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The $N = 2$ Supersymmetric Heavenly Equation and Its Super-Hydrodynamical Reduction

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Abstract

Manifest $N = 2$ supersymmetric Toda systems are constructed from the $sl(n, n+1)$ superalgebras by taking into account their complex structure. In the $n \to \infty$ continuum limit an $N = 2$ extension of the $(2 + 1)$-dimensional heavenly equation is obtained. The integrability is guaranteed by the existence of a supersymmetric Lax pair. We further analyze the properties of the $(1 + 1)$-dimensionally reduced system. Its bosonic sector is of hydrodynamical type. This is not the case for the whole supersymmetric system which, however, is super-hydrodynamical when properly expressed in terms of a supergeometry involving superfields and fermionic derivatives.

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1 Introduction

The so-called "Heavenly Equation" in (1+2) dimensions was at first introduced to describe solutions of the complexified Einstein equations [1]. It has received a lot of attention [2] for its remarkable integrability properties. Even if, to our knowledge, no attempt so far in the literature has been made to analyze the corresponding situation in the case of supergravity, the supersymmetric heavenly equation has been introduced and discussed in the context of superintegrable models. It appears [3] as the continuum limit of a system of equations known as "supersymmetric Toda lattice hierarchy", introduced in [4]. Such system of equations and their hidden $N = 2$ supersymmetric structures have been vastly investigated [5]. On the other hand, it has been recently pointed out that the dimensional reduction of the $(1 + 2)$-dimensional heavenly equation to a $(1 + 1)$-dimensional system is related to Benney-like integrable hierarchies of hydrodynamical equations [6]. Another very recent reference discussing the hydrodynamical properties of the reduced heavenly equation is [7].

In this paper we introduce the $N = 2$ supersymmetric heavenly equation, clarifying its integrability properties and its algebraic arising from the $n \to \infty$ limit of a class of $N = 2$ supersymmetric Toda equations. Further, we investigate its $(1+1)$ dimensional reduction which provides the supersymmetric hydrodynamical equations extending the Benney-like hierarchy of reference [6]. The integrability properties of such systems are induced by the integrability property of the $N = 2$ superheavenly equation and its Lax representation discussed below. This Lax representation is not in the form of supersymmetric dispersionless Lax operator, as we discuss at length later.

The plan of the paper is as follows, in the next section we introduce the discretized $N = 2$ Toda lattice hierarchy (as well as its continuum limit, the $N = 2$ Superheavenly Equation), as a supersymmetric Toda system based on the $sl(n|n + 1)$ superalgebra. It is worth to remark that these are the same superalgebras originally employed in the construction of [3]. On the other hand, the superalgebras of the $sl(n|n + 1)$ series admit a complex structure, allowing the construction of superToda systems based on $N = 2$ superfields, according to the scheme of [8]. For this reason, the next section results provide a generalization of those of [3]. The supersymmetric Lax pairs providing integrability of the systems (for any given n and in the limit $n \to \infty$) are explicitly constructed. They provide a dynamical formulation in the $x_\pm$ plane (without involving the extra-time direction τ of the superheavenly equation). In the following, we investigate the dimensional reduction of both the discretized and continuum systems from $(1 + 2)$ to $(1 + 1)$ dimensions. We obtain a supersymmetric system of equations with an interesting and subtle property. Unlike its purely bosonic subsector, the whole system involving fermions is not of hydrodynamical type. However, the same system, once expressed in terms of a supergeometry involving superfields and fermionic derivatives, satisfies a graded extension of the hydrodynamical property, which can be naturally named of "super-hydrodynamical type of equation". With this expression we mean that there is a way to recast the given $(1 + 1)$-dimensional supersymmetric equations into a system of non-linear equations for superfields which involves only first-order derivations w.r.t. the supersymmetric fermionic derivatives. This super-hydrodynamical system furnishes the supersymmetrization of the system introduced in [6].

