functions is not bounded at infinity and, therefore, is not appropriate for the entire range of the independent variable. Finally we regard other solutions and, in particular, the finite-series solutions for which there is no guarantee that the eigenvalues are real for all values of s.

The substitutions

$$\psi(u)=U(z),\quad z=\cosh^2\left(\frac{u}{2}\right),\quad (z\geq 1) \tag{57}$$

convert Eq. (56) into

$$z(z-1)\frac{d^2U}{dz^2}+\left(z-\frac{1}{2}\right)\frac{dU}{dz}+\left[\mathcal{E}+B\left(s+\frac{1}{2}\right)+2B\left(s+\frac{1}{2}\right)(z-1)-B^2z(z-1)\right]U=0,$$

which is a CHE (in fact, a Whittaker-Hill equation) with the parameters

$$z_0=1,\quad B_1=-\frac{1}{2},\quad B_2=1,\quad B_3=\mathcal{E}+B\left(s+\frac{1}{2}\right),$$

$$i\omega=-B,\quad i\eta=-s-\frac{1}{2}\quad\text{or}\quad i\omega=B,\quad i\eta=s+\frac{1}{2}. \tag{58}$$

Finite-series solutions. For the present case, we redefine the coefficients of the Barber-Hassé solutions given at the end of Sec. II.A according to

$$a_n^{(1)}=\frac{(-1)^np_n^{(1)}}{n!\Gamma[n+(1/2)]},\quad a_n^{(2)}=\frac{(-1)^np_n^{(2)}}{n!\Gamma[n+(1/2)]},\quad a_n^{(3)}=\frac{(-1)^np_n^{(3)}}{n!\Gamma[n+(3/2)]},\quad a_n^{(4)}=\frac{(-1)^np_n^{(4)}}{n!\Gamma[n+(3/2)]}.$$

Then, if s is a integer the solutions (34a) and (36a) in conjunction with Eqs. (57) and (58) give the finite-series expansions

$$\psi_1^{\text{barber}}(u)=e^{-B\cosh^2\frac{u}{2}}\sum_{n=0}^{s}\frac{(-1)^np_n^{(1)}}{n!\Gamma[n+(1/2)]}\sinh^{2n}\left(\frac{u}{2}\right),\qquad s=0,1,2,\cdots, \tag{59a}$$

$$\psi_3^{\text{barber}}(u)=\sinh u\; e^{-B\cosh^2\frac{u}{2}}\sum_{n=0}^{s-1}\frac{(-1)^np_n^{(3)}}{n!\Gamma[n+(3/2)]}\sinh^{2n}\left(\frac{u}{2}\right),\quad s=1,2,3,\cdots, \tag{59b}$$

which are even and odd solutions, respectively. The recurrence relations for the coefficients are ($p_{-1}^{(1)}=p_{-1}^{(3)}=0$)

$$p_{n+1}^{(1)}=\left[\mathcal{E}+Bs+n(n-2B)\right]p_n^{(1)}+2Bn\left(n-\frac{1}{2}\right)\left(n-s-1\right)p_{n-1}^{(1)}, \tag{59c}$$

$$p_{n+1}^{(3)} = \left[\mathcal{E} + Bs - B + 1 + n(n+2-2B)\right] p_n^{(3)} + 2Bn\left(n + \frac{1}{2}\right)\left(n - s\right) p_{n-1}^{(3)}, \tag{59d}$$

If s is a half-integer, the solutions (35a) and (37a) yield even and odd finite-series solutions, respectively,

$$\psi_2^{\text{barber}}(u) = \cosh\left(\frac{u}{2}\right) e^{-B\cosh^2\left(\frac{u}{2}\right)} \sum_{n=0}^{s-1/2} \frac{(-1)^n p_n^{(2)}}{n!\Gamma[n+(1/2)]} \sinh^{2n}\left(\frac{u}{2}\right), \quad s = \frac{1}{2}, \frac{3}{2}, \cdots, \tag{60a}$$

$$\psi_4^{\text{barber}}(u) = \sinh\left(\frac{u}{2}\right) e^{-B\cosh^2\left(\frac{u}{2}\right)} \sum_{n=0}^{s-1/2} \frac{(-1)^n p_n^{(4)}}{n!\Gamma[n+(3/2)]} \sinh^{2n}\left(\frac{u}{2}\right), \quad s = \frac{1}{2}, \frac{3}{2}, \cdots, \tag{60b}$$

where the coefficients satisfy ($p_{-1}^{(2)} = p_{-1}^{(4)} = 0$)

$$p_{n+1}^{(2)} = \left[\mathcal{E} + Bs + (1/4) + n(n+1-2B)\right] p_n^{(2)} + 2Bn\left[n - (1/2)\right]\left[n - s - (1/2)\right] p_{n-1}^{(2)}, \tag{60c}$$

$$p_{n+1}^{(4)} = \left[\mathcal{E} + Bs - B + n(n+1-2B)\right] p_n^{(4)} + 2Bn\left[n + (1/2)\right]\left[n - s - (1/2)\right] p_{n-1}^{(4)}. \tag{60d}$$

The previous solutions are bounded for any $u \in (-\infty, \infty)$ owing to the exponential factor. On the other hand, the energies are real because the coefficients of the recurrence relations are real and satisfy the relations (22) since $B > 0$. These eigenvalues can be computed by equating to zero the determinant of the matrices associated to the recurrence relations. However, there is a procedure due to Bender and Dunne [5] that, in addition to the energies, also gives the coefficients p_n as polynomials of degree n in the energy. As an example we consider the first solution. Taking $p_0^{(1)} = 1$ as initial condition and using the recurrence relations ($p_{-1}^{(1)} = 0$), we obtain

$$p_0^{(1)} = 1, \qquad p_1^{(1)} = E_s, \qquad p_2^{(1)} = [E_s]^2 + (1-2B)E_s - Bs,$$
$$p_3^{(1)} = [E_s]^3 + (5-6B)[E_s]^2 + \left(8B^2 + 4 - 6B - 7Bs\right)E_s - 4B(1-B)s,$$

and so on, where $E_s = \mathcal{E} + Bs$. Thence, since the summation terminates at $n = s$, the eigenvalues are obtained by requiring that $p_{s+1}^{(1)} = 0$. Thus, if $s = 1$ the equation $p_2^{(1)} = 0$ implies that

$$E_1^{\pm} = B - \frac{1}{2} \pm \sqrt{B^2 + \frac{1}{4}} \quad \Leftrightarrow \quad \mathcal{E}^{\pm} = -\frac{1}{2} \pm \sqrt{B^2 + \frac{1}{4}},$$

which correspond to the eigenfunctions

$$\psi_1^{\pm}(u) = e^{-B\cosh^2\frac{u}{2}}\left[1 - \frac{E_1^{\pm}}{\Gamma(3/2)}\sinh^2\frac{u}{2}\right].$$

Infinite-series solutions. The Barber-Hassé solutions which are given by finite series when s is integer (half-integer) become solutions given by infinite series when s is half-integer (integer). However these infinite-series expansions are not convergent at $u = \pm\infty$ ($z = \infty$), just at the point where $V(u) \to \infty$. For this reason, we consider the sets of one-sided infinite expansions in series of confluent hypergeometric functions. We will find no single solution bounded for any u but, in a given set, the solution in series of regular hypergeometric functions is bounded in the vicinity of $z = 1$ ($u = 0$), other solution vanishes at $z = \infty$ and the remaining solution (which is a linear combination of the others) is bounded in the intermediate domain ($z \neq 1$, $z \neq \infty$). Due to Eq. (24), these solutions lead to the same energies as the Barber-Hassé infinite-series solutions, that is, there is a correspondence among the energies of the following infinite-series solutions

$$\psi_i^{\text{barber}} \leftrightarrow (\psi_i, \psi_i^{\infty}, \bar{\psi}_i^{\infty}). \tag{61}$$

From the solutions (49a), if s is half-integer, we obtain the following set of even infinite-series solutions

$$\psi_1(u)=e^{B\cosh^2\frac{u}{2}}\sum_{n=0}^{\infty}(-1)^n b_n^{(1)}\tilde{\Phi}\left(s+1,n+1;-2B\cosh^2\frac{u}{2}\right),$$

$$\psi_1^{\infty}(u)=e^{B\cosh^2\frac{u}{2}}\sum_{n=0}^{\infty}(-1)^n b_n^{(1)}\Psi\left(s+1,n+1;-2B\cosh^2\frac{u}{2}\right),$$

$$\bar{\psi}_1^{\infty}(u)=e^{-B\cosh^2\frac{u}{2}}\sum_{n=0}^{\infty}c_n^{(1)}\Psi\left(n-s,n+1;2B\cosh^2\frac{u}{2}\right), \tag{62a}$$

where the $b_n^{(1)}$ satisfy ($b_{-1}^{(1)}=0$)

$$(n+1)(n-s)b_{n+1}^{(1)}+\left[\mathcal{E}+Bs+n(n-2B)\right]b_n^{(1)}-2B\left(n-\frac{1}{2}\right)b_{n-1}^{(1)}=0 \tag{62b}$$

and $c_n^{(1)}=\Gamma(n-s)b_n^{(1)}$. Since $a=s+1$, $c=n+1$ and $c-a=n-s$ are not negative integers, the formula (32) can be used to express one solution as a superposition of the others. The solutions ψ_1^{∞} and $\bar{\psi}_1^{\infty}$ converge for $\cosh^2(u/2)>1$ and, when $u\to\pm\infty$ ($z\to\infty$), we find

$$\lim_{u\to\pm\infty}\psi_1^{\infty}(u)\sim e^{B\cosh^2\frac{u}{2}}\left[\cosh\frac{u}{2}\right]^{-2s-2}\to\infty,\qquad \lim_{u\to\pm\infty}\bar{\psi}_1^{\infty}(u)\sim e^{-B\cosh^2\frac{u}{2}}\left[\cosh\frac{u}{2}\right]^{2s}\to 0.$$

Therefore, ψ_1 is bounded in the neighborhood $z=1$, $\bar{\psi}_1^{\infty}$ is bounded near $z=\infty$, whereas ψ_1^{∞} is bounded in the intermediate region ($z\neq 1, z\neq\infty$) . When s is half-integer, a set of odd solutions, having properties similar to the ones of the above set, is obtained from the third set given in Appendix C (with $\nu=0$), namely,

$$\psi_3(u)=e^{B\cosh^2\frac{u}{2}}\sinh u\sum_{n=0}^{\infty}(-1)^n b_n^{(3)}\tilde{\Phi}\left(s+2,n+3;-2B\cosh^2\frac{u}{2}\right),$$

$$\psi_3^{\infty}(u)=e^{B\cosh^2\frac{u}{2}}\sinh u\sum_{n=0}^{\infty}(-1)^n b_n^{(3)}\Phi\left(s+2,n+3;-2B\cosh^2\frac{u}{2}\right),$$

$$\bar{\psi}_3^{\infty}(u)=e^{-B\cosh^2\frac{u}{2}}\sinh u\sum_{n=0}^{\infty}c_n^{(3)}\Psi\left(n+1-s,n+3;2B\cosh^2\frac{u}{2}\right), \tag{63a}$$

where the recurrence relations for $b_n^{(3)}$ are ($b_{-1}^{(3)}=0$)

$$(n+1)(n+1-s)b_{n+1}^{(3)}+\left[\mathcal{E}+Bs-B+1+n(n+2-2B)\right]b_n^{(3)}-2B\left(n+\frac{1}{2}\right)b_{n-1}^{(3)}=0 \tag{63b}$$

and $c_n^{(3)}=\Gamma(n+1-s)b_n^{(3)}$.

The sets of solutions for integer values of s are obtained from the second and fourth sets given in Appendix C (with $\nu=0$). The even solutions, obtained from the second set, are

$$\psi_2(u)=e^{B\cosh^2\frac{u}{2}}\cosh\frac{u}{2}\sum_{n=0}^{\infty}(-1)^n b_n^{(2)}\tilde{\Phi}\left(s+\frac{3}{2},n+2;-2B\cosh^2\frac{u}{2}\right),$$

$$\psi_2^{\infty}(u) = e^{B\cosh^2\frac{u}{2}}\cosh\frac{u}{2}\sum_{n=0}^{\infty}(-1)^n b_n^{(2)}\Psi\left(s+\frac{3}{2}, n+2; -2B\cosh^2\frac{u}{2}\right),$$

$$\bar{\psi}_2^{\infty}(u) = e^{-B\cosh^2\frac{u}{2}}\cosh\frac{u}{2}\sum_{n=0}^{\infty}c_n^{(2)}\Psi\left(n+\frac{1}{2}-s, n+2; 2B\cosh^2\frac{u}{2}\right), \tag{64a}$$

where the $b_n^{(2)}$ satisfy $(b_{-1}^{(2)}=0)$

$$(n+1)\left(n-s+\frac{1}{2}\right)b_{n+1}^{(2)} + \left[\mathcal{E}+Bs+\frac{1}{4}+n(n+1-2B)\right]b_n^{(2)} - 2B\left(n-\frac{1}{2}\right)b_{n-1}^{(2)} = 0 \tag{64b}$$

and $c_n^{(2)} = \Gamma[n-s+(1/2)]b_n^{(2)}$. The odd solutions ($s$ =integer) are

$$\psi_4(u) = e^{B\cosh^2\frac{u}{2}}\sinh\frac{u}{2}\sum_{n=0}^{\infty}(-1)^n b_n^{(4)}\tilde{\Phi}\left(s+\frac{3}{2}, n+2; -2B\cosh^2\frac{u}{2}\right),$$

$$\psi_4^{\infty}(u) = e^{B\cosh^2\frac{u}{2}}\sinh\frac{u}{2}\sum_{n=0}^{\infty}(-1)^n b_n^{(4)}\Psi\left(s+\frac{3}{2}, n+2; -2B\cosh^2\frac{u}{2}\right),$$

$$\bar{\psi}_4^{\infty}(u) = e^{-B\cosh^2\frac{u}{2}}\sinh\frac{u}{2}\sum_{n=0}^{\infty}c_n^{(4)}\Psi\left(n+\frac{1}{2}-s, n+2; 2B\cosh^2\frac{u}{2}\right), \tag{65a}$$

where the $b_n^{(4)}$ satisfy $(b_{-1}^{(4)}=0)$

$$(n+1)\left(n-s+\frac{1}{2}\right)b_{n+1}^{(4)} + \left[\mathcal{E}+Bs-B+\frac{1}{4}+n(n+1-2B)\right]b_n^{(4)} - 2B\left(n+\frac{1}{2}\right)b_{n-1}^{(4)} = 0, \tag{65b}$$

and $c_n^{(4)} = \Gamma[n-s+(1/2)]b_n^{(4)}$.

