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Localized-Spins Ideal Paramagnet Within Nonextensive Statistical Mechanics

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Abstract

We numerically discuss the ideal paramagnet (N noninteracting localized spins in the presence of an external magnetic field) within the recently generalized statistical mechanics framework (canonical ensemble). Both specific heat (generalized Schottky anomaly) and isothermal magnetic susceptibility (generalized Curie law) are calculated, in particular, close to the thermodynamic limit ($N \gg 1$). Evidence for the existence of such limit is provided for the first time within the generalized statistics. In addition to this, within a molecular-field approximation, we have generalized the Curie-Weiss law.

Keywords: Generalized Entropy; Nonextensive Thermodynamics; Ideal Paramagnet; Curie-Weiss Law.

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1. Introduction

Since more than one decade, the need for alternative (nonextensive) thermodynamics is intensively developing in Physics. This is notorious in the areas of Cosmology and Gravitation (e.g., black holes, superstrings [1,2]), Astrophysics ($d = 3$ gravitational N-body problem [3]), Levy random walks [4], vortex problems [5] and, more generally speaking, in systems which involve long-ranged interactions (e.g., stability of a sandpile). Along this line, two (formal) constructions are (up to now independently) growing in the literature, namely Quantum Groups, and the Generalized Statistical Mechanics to which the present work is devoted.

Quantum Groups are generalized standard Lie groups and algebras [6-19], which constitute the formal basis for generalized mechanics. Although finite-temperature discussions are available (see [19] and references therein), the emphasis is put on mechanical (and not thermodynamical) grounds. They have found applications in the inverse scattering method, vertex models, anisotropic spin chain Hamiltonians, knot theory, conformed field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyons, and the discussion of the existence of dark matter. Within Quantum Groups, nonextensivity appears through the introduction of a (real or complex) parameter q_G (we shall use here q_G, instead of the traditional notation q, to avoid confusion with the q-parameter we shall soon introduce in the entropy) which yields q_G-deformations, q_G-oscillators, q_G-rotators, q_G-calculcus, etc. The standard extensive calculus is recovered in the $q_G \to 1$ limit. Also, Quantum Groups might be connected, for real $q_G \geq 1$, [13] to the possible existence of a discrete space-time, the minimal space step being proportional to $(q_G - 1)$ and the minimal time step being proportional to $(q_G - 1)^2$. Along this line, Quantum Groups might be relevant to the already vast literature exploring discontinuous (or continuous but not differentiable) space-time, discrete (classical, quantum, relativistic) mechanics and related matters [20-33].

Similar connections might exist with the other nonextensive formalism mentioned above, namely the Generalized Statistical Mechanics and Thermodynamics. Inspired by the fact that powers of the probabilities are relevant quantities for multifractals [34-39], one of us proposed [40] a generalized expression for the entropy, namely

$$S_q = k\frac{1-\sum_{s=1}^{W} p_s^q}{q-1} \quad ; \quad (q \in \Re) \quad , \tag{1}$$

where $\{p_s\}$ are the probabilities associated with the W microscopic configurations that might occur in the system, and k is a conventional positive constant. Eq. (1) recovers, for $q \to 1$, the well-known Shannon expression, $S_1 = -k_B \sum_s p_s \ln p_s$. Next, we briefly review the main properties that S_q satisfies.

(i) $S_q \geq 0 \quad , \quad \forall q \quad , \quad \forall \{p_s\}$.

(ii) Equiprobability (i.e., $p_s = 1/W$, $\forall s$) extremizes S_q , $\forall q$ (maximal for $q > 0$, minimal for $q < 0$, and constant for $q = 0$).

(iii) S_q is expansible for $q > 0$, i.e.,

$$S_q(p_1, p_2, \cdots, p_W, 0) = S_q(p_1, p_2, \cdots, p_W) \quad . \tag{2}$$

(iv) S_q is concave (convex) for all $\{p_s\}$ if $q > 0$ ($q < 0$), a fact which garantees thermodynamic stability for the system.

(v) H-theorem: under quite general conditions [41-43], $dS_q/dt \geq 0$, $= 0$ and ≤ 0, if $q > 0$, $= 0$ and < 0, respectively (t being the time).

(vi) If Σ and Σ' are two independent systems (i.e., $\hat{\rho}_{\Sigma\cup\Sigma'} = \hat{\rho}_{\Sigma}\hat{\rho}_{\Sigma'}$, where $\hat{\rho}$ denotes the density operator, whose eigenvalues are the $\{p_s\}$), S_q is pseudo-additive, i.e.,

$$\frac{S_q^{\Sigma\cup\Sigma'}}{k} = \frac{S_q^{\Sigma}}{k} + \frac{S_q^{\Sigma'}}{k} + (1-q)\frac{S_q^{\Sigma}}{k}\frac{S_q^{\Sigma'}}{k} \quad . \tag{3}$$

Consequently, the entropy is generically extensive if and only if $q = 1$. For $q < 1$ we have $S_q^{\Sigma\cup\Sigma'} > S_q^{\Sigma} + S_q^{\Sigma'}$, i.e., the entropy is superadditive, as desired in black hole and superstring problems [1,2].

(vii) Microcanonical ensemble: if $p_s = 1/W$, $\forall s$, we have [40]

$$S_q = k\frac{W^{1-q}-1}{1-q} \quad . \tag{4}$$

In the $q \to 1$ limit, this expression recovers the celebrated Boltzmann expression, $S_q = k_B \ln W$.

(viii) Canonical ensemble: the optimization of S_q under the constraints, $\mathrm{Tr}\hat{\rho} = 1$ and $\mathrm{Tr}(\hat{\rho}^q\hat{\mathcal{H}}) \equiv \langle\hat{\mathcal{H}}\rangle_q = U_q$ [44], yields, for $q < 1$ and $\beta \gtrless 0$, the generalized equilibrium distribution [40,44]

$$\hat{\rho} = \begin{cases} [1-\beta(1-q)\hat{\mathcal{H}}]^{1/(1-q)}/Z_q\ , & \text{if} \quad 1-\beta(1-q)\hat{\mathcal{H}} > 0\ , \\ 0\ , & \text{otherwise}\ , \end{cases} \tag{5a}$$

with

$$Z_q = \mathrm{Tr}[1-\beta(1-q)\hat{\mathcal{H}}]^{1/(1-q)} \quad , \tag{5b}$$

where $\beta \equiv 1/kT$ is a Lagrange parameter. In the $q \to 1$ limit, this expression recovers the Boltzmann-Gibbs distribution $\hat{\rho} = \exp(-\beta\hat{\mathcal{H}})/Z_1$. If $q > 1$ and $\beta \geq 0$, Eq. (5a) is replaced by

$$\hat{\rho} = \begin{cases} [1-\beta(1-q)\hat{\mathcal{H}}]^{1/(1-q)}/Z_q\ , & \text{if} \quad 1-\beta(1-q)E_l > 0\ , \\ \delta_{l,s}/g_l\ , & \text{otherwise}\ , \end{cases} \tag{5c}$$

where E_l is the lowest eigenvalue of $\hat{\mathcal{H}}$, g_l is the associated degeneracy, and $\delta_{l,s}$ equals unity if state s has eigenvalue E_l and equals zero otherwise. If $q > 1$ and $\beta \leq 0$, Eq. (5a) becomes

$$\hat{\rho} = \begin{cases} [1-\beta(1-q)\hat{\mathcal{H}}]^{1/(1-q)}/Z_q \,, & \text{if} \quad 1-\beta(1-q)E_h > 0 \,, \\ \delta_{h,s}/g_h \,, & \text{otherwise} \,, \end{cases} \tag{5d}$$

where E_h is the highest eigenvalue of $\hat{\mathcal{H}}$, g_h is the associated degeneracy, and $\delta_{h,s}$ equals unity if state s has eigenvalue E_h and equals zero otherwise. Excepting for extremely pathological cases (which are out of the scope of the present work) the negative (positive) temperature region is physically accessible only if E_h (E_l) is finite, i.e., if the energy spectrum $\{E_s\}$ has an upper (lower) bound. It can be shown [44] that, in general,

$$\frac{1}{T} = \frac{\partial S_q}{\partial U_q} \quad , \tag{6}$$

$$U_q = -\frac{\partial}{\partial \beta}\frac{Z_q^{1-q}-1}{1-q} \quad , \tag{7}$$

and

$$F_q \equiv U_q - TS_q = -\frac{1}{\beta}\frac{Z_q^{1-q}-1}{1-q} \quad . \tag{8}$$

Besides the above properties, the present generalized statistics has been shown to: (i) leave form-invariant, for all values of q, the Ehrenfest theorem [45] and the von Neumann equation [46]; (ii) satisfy Jaynes Information Theory duality relations [45] (necessary for the corresponding entropy to be considered as a measure of the (lack of) information); (iii) yield generalized Bogolyubov inequality [47], Langevin and Fokker-Planck equations [48], single-site Callen identity [49], quantum statistics [50], fluctuation-dissipation theorem [51], and a criterion for consistent nonparametric testing [52]. In addition to these properties, Plastino and Plastino have pointed [53] that $q \neq 1$ Thermodynamics overcomes the Boltzmann-Gibbs inability to provide finite mass for astrophysical systems within the polytropic model (studied by Chandrasekhar and others). An interesting connection has been recently established [54] between q and the fractal dimension of a d-dimensional Levy distribution. Also, through the generalization of the black-body radiation Planck law, a quantitative test for the nature of space-time has been recently suggested [55]. The possible relevance of this Generalized Statistical Mechanics for systems whose (linear) size is smaller than the range of the involved interactions has been discussed recently [56]; also it has the possibility of cancellation of a q_G-source of nonextensivity by a q-source of nonextensivity, thus providing extensive mean values [57]. Finally, the discussion of a variety of relatively simple systems is available: two-level system [40,58], free particle [59], Larmor precession [46], $d = 1$ Ising ferromagnet [60,61], $d = 2$ Ising ferromagnet [49,62,63].

However, none of these works has focused in detail the problem of the thermodynamic limit ($N \to \infty$) of a many-body problem. This is the central aim of the present paper, in which we discuss the ideal paramagnet (Section 2), and, as a straightforward consequence, we obtain the generalized Curie-Weiss law (Section 3).