2 The $N = 2$ Superheavenly equation.

The construction of a continuum limit (for $n \to \infty$) of a discretized superToda system requires a presentation of the system in terms of a specific Cartan matrix. The symmetric choice in [9] for the Cartan matrix of the superalgebra $sl(n|n+1)$ does not allow to do so. On the other hand [10], the Cartan matrix a_{ij} of $sl(n|n+1)$ can be chosen to be antisymmetric with the only non-vanishing entries given by $a_{ij} = \delta_{i,i+1} - \delta_{i,i-1}$.

The Cartan generators H_i and the fermionic simple roots $F_{\pm i}$ satisfy

$$\begin{aligned} [H_i, F_{\pm j}] &= \pm a_{ij} F_{\pm j}, \\ \{F_i, F_{-j}\} &= \delta_{ij}. \end{aligned} \tag{2.1}$$

The continuum limit of the [3] construction could have been performed for any superalgebra admitting an $n \to \infty$ limit, such as $sl(n|n)$, etc. On the other hand, the superalgebras of the series $sl(n|n+1)$ are special because they admit a complex structure and therefore the possibility of defining an $N = 2$ manifestly supersymmetric Toda system, following the prescription of [8]. This is the content of the present section.

At first we introduce the $N = 2$ fermionic derivatives $D_\pm$, $\overline{D}_\pm$, acting on the $x_\pm$ $2D$ spacetime ($\theta_\pm$ and $\overline{\theta}_\pm$ are Grassmann coordinates). The $2D$ spacetime can be either Euclidean ($x_\pm = x \pm t$) or Minkowskian ($x_\pm = x \pm it$).

We have

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial \theta_\pm} - i\overline{\theta}_\pm \partial_\pm, \\ \overline{D}_\pm &= -\frac{\partial}{\partial \overline{\theta}_\pm} + i\theta_\pm \partial_\pm. \end{aligned} \tag{2.2}$$

They satisfy the anticommutator algebra

$$\{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm \tag{2.3}$$

and are vanishing otherwise.

Chiral (Φ) and antichiral ($\overline{\Phi}$) $N = 2$ superfields are respectively constrained to fulfill the conditions

$$\begin{aligned} \overline{D}_\pm \Phi &= 0, \\ D_\pm \overline{\Phi} &= 0. \end{aligned} \tag{2.4}$$

Accordingly, a generic chiral superfield Φ is expanded in its bosonic φ, F (the latter auxiliary) and fermionic component fields (ψ_+, ψ_-) as

$$\Phi(\hat{x}_\pm, \theta_\pm) = \varphi + \theta_+ \psi_+ + \theta_- \psi_- + \theta_+ \theta_- F, \tag{2.5}$$

with φ, $\psi_\pm$ and F evaluated in $\hat{x}_\pm = x_\pm + i\overline{\theta}_\pm \theta_\pm$.

Similarly, the antichiral superfield $\overline{\Phi}$ is expanded as

$$\overline{\Phi}(\overline{x}_\pm, \overline{\theta}_\pm) = \overline{\varphi} + \overline{\theta}_+ \overline{\psi}_+ + \overline{\theta}_- \overline{\psi}_- + \overline{\theta}_+ \overline{\theta}_- \overline{F}, \tag{2.6}$$

with all component fields evaluated in $\overline{x}_\pm = x_\pm - i\overline{\theta}_\pm \theta_\pm$.

Due to the complex structure of $sl(n|n+1)$, its Cartan and its simple (positive and negative) root sector can be split into its conjugated parts

$$\begin{array}{ll} \mathcal{H} \equiv \{H_{2k-1}\}, & \overline{\mathcal{H}} \equiv \{H_{2k}\}, \\ \mathcal{F}_+ \equiv \{F_{2k-1}\}, & \mathcal{F}_- \equiv \{F_{-(2k-1)}\}, \\ \overline{\mathcal{F}}_+ \equiv \{F_{-2k}\}, & \overline{\mathcal{F}}_- \equiv \{F_{2k}\}, \end{array} \tag{2.7}$$

for $k = 1, 2, \ldots, n$.