Other solutions. We have considered only four (sets of) solutions for the CHE, but the transformation rules given in Appendix C lead to 16 (sets of) solutions. The other solutions are obtained by using the rules T_4 and T_3, which for the present problem are equivalent to the operations

$$T_4: \qquad (u, B) \to (u+i\pi, -B), \tag{66a}$$

$$T_3: \qquad (B, s) \to (-B, -s-1). \tag{66b}$$

We can check that these operations leave the Schrödinger equation (56) invariant.

Using the rule T_4, we can generate four finite-series solutions that do not satisfy the condition (22) which assures real energies for an arbitrary s, as noted at the end of Sec. II.A. One example is the bounded Heun polynomial

$$\psi_5^{\text{barber}}(u) = T_4\psi_1^{\text{barber}}(u) = e^{-B\sinh^2\frac{u}{2}}\sum_{n=0}^{s}\frac{p_n^{(5)}}{n!\Gamma[n+(1/2)]}\cosh^{2n}\left(\frac{u}{2}\right), \quad s=0,1,2,\cdots, \tag{67a}$$

where the series coefficients satisfy the recurrence relations

$$p_{n+1}^{(5)} = \left[\mathcal{E}-Bs+n(n+2B)\right]p_n^{(5)} - 2Bn\left(n-\frac{1}{2}\right)\left(n-s-1\right)p_{n-1}^{(5)}. \tag{67b}$$

In addition, T_3 and T_4 generate new infinite-series solutions when applied to the previous expansions in series of confluent hypergeometric functions. Probably some of the infinite-series solutions may be discarded by requiring real energies for any s.

III. THE WHITTAKER-INCE LIMIT OF THE CHE

In this section we show that for the Whittaker-Ince limit of the CHE, that is, for equation

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+[B_3+q(z-z_0)]\,U=0,\quad (q\neq 0), \tag{68}$$

the solutions of the CHE reduce to expansions in series of Bessel functions. The procedure is the same used in Ref. [17] for one-sided solutions, but we correct the following systematic error: the Bessel functions of the first kind J_λ which appear in Ref. [17] must be replaced by $(-1)^n J_\lambda$.

In Sec. III.A we write the first set of solutions for Eq. (68) but, using the transformations rules $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ given in Appendix D, we can generate new sets of solutions. This time the three solutions have exactly the same series coefficients and, in order to write one solution as a linear combination of the others, no restriction must be imposed on the parameters of Eq. (68). In Sec. III.B we show how these solutions are obtained from the solutions of the CHE by using the Whittaker-Ince limit (5). Finally, in Sec. III.C we use solutions of Eq. (68) to solve the Schrödinger equation for a quasi-exactly solvable potential given by Cho and Ho [8]. As far as we are aware, the only other example for Eq. (68) is a problem considered by Malmendier [29] and Mignemi [34].

III.A. The first set of solutions

The Whittaker-Ince limit of the fundamental set of solutions given in Eqs. (43a) yields

$$U_1(z)=\sum_n(-1)^n c_n^{(1)}\big(\sqrt{qz}\big)^{-(n+\nu+B_2-1)}J_{n+\nu+B_2-1}\big(2\sqrt{qz}\big),\quad \forall z,$$

$$U_1^{(i)}(z)=\sum_n(-1)^n c_n^{(1)}\big(\sqrt{qz}\big)^{-(n+\nu+B_2-1)}H^{(i)}_{n+\nu+B_2-1}\big(2\sqrt{qz}\big),\quad |z|>|z_0|,\quad (i=1,2) \tag{69a}$$

where the limits of the recurrence relations (43b) and (43c) are

$$(n+\nu+1)c_{n+1}^{(1)}+\big[(n+\nu)(n+\nu+B_2-1)+B_3\big]c_n^{(1)}+$$

$$qz_0\left(n+\nu+B_2+\frac{B_1}{z_0}-1\right)c_{n-1}^{(1)}=0. \tag{69b}$$

In these solutions $J_\lambda(x)$ denotes Bessel functions of the first kind of order λ, whereas $H_\lambda^{(1)}(x)$ and $H_\lambda^{(2)}(x)$ denote Hankel functions of first and second kind respectively. The solution $U_1(z)$ comes from the solution $U_1(z)$ of the CHE, while the solutions denoted by $U_i^{(i)}(z)$ follow either from $U_1^{(\infty)}(z)$ or $\bar{U}_1^{(\infty)}(z)$.

The Bessel functions which appear in the solutions are all independent since their Wronskians are [14]

$$W\big(J_\lambda(x),H_\lambda^{(1)}(x)\big)=\frac{2i}{\pi x},\quad W\big(J_\lambda(x),H_\lambda^{(2)}(x)\big)=-\frac{2i}{\pi x},\quad W\big(H_\lambda^{(1)}(x),H_\lambda^{(2)}(x)\big)=-\frac{4i}{\pi x}.$$

Then, the relation

$$J_\lambda(x)=\frac{1}{2}\big[H_\lambda^{(1)}(x)+H_\lambda^{(2)}(x)\big], \tag{70}$$

can be used to write each solution as a linear combination of the others in a region where the three solutions are valid.

On the other hand, for a fixed λ the asymptotic behaviors of the Bessel functions as $|x|\to\infty$ are [14]

$$J_\lambda(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{1}{2}\lambda\pi-\frac{1}{4}\pi\right),\qquad |\arg\ x|<\pi;$$

$$H_\lambda^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\ e^{i(x-\frac{1}{2}\lambda\pi-\frac{1}{4}\pi)}, \qquad -\pi < \arg\ x < 2\pi;$$

$$H_\lambda^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\ e^{-i(x-\frac{1}{2}\lambda\pi-\frac{1}{4}\pi)}, \qquad -2\pi < \arg\ x < \pi. \tag{71}$$

Thus, for the solutions $U_1^{(i)}$ we find

$$\lim_{z\to\infty} U_1^{(1)}(z) \sim e^{2i\sqrt{qz}}\ z^{(1/4)-(B_2/2)-(\nu/2)}, \qquad \lim_{z\to\infty} U_1^{(2)}(z) \sim e^{-2i\sqrt{qz}}\ z^{(1/4)-(B_2/2)-(\nu/2)}.$$

The behavior of U_1 when $z \to \infty$ is a linear combination of these due to Eq. (70).

Notice that, by setting

$$w(u) = U(z), \qquad z = \cos^2(\sigma u), \qquad (\sigma = 1, i),$$

$$z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = 1, \quad B_3 = \frac{k^2}{2} - \frac{a}{4}, \tag{72}$$

in Eq. (68), we obtain the Mathieu equation (12). Then, from the solutions (69a) and (69b) we get the following even solutions for the Mathieu equation

$$w_1(u) = \sum_n (-1)^n c_n^{(1)} \left[k\cos(\sigma u)\right]^{-n-\nu} J_{n+\nu}\big(2k\cos(\sigma u)\big), \quad \forall u,$$

$$w_1^{(i)}(u) = \sum_n (-1)^n c_n^{(1)} \left[k\cos(\sigma u)\right]^{-n-\nu} H_{n+\nu}^{(i)}\big(2k\cos(\sigma u)\big), \quad |\cos(\sigma u)| > 1, \tag{73a}$$

where the coefficients $c_n^{(1)}$ satisfy

$$(n+\nu+1)c_{n+1}^{(1)} + \left[(n+\nu)^2 + (k^2/2) - (a/4)\right]c_n^{(1)} + k^2\left[n+\nu-(1/2)\right]c_{n-1}^{(1)} = 0. \tag{73b}$$

In this set of two-sided infinity series solutions, the first solution converges for any u, in contrast with the usual two-sided solutions for the Mathieu equation which converge, all of them, only for $|\cos(\sigma u)| > 1$ [1, 33].

One-sided infinite series are obtained by putting $\nu = 0$ in the two-sided series solutions.

III.B. Derivation of the solutions

To compute the Whittaker-Ince limits, first we rewrite the solutions of the CHE in a form convenient for using the formulas (B8) and (B9). Thus, we rewrite the solutions (43a) as ($q = -2\eta\omega$)

$$U_1(z) = e^{-i\omega z} \sum_n \frac{(-1)^n c_n^{(1)}}{\Gamma(n+\nu+B_2)}\ \Phi\left(\frac{B_2}{2} - i\eta, n+\nu+B_2; \frac{qz}{i\eta}\right), \tag{74a}$$

$$U_1^{\infty}(z) = e^{-i\omega z} \sum_n D_n \Gamma\left(1 - i\eta - n - \nu - \frac{B_2}{2}\right) \Psi\left(\frac{B_2}{2} - i\eta, n+\nu+B_2; \frac{qz}{i\eta}\right), \tag{74b}$$

$$\bar{U}_1^{\infty}(z) = e^{i\omega z} \sum_n E_n (-qz)^{1-n-\nu-B_2} \Gamma\left(i\eta + n + \nu + \frac{B_2}{2}\right) \times$$

$$\Psi\left(1 + i\eta - \frac{B_2}{2}, 2 - n - \nu - B_2; -\frac{qz}{i\eta}\right), \tag{74c}$$

where the new coefficients are defined by

$$D_n = \frac{(-1)^n b_n^{(1)}}{\Gamma\left(1-n-\nu-i\eta-\frac{B_2}{2}\right)}, \quad E_n = \frac{(i\eta)^{n+\nu} c_n^{(1)}}{\Gamma\left(i\eta+n+\nu+\frac{B_2}{2}\right)}.$$

Inserting these relations into Eqs. (43b) and (43c) we find

$$(n+\nu+1)D_{n+1} + \beta_n^{(1)} D_n + 2i\omega z_0 \left(n+\nu+i\eta+\tfrac{B_2}{2}-1\right)\left(n+\nu+B_2+\tfrac{B_1}{z_0}-1\right) D_{n-1} = 0,$$

$$(n+\nu+1)\left[\tfrac{n+\nu+i\eta+(B_2/2)}{i\eta}\right] E_{n+1} + \beta_n^{(1)} E_n - 2\eta\omega z_0 \left(n+\nu+B_2+\tfrac{B_1}{z_0}-1\right) E_{n-1} = 0,$$

where

$$\beta_n^{(1)} = (n+\nu)\left(n+\nu+B_2-1+2i\omega z_0\right) + B_3 + i\omega z_0\left(B_2+\frac{B_1}{z_0}\right).$$

Thence, we find that the Whittaker-Ince limits of these relations are identical since

$$\lim c_n^{(1)} = \lim D_n = \lim E_n, \quad (\omega \to 0, \quad \eta \to \infty, \quad 2\eta\omega = -q).$$

Denoting by $c_n^{(1)}$ the above limits, we obtain the recurrence relations (69b).

Now, by letting $(-i\eta) \to \infty$ and using Eq. (B8) we find

$$\Phi\left(\frac{B_2}{2}-i\eta, n+\nu+B_2; \frac{qz}{i\eta}\right) \to \Gamma(n+\nu+B_2)\left(\sqrt{qz}\right)^{1-n-\nu-B_2} J_{n+\nu+B_2-1}\left(2\sqrt{qz}\right).$$

Thus the limit of Eq. (74a) is the solution $U_1(z)$ written in Eqs. (69a). On the other hand, to obtain the limit of the solution (74b) we define $L_1(z)$ by

$$L_1(z) := \Gamma\left(1-i\eta-n-\nu-\frac{B_2}{2}\right)\Psi\left(\frac{B_2}{2}-i\eta, n+\nu+B_2; \frac{qz}{i\eta}\right).$$

Then, when $(-i\eta) \to \infty$ the relation (B9) gives

$$L_1(z) \to 2\left(\pm i\sqrt{qz}\right)^{1-n-\nu-B_2} K_{n+\nu+B_2-1}\left(\pm 2i\sqrt{qz}\right) =$$

$$\begin{cases} -i\pi e^{i\pi(1-n-\nu-B_2)}\left(\sqrt{qz}\right)^{1-n-\nu-B_2} H^{(2)}_{n+\nu+B_2-1}\left(2\sqrt{qz}\right), \\ \\ i\pi e^{-i\pi(1-n-\nu-B_2)}\left(\sqrt{qz}\right)^{1-n-\nu-B_2} H^{(1)}_{n+\nu+B_2-1}\left(2\sqrt{qz}\right), \end{cases}$$

where in last step we have used the relations among the functions K_λ and $H_\lambda^{(i)}$ given in Eq. (B11). Inserting these into the limit of the solution (74b) and supressing multiplicative factors, we find the solutions $U_1^{(i)}(z)$ given in (69a). The same solutions follow from the limit of the solution (74c). Notice that these are formal derivations which involve some tricks. However, we may check the resulting solutions by inserting them into Eq. (68).

