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Minimal Closed Set of Observables in the Theory of Cosmological Perturbations II: Vorticity and Gravitational Waves

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Abstract

In a previous paper we analysed the theory of perturbation of Friedman-Robertson-Walker (FRW) cosmology exclusively in terms of observable quantities in the framework of Quasi-Maxwellian equations of gravitation. In that paper we limited ourselves to the case of irrotational perturbations for simplicity. We complete here the previous task by presenting the remaining cases of vector and tensor perturbations.

Following the same reasoning as for the scalar case, we show here that the vorticity Ω and the shear Σ constitute the two basic perturbed variables in terms of which all remaining observable quantities can be described for the vectorial case. The tensorial case can be described by the variables E and H, the electric and magnetic parts of Weyl conformal tensor. Einstein's equations of General Relativity reduce to a closed set of dynamical system for those pairs of variables. We then obtain a Hamiltonian treatment of the Perturbation Theory in FRW..

Key-words: Perturbation; Cosmology; Gravitational Waves.

1 Introduction

1.1 Introductory Remarks

As it has been discussed in our previous paper [1], we will make use of the perturbation formalism of Einstein's theory of gravitation based in gauge-independent and evident physically meaningful quantities, such as the vorticity, shear, electric and magnetic parts of the conformal Weyl tensor and so on. The advantage of this formalism (mainly in a conformally flat background, e.g. FRW universe geometry), as we have pointed out previously (see references therein), is that we can then restrict our analysis only to the quantities for which Stewart's Lemma [2, 3] is valid; the perturbed quantity will therefore be indeed a gauge-independent one.

1.2 Synopsis

We will apply the method of expansion of the perturbed variables in terms of the spatial harmonics in order to obtain a closed dynamical set, where its elements are the expansion coefficients. This will be presented separately for each kind of perturbation, pointing out their independent character and also to simplify the notation.

In **Section 2** we present the Harmonics fundamental equations that specify our assumption on the meaning for the words **scalar, vector** and **tensor**. We also insert there a set of useful equations suited to work within the Quasi-Maxwellian formalism.

In **Section 3** we obtain the fundamental equations for the kinematical (vorticity, acceleration and shear), matter (energy flux and anisotropic pressure) and geometrical (the electric and magnetic parts of Weyl tensor) variables that, we argue, completely describe the vectorial perturbations for the Friedman background. **Section 4** is devoted to the completeness of our basic formalism, stressing that there is no secondary constraint. This is an important point on the path to quantize the system.

Finally, **Section 5** presents the Hamiltonian treatment for the perturbations and uses it to get the explicit evolution of the "good variables" (that is, gauge-independent ones). This approach recovers the degree of instability of the Standard Cosmological Model (FRW), for the cases presented there. **Section 6** deals with three specific, physically motivated models which solve the dynamical system obtained: isotropic, irrotational and Stokesian fluid.

In the same way, the remaining of the paper applies the method to the case of tensorial perturbations: in **Section 7** we obtain the basic equations for the kinematical (shear), matter (anisotropic pressure) and geometrical (electric and magnetic parts of Weyl tensor) variables. **Section 8** then presents the Hamiltonian description of this type of perturbations and recovers the instability results.

Appendix A gives a summary of the perturbed Quasi-Maxwellian equations and **Appendix B** is devoted to the special case of perturbations in Milne's background. We present here, for completeness, the case of scalar perturbations in Milne, which was not dealt with in the first paper [1].

2 Definitions and Notations

In the scalar case [1] the convention was simplified in order to make the resulting system of dynamical equations easier. For the vectorial and tensorial cases, however, we feel that the convention set in [4, 5] is more adequate. Therefore, we will present it again here for the sake of completeness.

Greek indices run into the set $\{0,1,2,3\}$, while Latin indices run into the set $\{1,2,3\}$. The metric of the background is given in the standard Gaussian form

$$\begin{aligned} ds^2 &= g_{\alpha\beta}(x)dx^\alpha dx^\beta = dt^2 - h_{\alpha\beta}(x)dx^\alpha dx^\beta \\ &= dt^2 - A^2(t)\left[\frac{dr^2}{1-Kr^2} + d\Theta^2 + sin^2\Theta\, d\varphi^2\right] \end{aligned} \tag{1}$$

defining in this way a class of privileged observers $V^\alpha = \delta^\alpha_0$. The projector $h_{\mu\nu}$, defined in the 3-dimensional space, is given by

$$h_{\mu\nu} = g_{\mu\nu} - V_\mu V_\nu.$$

We will define the 3-dimensional quantities (projected by $h_{\mu\nu}$) with the symbol $\hat{()}$; thus,

$$\hat{X}_\alpha \equiv h_\alpha{}^\beta X_\beta,$$

denotes a projection into the 3-geometry. For the same reasoning, we define the operator $\hat{\nabla}_\alpha$ as the covariant derivative in the 3-geometry. This operator is given by

$$\begin{aligned} &\hat{\nabla}_\alpha X_\beta = h_\alpha{}^\mu h_\beta{}^\nu X_{\nu;\mu} \\ &\hat{\nabla}_\alpha \hat{\nabla}_\beta X_\gamma = h_\alpha{}^\mu h_\beta{}^\nu h_\gamma{}^\lambda (h_\nu{}^\varepsilon h_\lambda{}^\tau (h_\tau{}^\sigma X_\sigma)_{;\varepsilon})_{;\mu}. \end{aligned} \tag{2}$$

We denote by (;) the covariant 4-dimensional derivatives (respective to the metric $g_{\mu\nu}$) and by a dot $\dot{()}$ its projection onto the velocity V^μ. Partial derivatives will be written as a comma. The symmetry convention is defined as follows:

$$X_{(\alpha\beta)} \equiv X_{\alpha\beta} + X_{\beta\alpha}$$

$$X_{[\alpha\beta]} \equiv X_{\alpha\beta} - X_{\beta\alpha}.$$

Riemann, Ricci and curvature tensors are written, in the 3-geometry of the background, as:

$$\hat{R}_{\alpha\beta\mu\nu} = \tfrac{K}{A^2}\,(h_{\alpha\mu} h_{\beta\nu} - h_{\alpha\nu} h_{\beta\mu})$$

$$\hat{R}_{\beta\nu} = \tfrac{2K}{A^2}\, h_{\beta\nu}$$

$$\hat{R} = \tfrac{6K}{A^2},$$

and the relations of definition hold for the 3-geometry:

$$\hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{X}_\gamma - \hat{\nabla}_\beta \hat{\nabla}_\alpha \hat{X}_\gamma = -\hat{R}^\lambda{}_{\gamma\beta\alpha} \hat{X}_\lambda$$

$$\hat{\nabla}_\alpha \hat{\nabla}_\beta \hat{Y}_{\gamma\tau} - \hat{\nabla}_\beta \hat{\nabla}_\alpha \hat{Y}_{\gamma\tau} = -\hat{R}^\lambda{}_{\gamma\beta\alpha} \hat{Y}_{\lambda\tau} - \hat{R}^\lambda{}_{\tau\beta\alpha} \hat{Y}_{\gamma\lambda}.$$

The relation between the 3-dimensional Laplacian ($\hat{\nabla}^2$) and the 4-dimensional one is given as follows:

$$\hat{\nabla}^2 \hat{X}_\alpha = \left(\frac{\theta}{3}\right)^2 \hat{X}_\alpha + h_\alpha^\beta \nabla^2 \hat{X}_\beta.$$

The completely skewsymmetric tensor, $\eta_{\alpha\beta\mu\nu}$, is defined by the relations

$$\eta_{\alpha\beta\mu\nu} = \sqrt{-g}\,\varepsilon_{\alpha\beta\mu\nu}$$

$$\eta^{\alpha\beta\mu\nu} = -\frac{1}{\sqrt{-g}}\,\varepsilon^{\alpha\beta\mu\nu},$$

where $g \equiv det(g_{\mu\nu})$ and ε is the completely skewsymmetric pseudo-tensor. The following relations are also useful:

$$\eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\lambda\tau\sigma} = -\delta^{\alpha\beta\mu\nu}_{\gamma\lambda\tau\sigma}$$

$$\eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\lambda\tau\nu} = -\delta^{\alpha\beta\mu}_{\gamma\lambda\tau}$$

$$\eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\lambda\mu\nu} = -2\,\delta^{\alpha\beta}_{\gamma\lambda}$$

$$\eta^{\alpha\beta\mu\nu}\,\eta_{\gamma\beta\mu\nu} = -6\,\delta^{\alpha}_{\gamma}$$

$$\eta^{\alpha\beta\mu\nu}\,\eta_{\alpha\beta\mu\nu} = -24.$$

We then introduce the fundamental harmonic basis of the functions projected onto the 3-surface

$$\{\hat{Q}(x)\} \qquad \{\hat{P}_\alpha(x)\} \qquad \{\hat{U}_{\alpha\beta}(x)\}. \tag{3}$$

The scalar function $\hat{Q}(x)$ is defined by

$$\frac{\partial}{\partial t}\hat{Q} = 0 \qquad \hat{\nabla}^2 \hat{Q} = \frac{m}{A^2}\,\hat{Q}, \tag{4}$$

with

$$m = \begin{cases} -q^2 - 1, & 0 < q < \infty, \quad K = -1 \text{ (open)} \\ -q, & 0 < q < \infty, \quad K = \ \ 0 \text{ (plane)} \\ -n^2 + 1, & n = 1, 2, \ldots, \quad K = +1 \text{ (closed)} \end{cases} \tag{5}$$

where m is the wave number of the scalar eigenfunction, and $\hat{\nabla}^2 \equiv \hat{\nabla}^\alpha\,\hat{\nabla}_\alpha$ is the usual Laplacian operator, projected onto the 3-surface.