2. Localized-Spins Ideal Paramagnet

2.1 – Model and Formalism

Let us consider the Hamiltonian

$$\mathcal{H} = -\mu H \left(\sum_{i=1}^{N} S_i - B \right) \quad ; \quad (S_i = \pm 1) \quad , \tag{9}$$

with $\mu H > 0$ (μ being the elementary magneton, and H the external magnetic field) and B an arbitrary number (positive, negative or zero). The spectrum of this Hamiltonian is given by

$$\epsilon_n = -\mu H(N - 2n - B) \quad ; \quad (n = 0, 1, 2, \cdots, N) \quad , \tag{10}$$

and the associated degeneracy by

$$g_n = \frac{N!}{n!(N-n)!} \quad . \tag{11}$$

Consequently, the partition function is given (using Eq. (5b)) by

$$Z_q = \sum_{n=0}^{N}{}' \frac{N!}{n!(N-n)!} \left[1 + \frac{\mu H}{kT}(1-q)(N-2n-B)\right]^{1/(1-q)} \quad , \tag{12}$$

where $\sum'$ runs only over the terms satisfying the conditions appearing in Eqs. (5). It is clear that $Z_q(N, kT/\mu H, B) = Z_q(N, -kT/\mu H, -B)$. This property propagates to all thermodynamical quantities. In particular, the specific heat C_q satisfies

$$C_q\left(N, \frac{kT}{\mu H}, B\right) = C_q\left(N, -\frac{kT}{\mu H}, -B\right) \quad , \tag{13a}$$

hence

$$C_q\left(N, \frac{kT}{\mu H}, 0\right) = C_q\left(N, -\frac{kT}{\mu H}, 0\right) \quad . \tag{13b}$$

Also the (vanishing-field) isothermal magnetic susceptibility χ_q satisfies

$$\chi_q(N, T, B) = -\chi_q(N, -T, -B) \quad , \tag{14a}$$

hence

$$\chi_q(N, T, 0) = -\chi_q(N, -T, 0) \quad . \tag{14b}$$

Without loss of generality we shall herein discuss the $T \geq 0$ region.

The probabilities of states $\{n\}$ are given (using Eqs. (5)) by

$$p_n = [1 + (\mu H/kT)(1-q)(N - 2n - B)]^{1/(1-q)}/Z_q \quad . \tag{15}$$

For $q < 1$, the region $T \in [0, T_{Forb}]$ is thermally forbidden (physically inaccessible), the region $T \in (T_{Forb}, T_{Froz}]$ is thermally frozen (vanishing specific heat), and the region $T > T_{Froz}$ is thermally active; $kT_{Forb}/\mu H \equiv \sup\{(1-q)(B-N), 0\}$ and $kT_{Froz}/\mu H \equiv \sup\{(1-q)(B+2-N), 0\}$. For $q > 1$, the region $T \in [0, T_{Froz}]$ is thermally frozen (vanishing specific heat), and the region $T > T_{Froz}$ is thermally active; $kT_{Froz}/\mu H \equiv \sup\{(q-1)(N-B), 0\}$.

To calculate C_q we use the following fluctuation-form expression ([59]; see also [51,57])

$$\frac{C_q}{k} = \frac{qZ_q^{1-q}}{(kT)^2} \sum_{n=0}^{N}{}' g_n p_n \times$$

$$\times \left\{ \frac{\varepsilon_n}{1-\beta(1-q)\varepsilon_n} - \left[\sum_{n=0}^{N}{}' g_n p_n \frac{\varepsilon_n}{1-\beta(1-q)\varepsilon_n} \right] \right\}^2 \quad , \tag{16}$$

where g_n and ε_n are given by Eqs. (10) and (11) respectively. Eq. (16) can be rewritten as

$$\frac{C_q}{k} = \frac{q}{(kT)^2} \left\{ \left[\sum_{n=0}^{N}{}' g_n p_n^q \frac{\varepsilon_n^2}{1-\beta(1-q)\varepsilon_n} \right] \right.$$

$$\left. - \left[\sum_{n=0}^{N}{}' g_n p_n^q \varepsilon_n \right] \left[\sum_{n=0}^{N}{}' g_n p_n \frac{\varepsilon_n}{1-\beta(1-q)\varepsilon_n} \right] \right\} \quad , \tag{17}$$

which is quite convenient for numerical calculations. This expression can be rewritten as follows,

$$\frac{C_q}{k} = q \left(\frac{\mu H}{kT} \right)^2 f_q \left(N, \frac{\mu H}{kT}, B \right) \quad , \tag{18}$$

where the dimensionless quantity f_q is defined as

$$f_q\left(N,\frac{\mu H}{kT},B\right) = \sum_{n=0}^{N}{}' g_n p_n^q \frac{(N-2n-B)^2}{1+(\mu H/kT)(1-q)(N-2n-B)}$$

$$-\left[\sum_{n=0}^{N}{}' g_n p_n^q\,(N-2n-B)\right]\left[\sum_{n=0}^{N}{}' g_n p_n \frac{N-2n-B}{1+(\mu H/kT)(1-q)(N-2n-B)}\right] \qquad (19a)$$

$$= \left[Z_q\left(N,\frac{\mu H}{kT},B\right)\right]^{1-q}\left\{\sum_{n=0}^{N}{}' g_n p_n \left[\frac{N-2n-B}{1+(\mu H/kT)(1-q)(N-2n-B)}\right]^2\right.$$

$$\left.-\left[\sum_{n=0}^{N}{}' g_n p_n \frac{N-2n-B}{1+(\mu H/kT)(1-q)(N-2n-B)}\right]^2\right\} \quad . \qquad (19b)$$

In the limit $\mu H/kT \to 0$, we have $p_n = 1/2^N$ $(\forall n)$, hence

$$f_q(N,0,B) = 2^{(1-q)N}N \qquad (\forall B) \quad . \qquad (20)$$

The magnetic fluctuation-dissipation theorem ([56]; see also [51]) yields, for $q \le 1$, the following vanishing-field isothermal magnetic susceptibility,

$$\chi_q = \frac{q\mu^2}{kT}\, f_q(N,0,B) \quad ; \qquad (21)$$

hence (using Eq. (20)),

$$\chi_q = \frac{q\mu^2\, 2^{(1-q)N}N}{kT} \quad , \qquad (22)$$

which generalizes the well-known Curie's law, $\chi_1 = N\mu^2/kT$. The discussion is somewhat more complex for $q > 1$, and we are not addressing here the details.

At high temperatures ($kT/\mu H \gg 1$), Eqs. (18) and (20) imply

$$\frac{C_q}{k} \sim q\left(\frac{\mu H}{kT}\right)^2 2^{(1-q)N}N \equiv \frac{C_q^\infty}{k} \quad ; \quad (\forall q\ ,\ \forall B) \quad . \qquad (23)$$

Let us now introduce the dimensionless quantity,

$$\Delta_q \equiv \frac{C_q}{C_q^\infty} = \frac{C_q}{qk2^{(1-q)N}N}\left(\frac{kT}{\mu H}\right)^2 \qquad , \tag{24}$$

which will come up to be very appropriate for the discussion which follows. Indeed, Δ_q is universal (i.e., independent of q and B) for $kT/\mu H \gg 1$, whereas for $kT/\mu H \lesssim 1$, it strongly depends on both q and B.

2.2 – $q = 1$ (Boltzmann-Gibbs) Specific Heat

Eqs. (18) and (19) imply, for $q = 1$, the well-known result (Schottky anomaly)

$$\frac{C_1}{k_B} = N\left(\frac{\mu H}{k_B T}\right)^2 \mathrm{sech}^2\left(\frac{\mu H}{k_B T}\right) \qquad ; \qquad (\forall B) \qquad , \tag{25}$$

represented in Fig. 1.

2.3 – $q < 1$ Specific Heat

The results we have obtained are represented in Figs. 2 and 3 for $B = 0$, in Figs. 4 and 5 for $B = N$ and in Figs. 6 and 7 for $B = -N$.

a) $B = 0$: Fig. 2 illustrates (through the numerical data collapse) very clearly, that a well-defined thermodynamic limit ($N \to \infty$) does exist. One observes that: (i) the convergence to the thermodynamic limit is very rapid, such that $N = 40$ is already a very good realization of an infinite system (see Fig. 2(a)); (ii) the oscillations which appear for decreasing q tend to be washed away as N increases (see Figs. 2(b),(c)). This is the first time such a limit is exhibited within the generalized statistical mechanics ($q \neq 1$). As mentioned before, the quantity Δ_q defined in Eq. (24), presents two distinct regimes, namely one in which all values of q approach the $q = 1$ limit, and the other in which different values of q lead to distinct behaviour (see Figs. 3(a),(b)). Let us also comment on the role of N by discussing Fig. 3 in further detail. We define $t \equiv$

$kT/\mu H\sqrt{N}$, as well as t^*, by imposing $\Delta_q(t^*) = 1/2$. We have represented t^* as a function of $q \leq 1$ in Fig. 3(c). If $t \gg t^*(q)$, the $q = 1$ description is satisfactory (extensive thermodynamics; EXT) and there is no need to generalize Boltzmann-Gibbs statistics. On the contrary, if $t \lesssim t^*$, a crossover occurs and the results are sensibly q-dependent (nonextensive thermodynamics; NEXT). Consequently, if we had a physical system which would behave as assumed in Hamiltonian (9) "all the way long", the convenient region for searching for possible departures from Boltzmann-Gibbs behavior would be $N \gg [kT/\mu H t^*(q)]^2$, i.e., large systems (which precisely is the case suggested in [18], as well as the $N_{max} \to \infty$ case of [56]).

b) $B = N$: In Fig. 4, one observes a divergence in the scaled specific heat, characteristic of a phase transition. By following the position of the peak, for several values of q and N (up to $N = 900$), we conjecture the critical temperature as $\lim_{N\to\infty}(kT_c/\mu H N) = 1-q$. Such cooperative effect, present in a system defined by an independent-spins Hamiltonian, is introduced by the statistics, similarly to what happens in the Bose-Einstein condensation for the ideal gas. In Fig. 5 we plot Δ_q for several values of q (including $q > 1$); with $t = kT/\mu H N$, one can define now t^* (by imposing $\Delta_q(t^*) = 2$) such as to represent the crossover between the extensive and nonextensive regimes in Fig. 11.

c) $B = -N$: In Fig. 6 we present thermodynamic limits which resemble those shown in Fig. 2 ($B = 0$), but with some differences: (i) the curves are always smooth (no oscillations in the specific heat) for any values of q and N; (ii) there is a value q^* which splits distinct approachings to thermodynamic limit, namely the convergence to the maximum from below ($q > q^*$) as shown in Figs. 6(a),(b), and from above ($q < q^*$) as in Fig. 6(d), with increasing N. The crossover between these two tendencies happens for $q^* \sim 0.4$ (see Fig 6(c)). (iii) The position of the maximum occurs at $kT_{max}/\mu H N \cong (1-q)/2$. In Fig. 7(a) we plot Δ_q for several values of q (including $q > 1$). Similarly to what was done for the case $B = 0$, one can define $t \equiv kT/\mu H N$, as well as t^* (through $\Delta_q(t^*) = 1/2$), such as to represent t^* as a function of $q \leq 1$ (see Fig. 7(b)).