III.C. Schrödinger equation for the volcano-type potential

For the potential (17) given by Cho and Ho [8] the Schrödinger equation (15) is

$$\frac{d^2\psi}{du^2} + \left[\mathcal{E} + \frac{b^2}{4}\sinh^2 u + \frac{\left[(\ell+1)^2-(1/4)\right]}{\cosh^2 u}\right]\psi = 0 \tag{75}$$

where $b>0$ is a real constant. We transform this into the Whittaker-Ince limit (68) of the CHE and find bounded finite-series eigenstates corresponding to discrete energies, as in Cho and Ho. In fact, we prove that there are pairs of (even and odd) degenerate eigenfunctions for any value of the parameter ℓ. This degeneracy was first pointed out by Kar and Parwani [24], and by Koley and Kar [25] who have not provided a general proof for such degeneracy. We also find bounded infinite-series solutions.

In fact, the substitutions

$$\psi(u) = \left[\cosh u\right]^{-\ell-\frac{1}{2}} U(z), \qquad z = -\sinh^2 u \tag{76}$$

bring Eq. (75) to the form

$$z(z-1)\frac{d^2U}{dz^2} + \left[-\frac{1}{2} + \left(\frac{1}{2} - \ell\right) z\right]\frac{dU}{dz} + \left[\frac{\mathcal{E}}{4} - \frac{b^2}{16} + \left(\frac{1}{4} + \frac{\ell}{2}\right)^2 - \frac{b^2}{16}(z-1)\right] U = 0,$$

which is a particular case of Eq. (68), with the following set of parameters

$$z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{2} - \ell, \quad B_3 = \frac{\mathcal{E}}{4} - \frac{b^2}{16} + \left(\frac{1}{4} + \frac{\ell}{2}\right)^2, \quad q = -\frac{b^2}{16}. \tag{77}$$

Thus, if we denote by $\vec{\psi}_j = (\psi_j, \psi_j^{(1)}, \bar{\psi}_j^{(2)})$ the solutions of the Schrödinger equation corresponding to the each set of solutions $\vec{U}_j = (U_j, U_j^{(1)}, U_j^{(2)})$ of Eq. (68), we have

$$\vec{\psi}_j(u) = \left[\cosh u\right]^{-\ell-\frac{1}{2}} \vec{U}_j(z), \qquad z = -\sinh^2 u, \tag{78}$$

where on right-hand side we must use the parameters given in Eq. (77). In the present case it is sufficient to regard only the solutions ψ_j in series of Bessel functions of the first kind. First we study the solutions given by finite series and then the ones given by infinite series.

Finite-series solutions. The first and the second sets of solutions, given in Eqs. (69a) and (D4), afford solutions in finite series if $\nu = 0$. We redefine the series coefficients as

$$c_n^{(1)} = \frac{(-1)^n P_n}{n!}, \qquad c_n^{(2)} = \frac{(-1)^n Q_n}{n!}.$$

Thence from the first set (69a) we obtain the even solutions

$$\psi_1(u) = \left[\cosh u\right]^{-\ell-\frac{1}{2}} \sum_{n=0}^{\ell} \frac{P_n}{n!}\left(\frac{b}{4}\sinh u\right)^{-(n-\ell-\frac{1}{2})} J_{n-\ell-\frac{1}{2}}\left(\frac{b}{2}\sinh u\right), \quad \text{(even)} \tag{79a}$$

where the recurrence relations for P_n are

$$P_1 = E_\ell\, P_0, \qquad E_\ell := \frac{\mathcal{E}}{4} - \frac{b^2}{16} + \left(\frac{1}{4} + \frac{\ell}{2}\right)^2,$$

$$P_{n+1} = \left[E_\ell + n\left(n - \ell - \frac{1}{2}\right)\right] P_n + \frac{b^2}{16} n(n-\ell-1)P_{n-1}, \quad (n \geq 1). \tag{79b}$$

The second set (D4) yields the odd finite-series solutions

$$\psi_2(u) = \sinh u\left[\cosh u\right]^{-\ell-\frac{1}{2}} \sum_{n=0}^{\ell} \frac{Q_n}{n!}\left(\frac{b}{4}\sinh u\right)^{-(n-\ell+\frac{1}{2})} J_{n-\ell+\frac{1}{2}}\left(\frac{b}{2}\sinh u\right), \quad \text{(odd)} \tag{80a}$$

where the Q_n satisfy the recurrence relations

$$Q_1 = \left[E_\ell - \frac{\ell}{2}\right] Q_0, \quad Q_{n+1} = \left[E_\ell + (n-\ell)\left(n + \frac{1}{2}\right)\right] Q_n + \frac{b^2}{16} n(n-\ell-1)Q_{n-1}, \ (n \geq 1). \tag{80b}$$

These solutions are Heun polynomials because the coefficients of P_{n-1} and Q_{n-1} vanish for $n=\ell+1$ and, consequently, the series terminate at $n=\ell$. Further, $\psi_1(u)$ represents even solutions because the relations (B12) give

$$J_{n-\ell-\frac{1}{2}}(-x)=(-1)^{\ell-n-\frac{1}{2}}J_{n-\ell-\frac{1}{2}}(x)$$

and this implies that $\psi_1(-u)=\psi_1(u)$. Analogously we may show that $\psi_2(u)$ represents odd solutions.

On the other hand, using the fact that $J_\lambda(x)\sim(x/2)^\lambda$ when $x\to 0$, and the first of Eqs. (71) we find

$$\lim_{u\to 0}\psi_1(u)\sim\big[\cosh u\big]^{-\ell-\frac{1}{2}},\qquad \lim_{u\to\pm\infty}\psi_1(u)\sim\frac{[\tanh u]^\ell}{\sqrt{\cosh u}}\left[k_1\,e^{i\frac{b}{2}\sinh u}+k_2\,e^{-i\frac{b}{2}\sinh u}\right]\to 0,$$

where k_1 and k_2 are constants. Thus $\psi_1(u)$ is bounded for any u. The same holds for $\psi_2(u)$.

Notice that the coefficients $(\alpha_n^{(i)},\beta_n^{(i)},\gamma_n^{(i)})$ $(i=1,2)$ corresponding to (P_{n+1},P_n,P_{n-1}) and (Q_{n+1},Q_n,Q_{n-1}) are real and that

$$\alpha_n^{(i)}\gamma_{n+1}^{(i)}=\frac{b^2}{16}(n+1)(\ell-n)>0,\qquad 0\le n\le\ell-1,\tag{81}$$

where in the present case $\alpha_n^{(i)}=1$. Thus, the condition (22) is satisfied and consequently there are $\ell+1$ real and distinct eigenvalues.

Now we show that there are an even and an odd state corresponding to each energy eigenvalue, that is, the solutions are degenerate. First we rewrite the relations (79b) and (80b) as

$$P_{n+1}+\beta_n\,P_n+\gamma_n\,P_{n-1}=0,\qquad Q_{n+1}+\tilde{\beta}_n\,Q_n+\gamma_n\,Q_{n-1}=0,\qquad (P_{-1}=Q_{-1}=0)\tag{82a}$$

where

$$\beta_n=-\left[E_\ell+n\left(n-\ell-\frac{1}{2}\right)\right],\quad \tilde{\beta}_n=-\left[E_\ell+(n-\ell)\left(n+\frac{1}{2}\right)\right],\quad \gamma_n=-\frac{b^2}{16}n(n-\ell-1).\tag{82b}$$

The $(\ell+1)$-by-$(\ell+1)$ tridiagonal matrix $\mathbb{P}$ corresponding to the first system of equations is

$$\mathbb{P}=\begin{pmatrix}\beta_0 & 1 & & & & 0\\ \gamma_1 & \beta_1 & 1 & & & \\ & \gamma_2 & \beta_2 & \cdot & & \\ & & \cdot & & & \\ & & & \gamma_{\ell-1} & \beta_{\ell-1} & 1\\ 0 & & & & \gamma_\ell & \beta_\ell\end{pmatrix}.\tag{83a}$$

Since $\tilde{\beta}_0=\beta_\ell$, $\tilde{\beta}_1=\beta_{\ell-1}$, $\tilde{\beta}_2=\beta_{\ell-2}$, $\cdots$, $\tilde{\beta}_\ell=\beta_0$, the matrix $\mathbb{Q}$ of the second system is

$$\mathbb{Q}=\begin{pmatrix}\beta_\ell & 1 & & & & 0\\ \gamma_1 & \beta_{\ell-1} & 1 & & & \\ & \gamma_2 & \beta_{\ell-2} & \cdot & & \\ & & \cdot & & & \\ & & & \gamma_{\ell-1} & \beta_1 & 1\\ 0 & & & & \gamma_\ell & \beta_0\end{pmatrix}=\begin{pmatrix}\beta_\ell & 1 & & & & 0\\ \gamma_\ell & \beta_{\ell-1} & 1 & & & \\ & \gamma_{\ell-1} & \beta_{\ell-2} & \cdot & & \\ & & \cdot & & & \\ & & & \gamma_2 & \beta_1 & 1\\ 0 & & & & \gamma_1 & \beta_0\end{pmatrix},\tag{83b}$$

where the last equality results from the fact that $\gamma_1=\gamma_\ell$, $\gamma_2=\gamma_{\ell-1}$ and so forth, as inferred from the definition of γ_n. To prove the degeneracy, it is necessary to show that both matrices possess the same roots, that is, $\det\mathbb{P}=\det\mathbb{Q}$. For this, use the $(\ell+1)$-by-$(\ell+1)$ antidiagonal matrix $\mathbb{A}$ having 1's on the antidiagonal as the only nonzero elements, that is,

$$\mathbb{A}=\mathbb{A}^{-1}=\begin{pmatrix} & & 1\\ & \cdot & \\ 1 & & \end{pmatrix}.\tag{84a}$$

Then, using the last form given in Eq. (83b) for $\mathbb{Q}$, we find the similarity relation

$$\mathbb{P} = \mathbb{A}^{-1}\mathbb{Q}^T\mathbb{A} = \mathbb{A}\mathbb{Q}^T\mathbb{A}, \tag{84b}$$

where $\mathbb{Q}^T$ is the transpose of $\mathbb{Q}$. From the properties of the determinants, it follows that $\det\mathbb{P} = \det\mathbb{Q}$ and, therefore, the solutions are degenerate.

To determine the eigenvalues we can apply the procedure of Bender and Dunne [5]. For this we take $P_0 = Q_0 = 1$ as initial conditions and use the recurrence relations to generate the other coefficients. For a fixed ℓ the eigenvalues are obtained by imposing that $P_{\ell+1} = 0$ or $Q_{\ell+1} = 0$, since the series terminate at $n = \ell$. For example, the recurrence relations (79b) for P_n give

$$P_0 = 1, \qquad P_1 = E_\ell, \qquad P_2 = E_\ell^2 + \left(\frac{1}{2} - \ell\right) E_\ell - \frac{b^2}{16}\ell,$$

$$P_3 = E_\ell^3 + \left(\frac{7}{2} - 3\ell\right) E_\ell^2 + \left[\frac{b^2}{16}(2-3\ell) + (3-2\ell)\left(\frac{1}{2} - \ell\right)\right] E_\ell + \frac{b^2}{16}(2\ell - 3),$$

and so on. The recurrence relations for Q_n yield

$$Q_0 = 1, \qquad Q_1 = E_\ell - \frac{\ell}{2}, \qquad Q_2 = E_\ell^2 + \left(\frac{3}{2} - 2\ell\right) E_\ell + \frac{3}{4}\ell(\ell-1) - \frac{b^2}{16}\ell,$$

$$Q_3 = E_\ell^3 + \frac{1}{2}(13 - 9\ell) E_\ell^2 + \left[\frac{b^2}{16}(2-3\ell) + \frac{3}{4}\ell(\ell-1) + \frac{5}{2}(\ell-2)\left(2\ell - \frac{3}{2}\right)\right] E_\ell + \frac{b^2}{16}\ell\left(\frac{7\ell}{2} - 6\right) - \frac{15}{8}\ell(\ell-1)(\ell-2).$$

Thus, for $\ell = 0$ the energy that follows from the condition $P_1 = Q_1 = 0$ is

$$E_0 = 0 \quad \Rightarrow \quad \mathcal{E} = \frac{1}{4}\left[b^2 - 1\right], \tag{85a}$$

corresponding to the degenerate pair of eigenfunctions

$$\psi_1(u, \ell = 0) = \frac{2c_0^{(1)}}{\sqrt{\pi b \cosh u}} \cos\left(\frac{b}{2}\sinh u\right), \quad \psi_2(u, \ell = 0) = \frac{2c_0^{(2)}}{\sqrt{\pi b \cosh u}} \sin\left(\frac{b}{2}\sinh u\right), \tag{85b}$$

where the expressions (B15) for the Bessel functions have been used. For $\ell = 1$ the condition $P_2 = Q_2 = 0$ leads to the energies

$$E_1^{\pm} = \frac{1}{4}\left[1 \pm \sqrt{1 + b^2}\right] \quad \Rightarrow \quad \mathcal{E}^{\pm} = \frac{1}{4}\left[b^2 - 5\right] \pm \sqrt{1 + b^2} \tag{86a}$$

which correspond to the parity-paired degenerate wave functions

$$\psi_1^{\pm}(u, \ell = 1) = [\cosh u]^{-\frac{3}{2}} \left[-\frac{b}{4}(\sinh u)\sin\left(\frac{b}{2}\sinh u\right) + \left(E_1^{\pm} - \frac{1}{2}\right)\cos\left(\frac{b}{2}\sinh u\right)\right],$$

$$\psi_2^{\pm}(u, \ell = 1) = [\cosh u]^{-\frac{3}{2}} \left[\frac{b}{4}(\sinh u)\cos\left(\frac{b}{2}\sinh u\right) + \left(E_1^{\pm} - \frac{1}{2}\right)\sin\left(\frac{b}{2}\sinh u\right)\right], \tag{86b}$$

where normalization factors have been omitted.