Rigorous notation would demand us to write $\hat{Q}^{\{m\}}$ for $\hat{Q}$, and similarly for the vector $\hat{P}_\alpha$ and tensor $\hat{U}_{\alpha\beta}$ basis. However, our analysis will be restricted to **linear perturbations** of a **spatially homogeneous background**; therefore only the perturbations should be expanded in that basis, and no expression involving products of the basis will be considered. Then our expressions are valid for all m except when written explicitly; this allows us to drop such index throughout the paper (as it was done in [1]).

The scalar basis includes the vector and tensor quantities

$$\begin{gathered} \hat{Q}_\alpha(x) := \hat{Q}_{,\alpha}(x) \qquad \hat{Q}_{\alpha\beta}(x) := \hat{\nabla}_\beta\,\hat{Q}_\alpha(x) \\ \hat{M}_{\alpha\beta}(x) := \frac{A^2}{m}\hat{Q}_{\alpha\beta}(x) - \frac{1}{3}g_{\alpha\beta}(x)\hat{Q}(x), \end{gathered} \tag{6}$$

suited to expand spatial vectors, symmetric tensors and symmetric traceless tensors, respectively. We also note that these objects are spatial ones, and so the metric $g_{\alpha\beta}$ (or equivalently the projector $h_{\alpha\beta}$) raises and lowers their indices. The same mention applies to the remaining basis objects that follow.

The vector basis $\hat{P}_\alpha(x)$ is defined by the following relations:

$$\begin{aligned} &\hat{P}_\mu V^\mu = 0 \\ &\dot{\hat{P}}^\mu = 0 \\ &\hat{\nabla}^\alpha \hat{P}_\alpha = 0 \\ &\hat{\nabla}^2 \hat{P}_\alpha = \frac{m}{A^2} \hat{P}_\alpha, \end{aligned} \tag{7}$$

where the eigenvalue (again denoted by m, despite the fact that this eigenvalue and the scalar basis one have no relation at all) is given by

$$m = \begin{cases} -q^2 - 2, & 0 < q < \infty, \quad K = -1 \text{ (open)} \\ -q, & 0 < q < \infty, \quad K = \ \ 0 \text{ (plane)} \\ -n^2 + 2, & n = 2, 3, \ldots, \quad K = +1 \text{ (closed)} \end{cases} \tag{8}$$

From this basis it is possible to derive a pseudo-vector and a tensor:

$$\begin{aligned} &\hat{P}^{*\alpha} \equiv \eta^{\alpha\beta\mu\nu} V_\beta \hat{P}_{\mu\nu} \\ &\hat{P}_{\alpha\beta} \equiv \hat{\nabla}_\beta \hat{P}_\alpha \\ &\hat{P}^*_{\alpha\beta} \equiv \hat{\nabla}_\beta \hat{P}^*_\alpha, \end{aligned} \tag{9}$$

suited to developing pseudo-vectors and tensors.

The following vectorial relations are useful in obtaining the dynamical equations:

$$
\begin{aligned}
&\dot{\hat{P}}_{(\alpha\beta)} = -\tfrac{1}{3}\,\theta\,\hat{P}_{(\alpha\beta)}\\
&\dot{\hat{P}}^{*}_{(\alpha\beta)} = -\tfrac{2}{3}\,\theta\,\hat{P}^{*}_{(\alpha\beta)}\\
&\dot{\hat{P}}^{*}_{\alpha} = -\tfrac{1}{3}\,\theta\,\hat{P}^{*}_{\alpha}\\
&\hat{\nabla}^{\beta}\,\hat{P}_{(\alpha\beta)} = \tfrac{1}{A^2}\,(m-2K)\,\hat{P}_{\alpha}\\
&\hat{\nabla}^{\beta}\,\hat{P}^{*}_{(\alpha\beta)} = \tfrac{1}{A^2}\,(m-2K)\,\hat{P}^{*}_{\alpha}\\
&\hat{\nabla}^{2}\,\hat{P}^{*}_{\alpha} = \tfrac{m}{A^2}\,\hat{P}^{*}_{\alpha}\\
&\eta^{\alpha\beta\gamma\varepsilon}\,V_{\beta}\,\hat{P}^{*}_{\gamma;\varepsilon} = \tfrac{1}{A^2}\,(m+2K)\,\hat{P}^{\alpha}\\
&h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\eta_{\mu}{}^{\lambda\gamma\tau}\,V_{\tau}\,\hat{\nabla}_{\gamma}\,\hat{P}_{(\nu\lambda)} = h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\hat{P}^{*}_{\mu\nu}\\
&h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\eta_{\mu}{}^{\lambda\gamma\tau}\,V_{\tau}\,\hat{\nabla}_{\gamma}\,\hat{P}_{[\nu\lambda]} = -h^{\mu}_{(\alpha}\,h^{\nu}_{\beta)}\,\hat{P}^{*}_{\mu\nu}.
\end{aligned}
\tag{10}
$$

The tensorial basis $\hat{U}_{\alpha\beta}(x)$ is defined by the relations

$$
\begin{aligned}
&\dot{\hat{U}}_{\alpha\beta} = 0\\
&h^{\mu\nu}\,\hat{U}_{\mu\nu} = 0\\
&\hat{\nabla}^{\mu}\,\hat{U}_{\mu} = 0\\
&\hat{U}_{\alpha\beta} = \hat{U}_{\beta\alpha}\\
&\hat{\nabla}^{2}\,\hat{U}_{\alpha\beta} = \tfrac{m}{A^2}\,\hat{U}_{\alpha\beta},
\end{aligned}
\tag{11}
$$

where the new eigenvalue m has the following spectrum

$$
m = \begin{cases} -q^2-3, & 0<q<\infty, \quad K=-1 \text{ (open)}\\ -q, & 0<q<\infty, \quad K=\ \ 0 \text{ (plane)}\\ -n^2+3, & n=3,4,\ldots, \quad K=+1 \text{ (closed)} \end{cases}
\tag{12}
$$

Using the tensor basis we can define the dual tensor

$$
\hat{U}^{*}_{\mu\nu} \equiv \frac{1}{2}\,h^{\alpha}_{(\mu}\,h^{\beta}_{\nu)}\,\eta_{\beta}{}^{\lambda\varepsilon\gamma}\,V_{\lambda}\,\hat{\nabla}_{\varepsilon}\,\hat{U}_{\gamma\alpha}.
\tag{13}
$$

The tensorial relations below are employed in obtaining the dynamical equations system:

$$\dot{\hat{U}}^{*}_{\alpha\beta} = -\tfrac{1}{3}\,\theta\,\hat{U}^{*}_{\alpha\beta}$$
$$\hat{U}^{**}_{\alpha\beta} = \left(\tfrac{m}{A^2} + \rho - \tfrac{1}{3}\,\theta^2\right)\,\hat{U}_{\alpha\beta} = \tfrac{1}{A^2}\,(m + 3K)\,\hat{U}_{\alpha\beta}, \tag{14}$$

where we have used the constraint relation below,

$$\frac{K}{A^2} - \frac{1}{3}\,\rho + \left(\frac{\theta}{3}\right)^2 = 0, \tag{15}$$

valid in the FRW background. We remark that it involves the energy density ρ and the expansion θ, which satisfies:

$$\theta = 3\,\frac{\dot{A}}{A}. \tag{16}$$

The following auxiliary relations are also useful:

$$\dot{\theta} = -\tfrac{1}{3}\,\theta^2 - \tfrac{1}{2}\,(\rho + 3p)$$
$$\dot{\rho} = -\theta\,(\rho + p). \tag{17}$$

With the above basis we are able to expand any *good* perturbed quantity as

$$\begin{aligned}
\delta\omega_\alpha &= \Omega_{(1)}(t)\,\hat{P}^{*}_{\alpha}\\
\delta q_\alpha &= q_{(1)}(t)\,\hat{P}_{\alpha}\\
\delta a_\alpha &= \Psi_{(1)}(t)\,\hat{P}_{\alpha}\\
\delta V_\alpha &= V_{(1)}(t)\,\hat{P}_{\alpha}\\
\delta\sigma_{\alpha\beta} &= \Sigma_{(1)}(t)\,\hat{P}_{(\alpha\beta)} + \Sigma_{(2)}(t)\,\hat{U}_{\alpha\beta}\\
\delta H_{\alpha\beta} &= H_{(1)}(t)\,\hat{P}^{*}_{(\alpha\beta)} + H_{(2)}(t)\,\hat{U}^{*}_{\alpha\beta}\\
\delta E_{\alpha\beta} &= E_{(1)}(t)\,\hat{P}_{\alpha\beta} + E_{(2)}(t)\,\hat{U}_{\alpha\beta}\\
\delta\Pi_{\alpha\beta} &= \Pi_{(1)}(t)\,\hat{P}_{(\alpha\beta)} + \Pi_{(2)}(t)\,\hat{U}_{\alpha\beta},
\end{aligned} \tag{18}$$

where we insert auxiliary indices (1), (2) in order to distinguish between vector and tensor components, respectively. We also note that the relations (18) do not contain the scalar terms, already studied in [1].