2.4 - $q > 1$ Specific Heat

The results we have obtained are represented in Figs. 8 and 9 for $B = 0$, Figs. 10, 5 and 11 for $B = N$ and in Figs. 12 and 7(a) for $B = -N$.

a) $B = 0$: One now has a "Frozen region" for $0 \leq T \leq T_{Froz}$ ($kT_{Froz}/\mu HN = q - 1$). In Fig. 8 we present plots showing that a thermodynamic limit is reached in which the scaled specific heat touches the Frozen frontier (vertical dashed line) with a finite value. In Fig. 9 we plot Δ_q, showing again the two regimes discussed previously. As before, one may define $t = kT/\mu HN$ and t^* (such that $\Delta_q(t^*) = 2$); one sees from Fig. 9 that there is no solution for t^* in the Active-temperature region, and consequently, no crossover in Fig. 3(c) for $q > 1$.

b) $B = N$: In Fig. 10 we show thermodynamic limits which are qualitatively similar to those presented in Fig. 6 ($q < 1$; $B = -N$) in the sense that: (i) the curves are always smooth; (ii) the maximum is located at $kT_{max}/\mu HN \cong (q-1)/2$. Some basic differences with respect to the previous cases should be pointed out: (iii) the convergence to the thermodynamic limit is much slower in this case; in fact, small values of N lead to an unstable peak which collapses with the $T = 0$ axis (see Fig. 10(d)); only when N is sufficiently large (e.g., $N \gtrsim 40$), is that a stable curve can be visualized; a reasonable good realization of the thermodynamic limit is only reached for a much higher value of N ($N \sim 320$); (iv) the convergence is always from above, for increasing N (like in the cases $q < q^*$; $B = -N$), as shown in Figs. 10(a)-(c). The quantity Δ_q is shown in Fig. 5 for both $q < 1$ and $q > 1$; as before, one can define t^* ($\Delta_q(t^*) = 1/2$) and plot t^* versus q (see Fig. 11). One should notice that the straight line separating the EXT and NEXT regions for $B = N$ and $q \geq 1$ (Fig. 11) is a reflection with respect to $q = 1$ of the one obtained for $B = -N$ and $q \leq 1$ (Fig. 7(b)).

c) $B = -N$: The plots shown in Fig. 12 are qualitatively analogous to those for the case $q > 1$, $B = 0$, discussed before; the Frozen region now is delimited by $kT_{Froz}/\mu HN = 2(q-1)$. In Fig. 7(a) the quantity Δ_q is plotted for both cases $q > 1$ and $q < 1$. Again, by taking $\Delta_q(t^*) = 2$ one gets the straight line shown if Fig. 7(b), which is a reflection with respect to $q = 1$ of the one found for $B = N$ and $q \leq 1$.

3. Generalized Curie-Weiss Law

With the notation $\hat{S}_{total} \equiv \sum_{i=1}^{N} S_i$, the magnetization, $M_q \equiv \mu \langle \hat{S}_{total} \rangle_q \equiv \mu \mathrm{Tr} \hat{\rho}^q \hat{S}_{total}$, is given, for $H \to 0$, by

$$M_q \sim \chi_q H \quad . \tag{26}$$

Within a mean-field approach, above the critical temperature, we can now interpret H as given by

$$H = H_{ext} + \lambda M_q \quad , \tag{27}$$

where H_{ext} is the external field applied on the system and λM_q, the mean value coming from z ($\equiv$ coordination number) neighboring spins interacting through a coupling constant J. Replacing Eq. (27) into Eq. (26), implies

$$M_q \sim \frac{\chi_q}{1 - \lambda\, \chi_q} H_{ext} \quad . \tag{28}$$

Consequently, the effective susceptibility $\bar{\chi}_q \equiv \lim_{H \to 0}(M_q / H_{ext})$ is given by,

$$\bar{\chi}_q = \frac{\chi_q}{1 - \lambda\, \chi_q} \quad . \tag{29}$$

Hence (using Eq. (22)),

$$\bar{\chi}_q = \frac{q \mu^2\, 2^{(1-q)N} N}{k(T - T_c)} \qquad (T > T_c) \quad , \tag{30}$$

with

$$kT_c \equiv \lambda q \mu^2\, 2^{(1-q)N} N \quad . \tag{31}$$

But it is now known [49] that, within the mean-field framework, $kT_c/J = qz$. Replacing this into Eq. (30) yields

$$\bar{\chi}_q = \frac{q\mu^2\, 2^{(1-q)N} N}{k(T - Jqz)} \qquad (T > Jqz) \quad , \tag{32}$$

which generalizes, for arbitrary q, the well-known Curie-Weiss law, $\bar{\chi}_1 = \mu^2 N/k(T - Jz)$.

4. Conclusion

We have discussed, within the recently generalized formalism for statistical mechanics of a canonical ensemble, the specific heat and susceptibility of an ideal paramagnet of N localized spins (spins 1/2), thus extending, to arbitrary values of q, the Schottky anomaly, Curie law and Curie-Weiss law. The existence of a well-defined $N \to \infty$ thermodynamic limit is, for the first time, (numerically) established for arbitrary values of q. A variety of nonextensive effects are exhibited, and a crossover is pointed, which connects, on one side, a regime where Boltzmann-Gibbs ($q = 1$) statistics is satisfactory, with, on the other side, a regime where sensibly q-dependent effects are shown. The most striking nonextensive effect is the possibility of existence (for $q < 1$ and $B = N$) of a finite (scaled) temperature phase transition in spite of the fact that we are dealing with an ideal paramagnet. This fact obviously reminds the existence (for high enough dimensionality) of the Bose-Einstein condensation in an ideal bosonic gas. In both cases, the singularity is due to the statistics and not to the (energy) interactions.

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Figure Captions

Fig. 1: Boltzmann-Gibbs ($q = 1$) specific heat for an ideal paramagnet of N localized spins.

Fig. 2: Conveniently scaled C_q versus T for $B = 0$: (a) $q = 0.8$ ($N = 1,2,4,8,20,40,80,160,320,640$); (b) $q = 0.6$ ($N = 1,2,4,640$); (c) $q = 0.5$ ($N = 1,2,4,640$).

Fig. 3: (a) and (b): Scaled T-dependence of Δ_q for $B = 0$ and typical values of $q < 1$; for $q = 0.7$ we present a data collapse by plotting Δ_q for several values of N ($N = 40,80,160,320,640$), whereas for the other values of q we represent Δ_q for $N = 640$ only; (c) crossover between extensive (EXT) and nonextensive (NEXT) behaviors ($\Delta_q(t^*) = 1/2$, for $q < 1$); the solid curve was drawn by smoothly joining the points for $t^*(q)$, whereas the dashed part is purely speculative. The symmetric definition of $t^*(q)$ for $q > 1$, i.e., $\Delta_q(t^*) = 2$, presents no solution in the Active-temperature region, and so, within such proposal, no crossover exists for all $q > 1$ (see Fig. 9).

Fig. 4: Conveniently scaled C_q versus T for $B = N$: (a) $q = 0.8$ ($N = 2,4,10,20,40,80$); (b) $q = 0.6$ ($N = 2,4,10,20,40$); (c) $q = 0.5$ ($N = 4,10,40$).

Fig. 5: Scaled T-dependence of Δ_q for $B = N$ and typical values of $q < 1$ ($q = 0.5,0.7,0.9$; curves above $\Delta_q = 1$) and of $q > 1$ ($q = 1.1,1.2,1.5,2.0$; curves below $\Delta_q = 1$). For $q = 2.0$, Δ_q is plotted for $N = 160,320,640,900$, whereas for the other values of q, Δ_q is represented for $N = 640$ only.

Fig. 6: Conveniently scaled C_q versus T for $B = -N$: (a) $q = 0.8$ ($N = 1,2,4,10,20,40,80,160,320,640$); (b) $q = 0.5$ ($N = 1,2,4,10,20,40,80,160,320,640$); (c) $q = 0.4$ ($N = 1,4,10,20,40,80,160,320,640$); (d) $q = 0.2$ ($N = 2,4,10,20,40,80,160,320,640$).

Fig. 7: (a) Scaled T-dependence of Δ_q for $B = -N$ and typical values of $q < 1$ ($q =$ 0.4, 0.6, 0.8, 0.95; curves below $\Delta_q = 1$) and of $q > 1$ ($q = 1.05, 1.2, 1.5, 2.0$; curves above $\Delta_q = 1$); for $q = 0.4$, Δ_q is plotted for $N = 160, 320, 640, 900$, whereas for the other values of q, Δ_q is represented for $N = 640$ only; (b) crossover between extensive and nonextensive behaviors.

Fig. 8: Conveniently scaled C_q versus T for $B = 0$: (a) $q = 1.2$ ($N = 4, 10, 20, 40, 80, 160,$ $320, 640$); (b) $q = 1.5$ ($N = 2, 4, 20, 40, 80, 160, 320, 640$); (c) $q = 2.0$ ($N = 2, 10, 20,$ $40, 80, 160, 320, 640$). In each case, the dashed vertical line indicates the frontier between the Active ($T > T_{Froz}$) and Frozen ($T < T_{Froz}$) regions.

Fig. 9: Scaled T-dependence of Δ_q for $B = 0$ and typical values of $q > 1$ ($q =$ 1.01, 1.2, 1.5, 2.0); in each case, Δ_q is non-zero only in Active-temperature region ($T > T_{Froz}$).

Fig. 10: Conveniently scaled C_q versus T for $B = N$: (a) $q = 1.2$ ($N = 60, 80, 160, 320,$ $640, 900$); (b) $q = 1.5$ ($N = 40, 80, 160, 320, 640, 900$); (c) $q = 2.0$ ($N = 40, 80, 160,$ $320, 640, 900$); (d) $q = 2.0$ ($N = 2, 4, 8, 12, 16, 20, 24, 28$).

Fig. 11: Crossover between extensive and nonextensive behaviors for $B = N$; by comparing with Fig. 7(b), one sees clearly that these two figures are related to one another by a reflection with respect to $q = 1$.

Fig. 12: Conveniently scaled C_q versus T for $B = -N$: (a) $q = 1.2$ ($N = 4, 10, 20, 40, 80,$ $160, 320, 640$); (b) $q = 1.4$ ($N = 2, 4, 10, 20, 40, 80, 160, 320, 640$); (c) $q = 1.5$ ($N =$ $2, 4, 10, 20, 40, 80, 160, 320, 640$). In each case, the dashed vertical line indicates the frontier between the Active ($T > T_{Froz}$) and Frozen ($T < T_{Froz}$) regions.