The solutions found by Cho and Ho [8], for $\ell = 0$ and $\ell = 1$, have no definite parity and can be obtained from the expansions in series of Hankel functions. The solutions $\psi_1^{(i)}$ associated with ψ_1 are

$$\psi_1^{(i)}(u) = \left[\cosh u\right]^{-\ell-\frac{1}{2}} \sum_{n=0}^{\ell} \frac{P_n}{n!}\left(\frac{b}{4}\sinh u\right)^{-(n-\ell-\frac{1}{2})} H^{(i)}_{n-\ell-\frac{1}{2}}\left(\frac{b}{2}\sinh u\right), \quad (i = 1, 2). \tag{87a}$$

Notice that ψ_1 is a superposition of $\psi_1^{(1)}$ and $\psi_1^{(2)}$ due to Eq. (70). Likewise, the solution ψ_2 can be obtained as a combination of the solutions $\psi_2^{(1)}$ and $\psi_2^{(2)}$, given by

$$\psi_2^{(i)}(u) = \sinh u\big[\cosh u\big]^{-\ell-\frac{1}{2}} \sum_{n=0}^{\ell} \frac{Q_n}{n!} \left(\frac{b}{4}\sinh u\right)^{-(n-\ell+\frac{1}{2})} H^{(i)}_{n-\ell+\frac{1}{2}}\left(\frac{b}{2}\sinh u\right) \qquad (i=1,2). \tag{87b}$$

Infinite-series solutions. Supposing again that ℓ is a non-negative integer and using the third set of solutions (D6) with $\nu = 0$, we obtain the odd infinite-series solutions [odd due to Eq. (B12)]

$$\psi_3(u) = \sinh u\big[\cosh u\big]^{\ell+\frac{3}{2}} \sum_{n=0}^{\infty}(-1)^n c_n^{(3)} \left(\frac{b}{4}\sinh u\right)^{-n-\ell-\frac{5}{2}} J_{n+\ell+\frac{5}{2}}\left(\frac{b}{2}\sinh u\right), \quad \forall u, \tag{88a}$$

where the $c_n^{(3)}$ satisfy the recurrence relations ($c_{-1}^{(3)} = 0$)

$$(n+1)c_{n+1}^{(3)} + \left[E_\ell + \frac{\ell+1}{2} + (n+\ell+2)\left(n+\frac{1}{2}\right)\right] c_n^{(3)} - \frac{b^2}{16}(n+\ell+1)c_{n-1}^{(3)} = 0, \tag{88b}$$

being E_ℓ defined in Eq. (79b). From Eqs. (71) it follows that

$$\psi_3(u) \sim \frac{k_3\ e^{i\frac{b}{2}\sinh u} + k_4\ e^{-i\frac{b}{2}\sinh u}}{\sqrt{\sinh u}} \to 0, \quad (u\to\pm\infty),$$

where k_3 and k_4 are constants. Then, in addition of being bounded in the neighborhood of $u = 0$, this solution is bounded also at infinity. An even infinite-series solution, also regular for any u, is obtained by replacing $U_j(z)$ in Eq. (78) by the $U_4(z)$ given in Eqs. (D8) and (D9) with $\nu = 0$. We find

$$\psi_4(u) = \big[\cosh u\big]^{\ell+\frac{3}{2}} \sum_{n=0}^{\infty}(-1)^n c_n^{(4)} \left(\frac{b}{4}\sinh u\right)^{-n-\ell-\frac{3}{2}} J_{n+\ell+\frac{3}{2}}\left(\frac{b}{2}\sinh u\right), \tag{89a}$$

where the $c_n^{(4)}$ satisfy the recurrence relations

$$(n+1)c_{n+1}^{(4)} + \left[E_\ell + \frac{\ell+1}{2} + n\left(n+\ell+\frac{3}{2}\right)\right] c_n^{(4)} - \frac{b^2}{16}(n+\ell+1)c_{n-1}^{(4)} = 0. \tag{89b}$$

The solution ψ_3 can also be obtained by changing ℓ by $(-\ell-2)$ in the solutions $\psi_2(u)$ given in Eqs. (80a) and (80b) and allowing that the summation runs from zero to positive infinity, whereas ψ_4 is obtained by the substitution $\ell \to (-\ell-2)$ in the solution $\psi_1(u)$ given in Eq. (79a).

Other infinite-series solutions are obtained by using the solutions $U(z)$ which result from the four (sets of) solutions given in Sect. II.A and in Appendix D by means of the transformation $\mathscr{T}_3$ given in Eq. (D3). That transformation changes the independent variable $z = -\sinh^2 u$ into $1 - z = \cosh^2 u$ and also changes the parameters in accordance with Eq. (D3). Then, instead of Eq. (78), we can use

$$\vec{\psi}_{j+4}(u) = \big[\cosh u\big]^{-\ell-\frac{1}{2}}\ \vec{U}_j(z), \qquad z = \cosh^2 u, \quad (j = 1,\cdots,4) \tag{90a}$$

to construct the four additional solutions. Now, on the right-hand side, the parameters are

$$z_0 = 1, \qquad B_1 = \ell, \qquad B_2 = \frac{1}{2} - \ell, \qquad B_3 = \frac{\mathcal{E}}{4} + \left(\frac{1}{4}+\frac{\ell}{2}\right)^2, \qquad q = \frac{b^2}{16}. \tag{90b}$$

Again we use only the expansions in Bessel functions of the first kind, J_λ, with $\nu = 0$.

Thus, the solution U_1 given in Eq. (69a) yields the even infinite-series solution

$$\psi_5(u) = \sum_{n=0}^{\infty}(-1)^n c_n^{(5)} \left(\frac{b}{4}\cosh u\right)^{-n} J_{n-\ell-\frac{1}{2}}\left(\frac{b}{2}\cosh u\right), \tag{91a}$$

where the $c_n^{(5)}$ satisfy the recurrence relations

$$(n+1)c_{n+1}^{(5)}+\left[n\left(n-\ell-\frac{1}{2}\right)+\frac{\mathcal{E}}{4}+\left(\frac{1}{4}+\frac{\ell}{2}\right)^2\right]c_n^{(5)}+\frac{b^2}{16}\left(n-\frac{1}{2}\right)c_{n-1}^{(5)}=0. \tag{91b}$$

The solution $\psi_6(u)$ obtained from the U_2 given in Eqs. (D4) is also even and reads

$$\psi_6(u)=\sum_{n=0}^{\infty}(-1)^n c_n^{(6)}\left(\frac{b}{4}\cosh u\right)^{-n}J_{n+\ell+\frac{3}{2}}\left(\frac{b}{2}\cosh u\right), \tag{92a}$$

where the coefficients satisfy the relations

$$(n+1)c_{n+1}^{(6)}+\left[n\left(n+\ell+\frac{3}{2}\right)+\frac{\mathcal{E}}{4}+\left(\frac{1}{4}+\frac{\ell}{2}\right)^2+\frac{\ell+1}{2}\right]c_n^{(6)}+\frac{b^2}{16}\left(n-\frac{1}{2}\right)c_{n-1}^{(6)}=0. \tag{92b}$$

The solutions ψ_5 and ψ_6 are connected by the change $\ell\to-\ell-2$.

On the other hand, the solutions U_3 and U_4 given in Eqs. (D6) and (D8), respectively, lead to odd wave functions, namely,

$$\psi_7(u)=\tanh u\sum_{n=0}^{\infty}(-1)^n c_n^{(7)}\left(\frac{b}{4}\cosh u\right)^{-n}J_{n+\ell+\frac{5}{2}}\left(\frac{b}{2}\cosh u\right), \tag{93a}$$

where the recurrence relations for $c_n^{(7)}$ are

$$(n+1)c_{n+1}^{(7)}+\left[n\left(n+\ell+\frac{5}{2}\right)+\frac{\mathcal{E}}{4}+\left(\frac{1}{4}+\frac{\ell}{2}\right)^2+\ell+\frac{3}{2}\right]c_n^{(7)}+\frac{b^2}{16}\left(n+\frac{1}{2}\right)c_{n-1}^{(7)}=0, \tag{93b}$$

and the solution

$$\psi_8(u)=\tanh u\sum_{n=0}^{\infty}(-1)^n c_n^{(8)}\left(\frac{b}{4}\cosh u\right)^{-n}J_{n-\ell+\frac{1}{2}}\left(\frac{b}{2}\cosh u\right), \tag{94a}$$

where the $c_n^{(8)}$ satisfy

$$(n+1)c_{n+1}^{(8)}+\left[n\left(n-\ell+\frac{1}{2}\right)+\frac{\mathcal{E}}{4}+\left(\frac{1}{4}+\frac{\ell}{2}\right)^2-\frac{\ell}{2}\right]c_n^{(8)}+\frac{b^2}{16}\left(n+\frac{1}{2}\right)c_{n-1}^{(8)}=0, \tag{94b}$$

The solution ψ_6 can also be obtained by replacing ℓ by $(-\ell-2)$ in ψ_5 .

Notice that in the characteristic relations (20) for the solutions ψ_3 and ψ_4 the product $\alpha_i\gamma_{i+1}$ is negative, while for the other infinite-series solutions it is positive. We do not know if this fact implies some significant difference to the energy spectra which results in each case.

IV. THE DOUBLE-CONFLUENT HEUN EQUATION

In Sec. IV.A we obtain sets of solutions for the double-confluent Heun equation (3) by applying the Leaver limit ($z_0\to 0$) to solutions of the CHE. The expansions in series of regular hypergeometric functions converge for any z, whereas the expansions in series of irregular functions converge for $|z|>0$. We also give the conditions to express one solution in terms of the others and obtain one-sided series solutions by truncating the two-sided series on the left.

In Sec. IV.B we use the solutions of the DCHE to solve the Schrödinger equation with the quasi-exactly solvable potentials given in Eq. (16) when $C>0$ and Eq. (18). Finally, in Sec. IV.C we write the solutions for Eq. (7), that is, for the Whittaker-Ince limit of the DCHE.

IV.A. Solutions for the general case

For $z_0 \to 0$ the solutions (43a) are not affected formally, but their recurrence relations change. We find

$$U_1(z) = e^{-i\omega z}\sum_n (-1)^n b_n^{(1)}\ \widetilde{\Phi}\left(\frac{B_2}{2} - i\eta, n+\nu+B_2; 2i\omega z\right)$$

$$U_1^{\infty}(z) = e^{-i\omega z}\sum_n (-1)^n b_n^{(1)}\Psi\left(\frac{B_2}{2} - i\eta, n+\nu+B_2; 2i\omega z\right),$$

$$\bar{U}_1^{\infty}(z) = e^{i\omega z}\sum_n c_n^{(1)}\Psi\left(n+\nu+i\eta+\frac{B_2}{2}, n+\nu+B_2; -2i\omega z\right). \tag{95a}$$

The recurrence relations for the $c_n^{(1)}$ are

$$(n+\nu+1)c_{n+1}^{(1)} + \left[(n+\nu)\left(n+\nu+B_2-1\right) + i\omega B_1 + B_3\right]c_n^{(1)} +$$

$$2i\omega B_1\left(n+\nu+i\eta+\frac{B_2}{2}-1\right)c_{n-1}^{(1)} = 0, \tag{95b}$$

wherefrom we get relations for $b_n^{(1)}$ by means of

$$c_n^{(1)} = \Gamma\left(n+\nu+i\eta+\frac{B_2}{2}\right)b_n^{(1)}.$$

The behavior given in Eq. (46) for the CHE is also valid for the present solutions.

These three solutions are given by doubly infinite series if

$$\nu \quad \text{and} \quad \nu+i\eta+\frac{B_2}{2} \quad \text{are not integers.} \tag{96a}$$

If, besides this,

$$\nu+B_2 \quad \text{is not integer and} \quad \frac{B_2}{2}-i\eta \quad \text{is not zero or negative integer,} \tag{96b}$$

then any of these solutions can be written as a linear combination of the others by using Eq. (32). Again, as in the CHE, only the restriction on $(B_2/2)-i\eta$ cannot be satisfied by an suitable choice of ν. In addition, if $(B_2/2)-i\eta = -m$ $(m = 0, 1, 2, \cdots)$, the expansion U_1 becomes meaningless. For this case, instead of U_1, we can use

$$U_1^{\mathrm{p}}(z) = e^{-i\omega z}\sum_{n=0}^{\infty} d_n^{(1)}\left(\frac{z}{B_1}\right)^n, \qquad (|z| = \text{finite}) \tag{97a}$$

where recurrence relations for the coefficients are ($d_{-1}^{(1)} = 0$)

$$(n+1)d_{n+1}^{(1)} + \left[n\left(n+B_2-1\right) - i\omega B_1 + B_3\right]d_n^{(1)} - 2i\omega B_1\left(n - i\eta + \frac{B_2}{2} - 1\right)d_{n-1}^{(1)} = 0. \tag{97b}$$

This U_1^{p}, which reduces to a Heun polynomial when $(B_2/2)-i\eta = -m$, was obtained by letting $z_0 \to 0$ in Eq. (48a).