Following [8], we can introduce the $sl(n|n+1)$ $N = 2$ superToda dynamics, defined for the Cartan-valued chiral ($\mathbf{\Phi}$) and antichiral ($\overline{\mathbf{\Phi}}$) $N = 2$ superfields,

$$\begin{aligned} \mathbf{\Phi} &= \sum_{k=1}^{n} \Phi_k H_{2k-1}, \\ \overline{\mathbf{\Phi}} &= \sum_{k=1}^{n} \overline{\Phi}_k H_{2k}, \end{aligned} \tag{2.8}$$

through the Lax operators $\mathcal{L}_\pm$ and $\overline{\mathcal{L}}_\pm$, given by

$$\begin{aligned} \mathcal{L}_+ &= D_+\mathbf{\Phi} + e^{\overline{\mathbf{\Phi}}} F_+ e^{-\overline{\mathbf{\Phi}}}, \\ \mathcal{L}_- &= -F_- \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \overline{\mathcal{L}}_+ &= \overline{F}_+, \\ \overline{\mathcal{L}}_- &= \overline{D}_-\overline{\mathbf{\Phi}} + e^{\mathbf{\Phi}} \overline{F}_- e^{-\mathbf{\Phi}}, \end{aligned} \tag{2.10}$$

where

$$\begin{array}{ll} F_+ = \sum_k F_{2k-1}, & F_- = \sum_k F_{-(2k-1)}, \\ \overline{F}_+ = \sum_k F_{-(2k)}, & \overline{F}_- = \sum_k F_{2k}, \end{array} \tag{2.11}$$

(as before, the sum is over the positive integers up to n).

Explicitly, we have

$$\begin{aligned} \mathcal{L}_+ &= \sum_k (D_+\Phi_k H_{2k-1} + e^{\overline{\Phi}_{k-1}-\overline{\Phi}_k} F_{2k+1}), \\ \mathcal{L}_- &= -\sum_k F_{-(2k-1)}, \\ \overline{\mathcal{L}}_+ &= \sum_k F_{-2k}, \\ \overline{\mathcal{L}}_- &= \sum_k (\overline{D}_-\overline{\Phi}_k H_{2k} + e^{\Phi_k-\Phi_{k+1}} F_{2k}), \end{aligned} \tag{2.12}$$

Please notice that here and in the following we have formally set $\overline{\Phi}_0 \equiv 0$.

The zero-curvature equations are given by

$$\begin{aligned} D_+\mathcal{L}_- + D_-\mathcal{L}_+ + \{\mathcal{L}_+, \mathcal{L}_-\} &= 0, \\ \overline{D}_+\overline{\mathcal{L}}_- + \overline{D}_-\overline{\mathcal{L}}_+ + \{\overline{\mathcal{L}}_+, \overline{\mathcal{L}}_-\} &= 0, \end{aligned} \tag{2.13}$$

so that the following set of equations for the constrained (anti)chiral $N = 2$ superfields is obtained

$$
\begin{aligned}
D_+D_-\Phi_k &= -e^{\overline{\Phi}_{k-1}-\overline{\Phi}_k},\\
\overline{D}_+\overline{D}_-\overline{\Phi}_k &= -e^{\Phi_k-\Phi_{k+1}},
\end{aligned}
\tag{2.14}
$$

for the positive integers $k = 1, 2, \ldots, n$.

By setting,

$$
B_k = \Phi_k - \Phi_{k+1}, \qquad \overline{B}_k = \overline{\Phi}_k - \overline{\Phi}_{k+1},
\tag{2.15}
$$

we get the two systems of equations

$$
D_+D_-B_k = e^{\overline{B}_k} - e^{\overline{B}_{k-1}}, \qquad \overline{D}_+\overline{D}_-\overline{B}_k = e^{B_{k+1}} - e^{B_k},
\tag{2.16}
$$

for $k = 1, 2, \ldots, n$.

By identifying k as a discretized extra time-like variable τ we obtain, in the continuum limit for $n \to \infty$,

$$
D_+D_-B = \partial_\tau e^{\overline{B}}, \qquad \overline{D}_+\overline{D}_-\overline{B} = \partial_\tau e^{B},
\tag{2.17}
$$

which corresponds to the $N = 2$ extension of the superheavenly equation.

