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Recovery Operator for Local Errors Generated by Gauge Multiplets

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Abstract

The components of a quantum computer are quantum subsystems which have a complex internal structure. This structure is determined by short-range interactions which are appropriately described in terms of local gauge fields of the first kind. Any modification of the subsytems would produce, in general, a local error in the quantum state of the computer. We propose a general treatement of the local errors produced by a gauge multiplet in the framework of algebraic quantum field theory. A recovery operator is constructed from the first principles.

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The quantum computer is operational providing the errors in stocking and manipulating the information are known and can be controled. Classifying and and correcting the quantum errors is a crucial problem for practical reasons as well as for defining rigorously an Universal Quantum Computer. Most of the errors that alter the memory register or the output of the quantum computation are due to the small scale of the computer subsystems and to their sensitivity to the interaction with the environment and with each other. The decoherence of the quantum state of the computer and the uncertainty in the unitary evolution of it have been studied for spin $\frac{1}{2}$-systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and higher spin systems [12, 13, 14, 15]. Other effects that can alter the information as the quantum chaos [16, 17, 18], the self-interaction among various qubits [17, 19] and the changes in the continuous variables of the system [20, 21, 22, 23] have been considered and the codes to correct them have been written down. A general observation that can be drawn from the above studies is that long rage (electromagnetic) and/or non-linear interactions are responsible for the alterations in quantum states of the processing unit. This statement resumes the idea that the quantum computers are "small, sensitive and easily perturbed". However, one may ask whether the short-range interactions affect in any way the precission of quantum computations. While it is common to think that the "computer parts" (atoms, molecules or nuclei) are "unchangeble" subsystems (and for most of practical purposes this true), the fact is that they are not completly stable. Due to their internal structure, they can change their state (like through spontaneous emission or decayment) and thus alter the state of the registrer memory. The purpose of this letter is to put this question into a more formal frame and to discuss the possibility of constructing recovery operators for this type of errors.

The parts of a quantum computer are complex subsystems hold together by internal interactions. If we think to nuclei, which form subsystems of any quantum computer considered up to now, the most important role is played by gauge fields which produce local interactions and are nonlinear. In general, a modification of the state of the nuclei would induce a modification of the quantum state of the computer, as, for example, a spontaneous emission that would carry away spin quantum number. However, from the point of view of the computer, the change in its state is a local one as long as the long-range interactions are ignored. Therefore, it is natural to consider that the error is produced by local field operators and that the state of the computer is assigned to a vector in the Hilbert space of the field theory. The basic objects of the field theory are the observables which can be constructed starting from field operators. There are three basic principles that this theory should obey: *the principle of locality*, *the principle of causality* and *the principle of gauge invariance* which guarantee that the effects in the quantum computer are local, that they concearn the subsystems of the computer and that the quantum field theory we are dealing with is a known one. Formally, the principle of locality states that to any open and finite extended region of space-time Σ one can associate a *-algebra of observables $\mathcal{O}(\Sigma)$ and a *-algebra of fields $\mathcal{F}(\Sigma)$. (The involution * is necessary in order to define the conjugates of fields.) One can define global algebras over the full volume of the computer or any of its subsystems by considering the norm closed unions

$$\mathcal{O} = \cup\mathcal{O}(\Sigma) \quad , \quad \mathcal{F} = \cup\mathcal{F}(\Sigma). \tag{1}$$

The principle of causality states that for any two space-like separated regions of space-time Σ_1 and Σ_2 the corresponding algebras of observables commute with each other, i.e. $[\mathcal{O}(\Sigma_1), \mathcal{O}(\Sigma_2)] = 0$. The principle of gauge invariance tells us that there is a representation of a gauge group $\mathcal{G}$ (denoted for simplicity with the same letter) that acts on the algebra of fields $\mathcal{F}(\Sigma)$. The algebra $\mathcal{O}(\pm)$ is gauge invariant and since we assume that the observables are constructed out of fields, it is the gauge invariant part of the algebra $\mathcal{F}(\Sigma)$. If we want to describe the effects of short-range interactions we can limit ourselves to the gauge groups $SU(2)$ or $SU(3)$. However, we can keep the discussion more general by working with groups having the same property, i.e. compact groups. Therefore, in what follows $\mathcal{G}$ will be a compact gauge group that defines a gauge symmetry of first kind, i. e. which excludes the long range interactions. Under these assumptions the setting of the problem is the algebraic field theory [24]. As was shown in [25, 26, 27, 28] the algebra $\mathcal{F}$ is generated by a gauge multimplet $\{\Psi_i\}$, $i = 1, 2, \ldots, n$ and $\mathcal{O}$. From the principlest of locality and causality it follows that the multiplet should generate a Cuntz algebra O_d [29], i.e. it should satisfy the following relations [25, 26]