3 Vector Perturbations of FRW Geometry

The vector part of the perturbed geometry is obtained from (18) by assuming all intrinsically tensorial terms (indexed by 2) to vanish identically. This is possible because of linear independence of the harmonic basis. The remaining terms have the index (1), which will be dropped in this section. Also, in order to get simpler equations, we will approximate the thermodynamic equation,

$$\tau\,\dot{\Pi}_{\alpha\beta}+\Pi_{\alpha\beta}=\xi\,\sigma_{\alpha\beta},\tag{19}$$

to the limit of small relaxation time τ (adiabatic approximation) and constant viscosity coefficient ξ to get

$$\delta\Pi_{\alpha\beta}=\xi\,\delta\sigma_{\alpha\beta}\rightsquigarrow\Pi=\xi\,\Sigma.\tag{20}$$

The vorticity can be written in terms of the 3-velocity as

$$\delta\omega_{\alpha}=-\frac{1}{2}\delta V_{\alpha}\rightsquigarrow V=-2\,\Omega.\tag{21}$$

We will denote by $(\chi_r,\,\tilde{\Phi}_s)$ the fundamental dynamical and constraint equations, respectively. Introducing the reduced form of equations (18) and equations (20)–(21) into the perturbed Quasi-Maxwellian equations (see Appendix B) and making use of equations (7)–(10), we get

$$\begin{aligned}
\chi_1 &:= \dot{E}-\tfrac{1}{2}\xi\,\dot{\Sigma}+\tfrac{2}{3}\theta\,E+\tfrac{1}{2}(\rho+p)\,\Sigma+\tfrac{1}{2A^2}\,(m+2K)\,H+\tfrac{1}{4}q=0\\
\chi_2 &:= \dot{\Sigma}+\left(\tfrac{\theta}{3}+\tfrac{\xi}{2}\right)\,\Sigma+E-\tfrac{1}{2}\,\Psi=0\\
\chi_3 &:= \dot{\Omega}+\tfrac{1}{3}\theta\,\Omega+\tfrac{1}{2}\,\Psi=0\\
\chi_4 &:= \dot{H}+\tfrac{1}{3}\theta\,H-\tfrac{1}{2}\,E-\tfrac{1}{4}\xi\,\Sigma=0\\
\chi_5 &:= \dot{q}+\tfrac{4}{3}\theta\,q+\tfrac{1}{A^2}\,(m-2K)\,\xi\,\Sigma+2\dot{p}\,\Omega+(\rho+p)\,\Psi=0,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Phi}_1 &:= \Sigma+\Omega+2H=0\\
\tilde{\Phi}_2 &:= \tfrac{1}{A^2}\,(m-2K)\,E-\tfrac{1}{2A^2}\,(m-2K)\,\xi\,\Sigma+\tfrac{2}{3}\,\theta\,(\rho+p)\,\Omega-\tfrac{1}{3}\theta\,q=0\\
\tilde{\Phi}_3 &:= \tfrac{1}{A^2}\,(m-2K)\,H-(\rho+p)\,\Omega+\tfrac{1}{2}\,q=0\\
\tilde{\Phi}_4 &:= \tfrac{1}{A^2}\,(m-2K)\,\Sigma+\left\{\tfrac{1}{A^2}\,(m+2K)+4\,\left(\tfrac{\theta}{3}\right)^2+\tfrac{2}{3}\,(\rho+p)\right\}\,\Omega-q=0.
\end{aligned}$$

It can be easily shown, by making use of equations (15)–(17), that constraint $\tilde{\Phi}_4$ is not essential, since it is written in terms of $\tilde{\Phi}_1$ and $\tilde{\Phi}_3$. Indeed, we have:

$$\tilde{\Phi}_4=\frac{1}{A^2}\,(m-2K)\,\tilde{\Phi}_1-2\,\tilde{\Phi}_3.\tag{22}$$

We also note that we can write constraint $\tilde{\Phi}_2$ in a simpler form as

$$\Phi_2:=\frac{A^2}{(m-2K)}\,\left\{\tilde{\Phi}_2+\frac{2}{3}\,\theta\,\tilde{\Phi}_3\right\}=E-\frac{1}{2}\,\xi\,\Sigma+\frac{2}{3}\,\theta\,H=0.\tag{23}$$

The fundamental differential system is now written as

$$\chi_1 := \dot{E} - \tfrac{1}{2}\xi\,\dot{\Sigma} + \tfrac{2}{3}\theta\,E + \tfrac{1}{2}(\rho+p)\,\Sigma + \tfrac{1}{2A^2}(m+2K)\,H + \tfrac{1}{4}q = 0$$

$$\chi_2 := \dot{\Sigma} + \left(\tfrac{\theta}{3} + \tfrac{\xi}{2}\right)\,\Sigma + E - \tfrac{1}{2}\Psi = 0$$

$$\chi_3 := \dot{\Omega} + \tfrac{1}{3}\theta\,\Omega + \tfrac{1}{2}\Psi = 0 \tag{24}$$

$$\chi_4 := \dot{H} + \tfrac{1}{3}\theta\,H - \tfrac{1}{2}E - \tfrac{1}{4}\xi\,\Sigma = 0$$

$$\chi_5 := \dot{q} + \tfrac{4}{3}\theta\,q + \tfrac{1}{A^2}(m-2K)\,\xi\,\Sigma - 2\dot{p}\,\Omega + (\rho+p)\,\Psi = 0,$$

and

$$\Phi_1 := \Sigma + \Omega + 2H = 0$$

$$\Phi_2 := E - \tfrac{1}{2}\xi\,\Sigma + \tfrac{2}{3}\theta\,H = 0 \tag{25}$$

$$\Phi_3 := \tfrac{1}{A^2}(m-2K)\,H - (\rho+p)\,\Omega + \tfrac{1}{2}q = 0.$$

It could be argued that the acceleration Ψ should be eliminated from the dynamical system by using the definition

$$a_\alpha = \dot{V}_\alpha = V_{\alpha;\beta}\,V^\beta.$$

If this is done we obtain

$$\Psi\,\hat{P}_\alpha = \left(\dot{V} + \frac{\theta}{3}V\right)\,\hat{P}_\alpha - \delta\Gamma^0_{0\alpha}.$$

However, it is easily proven (see Reference [1]) that

$$\delta\Gamma^0_{0\alpha} = \frac{1}{2}\,(\delta\,g_{00})_{,\alpha} = (\delta\,V_0)_{,\alpha},$$

which is zero in the vector basis. Then we have, making use of Equation (21), the following relation:

$$\Psi = -2\,\dot{\Omega} - \frac{2}{3}\,\theta\,\Omega,$$

which is precisely the dynamical equation χ_3. The variable Ψ must then be eliminated by means of losing a degree of freedom. This way we get physically motivated (*i.e.*, by observation) algebraic relations between acceleration and other selected variables. We have three cases to consider separately.

The first possible choice is to admit an isotropic model for the cosmological perturbation. In such a case there is no shear, and from this the anisotropic pressure vanishes too. Therefore χ_2 becomes

$$\Psi = 2\,E. \tag{26}$$

The second possibility is to admit that no vorticity should be taken into account. As it has been known for long, the presence of a non vanishing vorticity usually brings together with it some troubles related to causality violation. So we motivate this case by eliminating the main source of causality breakdown. In this case we have

$$\Omega = 0,$$

and χ_3 then results

$$\Psi = 0. \tag{27}$$

Another possibility is to impose the physical source of curvature to be a Stokesian fluid. This means that the energy flux (heat flux in this case) vanishes. Despite the fact that we can always set this quantity to zero by a suitable choice of observers, it actually represents a true restriction, for our equations are written in such a way that no observer changes can be performed — that is, we have already fixed the observer by imposing the particle flux to vanish. Now χ_5 yields

$$\Psi = -\frac{1}{A^2}(m-2K)\frac{\xi}{(\rho+p)}\Sigma + 2\lambda\theta\Omega, \tag{28}$$

with

$$(\rho+p) = (1+\lambda)\rho \neq 0 \qquad \lambda \equiv const,$$

a relation that eliminates Ψ for all but the de Sitter background. All three possibilities will have their respective dynamics and Hamiltonian treatment investigated in a later section.