Indeed, the presence in the previous equations of the first derivative in τ is an artifact of the $N = 2$ superfield formalism. Once solved the equations at the level of the component fields and eliminated the auxiliary fields in terms of the equations of motion, we are left with a system of second-order equations.

We have, in terms of the component fields,

$$
\begin{aligned}
B &= \big(1 + i\overline{\theta}_+\theta_+\partial_+ + i\overline{\theta}_-\theta_-\partial_- - \overline{\theta}_+\theta_+\overline{\theta}_-\theta_-\partial_+\partial_-)\mathcal{C},\\
\overline{B} &= \big(1 - i\overline{\theta}_+\theta_+\partial_+ - i\overline{\theta}_-\theta_-\partial_- - \overline{\theta}_+\theta_+\overline{\theta}_-\theta_-\partial_+\partial_-\big)\overline{\mathcal{C}},
\end{aligned}
\tag{2.18}
$$

where

$$
\begin{aligned}
\mathcal{C} &= \big(b + \theta_+\psi_+ + \theta_-\psi_- + \theta_+\theta_-a\big),\\
\overline{\mathcal{C}} &= \big(\overline{b} + \overline{\theta}_+\overline{\psi}_+ + \overline{\theta}_-\overline{\psi}_- + \overline{\theta}_+\overline{\theta}_-\overline{a}\big),
\end{aligned}
\tag{2.19}
$$

with a, $\overline{a}$ bosonic auxiliary fields. All component fields are evaluated in $x_\pm$ only.

The equations of motion of the $N = 2$ superheavenly equation are explicitly given in components by

$$
\begin{aligned}
-a &= (e^{\overline{b}})_\tau,\\
2i\partial_-\psi_+ &= (\overline{\psi}_-e^{\overline{b}})_\tau,\\
-2i\partial_+\psi_- &= (\overline{\psi}_+e^{\overline{b}})_\tau,\\
-4\partial_+\partial_-b &= (\overline{a}e^{\overline{b}} - \overline{\psi}_+\overline{\psi}_-e^{\overline{b}})_\tau,
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
-\overline{a} &= (e^{b})_\tau,\\
2i\partial_-\overline{\psi}_+ &= (\psi_-e^{b})_\tau,\\
-2i\partial_+\overline{\psi}_- &= (\psi_+e^{b})_\tau,\\
-4\partial_+\partial_-\overline{b} &= (ae^{b} - \psi_+\psi_-e^{b})_\tau.
\end{aligned}
\tag{2.21}
$$

Eliminating the auxiliary fields we obtain the systems

$$\begin{aligned} 2i\partial_-\psi_+ &= (\overline{\psi}_- e^{\overline{b}})_\tau, \\ -2i\partial_+\psi_- &= (\overline{\psi}_+ e^{\overline{b}})_\tau, \\ 4\partial_+\partial_- b &= \Big((e^b)_\tau e^{\overline{b}} + (\overline{\psi}_+\overline{\psi}_- e^{\overline{b}}\Big)_\tau, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} 2i\partial_-\overline{\psi}_+ &= (\psi_- e^{b})_\tau, \\ -2i\partial_+\overline{\psi}_- &= (\psi_+ e^{b})_\tau, \\ 4\partial_+\partial_- \overline{b} &= \Big((e^{\overline{b}})_\tau e^{b} + (\psi_+\psi_- e^{b}\Big)_\tau. \end{aligned} \tag{2.23}$$

The bosonic component fields b, $\overline{b}$, as well as the fermionic ones $\psi_\pm$, $\overline{\psi}_\pm$, are all independent. The equations (2.22, 2.23) are a manifestly $N=2$ supersymmetric extension of the system introduced in [3].