$$\begin{aligned} \sum_i \Psi_i \Psi_i^\dagger &= 1 \\ \Psi_i^\dagger \Psi_j &= \delta_{ij}. \end{aligned} \tag{2}$$

There is a canonical endomorphism of the algebra $\mathcal{O}$ which defines the density matrix for any field $A \in \mathcal{F}$

$$\rho(A) = \sum_i \Psi_i A \Psi_i^\dagger \tag{3}$$

and which satisfy the following relation for any Ψ and any A

$$\Psi A = \rho(A)\Psi. \tag{4}$$

Let us assume that the computer is prepared in the initial pure state $|\phi_I\rangle$. The errors induced by the field are obtained by acting on the initial state with products of the field operators, i. e. elements of the algebra $\mathcal{F}$. For simplicity, we will consider in what follows only the action of the generators Ψ_i of O_d. Thus, the system evolves to a state ρ_F given by

$$\rho_F = \sum_i \Psi_i \rho_I \Psi_i^\dagger, \tag{5}$$

where ρ_I is the density matrix associated to the initial vector state. We remark from (2) that the set of operators Ψ_i form a superoperator. Due to this structure, it is apprpriate to correct the error by the recovery operator method, i.e. to construct a set of operators $\{R_a\}$ that projects the wrong state ρ_F back to the initial state [31]. Note firstly that one can associate a projector P_i to each field operator Ψ_i defined by the following relation

$$P_i = \Psi_i \Psi_i^\dagger \quad , \quad P_i^2 = P_i^\dagger = P_i, \tag{6}$$

with the following action on any density matrix

$$\rho \;\mapsto\; \rho_P = \sum_i P_i \rho P_i. \tag{7}$$

The set of projectors $\{P_i\}$ projects the state ρ_F to itself and map any other state $\rho \neq \rho_F$ to a different state. To determine the recovery operator we require that the fidelity of the state obtained by acting with the recovery operator on ρ_F be maximum, i.e. equal to the norm of $|\phi_I\rangle$. The fidelity of a state ρ is defined as the sqare of the norm of the element matrix of it in the inital state

$$F(\phi_I, \rho) = \langle \phi_I |\rho| \phi_I \rangle . \tag{8}$$

Consider the following linear combinations of the projectors P_i

$$R_a = \sum_i \alpha_{ai} P_i \tag{9}$$

where $a = 1, 2, \ldots, s$ belongs to a discrete set and α_{ai} are complex numbers. $\{R_a\}$ map ρ_F to the state ρ_R by the usual action of the operators on matrix densities

$$\rho_F \;\mapsto\; \rho_R = \sum_a R_a \rho_F R_a^\dagger . \tag{10}$$

The initial state is recovered by R_a's if

$$F(\phi_I, \rho_R) = \langle \phi_I |\rho_I| \phi_I \rangle . \tag{11}$$

Assume for simplicity that the norm of the initial state is one. Then (11) is equivalent to the following relation

$$\sum_a \sum_i |\alpha_{ai}|^2 |\Psi_{iI}|^2 = 1, \tag{12}$$

where $\Psi_{iI} = \langle \phi_I |\Psi_i| \phi_I \rangle$. The relation (12) represent a constraint on the moduli of the complex coefficients α_{ai} in terms of the known gauge multiplet and the initial state. If a hole subset C of the Hilbert space is recovered by the operators R_a's, then (12) should hold for any $|\phi_I\rangle$ from C. Suppose that C form a linear subspace of finite dimension k of the Hilbert space of the field theory. By picking up a basis $\{|\epsilon_A\rangle\}$ of it, one can determine the constants α_{ai}'s, and consequently the operators R_a's, up to some phase factors, by solving the corresponding linear system

$$\sum_a \sum_i |\alpha_{ai}|^2 |\Psi_{iA}|^2 = 1 \quad , \quad A = 1, 2, \ldots, k. \tag{13}$$

The number M of the operators R_a depends on the dimension k of C and one should seek for its minimum value $M = k$ for which there are sufficient equations in (13) to determine the moduli of the coefficients α_{ai}.