4 Permanence of Constraints

Since we obtained a constrained differential system, given by Equations (24) and (25), it is useful to consider wether constraints are automatically preserved or not. If one derives the expressions (25) and inserts into the results the relations (15)–(17), one gets directly

$$\begin{aligned}
\dot{\Phi}_1 &= \chi_2 + \chi_3 + 2\chi_4 - \tfrac{\theta}{3}\Phi_1 \\
\dot{\Phi}_2 &= \chi_1 - \tfrac{2}{3}\theta\chi_4 - \tfrac{1}{2}(\rho+p)\Phi_1 - \tfrac{\theta}{3}\Phi_2 + \tfrac{1}{2}\Phi_3 + \\
&\quad - \left[\tfrac{K}{A^2} + \left(\tfrac{\theta}{3}\right)^2 - \tfrac{1}{3}\rho\right](\Omega + 2H) \\
\dot{\Phi}_3 &= -(\rho+p)\chi_3 + \tfrac{1}{A^2}(m-2K)\chi_4 + \tfrac{1}{2}\chi_5 + \\
&\quad - \tfrac{1}{2A^2}(m-2K)\Phi_2 - \tfrac{2}{3}\theta\Phi_3.
\end{aligned} \tag{29}$$

Thus, it follows that no secondary constraint[1] (SC) appears in the case of vector perturbations. One should expect this, since this result reflects the fact that our basic (Quasi-Maxwellian) equations are dynamically equivalent to Einstein's field equations[2], which are complete.

[1] Terminology due to Bergmann relative to Dirac's [6] work on constrained systems.

[2] That is, if Einstein's equations are fulfilled in an arbitrary complete 3-dimensional spacelike surface, Quasi-Maxwellian equations propagate them throughout the whole spacetime.

5 Hamiltonian Treatment of the Vectorial Solution

If we keep all degrees of freedom, as we mentioned before, the simplest solution for Equations (24)–(25) is then to consider Ψ as a small arbitrary function of the background — *i.e.*, $\Psi = \Psi(t)$ — which can also be parametrized by the perturbation wavelength m.

The constraints can now be used to eliminate three of the five variables, and the most suited pair for this solution is (Σ, Ω). The resulting free dynamics is

$$\begin{aligned} \dot{\Sigma} &= -\left(\tfrac{2}{3}\theta + \xi\right)\Sigma - \tfrac{\theta}{3}\Omega + \tfrac{1}{2}\Psi \\ \dot{\Omega} &= -\tfrac{\theta}{3}\Omega - \tfrac{1}{2}\Psi, \end{aligned} \tag{30}$$

directly integrable as

$$\begin{aligned} \Sigma(t) &= A^{-2}(t)\,\mathrm{e}^{-\xi t}\left\{\alpha + \int_{-\infty}^{t} A^{2}(t')\,\mathrm{e}^{\xi t'}\left[\tfrac{\theta(t')}{3}\,\Omega(t') + \tfrac{1}{2}\,\Psi(t')\right] dt'\right\} \\ \Omega(t) &= A^{-1}(t)\left\{\beta - \int_{-\infty}^{t} \tfrac{1}{2}\,A(t')\,\Psi(t')\,dt'\right\}. \end{aligned} \tag{31}$$

Solution (31) can be thought of as a particular case of an arbitrary linear relation[3] between Ψ and the fundamental variables,

$$\Psi = y(t)\,Q + z(t)\,P + g(t), \tag{32}$$

where

$$y(t) := \frac{\partial\Psi}{\partial Q}, z(t) := \frac{\partial\Psi}{\partial P}$$

and (Q, P) is a pair of canonical variables (as we shall see) that describe the vector perturbations, given by

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \Sigma \\ \Omega \end{pmatrix} \qquad \begin{pmatrix} \Sigma \\ \Omega \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix} \tag{33}$$

where $\Delta \equiv ad - bc \neq 0$.

The choice for the above variables is motivated by traditional results of perturbations assuming a perfect fluid law; within this assumption both the vorticity and the shear are essential variables: none of them may vanish, or all the system turns out to be trivial (see Ref. [7]). In the more general case such result does not apply.

Introducing relation (32) into the dynamics given by Equations (24) they may be written in terms of (Q, P) as

$$\begin{aligned} \dot{Q} &= \left\{\dot{a} - \left(\tfrac{2}{3}\theta + \xi\right) a\right\}\Sigma + \left\{\dot{b} - \tfrac{\theta}{3}(a+b)\right\}\Omega + \tfrac{1}{2}(a-b)\,\Psi \\ \dot{P} &= \left\{\dot{c} - \left(\tfrac{2}{3}\theta + \xi\right) c\right\}\Sigma + \left\{\dot{d} - \tfrac{\theta}{3}(c+d)\right\}\Omega + \tfrac{1}{2}(c-d)\,\Psi. \end{aligned} \tag{34}$$

[3]Linearity is a requirement in order to preserve coherence with our basic assumption of linear perturbations approximation. For the understanding of the physical meaning of such a relation see the examples given in Section 3.

To ensure that we are actually working with canonically conjugated variables we write the Hamiltonian constraint

$$
\begin{aligned}
\Phi &:= \Delta\left(\frac{\partial\dot Q}{\partial Q}+\frac{\partial\dot P}{\partial P}\right)\\
&= \left\{\dot a-\left(\frac{2}{3}\theta+\xi\right)a\right\}d-\left\{\dot b-\frac{\theta}{3}(a+b)\right\}c-\left\{\dot c-\left(\frac{2}{3}\theta+\xi\right)c\right\}b\\
&+ \left\{\dot d-\frac{\theta}{3}(c+d)\right\}a+\frac{\Delta}{2}(a-b)\,y+\frac{\Delta}{2}(c-d)\,z\\
&= \dot\Delta-(\theta+\xi)\,\Delta+\frac{1}{2}\left[(a-b)\,y+(c-d)\,z\right]\Delta,
\end{aligned}
\tag{35}
$$

and set the solution of $\Phi=0$ as

$$
\Delta=\frac{A^3(t)}{A_0}\mathrm{e}^{\xi t},a=d=\Delta^{1/2},
\tag{36}
$$

with $A_0 = \mathit{const}$. It will indeed be a solution if

$$
(a-b)\,y+(c-d)\,z=0
$$

holds, which leads to the following three possibilities:

$$
\begin{array}{llllll}
i) & y\neq 0,\,\forall z & \rightarrow & f\equiv\frac{z}{y}, & b=a(1-f), & c=0;\\
ii) & y=0,\,z\neq 0 & \rightarrow & & b=0, & c=a;\\
iii) & y=z=0 & \rightarrow & & b=0, & c=0.
\end{array}
\tag{37}
$$

For the first case the dynamics results

$$
\begin{aligned}
\dot Q &= \left\{\frac{\dot a}{a}-\frac{2}{3}\theta-\xi+\frac{a}{2}z\right\}Q\\
&+ \left\{-\frac{\theta}{3}f+\xi\,(1-f)-\dot f-\frac{a}{2}f\,z\right\}P+\frac{a}{2}f\,g
\end{aligned}
\tag{38}
$$

$$
\dot P = -\frac{a}{2}y\,Q+\left\{\frac{\dot a}{a}-\frac{\theta}{3}-\frac{a}{2}z\right\}P-\frac{a}{2}g,
\tag{39}
$$

described by the Hamiltonian

$$
\begin{aligned}
\mathcal{H}(Q,P) &= \frac{a}{4}Q^2+\frac{1}{2}\left\{-\frac{\theta}{3}f+\xi\,(1-f)-\dot f-\frac{a}{2}f\,z\right\}P^2\\
&- \left\{\frac{\dot a}{a}-\frac{\theta}{3}-\frac{a}{2}z\right\}Q\,P+\frac{a}{2}(Q+f\,g\,P).
\end{aligned}
\tag{40}
$$

For the second case the dynamics becomes

$$
\begin{aligned}
&\dot Q=\left\{\tfrac{\dot a}{a}-\tfrac{\theta}{3}-\xi\right\}Q+\left\{-\tfrac{\theta}{3}+\tfrac{a}{2}z\right\}P+\tfrac{1}{2}a\,g\\
&\dot P=\left\{\tfrac{\dot a}{a}-\tfrac{2}{3}\theta\right\}P,
\end{aligned}
\tag{41}
$$

associated with the Hamiltonian

$$\mathcal{H}(Q,P) = \frac{1}{2}\left\{-\frac{\theta}{3}+\frac{a}{2}z\right\}P^2 - \left\{\frac{\dot a}{a}-\frac{2}{3}\theta\right\}QP + \frac{a}{2}g\,P. \tag{42}$$

The third case is equivalent to the situation given in Equations (24) with new variables, and can be written as

$$\begin{aligned}\dot Q &= \left\{\tfrac{\dot a}{a}-\tfrac{2}{3}\theta-\xi+\tfrac{a}{2}y\right\}Q+\left\{-\tfrac{\theta}{3}+\tfrac{a}{2}y\right\}P+\tfrac{a}{2}g\\ \dot P &= -\tfrac{a}{2}y\,Q+\left\{\tfrac{\dot a}{a}-\tfrac{\theta}{3}-\tfrac{a}{2}y\right\}P-\tfrac{a}{2}g.\end{aligned} \tag{43}$$

The Hamiltonian associated with this case is then

$$\begin{aligned}\mathcal{H}(Q,P) = {}&\tfrac{a}{4}y\,Q^2+\tfrac{1}{2}\left\{-\tfrac{\theta}{3}+\tfrac{a}{2}y\right\}P^2-\left\{\tfrac{\dot a}{a}-\tfrac{\theta}{3}-\tfrac{a}{2}y\right\}QP+\\ &+\tfrac{a}{2}g\,(Q+P).\end{aligned} \tag{44}$$

6 The Specific Solutions

We proceed to study the three particular cases presented in Section (3), where the acceleration Ψ was eliminated by an explicit losing of a degree of freedom.