3 Super-hydrodynamical reductions of the superheavenly equation.

The equations (2.22) and (2.23) correspond to a $(1+2)$-dimensional system, manifestly relativistic and $N=2$ supersymmetric w.r.t. the two-dimensional subspace spanned by the $x_\pm$ coordinates, while possessing an extra bosonic time-like dimension denoted as τ. A very interesting example of integrable system in $1+1$-dimension which is currently intensely investigated, see e.g. [6, 7], can be recovered by applying a dimensional-reduction to, let's say, the bosonic sector of the $(1+2)$-dimensional heavenly equation. We can refer to such a system, perhaps a bit pedantically, as the $(1+1)$-dimensionally reduced heavenly equation. It can be obtained by setting equal to zero the fermionic variables $\psi_\pm, \overline{\psi}_\pm \equiv 0$, while b, $\overline{b}$, can be consistently constrained as $\overline{b}=b$. The $x_\pm$ coordinates are identified, i.e. $x_+ = x_- = x$.

The resulting equation, by changing the normalizations (setting $f=2b$ and $t=2\tau$) can be conveniently written as

$$f_{tt} = (e^f)_{xx} \tag{3.24}$$

This equation has recently received a lot of attention in the literature. It is an example of a completely integrable, hydrodynamical-type equation. It admits a multihamiltonian structure and possesses an infinite number of conserved charges in involution. It has been recently shown that it can be recovered via a dispersionless Lax representation given by

$$L := p^{-1}\Big(1+gp^2+hp^4\Big)^{\frac{1}{2}}, \tag{3.25}$$

with f, g functions of x, t, while p is the classical momentum. The equations of motion are read from

$$\frac{\partial L}{\partial t} = \frac{p}{2}\Big\{L^2_{\leq 0}, L\Big\} \tag{3.26}$$

where $\{*,*\}$ denotes the usual Poisson brackets and $L^2_{\leq 0}$, defined in [6], is explicitly given by $L^2_{\leq 0} = p^{-2}+g$.

The equation (3.26) leads to

$$\begin{aligned}\frac{\partial h}{\partial t} &= hg_x \\ \frac{\partial g}{\partial t} &= h_x\end{aligned} \qquad (3.27)$$

The equation (3.24) is recovered after eliminating g from the previous system and setting $h = e^f$. In the (3.28) form, the "(1+1)-dimensionally reduced heavenly equation" is shown to be a hydrodynamical type of equation.

An interesting question, whose solution as we will see is non-trivial, is whether the $(1+1)$-dimensional reduction (for $x_+ = x_- = x$) of the full $N = 2$ supersymmetric heavenly equation is also of hydrodynamical type. The answer is subtle. The introduction of the fermionic fields $\psi_\pm$, $\overline{\psi}_\pm$, whose equations of motion are of first order in the extra-bosonic time, does not allow to represent the dimensionally reduced system from (2.22, 2.23) in hydrodynamical form, due to a mismatch with the second-order-equation satisfied by the bosonic fields b, $\overline{b}$. On the other hand, it is quite natural to expect that important properties are not spoiled by the supersymmetrization. This is indeed the case with the hydrodynamical reduction, when properly understood. The key issue here is the fact that the nice features of the supersymmetry are grasped when inserted in the proper context of a supergeometry, which must be expressed through the use of a superfield formalism. It is in this framework that a super-hydrodynamical reduction of the dimensionally reduced system from (2.22, 2.23) becomes possible. Indeed, for $x_+ = x_- = x$, we can write down the (2.17) system through the set of, first-order in the fermionic derivatives, *superfield* equations

$$\begin{aligned}\mathcal{D}_-\mathcal{B} &= N_\tau, \\ \mathcal{D}_+N &= e^{\overline{B}}\end{aligned} \qquad (3.28)$$

and

$$\begin{aligned}\overline{\mathcal{D}}_-\overline{\mathcal{B}} &= \overline{N}_\tau, \\ \overline{\mathcal{D}}_+\overline{N} &= e^{B}\end{aligned} \qquad (3.29)$$

with N and $\overline{N}$ subsidiary fermionic superfields . Taking into account the following expansions