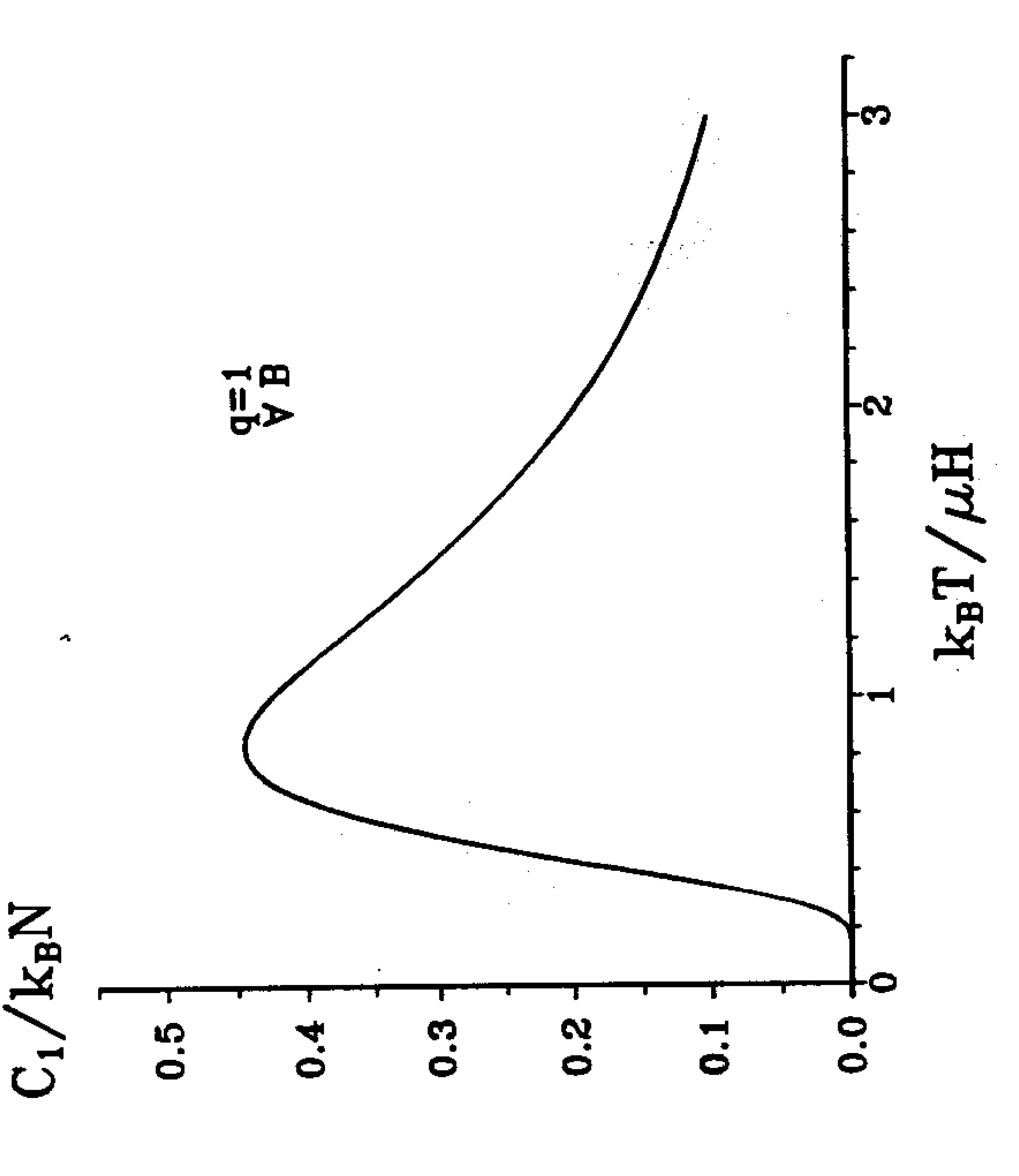

Fig. 1
Nobre and Tsallis

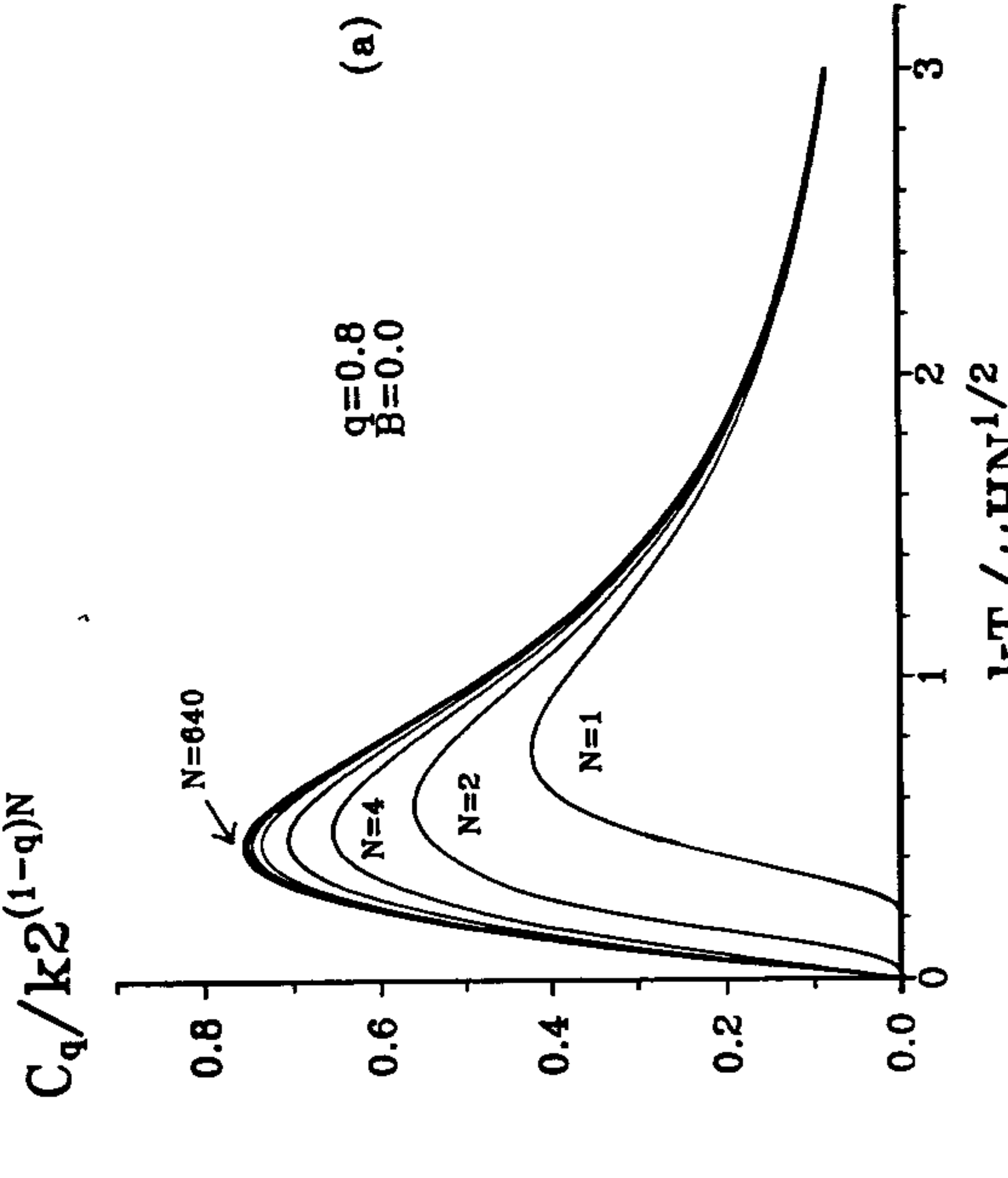

Fig. 2(a)
Nobre and Tsallis

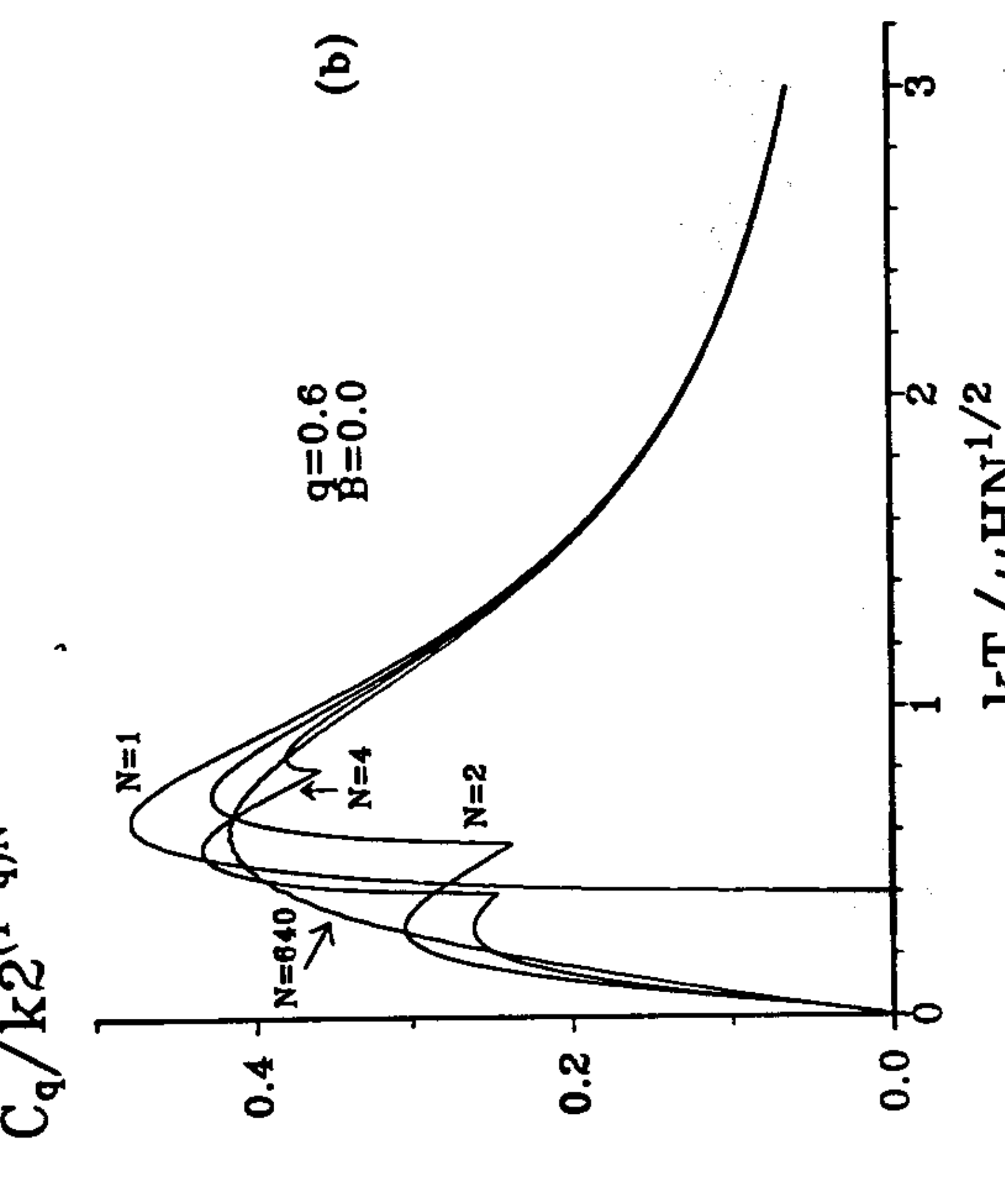

Fig. 2(b)