A second set follows from the solutions (C10) and the corresponding recurrence relations (C11). Using the L'Hospital rule we find that

$$z^{1+\frac{B_1}{z_0}}(z-z_0)^{1-B_2-\frac{B_1}{z_0}} = z(z-z_0)^{1-B_2}\left(1-\frac{z_0}{z}\right)^{-\frac{B_1}{z_0}} \to z^{2-B_2}e^{B_1/z}, \quad (z_0 \to 0) \tag{98}$$

and, hence, we obtain the solutions

$$U_2(z) = z^{2-B_2} e^{-i\omega z+\frac{B_1}{z}} \sum_n (-1)^n b_n^{(2)} \widetilde{\Phi}\left(2 - i\eta - \frac{B_2}{2}, n+\nu+4-B_2; 2i\omega z\right),$$

$$U_2^{\infty}(z) = z^{2-B_2} e^{-i\omega z+\frac{B_1}{z}} \sum_n (-1)^n b_n^{(2)} \Psi\left(2 - i\eta - \frac{B_2}{2}, n+\nu+4-B_2; 2i\omega z\right),$$

$$\bar{U}_2^{\infty}(z) = z^{2-B_2} e^{i\omega z+\frac{B_1}{z}} \sum_n c_n^{(2)} \Psi\left(n+\nu+2+i\eta - \frac{B_2}{2}, n+\nu+4-B_2; -2i\omega z\right), \tag{99a}$$

where the coefficients $c_n^{(2)}$ obey

$$(n+\nu+1)c_{n+1}^{(2)} + \Big[(n+\nu)(n+\nu+3-B_2) + B_3 + 2 - B_2 - i\omega B_1\Big] c_n^{(2)} -$$

$$2i\omega B_1 \left(n+\nu+i\eta+1-\frac{B_2}{2}\right) c_{n-1}^{(2)} = 0. \tag{99b}$$

The recurrence relations for the $b_n^{(2)}$ are derived from these via the relation

$$c_n^{(2)} = \Gamma\left(n+\nu+i\eta+2-\frac{B_2}{2}\right) b_n^{(2)}.$$

Now the conditions to assure that the three solutions are given by doubly infinite series are

$$\nu \quad \text{and} \quad \nu + i\eta - \frac{B_2}{2} \quad \text{cannot be integers,} \tag{100a}$$

and if, besides this,

$$\nu + B_2 \quad \text{is not integer and} \quad i\eta + \frac{B_2}{2} \quad \text{is not zero or negative integer,} \tag{100b}$$

then Eq. (32) can be used to express one solution in terms of the others. If $i\eta + (B_2/2) = 2, 3, \cdots$, instead of U_2 we can use the solution obtained from (104) by the substitution $(\eta, \omega) \rightarrow (-\eta, -\omega)$.

One-sided series solutions are obtained by putting $\nu = 0$ in the two-sided solutions, in the same way as in Sect. II.B. We write only the first set together with the limit of the Barber-Hassé solution (34a), since this last will be used next.

$$U_1(z) = e^{-i\omega z} \sum_{n=0}^{\infty} \frac{(-1)^n c_n^{(1)}}{\Gamma(n+B_2)} \Phi\left(\frac{B_2}{2} - i\eta, n+B_2; 2i\omega z\right),$$

$$U_1^{\infty}(z) = e^{-i\omega z} \sum_{n=0}^{\infty} (-1)^n b_n^{(1)} \Psi\left(\frac{B_2}{2} - i\eta, n+B_2; 2i\omega z\right),$$

$$\bar{U}_1^{\infty}(z) = e^{i\omega z} \sum_{n=0}^{\infty} c_n^{(1)} \Psi\left(n + i\eta + \frac{B_2}{2}, n+B_2; -2i\omega z\right),$$

$$U_1^{\text{barber}}(z) = e^{i\omega z} \sum_{n=0}^{\infty} c_n^{(1)} \left(\frac{z}{B_1}\right)^n, \tag{101a}$$

where the recurrence relations for $b_n^{(1)}$ and $c_n^{(1)}$ are ($b_{-1}^{(1)}=c_{-1}^{(1)}=0$)

$$(n+1)\left(n+i\eta+\frac{B_2}{2}\right)b_{n+1}^{(1)}+\left[n\left(n+B_2-1\right)+i\omega B_1+B_3\right]\,b_n^{(1)}+2i\omega B_1\;b_{n-1}^{(1)}=0, \tag{101b}$$

$$(n+1)c_{n+1}^{(1)}+\left[n\left(n+B_2-1\right)+i\omega B_1+B_3\right]c_n^{(1)}+2i\omega B_1\left(n+i\eta+\frac{B_2}{2}-1\right)c_{n-1}^{(1)}=0. \tag{101c}$$

Notice that the solution $U_1^{\infty}(z)$ does not admit finite-series solutions, while the other solutions do. However, for finite-series it is sufficient to consider only Barber-Hassé solution. On the other hand, the condition in order that the three expansions in series of confluent hypergeometric functions are given by infinite series ($n\geq 0$) is

$$i\eta+\frac{B_2}{2}\quad\text{is not zero or negative integer,} \tag{102}$$

since in this case the series do not truncate on the left or on right. In addition, if

$$B_2\quad\text{and}\quad\frac{B_2}{2}-i\eta\quad\text{are not zero or negative integer,} \tag{103}$$

then Eq. (32) can be used to connect the three one-sided infinite-series solutions.

The Barber-Hassé solution of the second set, obtained from solution (36a), is

$$U_2^{\text{barber}}(z)=e^{i\omega z+(B_1/z)}z^{2-B_2}\sum_{n=0}^{\infty}c_n^{(2)}\left(-\frac{z}{B_1}\right)^n, \tag{104}$$

where the recurrence relations for $c_n^{(2)}$ are obtained by putting $\nu=0$ in Eq. (99b). The corresponding solutions in series of hypergeometric functions can be obtained by taking $\nu=0$ in the solutions (99a).

IV.B. Applications to the Schrödinger equation

First we consider the QES potential (18) related to quasinormal modes and, after this, the asymmetric double-Morse potential (16). In addition to the finite-series solutions, once more we look for infinite-series solutions. We find that for the first case it is sufficient to use one-sided infinite series and in the second case it is necessary to use two-sided infinite series in order to get infinite-series solutions bounded for all values of u.

The potential of Cho and Ho. Inserting the potential (18) into Schrödinger equation (15) we find

$$\frac{d^2\psi}{du^2}+\left[\mathcal{E}+\frac{b^2}{4}e^{2u}+(\ell+1)de^{-u}-\frac{d^2}{4}e^{-2u}\right]\psi=0.$$

We will show that for $d>0$ complex energies corresponding to bounded finite-series solutions are obtained by treating this equation as a DCHE; in this case infinite-series solutions are not regular at $u=\pm\infty$. However, if $d<0$ we find bounded infinite-series solutions given by expansions in regular hypergeometric functions; in this case the finite-series solutions are unbounded.

The substitutions

$$z=e^u,\qquad \psi(u)=e^{-d/(2z)}\;z^{-(\ell+\frac{1}{2})}\;U(z),\quad z\in[0,\infty) \tag{105}$$

transform this equation into a particular case of the DCHE (3), namely,

$$z^2\frac{d^2U}{dz^2}+[d-2\ell z]\frac{dU}{dz}+\left[\mathcal{E}+\left(\frac{1}{2}+\ell\right)^2+\frac{b^2}{4}z^2\right]U=0,$$

and then we choose the following set of parameters for the DCHE

$$B_1=d,\quad B_2=-2\ell,\quad B_3=\mathcal{E}+\left(\frac{1}{2}+\ell\right)^2,\quad \omega=\frac{b}{2},\quad \eta=0 \tag{106}$$

(If we take $\omega=-b/2$ the solutions are obtained by changing b by $-b$ in the following ones). Hence, from Eq. (105) the solutions $\psi(u)$ can be constructed by writing

$$\psi(u)=\exp\left[-\frac{d}{2}e^{-u}-\left(\ell+\frac{1}{2}\right)u\right]U(z=e^{u}), \tag{107}$$

where $U(z)$ are solutions of the DCHE with the parameters specified in Eq. (106). To get finite-series solutions we take the Barber-Hassé solution given in Eq. (101a). This yields

$$\psi_1^{\rm barber}(u)=\exp\left[i\frac{b}{2}e^{u}-\frac{d}{2}e^{-u}-\left(\ell+\frac{1}{2}\right)u\right]\sum_{n=0}^{\ell}\frac{(-1)^nP_n}{n!}\left(\frac{e^u}{d}\right)^n,\qquad d>0, \tag{108a}$$

where the recurrence relations for P_n are

$$P_1=E_\ell P_0,\qquad E_\ell:=\mathcal{E}+\left(\frac{1}{2}+\ell\right)^2+\frac{i}{2}bd,$$

$$P_{n+1}=\left[E_\ell+n(n-2\ell-1)\right]P_n-ibd\,n(n-\ell-1)P_{n-1},\qquad (n\geq 1). \tag{108b}$$

Only if $d>0$ this wave function is bounded for any u and, in particular, at $u=-\infty$ where $V(u)\to-\infty$. As the coefficients of these recurrence relations involve imaginary terms, there no guarantee of real eigenvalues. In effect, taking $P_0=1$ as initial condition, we find

$$P_0=1,\qquad P_1=E_\ell,\qquad P_2=E_\ell^2-2\ell E_\ell+ibd\ell,$$

$$P_3=E_\ell^3+(2-6\ell)E_\ell^2+\left[4\ell(2\ell-1)-ibd(2-3\ell)\right]E_\ell-2ibd\ell(2\ell-1),$$

and so on. Then, by requiring that $P_1=0$ for $\ell=0$ we obtain

$$\mathcal{E}=-\frac{1}{4}-\frac{i}{2}bd\quad\Rightarrow\quad\psi_1^{\rm barber}(u)=\exp\left[i\frac{b}{2}e^{u}-\frac{d}{2}e^{-u}-\frac{1}{2}u\right], \tag{109a}$$

and requiring that $P_2=0$ for $\ell=1$ we find

$$\mathcal{E}^{\pm}=-\frac{5}{4}-\frac{i}{2}bd\pm\sqrt{1-ibd}\;\Rightarrow\;\psi_1^{\pm}(u)=\left[1-\frac{1\pm\sqrt{1-ibd}}{d}e^{u}\right]\exp\left[i\frac{b}{2}e^{u}-\frac{d}{2}e^{-u}-\frac{3}{2}u\right]. \tag{109b}$$

For $d>0$, solutions given by infinite series are discarded because they are not bounded for all values of u. In fact, we have found no solution regular at $u=-\infty$ $(z=0)$ where the potential tends to positive infinity.

In the following we show that, if $d<0$, the two-sided solutions (99a) for DCHE with $\nu=0$ lead to solutions bounded for all values of u (the solution ψ_1 is not valid for this problem). For this, we insert the solutions (99a) into Eq. (107). This yields

$$\psi_2(u)=\exp\left[-i\frac{b}{2}e^{u}+\frac{d}{2}e^{-u}+\left(\ell+\frac{3}{2}\right)u\right]\sum_{n=0}^{\infty}(-1)^nb_n^{(2)}\;\tilde{\Phi}\left(\ell+2,n+2\ell+4;ibe^{u}\right),$$

$$\psi_2^{\infty}(u)=\exp\left[-i\frac{b}{2}e^{u}+\frac{d}{2}e^{-u}+\left(\ell+\frac{3}{2}\right)u\right]\sum_{n=0}^{\infty}(-1)^nb_n^{(2)}\;\Psi\left(\ell+2,n+2\ell+4;ibe^{u}\right),$$

$$\bar{\psi}_2^{\infty}(u)=\exp\left[i\frac{b}{2}e^{u}+\frac{d}{2}e^{-u}+\left(\ell+\frac{3}{2}\right)u\right]\sum_{n=0}^{\infty}c_n^{(2)}\;\Psi\left(n+\ell+2,n+2\ell+4;-ibe^{u}\right), \tag{110a}$$

where $c_n^{(2)} = \Gamma(n+\ell+2)b_n^{(2)}$, being the recurrence relations for $c_n^{(2)}$ given by

$$(n+1)c_{n+1}^{(2)} + \left[n(n+3+2\ell)+\mathcal{E}+\left(\ell+\frac{1}{2}\right)^2+2(\ell+1)-\frac{i}{2}bd\right]c_n^{(2)} - ibd(n+\ell+1)c_{n-1}^{(2)} = 0. \tag{110b}$$

In the function $\tilde{\Phi}(a,c;y)$ the parameters $a=\ell+2$, $c=n+2\ell+4$ and $c-a=n+\ell+2$ are not zero or negative integers. For this reason, in a region where the three solutions are valid, one solution can be written as a linear combination of the others by means of Eq. (32). On the other hand, the solutions $\psi_2(u)$ converges for any u and from $\Phi(a,c;0)=1$ we find

$$\lim_{u\to-\infty}\psi_2(u) \sim \exp\left[-i\frac{b}{2}e^u+\frac{d}{2}e^{-u}+\left(\ell+\frac{3}{2}\right)u\right] \to 0.$$

The solutions $\psi_2^\infty(u)$ and $\bar{\psi}_2^\infty(u)$ converge only for $e^u>0$. From Eq. (B2) we find

$$\lim_{u\to\infty}\psi_2^\infty(u) \sim \exp\left[-i\frac{b}{2}e^u+\frac{d}{2}e^{-u}-\frac{1}{2}u\right] \to 0, \quad \lim_{u\to\infty}\bar{\psi}_2^\infty(u) \sim \exp\left[i\frac{b}{2}e^u+\frac{d}{2}e^{-u}-\frac{1}{2}u\right] \to 0.$$

Thus, ψ_2 vanishes for $u\to\infty$ too, since it can be written as a linear combination of ψ_2^∞ and $\bar{\psi}_2^\infty$. Therefore, we have found a case where the solution $\psi_2(u)$ in series of regular hypergeometric functions is convergent and bounded for any u provided that the energies satisfy the characteristic equation which follows from the recurrence relations.