$$\begin{aligned}e^{\overline{B}} &= (1 - i\overline{\theta}_+\theta_+\partial_+ - i\overline{\theta}_-\theta_-\partial_- - \overline{\theta}_+\theta_+\overline{\theta}_-\theta_-\partial_+\partial_-)e^{\overline{\mathcal{C}}} = \\ &\quad \mathcal{D}_+\theta_+\Big(1 - i\overline{\theta_-}\theta_-\partial_-\Big)e^{\overline{\mathcal{C}}} \\ e^{B} &= (1 + i\overline{\theta}_+\theta_+\partial_+ + i\overline{\theta}_-\theta_-\partial_- - \overline{\theta}_+\theta_+\overline{\theta}_-\theta_-\partial_+\partial_-)e^{\mathcal{C}} = \\ &\quad -\overline{\mathcal{D}}_+\overline{\theta}_+\Big(1 + i\overline{\theta}_-\theta_-\partial_-\Big)e^{\mathcal{C}}\end{aligned} \qquad (3.30)$$

we easily obtain the following solutions for N, $\overline{N}$

$$\begin{aligned}N &= \Big(\theta_+ - i\theta_+\overline{\theta}_-\theta_-\partial_-\Big)e^{\overline{\mathcal{C}}} + \mathcal{D}_+\Omega_1 \\ \overline{N} &= -\Big(\overline{\theta}_+ + i\overline{\theta}_+\overline{\theta}_-\theta_-\partial_-\Big)e^{\mathcal{C}} + \overline{\mathcal{D}}_+\overline{\Omega}_1\end{aligned} \qquad (3.31)$$

in terms of arbitrary Ω_1, Ω_2 bosonic superfunctions.

If we set, as we are free to choose,

$$\begin{aligned}\Omega_1 &= \theta_+\Big(\Psi_- - 2i\overline{\theta}_- b_- - i\overline{\theta}_-\theta_-\Psi_{-,-}\Big)\\ \overline{\Omega}_1 &= \overline{\theta}_+\Big(\overline{\Psi}_- - 2i\theta_-\overline{b}_- + i\overline{\theta}_-\theta_-\overline{\Psi}_{-,-}\Big)\end{aligned} \tag{3.32}$$

and substitute the values of N, $\overline{N}$ given in (3.31) back to (3.28) and (3.29), we obtain the equivalence of this system of equations w.r.t. the $x_+ = x_- = x$ dimensional reduction of the (2.22) and (2.23) equations. This proves the existence of a super-hydrodynamical reduction expressed in a superfield formalism. If we express in terms of the component fields this set of equations, as mentioned before, we are not obtaining a hydrodynamical type equation. This fact should not be regarded as a vicious feature of our system, but rather as a virtue of the supersymmetry. In several examples, this is one, the introduction of the super-formalism allows extending both the properties of the systems and the techniques used to investigate them, to cases for which the ordinary methods are of no help. In a somehow related area we can cite, e.g., the derivation of bosonic integrable hierarchies associated with the bosonic sector of super-Lie algebras [11]. They are outside the classification based on and cannot be produced from ordinary Lie algebras alone.

Let us finally discuss the restriction from $N = 2 \rightarrow N = 1$, i.e. down to the $N = 1$ supersymmetry. It is recovered from setting

$$\begin{aligned}\overline{\theta}_+ = -\theta_+, &\qquad \overline{\theta}_- = -\theta_-,\\ \overline{\mathcal{D}}_+ = -\mathcal{D}_+, &\qquad \overline{\mathcal{D}}_- = -\mathcal{D}_-.\end{aligned} \tag{3.33}$$

The equations (2.22) and (2.23) now read

$$\mathcal{D}_+\mathcal{D}_-\mathcal{C} = \partial_\tau e^{\overline{\mathcal{C}}}, \qquad \mathcal{D}_+\mathcal{D}_-\overline{\mathcal{C}} = \partial_\tau e^{\mathcal{C}}, \tag{3.34}$$

with

$$D_\pm = \frac{\partial}{\partial\theta_\pm} + i\theta_\pm\partial_\pm \tag{3.35}$$