Nobre and Tsallis

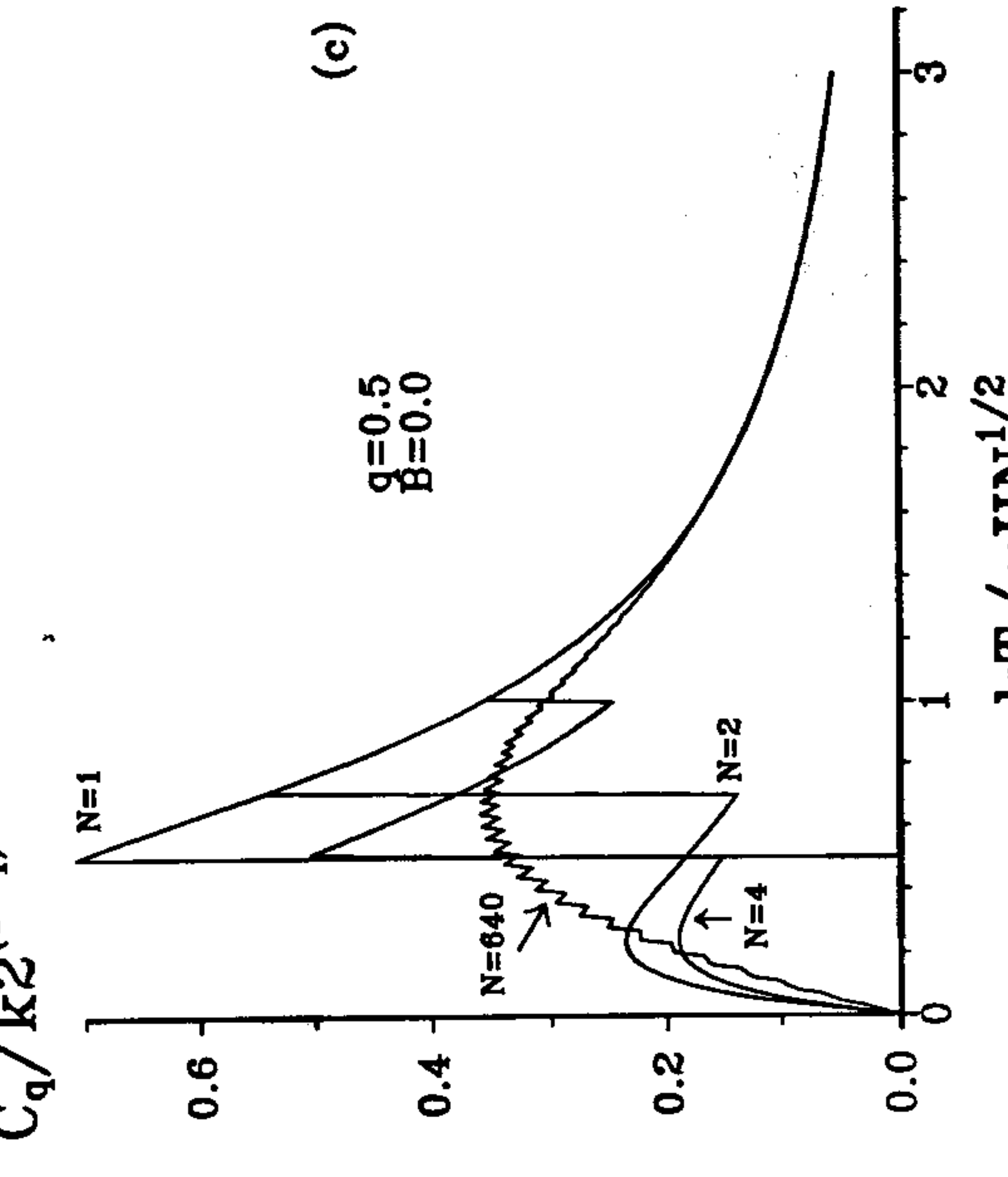

Fig. 2(c)
Nobre and Tsallis

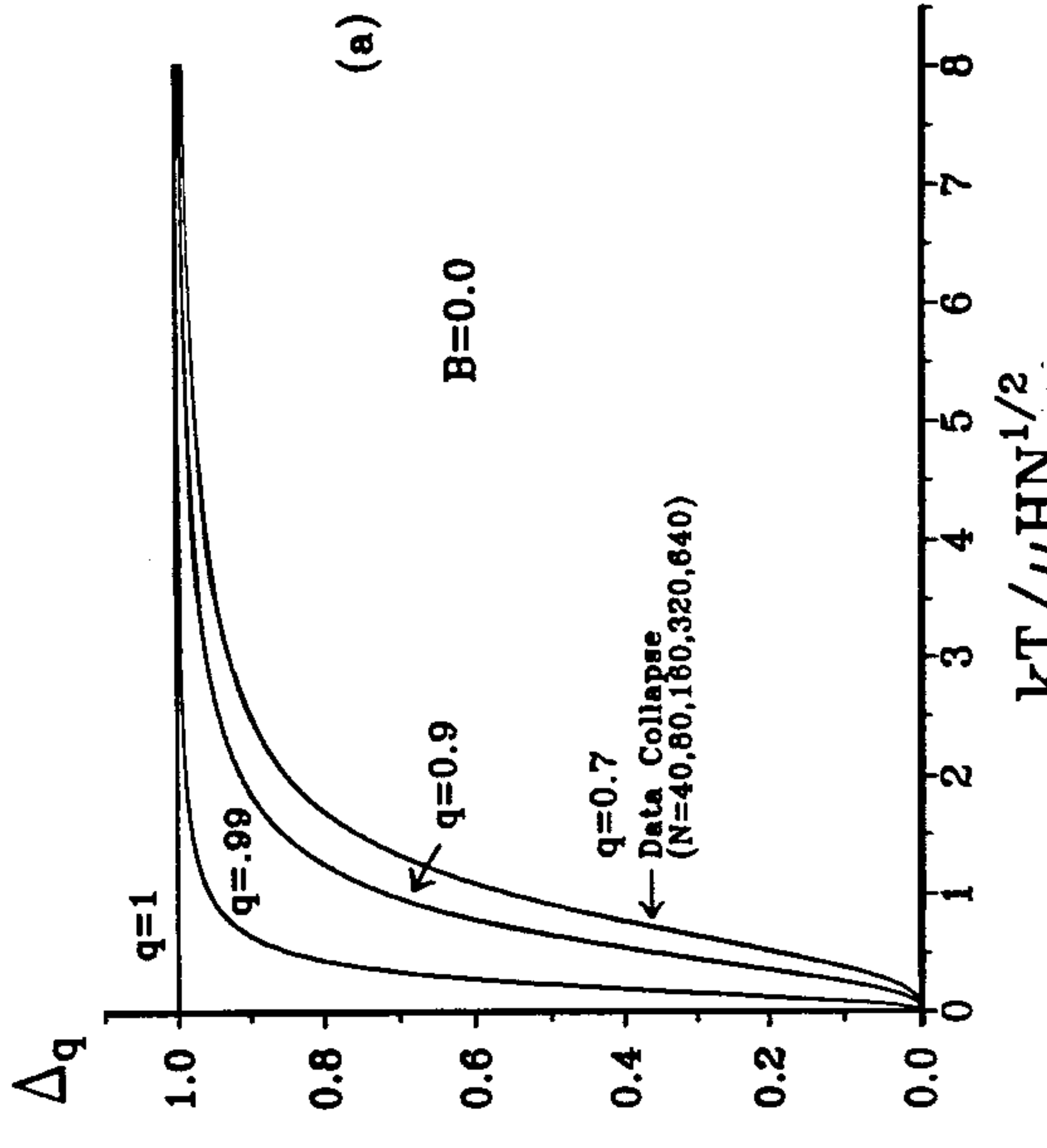

Fig. 3(a)
Nobre and Tsallis

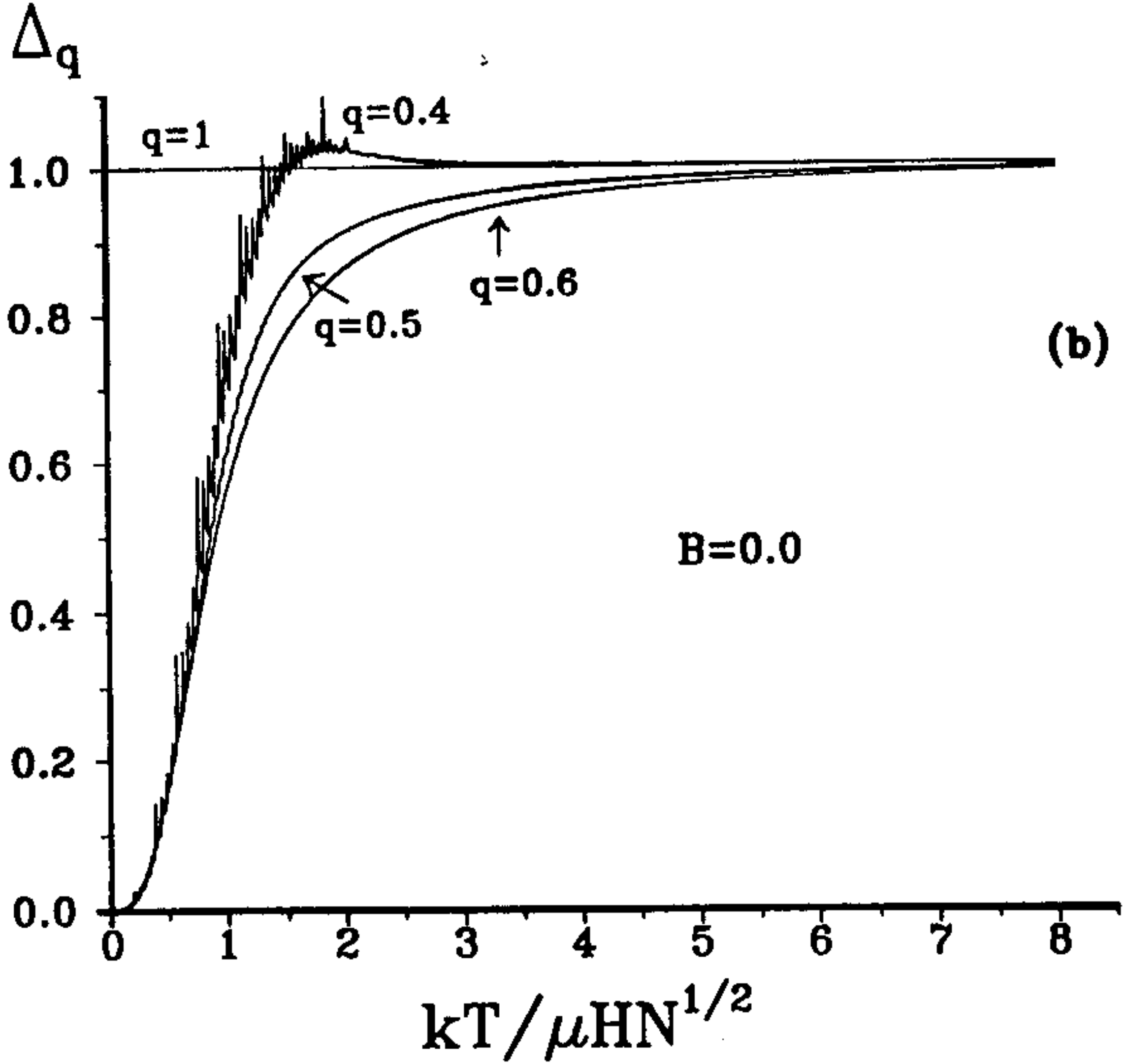