The asymmetric double-Morse potential. For the asymmetric double-Morse potential (16) the Schrödinger equation becomes

$$\frac{d^2\psi}{du^2}+\left[\mathcal{E}-\frac{C^2}{4}+\frac{B^2}{8}-\frac{B^2}{16}e^{-2u}-\frac{B}{2}\left(\frac{C}{2}-\frac{1}{2}-s\right)e^{-u}+\frac{B}{2}\left(\frac{C}{2}+\frac{1}{2}+s\right)e^{u}-\frac{B^2}{16}e^{2u}\right]\psi=0.$$

The substitutions [16, 28]

$$\psi(u) = e^{-B/(4z)}z^{(C/2)-s}U(z), \quad z=e^u, \quad z\in[0,\infty) \tag{111}$$

transform the previous equation into the DCHE

$$z^2\frac{d^2U}{dz^2}+\left[\frac{B}{2}+(1+C-2s)z\right]\frac{dU}{dz}+\left[\mathcal{E}+\frac{B^2}{8}+s^2-Cs+\frac{B}{2}\left(\frac{C}{2}+\frac{1}{2}+s\right)z-\frac{B^2}{16}z^2\right]U=0,$$

and then we can choose the following set of parameters

$$B_1=\frac{B}{2},\ B_2=1+C-2s,\ B_3=\mathcal{E}+\frac{B^2}{8}+s^2-sC,\ i\omega=-\frac{B}{4},\ i\eta=-\frac{C}{2}-\frac{1}{2}-s. \tag{112}$$

Hence, $\psi(u)$ is constructed by inserting into Eq. (111) the solutions of the DCHE with the parameters specified in Eq. (112) and demanding that these eigenfunctions are bounded for all values of z. Finite-series solutions are obtained from the Barber-Hasseé expansions as in the previous case. However, in order to get infinite-series solutions bounded for any value of u, now we have to use at least two expansions given by two-sided infinite series, in opposition to the previous case.

The first Baber-Hassé solution, given in Eqs. (101a), yields the bounded finite-series solution

$$\psi_1^{\text{barber}}(u) = e^{-\frac{B}{2}\cosh u+\left(\frac{C}{2}-s\right)u}\sum_{n=0}^{2s}\frac{p_n^{(1)}}{n!}\left(-\frac{2}{B}e^u\right)^n, \tag{113a}$$

whose recurrence relations for $p_n^{(1)}$ are ($p_{-1}^{(1)}=0$)

$$p_{n+1}^{(1)} = \left[\mathcal{E}+s^2-sC+n\left(n+C-2s\right)\right]p_n^{(1)}+\frac{B^2}{4}\left(n-2s-1\right)p_{n-1}^{(1)}. \tag{113b}$$

The other Barber-Hassé solution gives an infinite series which is not convergent for $u = \infty$.

On the other hand, in each set of expansions in series of hypergeometric confluent functions, there are two solutions which allow to cover the entire range of u. Thus, from the solutions (95a) we obtain

$$\psi_1(u) = e^{\frac{B}{2}\sinh u + (\frac{C}{2}-s)u} \sum_n (-1)^n b_n^{(1)}\ \tilde{\Phi}\left(1+C, n+\nu+C-2s; -\frac{B}{2}e^u\right),$$

$$\psi_1^\infty(u) = e^{\frac{B}{2}\sinh u + (\frac{C}{2}-s)u} \sum_n (-1)^n b_n^{(1)}\ \Psi\left(1+C, n+\nu+C-2s; -\frac{B}{2}e^u\right),$$

$$\bar{\psi}_1^\infty(u) = e^{-\frac{B}{2}\cosh u + (\frac{C}{2}-s)u} \sum_n c_n^{(1)} \Psi\left(n+\nu-2s, n+\nu+1+C-2s; \frac{B}{2}e^u\right), \tag{114a}$$

where the $c_n^{(1)}$ satisfy the recurrence relations

$$(n+\nu+1)c_{n+1}^{(1)} + \left[(n+\nu)(n+\nu+C-2s) + s^2 - sC + \mathcal{E}\right] c_n^{(1)} -$$

$$\frac{B^2}{4}(n+\nu-1-2s)c_{n-1}^{(2)} = 0, \tag{114b}$$

being $c_n^{(1)} = \Gamma(n+\nu-2s)b_n^{(1)}$. These solutions can be connected through Eq. (32) if ν and $\nu + C$ are not integer; such conditions also assure that the series are two-sided infinite series. In addition, the solution ψ_1 is bounded at $u = -\infty$ ($z = 0$), whereas for $u \to \infty$ ($z = \infty$) we find

$$\lim_{u\to\infty} \psi_1^\infty(u) \sim e^{\frac{B}{2}\sinh u - (\frac{C}{2}+s-1)u} \to \infty, \qquad \lim_{u\to\infty} \bar{\psi}_1^\infty(u) \sim e^{-\frac{B}{2}\cosh u + (\frac{C}{2}+s-\nu)u} \to 0.$$

Therefore, ψ_1 and $\bar{\psi}_1^\infty$ cover the entire range of u. From the solutions (99a), we get

$$\psi_2(u) = e^{\frac{B}{2}\cosh u + (1+s-\frac{C}{2})u} \sum_n (-1)^n b_n^{(2)}\ \tilde{\Phi}\left(2+2s, n+\nu+3+2s-C; -\frac{B}{2}e^u\right),$$

$$\psi_2^\infty(u) = e^{\frac{B}{2}\cosh u + (1+s-\frac{C}{2})u} \sum_n (-1)^n b_n^{(2)}\ \Psi\left(2+2s, n+\nu+3+2s-C; -\frac{B}{2}e^u\right),$$

$$\bar{\psi}_2^\infty(u) = e^{-\frac{B}{2}\sinh u + (1+s-\frac{C}{2})u} \sum_n c_n^{(2)} \Psi\left(n+\nu+1-C, n+\nu+3+2s-C; \frac{B}{2}e^u\right), \tag{115a}$$

where the $c_n^{(2)}$ satisfy the recurrence relations

$$(n+\nu+1)c_{n+1}^{(2)} + \left[(n+\nu)(n+\nu+2+2s-C) + \mathcal{E} + \frac{B^2}{4} + s^2 - sC - C + 2s + 1\right] c_n^{(2)} -$$

$$\frac{B^2}{4}(n+\nu-C)c_{n-1}^{(2)} = 0, \tag{115b}$$

being $b_n^{(2)}$ given by $c_n^{(2)} = \Gamma(n+\nu+1-C)b_n^{(2)}$. This time we find

$$\lim_{u\to\infty} \psi_2^\infty(u) \sim e^{\frac{B}{2}\cosh u - (\frac{C}{2}+s+1)u} \to \infty, \qquad \lim_{u\to\infty} \bar{\psi}_2^\infty(u) \sim e^{-\frac{B}{2}\sinh u + (\frac{C}{2}+s-\nu)u} \to 0.$$

If ν is chosen such that ν and $\nu - C$ are not integer, the solutions of this second set are given by infinite series and can be connected by using Eq. (32).

The solutions obtained by choosing $i\omega = B/4$ and $i\eta = (C+1+2s)/2$ must be discarded. Actually, we would find that the first parameter in the regular hypergeometric function given in Eq. (95a) is $a = -2s$ and, therefore, that solution is not valid. Similarly, the first parameter in the regular hypergeometric function given in Eq. (99a) is $a = 1 - C$ and, so, the solution would not be valid if C is a non-negative integer.

IV.C. Solutions for the Whittaker-Ince limit of the DCHE

For the Whittaker-Ince limit (7) of DCHE, namely,

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+(B_3+qz)\,U=0,\qquad (q\neq 0,\ B_1\neq 0),$$

there is no finite-series solutions. For this reason, the equation is irrelevant for quasi-exactly solvable problems. However, its solutions may be important for studying the scattering of ions which induce dipole and quadrupole moments in polarizable targets [16]. Solutions for the above equation can be found by applying either the Whittaker-Ince limit to the solutions given in Sec. IV.A or the Leaver limit to the solutions (69a-b) and (D6-7). We use the latter approach because it is the easiest.

In effect the Leaver limit $z_0\to 0$ does not modify the solutions (69a), which again read

$$U_1(z)=\sum_n(-1)^n c_n^{(1)}\big(\sqrt{qz}\big)^{-(n+\nu+B_2-1)}J_{n+\nu+B_2-1}\big(2\sqrt{qz}\big),\qquad \forall z,$$

$$U_1^{(i)}(z)=\sum_n(-1)^n c_n^{(1)}\big(\sqrt{qz}\big)^{-(n+\nu+B_2-1)}H^{(i)}_{n+\nu+B_2-1}\big(2\sqrt{qz}\big),\quad |z|>0, \tag{116a}$$

but changes their recurrence relations to

$$(n+\nu+1)c_{n+1}^{(1)}+\big[(n+\nu)(n+\nu+B_2-1)+B_3\big]c_n^{(1)}+qB_1c_{n-1}^{(1)}=0. \tag{116b}$$

These three solutions are really doubly infinite if ν is not integer. Furthermore, each solution can be expressed as a linear combination of the others by means of the relation (70).

On the other side, the only detail to get the Leaver limit of the solutions given in Eqs. (D6) is the use the limit written in Eq. (98). The results are

$$U_2(z)=e^{B_1/z}\ z^{2-B_2}\sum_n(-1)^n c_n^{(2)}\ \big(\sqrt{qz}\big)^{-n-\nu-3+B_2}\ J_{n+\nu+3-B_2}\big(2\sqrt{qz}\big),\qquad \forall z;$$

$$U_2^{(i)}(z)=e^{B_1/z}\ z^{2-B_2}\sum_n(-1)^n c_n^{(2)}\ \big(\sqrt{qz}\big)^{-n-\nu-3+B_2}\ H^{(i)}_{n+\nu+3-B_2}\big(2\sqrt{qz}\big),\qquad |z|>0; \tag{117a}$$

$$(n+\nu+1)c_{n+1}^{(2)}+[(n+\nu)(n+\nu+3-B_2)+B_3+2-B_2]\,c_n^{(2)}-qB_1c_{n-1}^{(2)}=0. \tag{117b}$$

V. CONCLUSIONS

We have started with a set of solutions for the confluent Heun equation (CHE) given by series of confluent hypergeometric functions and, by means of Leaver and the Whittaker-Ince limits, we have derived sets of solutions for all the equations included in the diagram described in the first section. The Barber-Hassé expansions in power series also admit the both limits and have been used to provide solutions for the cases in which the solutions in hypergeometric functions are not valid, as well as to get finite-series solutions for the confluent and double-confluent Heun equations.

In the fundamental set of two-sided solutions $(U_1, U_1^{\infty}, \bar{U}_1^{\infty})$, the first and the second solutions are, respectively, Leaver's expansions in series of regular and irregular confluent hypergeometric functions for the CHE [27], while $\bar{U}_1^{\infty}$ is the two-sided version of the expansion in series of irregular functions given in Ref. [17]. We have seen that, although the solution U_1 is not valid if there is a certain constraint between two parameters of the equation, we can use the transformation rules to find a two-sided solution valid for that case. We have also established the conditions to express each of the three solutions as a linear combination of the others.

The fact that the expansion U_1 in series of regular functions converges for any z distinguishes the present solutions from the Leaver expansions in series of Coulomb wave functions [27], since the latter converge only for $|z| > |z_0|$. On the other hand, we have seen that

$$\lim_{z\to\infty} U_1^{\infty}(z) \sim e^{-i\omega z} z^{i\eta-\frac{B_2}{2}}, \qquad \lim_{z\to\infty} \bar{U}_1^{\infty}(z) \sim e^{i\omega z} z^{-i\eta-\nu-\frac{B_2}{2}},$$

where ν is the characteristic parameter of the two-sided series. The behavior of U_1 when $z \to \infty$ is given by a combination of these. This explains why U_1 is sometimes inappropriate when the variable z tends to infinity.

All the solutions for the double-confluent Heun equation (DCHE), obtained from solutions of the CHE through Leaver limit ($z_0 \to 0$), are also given by expansions in series of confluent hypergeometric functions. Then, the analysis used for the solutions of the CHE has been promptly adapted to the solutions of the DCHE, including the conditions which allow to write one solution of a fixed set as a linear combination of the others.

We have found one-sided infinite-series solutions for the CHE and DCHE, by truncating the two-side infinite series on the left. These one-sided solutions can afford finite-series solutions, but the latter are also suplied by the Barber-Hassé expansions in power series. Then, the one-sided infinite series become useful only when we need expansions in infinite series, as illustrated in Sec. II.D.

The Whittakker-Ince limit (5) of the CHE and DCHE have generated expansions in series of Bessel functions of the first kind and in series of the first and second Hankel functions. In this case, each solution belonging to fixed set can be written as a linear combination of the others without the need of imposing restrictions on the parameters of the differential equations.

We have applied solutions of the Heun equations to the one-dimensional Schrödinger equation with quasi-exactly solvable potentials. In addition to the expected finite-series solutions, we have sought bounded infinite-series solutions. For the volcano-type potential (17) given by Cho and Ho [8], in Sec. III.C we have found infinite-series wavefunctions which are bounded for any value of the independent variable, but we have not solved the eigenvalue equations. We have also proved that there is a pair of degenerate finite-series eigenstates for any value of the parameter ℓ responsible for the quasi-exact solvability.

For the potential (18) associated to quasinormal modes, in Sec. IV.B we have found unbounded infinite-series solutions and bounded finite-series eigenstates for $d > 0$, as in Cho and Ho [9]. However, for $d < 0$ we have found an opposite result: bounded infinite-series solutions and unbounded finite-series eigenstates.

For the double-Morse potentials (16) we have found that no single infinite-series wavefunction is bounded for the entire range of the independent variable. Despite this, in Secs. II.D and IV.B we have seen that it is possible to get a pair of infinite-series solutions, both solutions possessing the same eigenvalue equation, which covers all the values of the independent variable. A similar result has already been reported in the literature [15, 17]. However, now we have found that, in the intermediate region, there is a third solution which can be expressed as a superposition of the other solutions. This fact may be relevant to match solutions bounded over different regions. On the other side, it is necessary to investigate the solutions of the characteristic equations in order to know if in fact we can determine the remaining part of the energy spectrum by using such infinite-series solutions.