The equations in (3.34) have a similar structure in components as the equations (2.22) and (2.23). We have indeed

$$\begin{aligned}\partial_-\psi_+ &= i(\overline{\psi}_- e^{\overline{b}})_\tau,\\ \partial_+\psi_- &= -i(\overline{\psi}_+ e^{\overline{b}})_\tau,\\ \partial_+\partial_- b &= \Big((e^b)_\tau e^{\overline{b}} + (\overline{\psi}_+\overline{\psi}_- e^{\overline{b}}\Big)_\tau,\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}\partial_-\overline{\psi}_+ &= i(\psi_- e^{b})_\tau,\\ \partial_+\overline{\psi}_- &= -i(\psi_+ e^{b})_\tau,\\ \partial_+\partial_- \overline{b} &= \Big((e^{\overline{b}})_\tau e^{b} + (\psi_+\psi_- e^{b}\Big)_\tau.\end{aligned} \tag{3.37}$$

If we further dimensionally reduce to $(1 + 1)$-dimension our $N = 1$ system, by constraining the variables $x_\pm$ ($x_+ = -x_- = it$) and setting $\overline{\mathcal{C}} = \mathcal{C}$, we obtain the system

$$\begin{aligned}\partial_t\psi_+ &= \partial_\tau\Big(e^b\psi_-\Big)\\ \partial_t\psi_- &= \partial_\tau\Big(e^b\psi_+\Big)\\ \partial_t^2 b &= \partial_\tau\Big(\frac{1}{2}(e^{2b})_\tau + e^b\psi_+\psi_-\Big).\end{aligned} \tag{3.38}$$

As its $N = 2$ counterpart, this system is not of hydrodynamical type. However, it can be easily shown, with a straightforward modification of the procedure previously discussed for the $N = 2$ case, to be super-hydrodynamical when expressed in terms of superfields and superderivatives.

4 Conclusions.

In this paper we have at first introduced manifest $N = 2$ supersymmetric discrete Toda equations in association with the complex-structure superalgebras of the $sl(n, n+1)$ series. We proved that in the $n \to \infty$ limit the discrete variable can be regarded as a continuum extra bosonic time τ. The corresponding system is a manifestly $N = 2$ supersymmetric extension of the standard heavenly equation in $(2 + 1)$ dimensions, the supersymmetry being associated with the plan spanned by the bosonic $x_\pm$ coordinates and their fermionic counterparts, which are naturally expressed in an $N = 2$ superspace formalism. A supersymmetric Lax-pair, proving the integrability of the system, was constructed. It expresses the dynamics w.r.t. the $\pm$ superspace coordinates (τ in this respect can be considered either as an auxiliary parameter or as the discrete label of the original formulation).

We further investigated the dimensional reduction of the above systems from the (2+1)-dimensional to the (1+1)-dimensional case. In the purely bosonic case the reduced systems of equations are of a special type, they are hydrodynamical systems of non-linear equations. The supersymmetric case is much subtler. This system is not of hydrodynamical type since the presence of the fermions spoils this property. However, when properly expressed in terms of a supergeometry (essentially, superfields and fermionic derivatives) is of super-hydrodynamical type (the precise meaning of this term has been explained in the Introduction). More than a vice, this can be considered as a virtue of the supersymmetry. It allows extending the notion of hydrodynamical equations beyond the realm of the systems ordinary allowed.

Finally, it is worth mentioning a formal, however challenging problem, concerning the supersymmetrization. While the integrability of the systems under consideration is automatically guaranteed by the given supersymmetric and relativistic Lax-pairs in the $\pm$ plane mentioned above, one can wonder whether a dispersionless Lax operator, directly expressing the τ dynamics (i.e., in terms of the bosonic extra-time) could be found. The answer is indeed positive for the purely bosonic sector. In the supersymmetric case, however, no closed dispersionless Lax operator is known at present. It is repeated here an analogous situation already encountered for the polytropic gas systems [12]. The problem of constructing supersymmetric dispersionless Lax operators for these related systems is still open.

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