There is one more constraint that can be imposed naturally on the recovery operator and it comes from the gauge structure of the theory. Firstly, note that the fidelity is gauge invariant

$$F(\phi'_I, \rho'_R) = F(\phi_I, \rho_R), \tag{14}$$

where the transformations of the states and fields given by the following relations

$$|\psi\rangle \;\mapsto\; |\psi'\rangle = g\,|\phi_I\rangle \tag{15}$$

$$\Psi_i \;\mapsto\; \Psi'_i = g \Psi_i g^{-1} \tag{16}$$

for any $g \in \mathcal{G}$. Since the norm of the initial state is also gauge invariant, the equations (12) and (13) are also gauge invariant as they should be. Consider next the following isometry of the algebra $\mathcal{F}$ [30]

$$S = \frac{1}{\sqrt{d!}} \sum_{q \in P(d)} \text{sign}(q) \Psi_{q(1)} \Psi_{q(2)} \ldots \Psi_{q(d)}, \tag{17}$$

where $P(d)$ is the permutation group of d elements. S is a gauge invariant object of $\mathcal{F}$ [28, 30] and it acts on the initial state as

$$\rho_I \mapsto \rho_S = S \rho_I S^{\dagger}. \tag{18}$$

Now consider the following transformations: act firstly with a gauge transformation (15) and (16) and then apply the gauge invariant isometry (17). Since S is an gauge invariant operator, we would like that the recovery operator R'_a obtained from (16) maximizes the fidelity on the gauge transformed initial state $|\psi'\rangle$

$$F(\phi'_I, \rho'_{RS}) = \langle \phi'_I | \rho'_S | \phi'_I \rangle . \tag{19}$$

It is an elementary exercise to show that (19) is equivalent to the same equation for the untrasformed objects. After a simple algebra we obtain the following relation from (19)

$$\sum_{a=1}^{M} \sum_{q,r \in P(d)} \frac{1}{d!} \text{sign}(q) \text{sign}(r) \alpha_{aq(1)} \alpha^*_{ar(1)} \times \left\langle \Psi_{q(1)} \cdots \Psi_{q(d)} \right\rangle_I \left\langle \Psi^{\dagger}_{q(d)} \cdots \Psi^{\dagger}_{q(1)} \right\rangle_I = 1, \tag{20}$$

where $\langle \cdots \rangle_I$ is a shorthand notation for $\langle \phi_I | \cdots | \phi_I \rangle$. We interpret (20) as a constraint on the complex coefficients α_{aI}.

The equations (12), (13) and (20) represent the main result of the present work. They are based on the Ansatz in (9). While (12) and (13) provide the linear equations for determining the coefficents in the recovery operator $\mathcal{R} = \{R_a\}$ for $a = 1, 2, \ldots, M = k$ up to some phase factors, the relation (20) is obtained from the gauge symmetry of the field operators. It states that under an arbitrary gauge transformation and a gauge invariant transformation the recovery operator continues the remain so. The dimension of the code space $\mathcal{C}$ fixes the number of operators R_a. The equations (12), (13) and (20) were determined from general principles of the Algebraic Quantum Field Theory and from the requirement that the fidelity is maximized for any state from the code space. The interpretation of $\mathcal{R}$ is that of the recovery operator for the internal errors produced by local gauge fields. They alter the state of the computer either by changing the states of quantum subsystems or by changing locally the general state of memory registrer. The physics behind the present description is that of modifications in the internal state of the subsystems of the computer induced by short-distance interactions. Mathematically, we have been working in the frame of the algebraic quantum field theory.

It remains open the discussion about the errors induced by the algebra of observables $\mathcal{O}$. On general grounds, we would expect that they produce similar errors as the local fields. However, the algebra of observables does not have, in general, the structure of a Cuntz algebra. Another interesting class of internal error is that produced by non-linearities in the fields that are responsible for the internal structure of the quantum components of the computer. We hope clarify these issues somewhere else [37].

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