In the first case (the isotropic model), we have $\Sigma = 0$ and, using Equation (26) in the system (24)–(25) we obtain the following results:

$$\begin{aligned}H(t) &= \alpha\,A^{-2}(t)\\ E(t) &= -\tfrac{2\alpha}{3}\,\theta\,A^{-2}(t)\\ \Omega(t) &= -2\,\alpha\,A^{-2}(t)\\ \Psi(t) &= -\tfrac{4\alpha}{3}\,\theta\,A^{-2}(t)\\ q(t) &= -2\alpha\,A^{-2}(t)\,[(m-2K)\,A^{-2}(t)+2\,(\rho+p)],\end{aligned} \tag{45}$$

where α is an integration constant.

The second case (irrotational model, $\Omega = 0$) gives, upon substitution of Equation (27) in (24)–(25), the results below:

$$\begin{aligned}\Sigma(t) &= \beta\,A^{-2}(t)\,\exp^{-\xi t}\\ H(t) &= -\tfrac{\beta}{2}\,A^{-2}(t)\,\exp^{-\xi t}\\ E(t) &= \beta\,\left(\tfrac{\theta}{3}+\tfrac{\xi}{2}\right)\,A^{-2}(t)\,\exp^{-\xi t}\\ q(t) &= \beta\,(m-2K)\,A^{-4}(t)\,\exp^{-\xi t},\end{aligned} \tag{46}$$

where β is another integration constant.

Finally, for the third case (Stokesian fluid), with $q = 0$ and Equation (28) holding, the system (24)–(25) gives the reduced dynamics

$$\begin{aligned}\dot\Sigma &= -\left\{\tfrac{2}{3}\theta+\xi\left(1+\tfrac{1}{2A^2}\tfrac{(m-2K)}{(\rho+p)}\right)\right\}\Sigma-(1-3\lambda)\,\tfrac{\theta}{3}\,\Omega\\ \dot\Omega &= \tfrac{1}{2A^2}\tfrac{(m-2K)}{(\rho+p)}\,\xi\,\Sigma-(1+3\lambda)\,\tfrac{\theta}{3}\,\Omega.\end{aligned} \tag{47}$$

We again seek a Hamiltonian description with variables $(Q,\, P)$, using the same transformation given in Equation (33). Differentiating these expressions we find that Equations (47) can be written as

$$
\begin{aligned}
\dot{Q} &= \left\{\dot{a} - \left[\tfrac{2}{3}\,\theta + \xi\,\left(1 + \tfrac{1}{2A^2}\,\tfrac{(m-2K)}{(\rho+p)}\right)\right]\,a + \right.\\
&+ \left.\tfrac{1}{2A^2}\,\tfrac{(m-2K)}{(\rho+p)}\,\xi\,b\right\}\,\tfrac{1}{\Delta}\,(d\,Q - b\,P)\\
&+ \left\{\dot{b} - [a\,(1-3\lambda) + b\,(1+3\lambda)]\,\tfrac{\theta}{3}\right\}\,\tfrac{1}{\Delta}\,(-c\,Q + a\,P)\\
\dot{P} &= \left\{\dot{c} - \left[\tfrac{2}{3}\,\theta + \xi\,\left(1 + \tfrac{1}{2A^2}\,\tfrac{(m-2K)}{(\rho+p)}\right)\right]\,c\right.\\
&+ \left.\tfrac{1}{2A^2}\,\tfrac{(m-2K)}{(\rho+p)}\,\xi\,d\right\}\,\tfrac{1}{\Delta}\,(d\,Q - b\,P)\\
&+ \left\{\dot{d} - [c\,(1-3\lambda) + d\,(1+3\lambda)]\,\tfrac{\theta}{3}\right\}\,\tfrac{1}{\Delta}\,(-c\,Q + a\,P).
\end{aligned}
\tag{48}
$$

From Equations (48) we read the Hamiltonian constraint

$$
\begin{aligned}
\Phi &:= \Delta\,\left(\frac{\partial\dot{Q}}{\partial Q} + \frac{\partial\dot{P}}{\partial P}\right)\\
&= \dot{\Delta} - \left\{\frac{\theta}{3} + \xi\,\left(1 + \frac{1}{2A^2}\,\frac{(m-2K)}{(\rho+p)}\right)\right\}\,\Delta + \lambda\,\theta\,\Delta = 0,
\end{aligned}
\tag{49}
$$

whose solution is given by

$$
\Delta(t) = A^{(1-3\lambda)}(t)\,\mathrm{e}^{\xi\,\int_{-\infty}^{t}\left\{1+\frac{1}{2A^2}\,\frac{(m-2K)}{(1+\lambda)\,\rho(t')}\right\}\,dt'}\,, \tag{50}
$$

omitting the integration constant. We now set the Hamiltonian variables $(Q,\, P)$ as given by Equation (33) with

$$
a = d = \Delta^{1/2}
$$

$$
b = c = 0,
$$

where Δ is given by Equation (50). Therefore we finally obtain the dynamics

$$
\begin{aligned}
\dot{Q} &= \left\{\tfrac{\dot{a}}{a} - \tfrac{2}{3}\,\theta - \xi\,\left[1 + \tfrac{1}{2A^2}\,\tfrac{(m-2K)}{(\rho+p)}\right]\right\}\,Q - \left\{(1-3\lambda)\,\tfrac{\theta}{3}\right\}\,P\\
\dot{P} &= \left\{\tfrac{1}{2A^2}\,\tfrac{(m-2K)}{(\rho+p)}\,\xi\right\}\,Q + \left\{\tfrac{\dot{a}}{a} - (1-3\lambda)\,\tfrac{\theta}{3}\right\}\,P,
\end{aligned}
\tag{51}
$$

submitted to the constraint of vanishing heat flux

$$
Q = -\left\{1 + \frac{2A^2}{(m-2K)}\,(\rho+p)\right\}\,P. \tag{52}
$$

The associated Hamiltonian is then given by

$$
\begin{aligned}
\mathcal{H}(Q,P) = &-\frac{1}{2}\,\left\{\frac{1}{2A^2}\,\frac{(m-2K)}{(\rho+p)}\,\xi + \frac{2}{3}\,\frac{(1+3\lambda)\,\theta}{1 + \frac{2A^2}{(m-2K)}\,(\rho+p)}\right\}\,Q^2 + \\
&-(1-3\lambda)\,\frac{\theta}{3}\,P^2 - \left\{(1+\lambda)\,\frac{\theta}{2} + \frac{\xi}{2}\,\left[1 + \frac{1}{2A^2}\,\frac{(m-2K)}{(\rho+p)}\right]\right\}\,Q\,P.
\end{aligned}
\tag{53}
$$

As an example, equations of motion (51) can be explicitly integrated by taking into account Equation (52). Thus the system evolution follows

$$\dot{P} = -\left\{(1+9\lambda)\,\frac{\theta}{6} + \frac{\xi}{2}\left[1 + \frac{1}{2A^2}\,\frac{(m-2K)}{(\rho+p)}\right]\right\} P, \tag{54}$$

which can be readily integrated, and we finally find

$$\begin{aligned} Q &= -\gamma\,\left\{1 + \tfrac{2A^2}{(m-2K)}\,(\rho+p)\right\}\,A^{-\frac{(1+9\lambda)}{2}}\,\mathrm{e}^{-\frac{\xi}{2}\int_{-\infty}^{t}\left\{1+\frac{1}{2A^2(t')}\,\frac{(m-2K)}{(1+\lambda)\,\rho(t')}\right\}dt'} \\ P &= \gamma\,A^{-\frac{(1+9\lambda)}{2}}\,\mathrm{e}^{-\frac{\xi}{2}\int_{-\infty}^{t}\left\{1+\frac{1}{2A^2(t')}\,\frac{(m-2K)}{(1+\lambda)\,\rho(t')}\right\}dt'}, \end{aligned} \tag{55}$$

where γ is an integration constant. Returning to the physically relevant variables we find particularly that

$$\Omega(t) = \gamma\,A^{-(1+3\lambda)}\,\mathrm{e}^{-\xi\int_{-\infty}^{t}\left\{1+\frac{\lambda}{2A^2}\,\frac{(m-2K)}{(1+\lambda)\,\rho(t')}\right\}dt'}. \tag{56}$$

The perturbation in vorticity appears to diverge — thus breaking down our fundamental approach of the linear treatment — for perturbation wavelengths such that

$$m > 2\,K + 2\,(1+\lambda)\,A^2\,\rho. \tag{57}$$

The lower bound of the last term on the left hand side of Equation (57) is zero. Even when this is assumed to be true, however, Equation (8) shows that we always have $m < 2\,K$, and from this Ω goes to zero. Such a result could also be expected from the angular momentum conservation law.