Finally, we mention that the problem of matching different solutions of the CHE appears also in relativistic astrophysics, more specifically, in the study of the Teukolsky equations in Kerr spacetimes [30, 31]. Thus, the solutions of the present paper may be important in this context. Inversely, the procedures used in astrophysics could be useful for the problems considered here.

APPENDIX A: HEUN EQUATIONS AND QUASI-EXACT SOLVABILITY

We consider the Heun equations in their normal or Schrödinger form,

$$\left[\frac{d^2}{dz^2} + Q(z)\right] y(z) = 0, \tag{A1}$$

that is, in the form where there is no first-order derivative term [11]. The function $Q(z)$ for the general Heun equation and its confluent cases is given in the following [36], where it is understood that, in general, there exists some constraint among the parametres A, B, and so on. In each case we indicate the singular points of the equation.

General Heun equation. Four regular singular points at $z = 0, 1, a, \infty$:

$$Q(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-a} + \frac{D}{z^2} + \frac{E}{(z-1)^2} + \frac{F}{(z-a)^2}, \qquad (a \neq 0 \text{ or } 1). \tag{A2}$$

Confluent Heun equation (or generalized spheroidal wave equation). Two regular points at $z = 0, 1$ and one irregular at $z = \infty$:

$$Q(z) = A + \frac{B}{z} + \frac{C}{z-1} + \frac{D}{z^2} + \frac{E}{(z-1)^2}. \tag{A3}$$

Double-confluent Heun equation. Two irregular points at $z = 0, \infty$:

$$Q(z) = A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \frac{E}{z^4}. \tag{A4}$$

Biconfluent Heun equation. One regular point at $z = 0$ and one irregular point at $z = \infty$:

$$Q(z) = Az^2 + Bz + C + \frac{D}{z} + \frac{E}{z^2}. \tag{A5}$$

Triconfluent Heun equation. One irregular point at $z = \infty$:

$$Q(z) = Az^4 + Bz^3 + Cz^2 + Dz + E. \tag{A6}$$

Other normal forms are given in the tables constructed by Lemieux and Bose [28] who, however, have not considered the triconfluent equation. These tables are helpful to recognize whether a given equation is of the Heun type.

By using the Lemieux-Bose tables in addition to the normal forms written above, it is straightforward to establish relations among the Heun and the Schrödinger equations for some quasi-exactly solvable potentials. We find: (i) a triconfluent Heun equation for the quartic potential given in Eq. (5.34) of González-López, Kamran and Olver [18]; (ii) biconfluent Heun equations for the sextic potential $V_1z^6 + V_2z^4 + V_3z^2 + V_4 + V_5/z^2$ given by Turbiner [40] and Ushveridze [43], and for the potentials II, III and VIII given in Turbiner's list [40]; (iii) double-confluent Heun equations for the inverse fourth-power potential $V(r) = V_1r^{-4} + V_2r^{-3} + V_3r^{-2} + V_4r^{-1}$ [39, 40], and for the asymmetric double-Morse potential given by Zaslavskii and Ulyanov [46]; (iv) confluent Heun equations for the trigonometric and hyperbolic potentials given by Ushveridze [42]; (v) general Heun equations in the Darboux elliptic form [10] for the first and second Ushveridze's elliptic potentials [42].

APPENDIX B: PROPERTIES OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

The regular and irregular confluent hypergeometric functions, denoted by $\Phi(a, c; y)$ and $\Psi(a, c; y)$, satisfy the Kummer transformations

$$\Phi(\mathrm{a}, \mathrm{c}; y) = e^y\Phi(c - a, c; -y), \qquad \Psi(a, c; y) = y^{1-c}\Psi(1 + a - c, 2 - c; y). \tag{B1}$$

The behavior of of $\Psi(a, c; y)$ and $\Phi(a, c; y)$ when $y \to \infty$ is given by

$$\lim_{y\to\infty} \Psi(a, c; y) \sim y^{-a}[1 + O(|y|^{-1})], \qquad \left(-\frac{3\pi}{2} < \arg y < \frac{3\pi}{2}\right) \tag{B2}$$

$$\lim_{y\to\infty} \Phi(a, c; y) = \begin{cases} \dfrac{\Gamma(c)}{\Gamma(a)} e^y y^{a-c}[1 + O(|y|^{-1})], & (\Re y > 0) \\[2ex] \dfrac{\Gamma(c)}{\Gamma(c-a)}(-y)^{-a}[1 + O(|y|^{-1})], & (\Re y < 0). \end{cases} \tag{B3}$$

These can be used to get the limits of the series expansions as $y \to \infty$. For $\Re y = 0$, the limit of $\Phi(a, c; y)$ is a combination of the limits on the right-hand side of the previous expression [13].

To find the expansions in series of irregular confluent hypergeometric functions for the CHE, in addition to Eq. (28) we need the relations

$$y\frac{d\Psi(a,c;y)}{dy} = (1-c)\Psi(a,c;y) + (c-a-1)\Psi(a,c-1;y), \tag{B4}$$

$$\frac{d\Psi(a,c;y)}{dy} = \Psi(a,c;y) - \Psi(a,c+1;y), \tag{B5}$$

$$[c-a-1]\Psi(a,c-1;y) + [1-c-y]\Psi(a,c;y) + y\Psi(a,c+1;y) = 0, \tag{B6}$$

where the last relation results from the first and second ones. These relations also hold for the functions $\tilde{\Phi}(a, b; y)$ defined in Eq. (42). On the other hand, if $c \to \infty$, while a and y remain bounded, we have [13]

$$\Psi(a,c;y) = (-c)^{-a}\left[1 + O(|c|^{-1}\right] + \frac{\sqrt{2\pi}}{\Gamma(a)}\, c^{c-(3/2)} y^{1-c} e^{y-c}\left[1 + O(|c|^{-1}\right] \tag{B7}$$

which, in conjunction with Eq. (B6), is useful in the study of the convergence of the series solutions for the CHE.

The Whittaker-Ince limit of the expansions in series of confluent hypergeometric functions for the CHE and DCHE yields expansions in series of Bessel functions for Eqs. (6) and (7) by means of the limits [13]

$$\lim_{a\to\infty} \Phi\left(a, c; -\frac{y}{a}\right) = \Gamma(c)\; y^{(1-c)/2} J_{c-1}\big(2\sqrt{y}\big), \tag{B8}$$

$$\lim_{a\to\infty} \left[\Gamma(a+1-c)\;\Psi\left(a, c; \frac{y}{a}\right)\right] = 2y^{(1-c)/2} K_{c-1}\big(2\sqrt{y}\big), \tag{B9}$$

$$\lim_{a\to\infty} \left[\Gamma(a+1-c)\;\Psi\left(a, c; -\frac{y}{a}\right)\right] = \begin{cases} -i\pi e^{i\pi c} y^{(1-c)/2} H^{(1)}_{c-1}\big(2\sqrt{y}\big), & \text{Im } y > 0, \\ i\pi e^{-i\pi c} y^{(1-c)/2} H^{(2)}_{c-1}\big(2\sqrt{y}\big), & \text{Im } y < 0, \end{cases} \tag{B10}$$

where J_λ denotes the Bessel functions of the first kind of order λ, K_λ are modified Bessel functions of the third kind, whereas $H^{(1)}_\lambda$ and $H^{(2)}_\lambda$ denote the first and second Hankel functions [14], respectively.

Some relations useful for Sec. III.B are

$$H^{(1)}_{-\lambda}(x) = e^{i\pi\lambda} H^{(1)}_\lambda(x), \quad H^{(2)}_{-\lambda}(x) = e^{-i\pi\lambda} H^{(2)}_\lambda(x),$$

$$K_\lambda(-ix) = \frac{1}{2}\pi i e^{\frac{1}{2}\pi i\lambda} H^{(1)}_\lambda(x), \quad K_\lambda(ix) = -\frac{1}{2}\pi i e^{-\frac{1}{2}\pi i\lambda} H^{(2)}_\lambda(x). \tag{B11}$$

Bessel functions whose order is half of an odd integer reduce to combinations of elementary functions. The following formulas [14] are used in Sec. III.C ($m = 0, 1, 2, \cdots$):

$$J_{m+\frac{1}{2}}(x) = (-1)^m \sqrt{\frac{2}{\pi x}}\; x^{m+1} \left(\frac{d}{xdx}\right)^m \frac{\sin x}{x}, \quad J_{-m-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\; x^{m+1} \left(\frac{d}{xdx}\right)^m \frac{\cos x}{x}; \tag{B12}$$

$$H^{(1)}_{m+\frac{1}{2}}(x) = -i(-1)^m \sqrt{\frac{2}{\pi x}}\; x^{m+1} \left(\frac{d}{xdx}\right)^m \frac{e^{ix}}{x}, \qquad H^{(1)}_{-m-\frac{1}{2}}(x) = i(-1)^m H^{(1)}_{m+\frac{1}{2}}(x); \tag{B13}$$

$$H^{(2)}_{m+\frac{1}{2}}(x) = i(-1)^m\sqrt{\frac{2}{\pi x}}\, x^{m+1}\left(\frac{d}{xdx}\right)^m \frac{e^{-ix}}{x}, \qquad H^{(2)}_{-m-\frac{1}{2}}(x) = -i(-1)^m H^{(2)}_{m+\frac{1}{2}}(x). \tag{B14}$$

Some special cases are

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x, \quad H^{(1)}_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, e^{ix}, \quad H^{(2)}_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-ix}, \tag{B15}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x, \quad H^{(1)}_{\frac{1}{2}}(x) = -i\sqrt{\frac{2}{\pi x}}\, e^{ix}, \quad H^{(2)}_{\frac{1}{2}}(x) = i\sqrt{\frac{2}{\pi x}}\, e^{-ix}. \tag{B16}$$

APPENDIX C: OTHER SOLUTIONS FOR THE CONFLUENT HEUN EQUATION

In order to generate other solutions for the CHE we use the transformations rules resulting from substitutions of variables which leave the form of the CHE unaltered but change its parameters. Thus, if $U(z) = U(B_1, B_2, B_3; z_0, \omega, \eta; z)$ denotes one solution of the CHE in the Leaver form (3), we have the rules T_1, T_2, T_3 and T_4 which operate as [17]

$$T_1U(z) = z^{1+B_1/z_0}U(C_1, C_2, C_3; z_0, \omega, \eta; z), \qquad z_0 \neq 0, \tag{C1}$$

$$T_2U(z) = (z - z_0)^{1-B_2-B_1/z_0}U(B_1, D_2, D_3; z_0, \omega, \eta; z), \qquad z_0 \neq 0, \tag{C2}$$

$$T_3U(z) = U(B_1, B_2, B_3; z_0, -\omega, -\eta; z), \qquad \forall z_0, \tag{C3}$$

$$T_4U(z) = U(-B_1 - B_2z_0, B_2, B_3 + 2\eta\omega z_0; z_0, -\omega, \eta; z_0 - z), \qquad \forall z_0, \tag{C4}$$

where

$$C_1 = -B_1 - 2z_0, \;\; C_2 = 2 + B_2 + \frac{2B_1}{z_0}, \; C_3 = B_3 + \left(1 + \frac{B_1}{z_0}\right)\left(B_2 + \frac{B_1}{z_0}\right),$$

$$D_2 = 2 - B_2 - \frac{2B_1}{z_0}, \; D_3 = B_3 + \frac{B_1}{z_0}\left(\frac{B_1}{z_0} + B_2 - 1\right). \tag{C5}$$

Applying these rules to the basic set (43a) we may generate a group containing 16 sets of solutions for the CHE, while for the two-sided expansions in series Coulomb wave functions [27] and in series of hypergeometric functions [15] we can generate only 8 solutions. We will write only the subgroup obtained by using the rules T_1 and T_2 in this order:

$$\left(U_1, U_1^{\infty}, \bar{U}_1^{\infty}\right) \xleftrightarrow{T_1} \left(U_2, U_2^{\infty}, \bar{U}_2^{\infty}\right) \xleftrightarrow{T_2} \left(U_3, U_3^{\infty}, \bar{U}_3^{\infty}\right) \xleftrightarrow{T_1} \left(U_4, U_4^{\infty}, \bar{U}_4^{\infty}\right) \tag{C6}$$

where $\left(U_1, U_1^{\infty}, \bar{U}_1^{\infty}\right)$ denotes the first set of solutions (43a).

In each of the following sets, the conditions for doubly infinite series are obtained by choosing ν such that the coefficients of $c_n^{(i)}$ and $c_{n-1}^{(i)}$ do not vanish in the recurrence relations. Besides this, one solution can be expressed as a linear combination of the others, if a, c and $c - a$ are not negative integers, where a and c are respectively the first and the second parameters of the confluent hypergeometric functions: these conditions lead to some restrictions on the values of ν as well as on the parameters of the CHE. Despite the notation, it is understood that the parameter ν may be different in each set of solutions.