Fig. 3(b)
Nobre and Tsallis

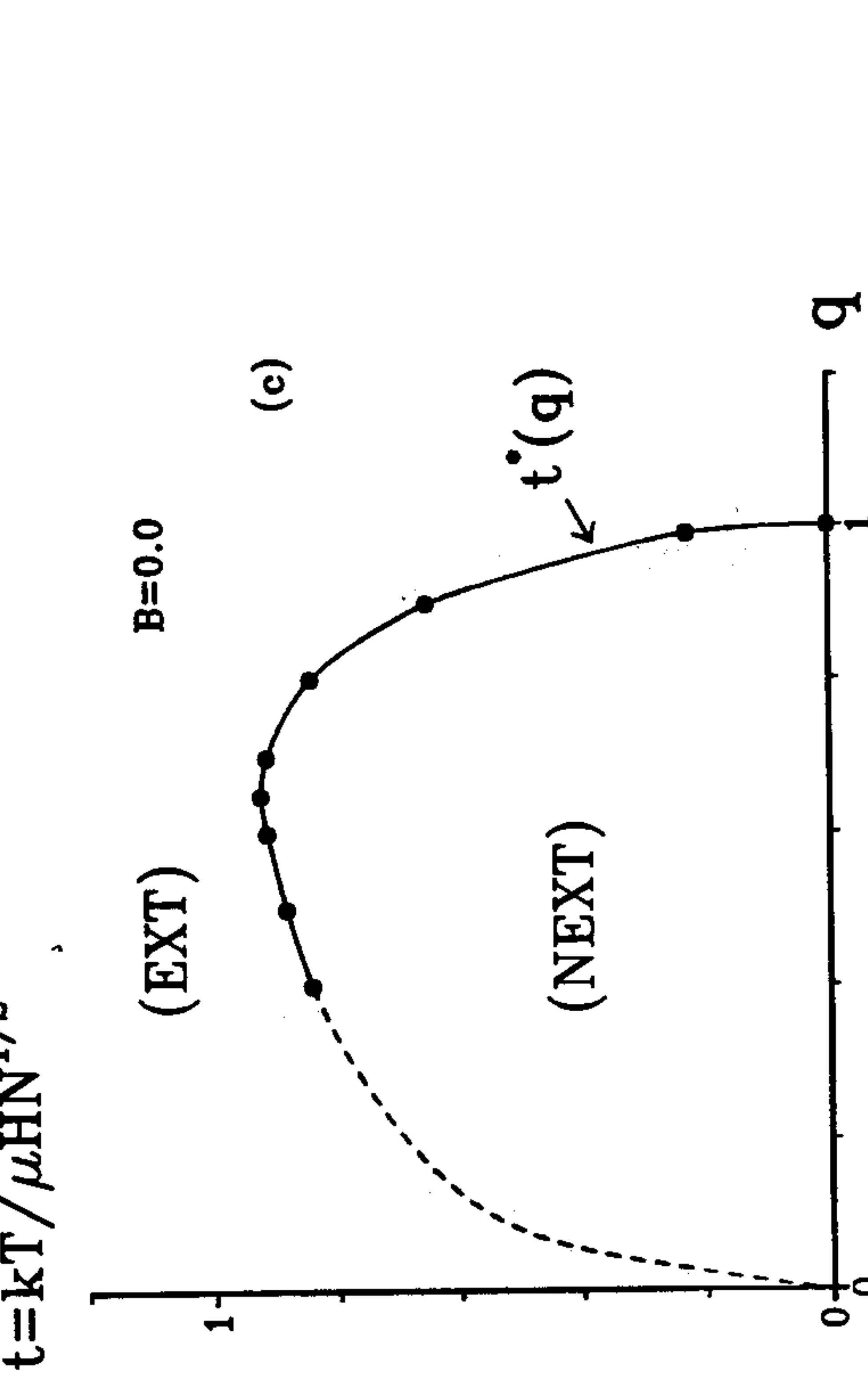

Fig. 3(c)
Nobre and Tsallis

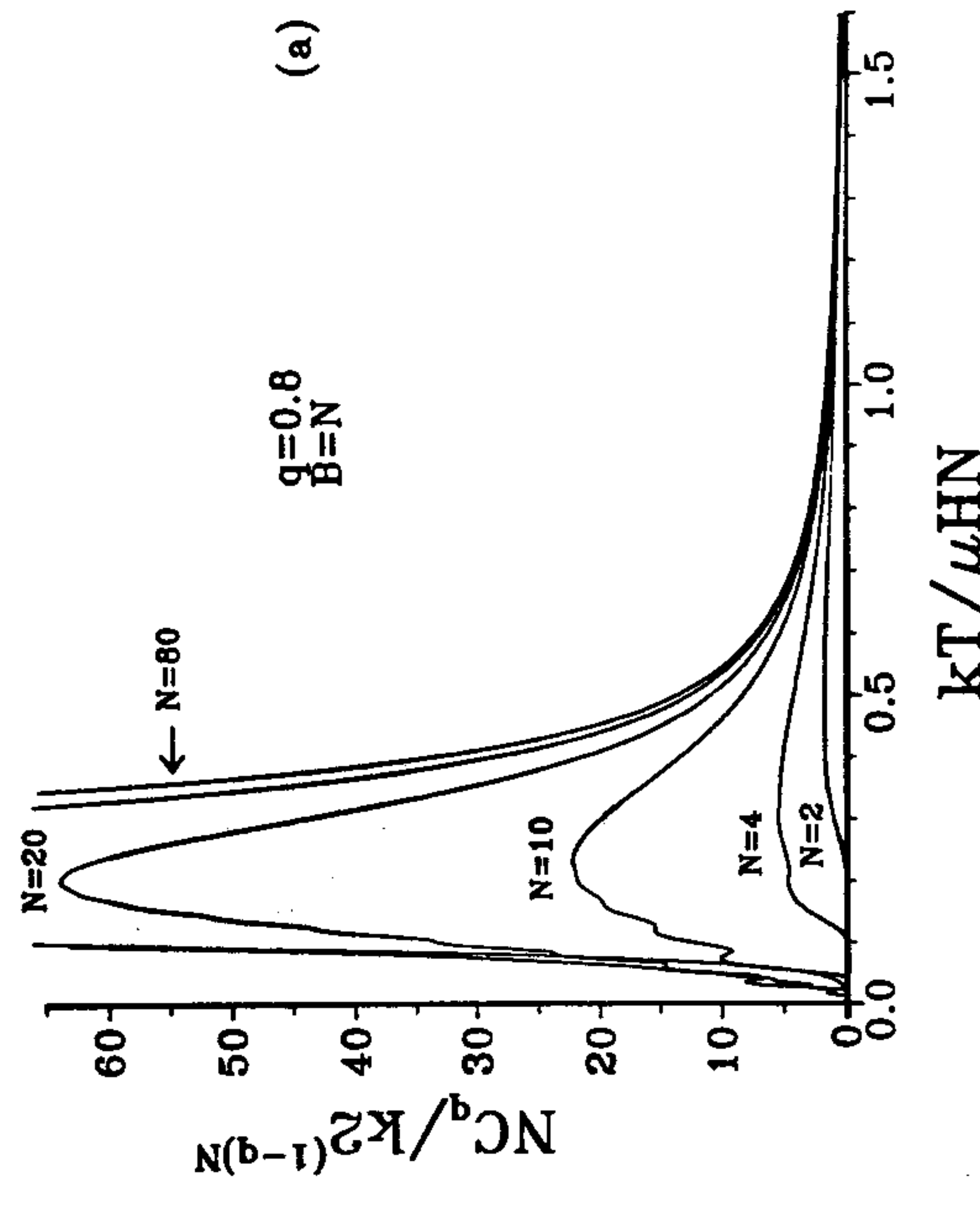

Fig. 4(a)
Nobre and Tsallis

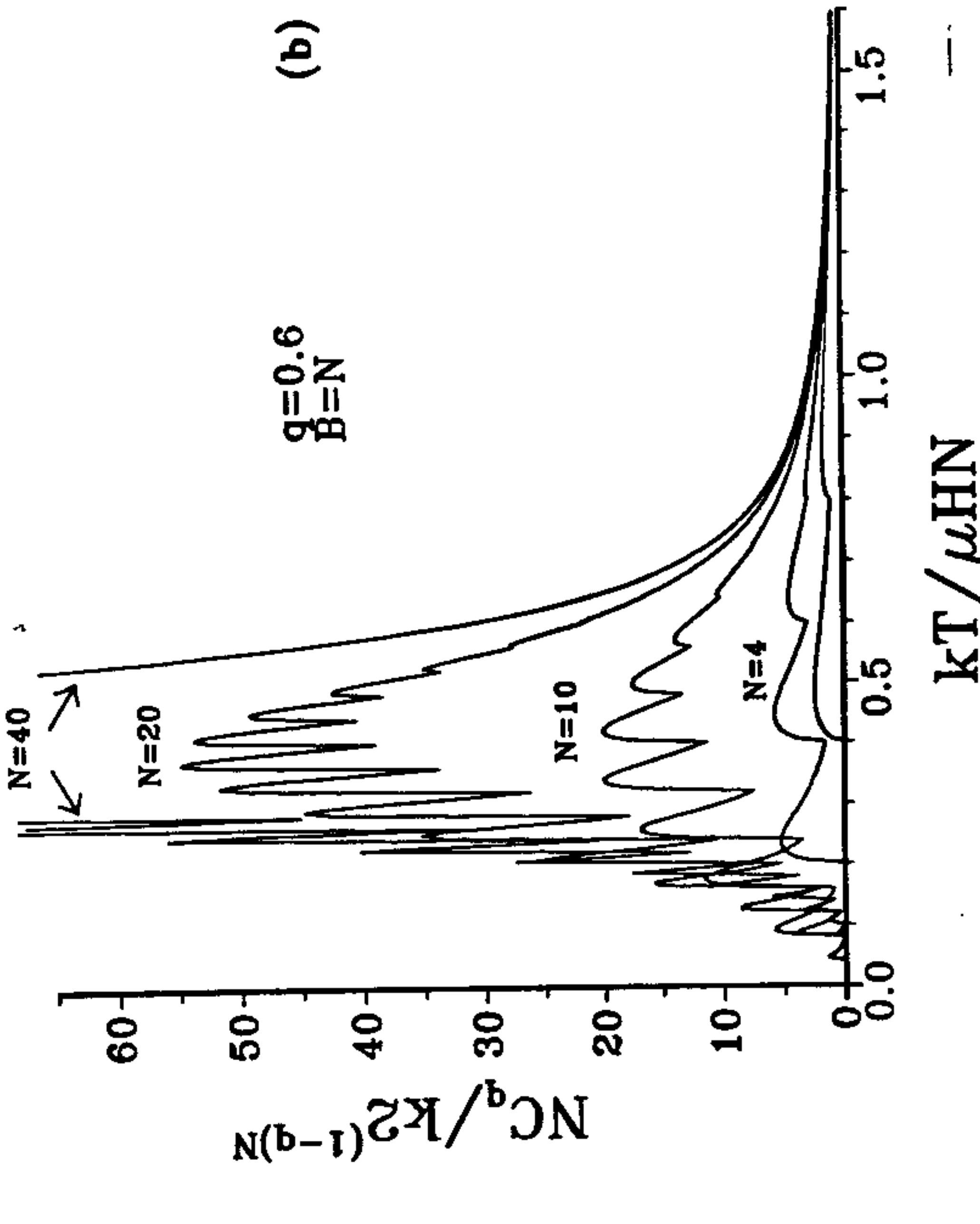