7 Tensor Perturbations of FRW Geometry

Here we will proceed as in Section 3 in order to get an ordinary differential system which describes tensorial perturbations in terms of *good* variables. That is, we again consider decomposition (18), but now setting intrinsically vectorial terms (those indexed by 1) to zero. All the remaining terms are then indexed by 2, which will accordingly be dropped here as it was done for the vectorial case.

Under the properties (11)–(14) and again making use of relation (20) Quasi-Maxwellian equations (Appendix B) are written as:

$$\begin{aligned} \chi_1 &:= \dot{E} - \tfrac{\xi}{2}\,\dot{\Sigma} + \theta\,E - \tfrac{1}{2}\,\left[\tfrac{\theta}{3}\,\xi - (\rho+p)\right]\,\Sigma - \tfrac{1}{A^2}\,(m+3K)\,H = 0 \\ \chi_2 &:= \dot{H} + \tfrac{2}{3}\,\theta\,H + E + \tfrac{\xi}{2}\,\Sigma = 0 \\ \chi_3 &:= \dot{\Sigma} + \left(\tfrac{2}{3}\,\theta + \tfrac{\xi}{2}\right)\,\Sigma + E = 0, \end{aligned} \tag{58}$$

constrained to

$$\Phi_1 := H - \Sigma = 0. \tag{59}$$

We also know that Φ_1 is dynamically preserved as

$$\dot{\Phi}_1 = \chi_2 - \chi_3 - \frac{2}{3}\theta\,\Phi_1, \tag{60}$$

and from this we are, therefore, authorized to insert it into dynamics. So proceeding we get the unconstrained coupled differential system

$$\begin{aligned}
\chi_1 \;&:=\; \dot{E} + \left(\theta + \frac{\xi}{2}\right) E + \left\{\frac{1}{2}\left[\xi\,\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + (\rho + p)\right] + \right. \\
&\qquad \left. - \frac{1}{A^2}(m + 3K)\right\} H = 0 \\
\chi_2 \;&:=\; \dot{H} + \left(\frac{2}{3}\theta + \frac{\xi}{2}\right) H + E = 0.
\end{aligned} \tag{61}$$

It should be remarked that the coefficient H in χ_1 in the de Sitter background yields a positive [4] constant leading term, for times such that $\frac{1}{A^2} \simeq 0$. This feature will be important in Section 8.

We also stress that Equations (61) have no non trivial solution unless both $(E,\, H)$ are assumed to be non zero. That is, both variables are essential in describing tensor perturbations — it should be remembered that these variables constitute the electric and magnetic parts of Weyl tensor, allowing one to write "gravitational waves" for "tensor perturbations".

8 Hamiltonian Treatment of the Tensorial Solution

The basic system given by Equations (61) can be described in the Hamiltonian language, which provides a more elegant interpretation of the dynamical role of our variables. The link between it and perturbation theory has worth for its own. We thus introduce new variables

$$\begin{pmatrix} Q \\ P \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}, \tag{62}$$

where we suppose

$$\Delta := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv ad - bc \neq 0,$$

which is proven ***a posteriori*** to be actually correct. Therefore we can use the set $(Q,\, P)$ for $(E,\, H)$ in order to characterize the gravitational waves. Inserting definitions (62) into

[4] The Hubble constant, here translated to θ, is positive from astronomical observations, despite the fact that its magnitude is not universally agreed upon. Thermodynamical reasoning ensures the nonnegativeness of parameter ξ.

Equations (61) we eventually get

$$
\begin{aligned}
\dot{Q} &= \left\{\dot{a} - a\,\left(\theta + \tfrac{\xi}{2}\right) - b\right\}\,E + \left\{\dot{b} - b\,\left(\tfrac{2}{3}\theta + \tfrac{\xi}{2}\right)\right.\\
&\quad \left. -a\,\left(\tfrac{1}{2}\,\left[\xi\,\left(\tfrac{\theta}{3} + \tfrac{\xi}{2}\right) + (\rho + p)\right] - \tfrac{1}{A^2}\,(m + 3K)\right)\right\}\,H\\
\dot{P} &= \left\{\dot{c} - c\,\left(\theta + \tfrac{\xi}{2}\right) - d\right\}\,E + \left\{\dot{d} - d\,\left(\tfrac{2}{3}\theta + \tfrac{\xi}{2}\right)\right.\\
&\quad \left. -c\,\left(\tfrac{1}{2}\,\left[\xi\,\left(\tfrac{\theta}{3} + \tfrac{\xi}{2}\right) + (\rho + p)\right] - \tfrac{1}{A^2}\,(m + 3K)\right)\right\}\,H.
\end{aligned}
\tag{63}
$$

We also need to show that our variables are, in fact, canonically conjugated to each other, as suggested by notation. That is, we again make use of the Hamiltonian constraint,

$$
\Phi := \Delta\,\left(\frac{\partial \dot{Q}}{\partial Q} + \frac{\partial \dot{P}}{\partial P}\right) = \dot{\Delta} - \left(\frac{5}{3}\theta + \xi\right)\,\Delta = 0. \tag{64}
$$

A particular solution of Equation (64) is

$$
\Delta(t) = A^5(t)\,e^{\xi t}, \tag{65}
$$

and we then set

$$
\begin{aligned}
a &= \Delta^{\omega}\\
d &= \Delta^{(1-\omega)}\\
b &= c = 0,
\end{aligned}
\tag{66}
$$

where ω is an arbitrary constant.

With the choice (66), and using solution (65), system (63) becomes

$$
\begin{aligned}
\dot{P} &= -\left[\left(\tfrac{5}{3}\omega - 1\right)\,\theta + \left(\omega - \tfrac{1}{2}\right)\,\xi\right]\,P\\
&\quad -\Delta^{(1-2\omega)}\,Q\\
\dot{Q} &= -\left[\tfrac{\xi}{2}\,\left(\tfrac{\theta}{3} + \tfrac{\xi}{2}\right) - \tfrac{1}{2}(\rho + p) - \tfrac{1}{2A^2}\,(m + 3K)\right]\,\Delta^{(2\omega-1)}\,P +\\
&\quad + \left[\omega\,\left(\tfrac{5}{3}\theta + \xi\right) - \left(\theta + \tfrac{\xi}{2}\right)\right]\,Q.
\end{aligned}
\tag{67}
$$

From this we read directly the Hamiltonian

$$
\begin{aligned}
\mathcal{H}(Q,\,P) = \; & -\frac{1}{2}\,\Delta^{(2\omega-1)}\,\left[\frac{\xi}{2}\,\left(\frac{\theta}{3} + \frac{\xi}{2}\right) + \frac{1}{2}(\rho + p) - \frac{1}{A^2}(m + 3K)\right]\,P^2 +\\
& + \frac{1}{2}\,\Delta(1 - 2\omega)\,Q^2 + \left[\left(\frac{5}{3}\omega - 1\right)\,\theta + \left(\omega - \frac{1}{2}\right)\,\xi\right]\,PQ.
\end{aligned}
\tag{68}
$$

This result shows that de Sitter (θ = *const.*) geometry admits a tensor perturbation Hamiltonian of a typical harmonic oscillator with imaginary mass, which evidencies instability. This is obtained by setting the arbitrary constant parameter

$$
\omega = \frac{3}{2}\,\frac{(2\theta + \xi)}{(5\theta + 3\xi)},
$$

in the Hamiltonian (68). We thus recover the well known result of instability of de Sitter solution. The above result also shows, however, that the same remark applies to arbitrary Friedman-like background with no tensorial perturbation in anisotropic pressure tensor, $\xi = 0$. In such cases we set $\omega = 3/5$ to get

$$\mathcal{H}(Q,\, P)\,|_{\xi=0} = -\frac{1}{2}\,\Delta^{1/5}\left[(\rho + p) - \frac{1}{A^2}(m + 3K)\right] P^2 + \frac{1}{2}\,\Delta^{-1/5}\,Q^2, \tag{69}$$

where $(Q,\, P)$ are given by

$$\begin{aligned} Q &= A^3(t)\,\mathrm{e}^{\frac{3}{5}\xi t}\,E \\ P &= A^2(t)\,\mathrm{e}^{\frac{2}{5}\xi t}\,H. \end{aligned} \tag{70}$$

9 Conclusions

In this paper we continue our analysis of the Perturbations Theory of FRW universes using the Quasi-Maxwellian approach. As it was seen, this method is more convenient than the traditional one, which deals with Einstein equations in its standard form. The reason for this is the property of conformal flatness of FRW geometry, as discussed before (see [1]).

In anology with the study in [1] we have found that there is a complete set of *good* perturbed variables for each mode:

$$\mathcal{M}_A^{vector} = \{\Sigma, \Omega, q, \Psi\},$$

and

$$\mathcal{M}_A^{tensor} = \{E, H\}.$$

Each of them constitutes an independent dynamical system. However, the first system is not closed, since the variable Ψ cannot be written in terms of the other ones. In order to solve this system we have then to eliminate one of the variables involved, thus losing a degree of freedom.