The second set of solutions admits the Whittaker-Ince limit but does not admit the Leaver limit. It reads

$$U_2(z) = e^{-i\omega z}z^{1+\frac{B_1}{z_0}}\sum_n (-1)^n b_n^{(2)}\widetilde{\Phi}\left(1 - i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, n + \nu + 2 + B_2 + \frac{2B_1}{z_0}; 2i\omega z\right),$$

$$U_2^{\infty}(z) = e^{-i\omega z} z^{1+\frac{B_1}{z_0}} \sum_n (-1)^n b_n^{(2)} \Psi\left(1 - i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, n+\nu+2+B_2+\frac{2B_1}{z_0}; 2i\omega z\right),$$

$$\bar{U}_2^{\infty}(z) = e^{i\omega z} z^{1+\frac{B_1}{z_0}} \sum_n c_n^{(2)} \Psi\left(n+\nu+1+i\eta + \frac{B_1}{z_0} + \frac{B_2}{2}, n+\nu+2+B_2+\frac{2B_1}{z_0}; -2i\omega z\right), \quad \text{(C7)}$$

where the recurrence relations for $c_n^{(2)}$ are

$$(n+\nu+1)c_{n+1}^{(2)} + \left[(n+\nu)\left(n+\nu+1+2i\omega z_0+B_2+\frac{2B_1}{z_0}\right) + B_3 + \left(1+\frac{B_1}{z_0}\right)\left(B_2+\frac{B_1}{z_0}\right)\right.$$

$$\left. + i\omega z_0\left(B_2+\frac{B_1}{z_0}\right)\right]c_n^{(2)} + 2i\omega z_0\left(n+\nu+B_2+\frac{B_1}{z_0}-1\right)\left(n+\nu+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)c_{n-1}^{(2)} = 0. \quad \text{(C8)}$$

The recurrence relations for $b_n^{(2)}$ are obtained from the previous ones by

$$c_n^{(2)} = \Gamma\left(n+\nu+1+i\eta+\frac{B_1}{z_0}+\frac{B_2}{2}\right)b_n^{(2)}. \quad \text{(C9)}$$

The third set of solutions, which admits both the Leaver and Ince-Whittaker limits, is

$$U_3(z) = e^{-i\omega z}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\; z^{1+\frac{B_1}{z_0}} \sum_n (-1)^n b_n^{(3)} \tilde{\Phi}\left(2 - i\eta - \frac{B_2}{2}, n+\nu+4-B_2; 2i\omega z\right),$$

$$U_3^{\infty}(z) = e^{-i\omega z}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\; z^{1+\frac{B_1}{z_0}} \sum_n (-1)^n b_n^{(3)} \Psi\left(2 - i\eta - \frac{B_2}{2}, n+\nu+4-B_2; 2i\omega z\right),$$

$$\bar{U}_3^{\infty}(z) = e^{i\omega z}(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\; z^{1+\frac{B_1}{z_0}} \sum_n c_n^{(3)} \times$$

$$\Psi\left(n+\nu+2+i\eta-\frac{B_2}{2}, n+\nu+4-B_2; -2i\omega z\right), \quad \text{(C10)}$$

where the $c_n^{(3)}$ satisfy

$$(n+\nu+1)c_{n+1}^{(3)} + \left[(n+\nu)(n+\nu+3-B_2+2i\omega z_0) + i\omega z_0\left(2-B_2-\frac{B_1}{z_0}\right) + \right.$$

$$\left. B_3 + 2 - B_2\right]c_n^{(3)} + 2i\omega z_0\left(n+\nu+1-B_2-\frac{B_1}{z_0}\right)\left(n+\nu+1+i\eta-\frac{B_2}{2}\right)c_{n-1}^{(3)} = 0 \quad \text{(C11)}$$

and the recurrence relations for $b_n^{(3)}$ follow from

$$c_n^{(3)} = \Gamma\left(n+\nu+i\eta+2-\frac{B_2}{2}\right)b_n^{(3)}. \quad \text{(C12)}$$

The fourth set of solutions is

$$U_4(z) = f(z)\; e^{-i\omega z} \sum_n (-1)^n b_n^{(4)} \tilde{\Phi}\left(1 - i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, n+\nu+2-B_2-\frac{2B_1}{z_0}; 2i\omega z\right),$$

$$U_4^{\infty}(z) = f(z)\ e^{-i\omega z}\sum_n (-1)^n b_n^{(4)}\Psi\left(1 - i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, n+\nu+2-B_2-\frac{2B_1}{z_0}; 2i\omega z\right),$$

$$\bar{U}_4^{\infty}(z) = f(z)\ e^{i\omega z}\sum_n c_n^{(4)}\Psi\left(n+\nu+1+i\eta - \frac{B_1}{z_0} - \frac{B_2}{2}, n+\nu+2-B_2-\frac{2B_1}{z_0}; -2i\omega z\right), \quad \text{(C13)}$$

where

$$f(z) = (z-z_0)^{1-B_2-\frac{B_1}{z_0}};$$

and the recurrence relations for $c_n^{(4)}$ are

$$(n+\nu+1)c_{n+1}^{(4)} + \Big[(n+\nu)\left(n+\nu+1-B_2-\tfrac{2B_1}{z_0}+2i\omega z_0\right) + B_3 + \tfrac{B_1}{z_0}\left(B_2+\tfrac{B_1}{z_0}-1\right) +$$

$$i\omega z_0\left(2-B_2-\tfrac{B_1}{z_0}\right)\Big]c_n^{(4)} + 2i\omega z_0\left(n+\nu+1-B_2-\tfrac{B_1}{z_0}\right)\left(n+\nu+i\eta-\tfrac{B_2}{2}-\tfrac{B_1}{z_0}\right)c_{n-1}^{(4)} = 0. \quad \text{(C14)}$$

The $b_n^{(4)}$ and $c_n^{(4)}$ are connected by

$$c_n^{(4)} = \Gamma\left(n+\nu+i\eta+1-\tfrac{B_2}{2}-\tfrac{B_1}{z_0}\right)b_n^{(4)}. \quad \text{(C15)}$$

APPENDIX D: OTHER SOLUTIONS FOR THE WHITTAKER-INCE LIMIT OF THE CHE

If $U(z) = U(B_1, B_2, B_3; z_0, q; z)$ denotes one solution (or set of solutions) for Whittaker-Ince limit (6) of the CHE, then the rules $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ given by

$$\mathscr{T}_1 U(z) = z^{1+B_1/z_0}U(C_1, C_2, C_3; z_0, q; z), \quad z_0 \neq 0, \quad \text{(D1)}$$

$$\mathscr{T}_2 U(z) = (z-z_0)^{1-B_2-B_1/z_0}U(B_1, D_2, D_3; z_0, q; z), \quad z_0 \neq 0, \quad \text{(D2)}$$

$$\mathscr{T}_3 U(z) = U(-B_1 - B_2 z_0, B_2, B_3 - qz_0; z_0, -q; z_0 - z), \quad \text{(D3)}$$

can generate a group having 8 solutions. The constants C_i and D_i are defined in Eqs. (C5). Next we use only $\mathscr{T}_1$ and $\mathscr{T}_2$, following the sequence

$$\left(U_1, U_1^{(i)}\right) \overset{\mathscr{T}_1}{\longleftrightarrow} \left(U_2, U_2^{(i)}\right) \overset{\mathscr{T}_2}{\longleftrightarrow} \left(U_3, U_3^{(i)}\right) \overset{\mathscr{T}_1}{\longleftrightarrow} \left(U_4, U_4^{(i)}\right) \overset{\mathscr{T}_2}{\longleftrightarrow} \left(U_1, U_1^{(i)}\right),$$

where $(U_1, U_1^{(i)})$ denotes the set of solutions written in Eqs. (69a) and (69b). The conditions for having doubly infinite series are obtained by choosing ν such that the coefficients of $c_n^{(i)}$ and $c_{n-1}^{(i)}$ do not vanish in the recurrence relations, as in the case of the CHE. Besides this, according to Eq. (70) one solution can be expressed as a linear combination of the other solutions. The solutions for the Mathieu equation are derived by using Eq. (72).

The second set of solutions resulting from previous procedure is

$$U_2(z) = z^{1+\frac{B_1}{z_0}}\sum_n (-1)^n c_n^{(2)}\left(\sqrt{qz}\right)^{-n-\nu-1-B_2-\frac{2B_1}{z_0}} J_{n+\nu+1+B_2+\frac{2B_1}{z_0}}\left(2\sqrt{qz}\right), \quad \forall z,$$

$$U_2^{(i)}(z) = z^{1+\frac{B_1}{z_0}}\sum_n (-1)^n c_n^{(2)}\left(\sqrt{qz}\right)^{-n-\nu-1-B_2-\frac{2B_1}{z_0}} H^{(i)}_{n+\nu+1+B_2+\frac{2B_1}{z_0}}\left(2\sqrt{qz}\right), \quad |z| > |z_0|; \quad \text{(D4)}$$

$$(n+\nu+1)c_{n+1}^{(2)}+\left[(n+\nu)\left(n+\nu+B_2+1+\frac{2B_1}{z_0}\right)+B_3+\left(1+\frac{B_1}{z_0}\right)\left(B_2+\frac{B_1}{z_0}\right)\right]c_n^{(2)}+$$

$$qz_0\left(n+\nu+B_2+\frac{B_1}{z_0}-1\right)c_{n-1}^{(2)}=0. \tag{D5}$$

For the Mathieu equation the corresponding solutions are even, as those of the first set, since

$$w_2(u)=\cos(\sigma u)\sum_n(-1)^n c_n^{(2)}\left[k\cos(\sigma u)\right]^{-n-\nu-1}J_{n+\nu+1}\big(2k\cos(\sigma u)\big),\quad \forall u,$$

$$w_2^{(i)}(u)=\cos(\sigma u)\sum_n(-1)^n c_n^{(2)}\left[k\cos(\sigma u)\right]^{-n-\nu-1}H_{n+\nu+1}^{(i)}\big(2k\cos(\sigma u)\big),\quad |\cos(\sigma u)|>1;$$

$$(n+\nu+1)c_{n+1}^{(2)}+\left[(n+\nu)(n+\nu+1)+\frac{k^2}{2}+\frac{1-a}{4}\right]c_n^{(2)}+k^2\left[n+\nu-\frac{1}{2}\right]c_{n-1}^{(2)}=0.$$

Third set, which admits the Leaver limit $z_0\to 0$ as the first set, reads

$$U_3(z)=z^{1+\frac{B_1}{z_0}}\,(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_n(-1)^n c_n^{(3)}\,\left(\sqrt{qz}\right)^{-n-\nu-3+B_2}\,J_{n+\nu+3-B_2}\big(2\sqrt{qz}\big),\quad \forall z,$$

$$U_3^{(i)}(z)=z^{1+\frac{B_1}{z_0}}\,(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_n(-1)^n c_n^{(3)}\,\left(\sqrt{qz}\right)^{-n-\nu-3+B_2}\,H_{n+\nu+3-B_2}^{(i)}\big(2\sqrt{qz}\big),$$
$$|z|>|z_0|; \tag{D6}$$

$$(n+\nu+1)c_{n+1}^{(3)}+\left[(n+\nu)(n+\nu+3-B_2)+B_3+2-B_2\right]c_n^{(3)}+$$

$$qz_0\left(n+\nu+1-B_2-\frac{B_1}{z_0}\right)c_{n-1}^{(3)}=0. \tag{D7}$$

For the Mathieu equation, these give odd solutions:

$$w_3(u)=\sin(2\sigma u)\sum_n(-1)^n c_n^{(3)}\left[k\cos(\sigma u)\right]^{-n-\nu-2}J_{n+\nu+2}\big(2k\cos(\sigma u)\big),\quad \forall u,$$

$$w_3^{(i)}(u)=\sin(2\sigma u)\sum_n(-1)^n c_n^{(3)}\left[k\cos(\sigma u)\right]^{-n-\nu-2}H_{n+\nu+2}^{(i)}\big(2k\cos(\sigma u)\big),\quad |\cos(\sigma u)|>1;$$

$$(n+\nu+1)c_{n+1}^{(3)}+\left[(n+\nu)(n+\nu+2)+\frac{k^2}{2}-\frac{a}{4}+1\right]c_n^{(3)}+k^2\left[n+\nu+\frac{1}{2}\right]c_{n-1}^{(3)}=0.$$

The fourth set of solutions is

$$U_4(z)=(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_n(-1)^n c_n^{(4)}\,\left(\sqrt{qz}\right)^{-n-\nu-1+B_2+\frac{2B_1}{z_0}}\,J_{n+\nu+1-B_2-\frac{2B_1}{z_0}}\big(2\sqrt{qz}\big),\quad \forall z,$$

$$U_4^{(i)}(z)=(z-z_0)^{1-B_2-\frac{B_1}{z_0}}\sum_n(-1)^n c_n^{(4)}\,\left(\sqrt{qz}\right)^{-n-\nu-1+B_2+\frac{2B_1}{z_0}}\,H^{(i)}_{n+\nu+1-B_2-\frac{2B_1}{z_0}}\big(2\sqrt{qz}\big),$$
$$|z|>|z_0|; \tag{D8}$$

$$(n+\nu+1)c_{n+1}^{(4)}+\left[(n+\nu)\left(n+\nu+1-B_2-\frac{2B_1}{z_0}\right)+B_3+\frac{B_1}{z_0}\left(B_2+\frac{B_1}{z_0}-1\right)\right]c_n^{(4)}+$$

$$qz_0\left(n+\nu+1-B_2-\frac{B_1}{z_0}\right)c_{n-1}^{(4)}=0. \tag{D9}$$

For the Mathieu equation, these also give odd solutions:

$$w_4(u)=\sin(\sigma u)\sum_n(-1)^n c_n^{(4)}\left[k\cos(\sigma u)\right]^{-n-\nu-1}J_{n+\nu+1}\big(2k\cos(\sigma u)\big),\quad \forall u,$$

$$w_4^{(i)}(u)=\sin(\sigma u)\sum_n(-1)^n c_n^{(4)}\left[k\cos(\sigma u)\right]^{-n-\nu-1}H_{n+\nu+1}^{(i)}\big(2k\cos(\sigma u)\big),\quad |\cos(\sigma u)|>1;$$

$$(n+\nu+1)c_{n+1}^{(4)}+\left[(n+\nu)(n+\nu+1)+\frac{k^2}{2}-\frac{a-1}{4}\right]c_n^{(4)}+k^2\left[n+\nu+\frac{1}{2}\right]c_{n-1}^{(4)}=0.$$

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