Fig. 4(b)
Nobre and Tsallis

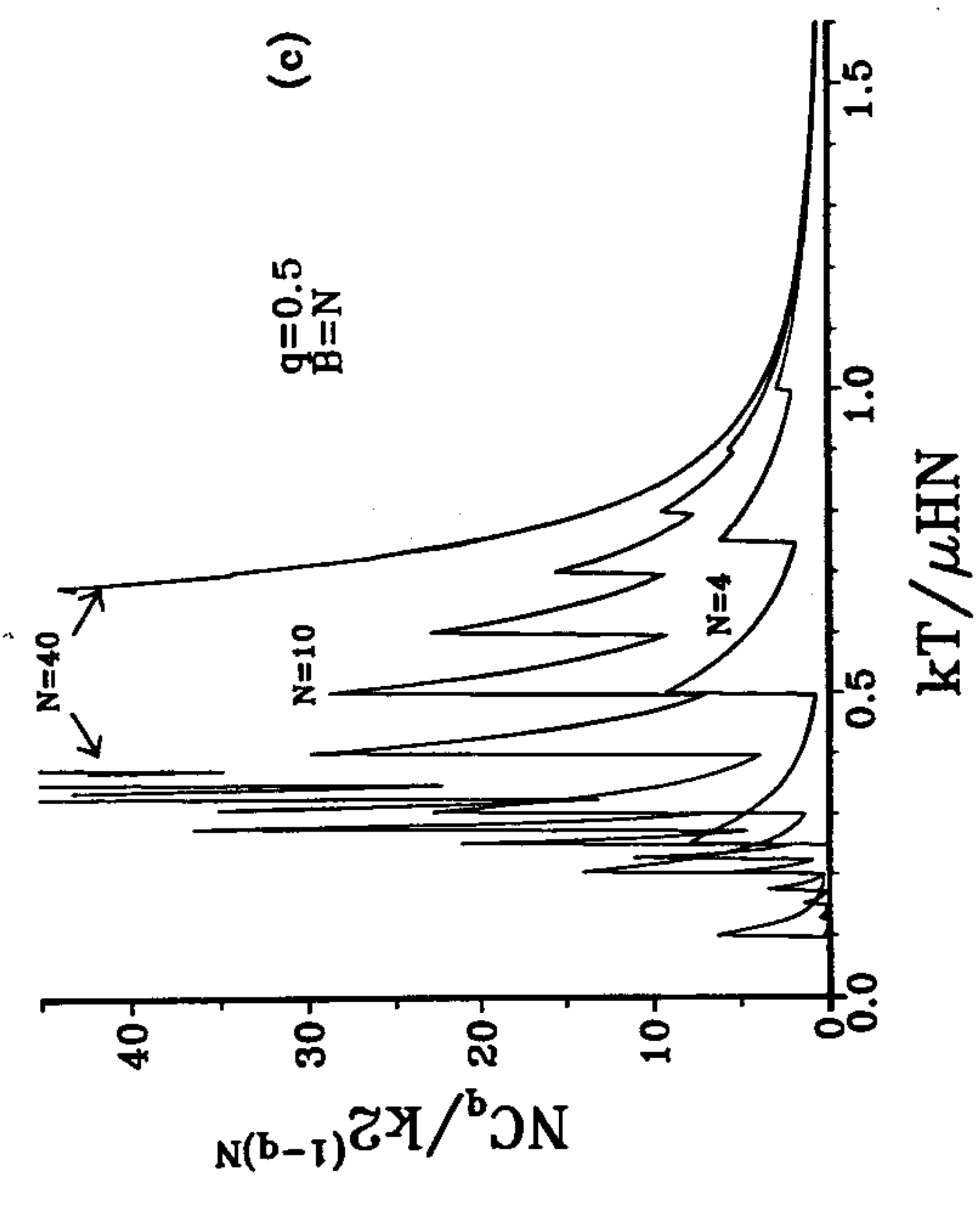

Fig. 4(c)
Nobre and Tsallis

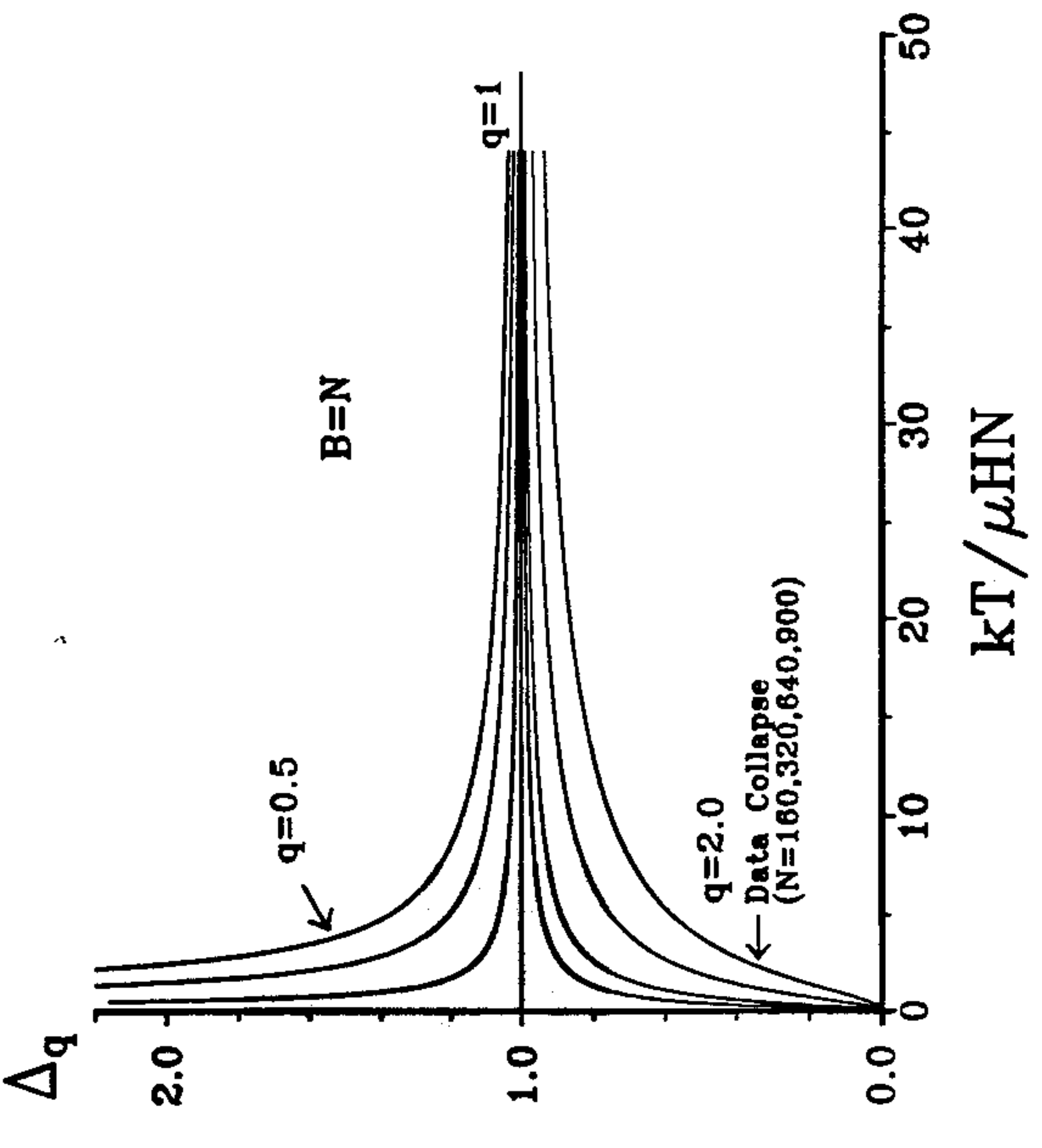

Fig. 5
Nobre and Tsallis