We have also obtained the Hamiltonian formulation for both cases, and we address the possibility to canonically quantize the cosmological perturbations of FRW universes. This analysis is now under progress.

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Appendix A: Perturbed Quasi-Maxwellian Equations

The Quasi-Maxwellian equations for gravity were already shown in Reference [1]. We state here their perturbed linearized form only for the sake of completeness.

$$\begin{aligned}(\delta E^{\mu\nu})^{\cdot}\, h_\mu{}^\alpha h_\nu{}^\beta &+ \Theta\,(\delta E^{\alpha\beta}) - \frac{1}{2}(\delta E_\nu{}^{(\alpha})h^{\beta)}{}_\mu\, V^{\mu;\nu} \\ &+ \frac{\Theta}{3}\eta^{\beta\nu\mu\epsilon}\,\eta^{\alpha\gamma\tau\lambda}\, V_\mu V_\tau (\delta E_{\epsilon\lambda})\, h_{\gamma\nu} \\ &- \frac{1}{2}(\delta H_\lambda{}^\mu)_{;\gamma}\, h_\mu{}^{(\alpha}\eta^{\beta)\tau\gamma\lambda} V_\tau \\ &= -\frac{1}{2}(\rho+p)\,(\delta\sigma^{\alpha\beta}) \\ &+ \frac{1}{6}\,h^{\alpha\beta}\,(\delta q^\mu)_{;\mu} - \frac{1}{4}\,h^{\mu(\alpha}h^{\beta)\nu}\,(\delta q_\mu)_{;\nu} \\ &+ \frac{1}{2}\,h^{\mu\alpha}h^{\beta\nu}\,(\delta\Pi_{\mu\nu})^{\cdot} + \frac{1}{6}\,\Theta\,(\delta\Pi^{\alpha\beta})\end{aligned} \tag{71}$$

$$\begin{aligned}(\delta H^{\mu\nu})^{\cdot}\, h_\mu{}^\alpha h_\nu{}^\beta &+ \Theta\,(\delta H^{\alpha\beta}) - \frac{1}{2}(\delta H_\nu{}^{(\alpha})h^{\beta)}{}_\mu\, V^{\mu;\nu} \\ &+ \frac{\Theta}{3}\eta^{\beta\nu\mu\epsilon}\,\eta^{\alpha\lambda\tau\gamma}\, V_\mu V_\tau\,(\delta H_{\epsilon\gamma})\, h_{\lambda\nu} \\ &- \frac{1}{2}(\delta E_\lambda{}^\mu)_{;\tau}\, h_\mu{}^{(\alpha}\eta^{\beta)\tau\gamma\lambda}\, V_\gamma \\ &= \frac{1}{4}\,h^{\nu(\alpha}\eta^{\beta)\epsilon\tau\mu}\, V_\mu\,(\delta\Pi_{\nu\epsilon})_{;\tau}\end{aligned} \tag{72}$$

$$(\delta H_{\alpha\mu})_{;\nu}h^{\alpha\epsilon}h^{\mu\nu} = (\rho+p)\,(\delta\omega^\epsilon) - \frac{1}{2}\,\eta^{\epsilon\alpha\beta\mu}\, V_\mu\,(\delta q_\alpha)_{;\beta} \tag{73}$$

$$\begin{aligned}(\delta E_{\alpha\mu})_{;\nu}h^{\alpha\epsilon}\, h^{\mu\nu} &= \frac{1}{3}(\delta\rho)_{,\alpha}h^{\alpha\epsilon} - \frac{1}{3}\dot\rho\,(\delta V^\epsilon) \\ &- \frac{1}{3}\,\rho_{,0}\,(\delta V^0)\, V^\epsilon \\ &+ \frac{1}{2}\,h^\epsilon{}_\alpha\,(\delta\Pi^{\alpha\mu})_{;\mu} + \frac{\Theta}{3}\,(\delta q^\epsilon)\end{aligned} \tag{74}$$

$$(\delta\Theta)^{\cdot} + \dot\theta\,(\delta V^0) + \frac{2}{3}\Theta\,(\delta\Theta) - (\delta a^\alpha)_{;\alpha} = -\frac{(1+3\lambda)}{2}\,(\delta\rho) \tag{75}$$

$$\begin{aligned}(\delta\sigma_{\mu\nu})^{\cdot} &+ \frac{1}{3}h_{\mu\nu}(\delta a^\alpha)_{;\alpha} - \frac{1}{2}(\delta a_{(\alpha})_{;\beta)}\, h_\mu{}^\alpha\, h_\nu{}^\beta \\ &+ \frac{2}{3}\Theta\,(\delta\sigma_{\mu\nu}) = -(\delta E_{\mu\nu}) - \frac{1}{2}(\delta\Pi_{\mu\nu})\end{aligned} \tag{76}$$

$$(\delta\omega^{\mu})^{\cdot} + \frac{2}{3}\Theta\,(\delta\omega^{\mu}) = \frac{1}{2}\eta^{\alpha\mu\beta\gamma}\,(\delta a_{\beta})_{;\gamma}\,V_{\alpha} \tag{77}$$

$$\begin{aligned} \frac{2}{3}(\delta\Theta)_{,\lambda}\,h^{\lambda}{}_{\mu} &- \frac{2}{3}\dot{\Theta}\,(\delta V_{\mu}) + \frac{2}{3}\,\dot{\Theta}\,(\delta V^{0})\,\delta_{\mu}{}^{0} \\ &- (\delta\sigma^{\alpha}{}_{\beta} + \delta\omega^{\alpha}{}_{\beta})_{;\alpha} h^{\beta}{}_{\mu} = -(\delta q_{\mu}) \end{aligned} \tag{78}$$

$$(\delta\omega^{\alpha})_{;\alpha} = 0 \tag{79}$$

$$(\delta H_{\mu\nu}) = -\frac{1}{2}\,h^{\alpha}{}_{(\mu}\,h^{\beta}{}_{\nu)}((\delta\sigma_{\alpha\gamma})_{;\lambda} + (\delta\omega_{\alpha\gamma})_{;\lambda})\,\eta_{\beta}{}^{\epsilon\gamma\lambda}\,V_{\epsilon} \tag{80}$$

$$(\delta\rho)^{\cdot} + \dot{\rho}\,(\delta V^{0}) + \Theta\,(\delta\rho + \delta p) + (\rho + p)\,(\delta\Theta) + (\delta q^{\alpha})_{;\alpha} = 0 \tag{81}$$

$$\begin{aligned} \dot{p}\,(\delta V_{\mu}) &+ p_{,0}\,(\delta V^{0})\,\delta_{\mu}{}^{0} - (\delta p)_{,\beta}\,h^{\beta}{}_{\mu} + (\rho + p)\,(\delta a_{\mu}) \\ &+ h_{\mu\alpha}(\delta q^{\alpha})^{\cdot} + \frac{4}{3}\Theta\,(\delta q_{\mu}) + h_{\mu\alpha}\,(\delta\pi^{\alpha\beta})_{;\beta} = 0. \end{aligned} \tag{82}$$

Appendix B: Milne Background

There is a particular class of FRW geometries dealt with in this paper that seems worth to be explicitly examined. This is the case analysed by Milne and contains a portion of Minkowski geometry. The metric is then FRW-type, where the radius of the universe, the 3-curvature and the expansion are given respectively by:

$$\begin{aligned} A(t) &= t \\ K &= -1 \\ \theta &= \tfrac{3}{t}. \end{aligned} \tag{83}$$

We will present only the results which follow:

I- *Scalar Perturbations:*

If we consider the case of scalar perturbations, the vorticity should vanish, which implies that the magnetic part of Weyl conformal tensor will also be zero (as it was proved in [1]); thus we have:

$$\begin{aligned} \delta\omega_{ij} &= 0 \\ \delta H_{ij} &= 0. \end{aligned} \tag{84}$$

Following the notation used in [1], the other perturbed quantities are listed below:

(i) *Geometric Quantity:*

$$\delta E_{ij} = E(t)\,\hat{Q}_{ij}(\vec{x}).$$

(ii) *Kinematic Quantities:*

$$\begin{aligned} \delta V_0 &= -\delta V^0 = \tfrac{1}{2}\,\delta g_{00} = \tfrac{1}{2}\,\beta(t)\,Q(\vec{x}) + \tfrac{1}{2}\,Y(t) \\ \delta V_k &= V(t)\,Q_k(\vec{x}) \\ \delta a_k &= \Psi(t)\,Q_k(\vec{x}) \\ \delta\sigma_{ij} &= \Sigma(t)\,\hat{Q}_{ij}(\vec{x}) \\ \delta\theta &= B(t)\,Q(\vec{x}) + Z(t). \end{aligned}$$

(iii) *Matter Quantities:*

$$\begin{aligned} \delta\rho &= N(t)\,Q(\vec{x}) + L(t) \\ \delta\Pi_{ij} &= \xi\,\delta\sigma_{ij} = \xi\,\Sigma(t)\,\hat{Q}_{ij}(\vec{x}) \\ \delta p &= \lambda\,\delta\rho \\ \delta q_k &= q(t)\,Q_k(\vec{x}), \end{aligned}$$

where we have used again the proportionality relation between the perturbed anisotropic pressure and the shear; we also consider the standard formulation in which the perturbed pressure is proportional to the density. The quantity $\beta(t)$ is gauge-dependent and $Y(t)$, $Z(t)$ and $L(t)$ are homogeneous terms.

Making use of the Quasi-Maxwellian equations we obtain the system for the above quantities as:

$$\dot{E} = -\frac{\xi}{2} + \frac{\theta}{3} E + \frac{\xi\theta}{6} \Sigma + \frac{m}{2} q = 0 \tag{85}$$

$$\frac{2\theta^2}{3} \left(\frac{1}{3} - \frac{K}{m}\right) \left[E - \frac{\xi}{2} \Sigma\right] + N + \theta\, q = 0 \tag{86}$$

$$\dot{B} + \frac{2\theta}{3} B + \frac{\theta^2}{6} \beta(t) + \frac{\theta^2}{9} m\, \Psi + \frac{(1+3\lambda)}{2} N = 0 \tag{87}$$

$$\dot{\Sigma} + E + \frac{\xi}{2} \Sigma - m\, \Psi = 0 \tag{88}$$

$$V = \left(\frac{1}{3} - \frac{K}{m}\right) \Sigma - \frac{3}{\theta^2} B - \frac{9}{2\theta^2} q \tag{89}$$

$$\dot{N} + (1+\lambda)\, \theta\, N - \frac{\theta^2}{9} q = 0 \tag{90}$$

$$\dot{q} + \theta\, q - \lambda\, N - \frac{2\xi\theta^2}{9} \left(\frac{1}{3} - \frac{K}{m}\right) \Sigma = 0. \tag{91}$$

The dynamical equations on the homogeneous terms $Z(t)$ and $L(t)$ are written as:

$$\dot{Z} + \frac{2\theta}{3} Z + \frac{(1+3\lambda)}{2} L + \frac{\theta^2}{6} Y = 0 \tag{92}$$

$$\dot{L} + (1+\lambda)\, \theta\, L = 0. \tag{93}$$

Let us solve this system for the special simple case where $q = 0$. We have then (from eq. (91), the dynamical equation for q),

$$-\lambda\, N - \frac{2\xi\theta^2}{9} \left(\frac{1}{3} - \frac{K}{m}\right) \Sigma = 0. \tag{94}$$

Equations (83) and (90) give

$$N(t) = N_0\, t^{-3(1+\lambda)}, \tag{95}$$

where N_0 is a constant. From (94) and (95) we obtain

$$\Sigma(t) = -\frac{\lambda N_0}{2\xi} \left(\frac{1}{3} - \frac{K}{m}\right)^{-1} t^{-(1+3\lambda)}. \tag{96}$$

These results applied in equation (86) give

$$E(t) = -\frac{N_0}{6} \left(1 + \frac{3\lambda}{2}\right) \left(\frac{1}{3} - \frac{K}{m}\right)^{-1} t^{-(1+3\lambda)}. \tag{97}$$

Equation (85) is automatically valid if we make use of the above results for $N(t)$, $\Sigma(t)$ and $E(t)$. Equation (88) then gives $\Psi(t)$ as:

$$\Psi(t) = \frac{N_0}{2m}\left(\frac{1}{3} - \frac{K}{m}\right)^{-1}\left[\frac{\lambda(1+3\lambda)}{\xi}\,t^{-1} - \frac{(2+9\lambda)}{6}\right]t^{-(1+3\lambda)}. \tag{98}$$

It must be noted that the constant N_0 cannot be zero, since this would give a trivial result. Equations (87) and (89) give the quantities $B(t)$ and $V(t)$ in terms of $N(t)$, $\Psi(t)$ and $\Sigma(t)$, $B(t)$ respectively. Both quantities may be obtained if the gauge-dependent function $\beta(t)$ is chosen. They are therefore "bad"quantities to analyse. The minimal closed set of quantities for perturbations in Milne universe is

$$\mathcal{M}^{scalar}_{[A]} = \{E, \Sigma, N, \Psi\}.$$

The homogeneous part of $(\delta\rho)$, $L(t)$, is directly determined by equation (92):

$$L(t) = L_0\,t^{-3(1+\lambda)}, \tag{99}$$

where again L_0 denotes a constant. The function $Z(t)$, whose dynamics is given by equation (92), can only be integrated by choosing another homogeneous term ($Y(t)$). That completes the solution for the case $q = 0$.

We can analyse the behaviour of the solution above for different values of λ. The results are as follows:

1. $\lambda > -\frac{1}{3}$:

   E, Σ, N and Ψ go to zero when $t \longrightarrow \infty$

2. $\lambda = -\frac{1}{3}$:

   E, Σ and Ψ are constant; N goes to zero when $t \longrightarrow \infty$

3. $-1 < \lambda < -\frac{1}{3}$:

   E, Σ and Ψ diverge when $t \longrightarrow \infty$ and N goes to zero

4. $\lambda = -1$ (vacuum, cosmological constant Λ):

   E, Σ and Ψ diverge when $t \longrightarrow \infty$ and N is constant

5. $\lambda < -1$ (unphysical situation):

   E, Σ, N and Ψ diverge when $t \longrightarrow \infty$.

II- *Vector Perturbations:*

In this case the original dynamical system, Equations (24)–(25) give

$$\begin{aligned}
\chi_1 &:= \dot{E} - \tfrac{\xi}{2}\dot{\Sigma} + \tfrac{2}{3}\theta\,E + \tfrac{1}{2A^2}(m+2K)\,H + \tfrac{1}{4}q = 0\\
\chi_2 &:= \dot{\Sigma} + \left(\tfrac{\theta}{3} + \tfrac{\xi}{2}\right)\Sigma + E - \tfrac{1}{2}\Psi = 0\\
\chi_3 &:= \dot{\Omega} + \tfrac{\theta}{3}\Omega + \tfrac{1}{2}\Psi = 0\\
\chi_4 &:= \dot{H} + \tfrac{\theta}{3}H - \tfrac{1}{2}E - \tfrac{\xi}{4}\Sigma = 0\\
\chi_5 &:= \dot{q} + \tfrac{4}{3}\theta\,q + \tfrac{1}{A^2}(m-2K)\,\xi\,\Sigma = 0,
\end{aligned} \tag{100}$$

and

$$\begin{aligned}\Phi_1 &:= \Sigma + \Omega + 2\,H = 0\\ \Phi_2 &:= E - \tfrac{\xi}{2}\,\Sigma + \tfrac{2}{3}\,\theta\,H = 0\\ \Phi_3 &:= \tfrac{1}{A^2}\,(m - 2K)\,H + \tfrac{1}{2}\,q = 0.\end{aligned} \tag{101}$$

We will present here only the three cases dealt with in Section (6): isotropic, irrotational and Stokesian fluid. The results are as follows:

(i)*Isotropic Model* ($\Sigma = 0$):

$$\begin{aligned}E(t) &= \mu\, t^{-2}\\ H(t) &= \tfrac{\mu}{2}\, t^{-1}\\ \Psi(t) &= 2\,\mu\, t^{-2}\\ q(t) &= -(m+2)\,\mu\, t^{-3},\end{aligned} \tag{102}$$

where μ is an integration constant and we used $K = -1$. These functions of t diverge when t goes to zero and become null for infinite values of t.

(ii)*Irrotational Model* ($\Omega = 0$):

In this case the acceleration Ψ is also zero and

$$\begin{aligned}\Sigma(t) &= \nu\, t^{-2}\, \exp^{-\xi\, t}\\ E(t) &= \nu\, \exp^{-\xi\, t}\, t^{-2}\, \left(\tfrac{1}{t} + \tfrac{\xi}{2}\right)\\ H(t) &= -\tfrac{\nu}{2}\, t^{-2}\, \exp^{-\xi\, t}\\ q(t) &= \nu\,(m+2)\, t^{-4}\, \exp^{-\xi\, t}.\end{aligned} \tag{103}$$

These functions also diverge when t goes to zero and become zero when t goes to infinity.

(iii)*Stokesian fluid* ($q = 0$):

In this case the only possible solution is trivially zero. We conclude therefore that vector perturbations in Milne universes must have a non zero heat flux.

III-*Tensor Perturbations:*

The original Equations (58) give a closed dynamical system in the variables (E, Σ):

$$\begin{aligned}&\dot{E} + \left(\theta + \tfrac{\xi}{2}\right)\, E + \left[\tfrac{\xi}{2}\, \left(\tfrac{\theta}{3} + \tfrac{\xi}{2}\right) - \tfrac{1}{A^2}\,(m + 3K)\right]\, \Sigma = 0\\ &\dot{\Sigma} + \left(\tfrac{2}{3}\,\theta + \tfrac{\xi}{2}\right)\, \Sigma + E = 0,\end{aligned} \tag{104}$$

where H is given by the constraint

$$\Sigma = H.$$

We have then the following set of *good* quantities for tensorial perturbations in Milne background:

$$\mathcal{M}_A = \{E, H\}.$$

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