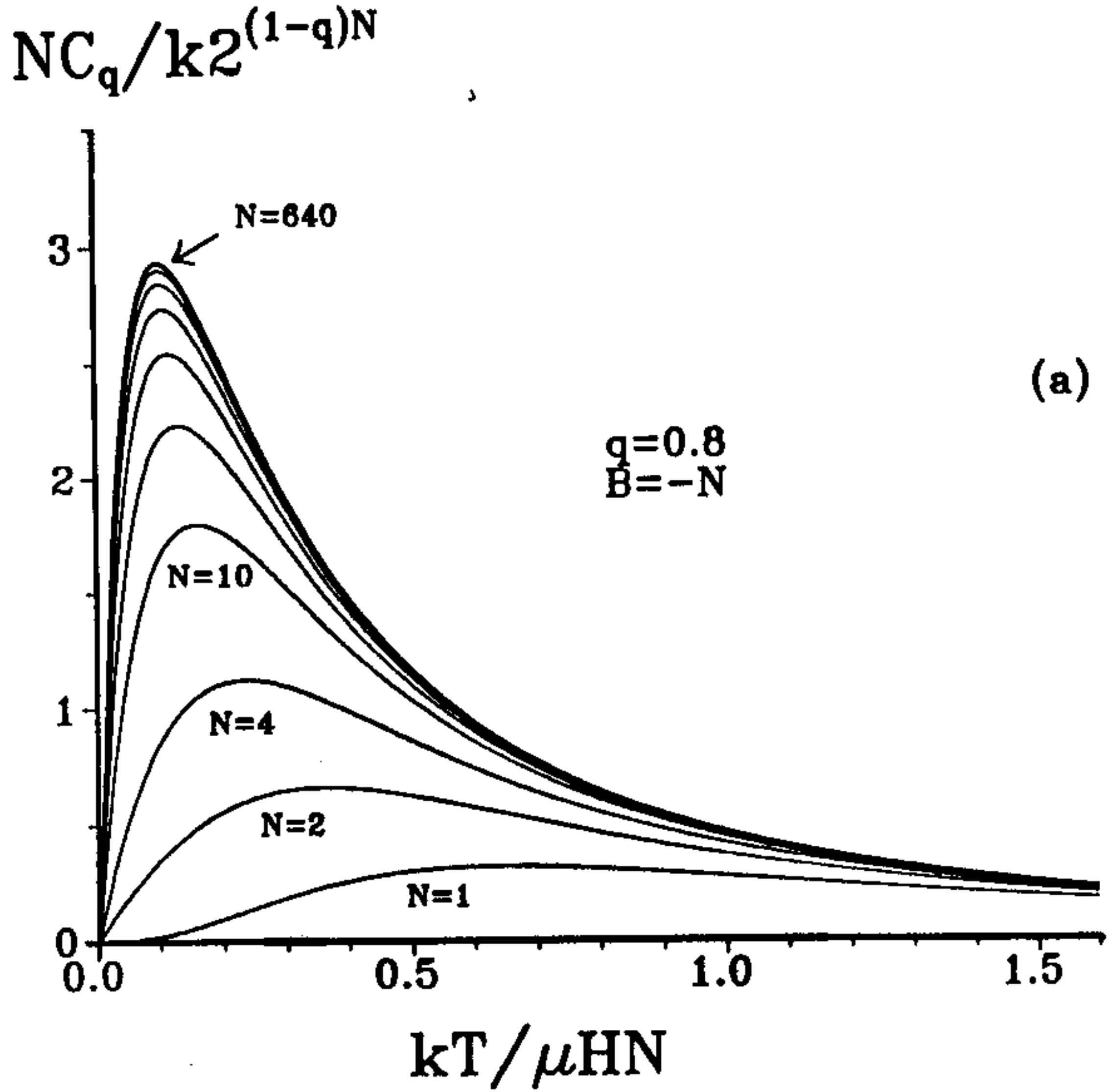

Fig. 6(a)
Nobre and Tsallis

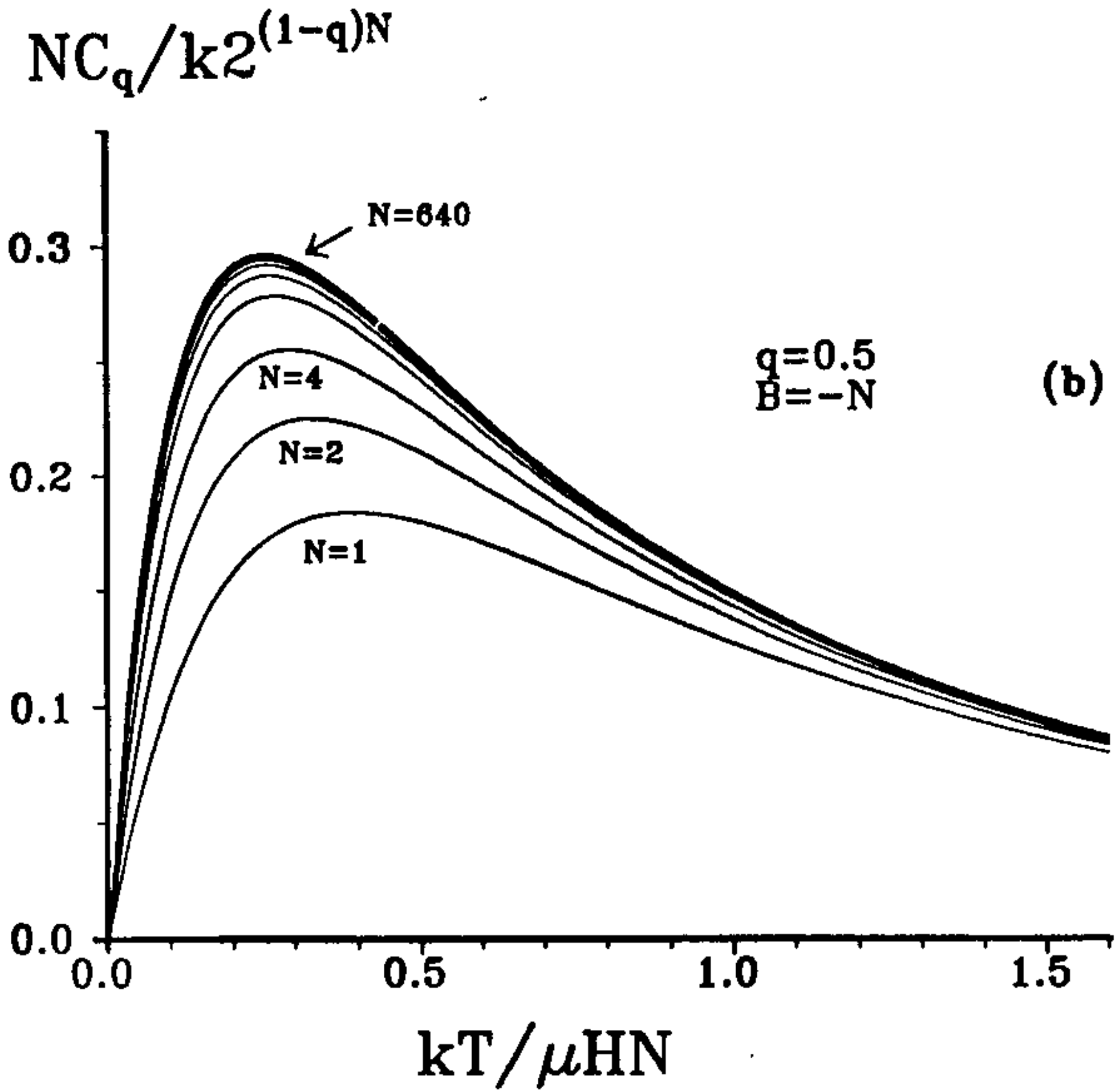

Fig. 6(b)
Nobre and Tsallis

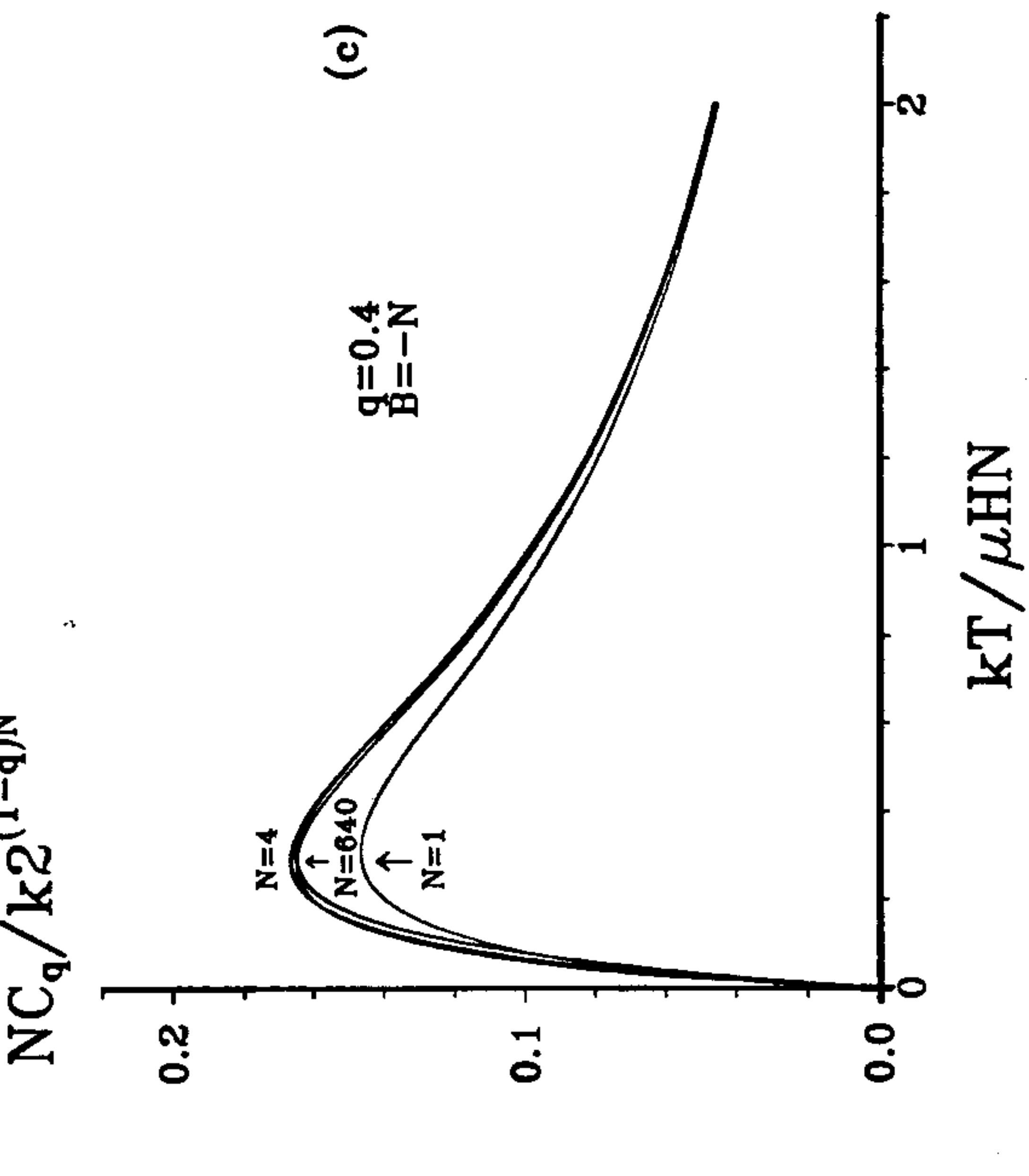

Fig. 6(c)
Nobre and Tsallis

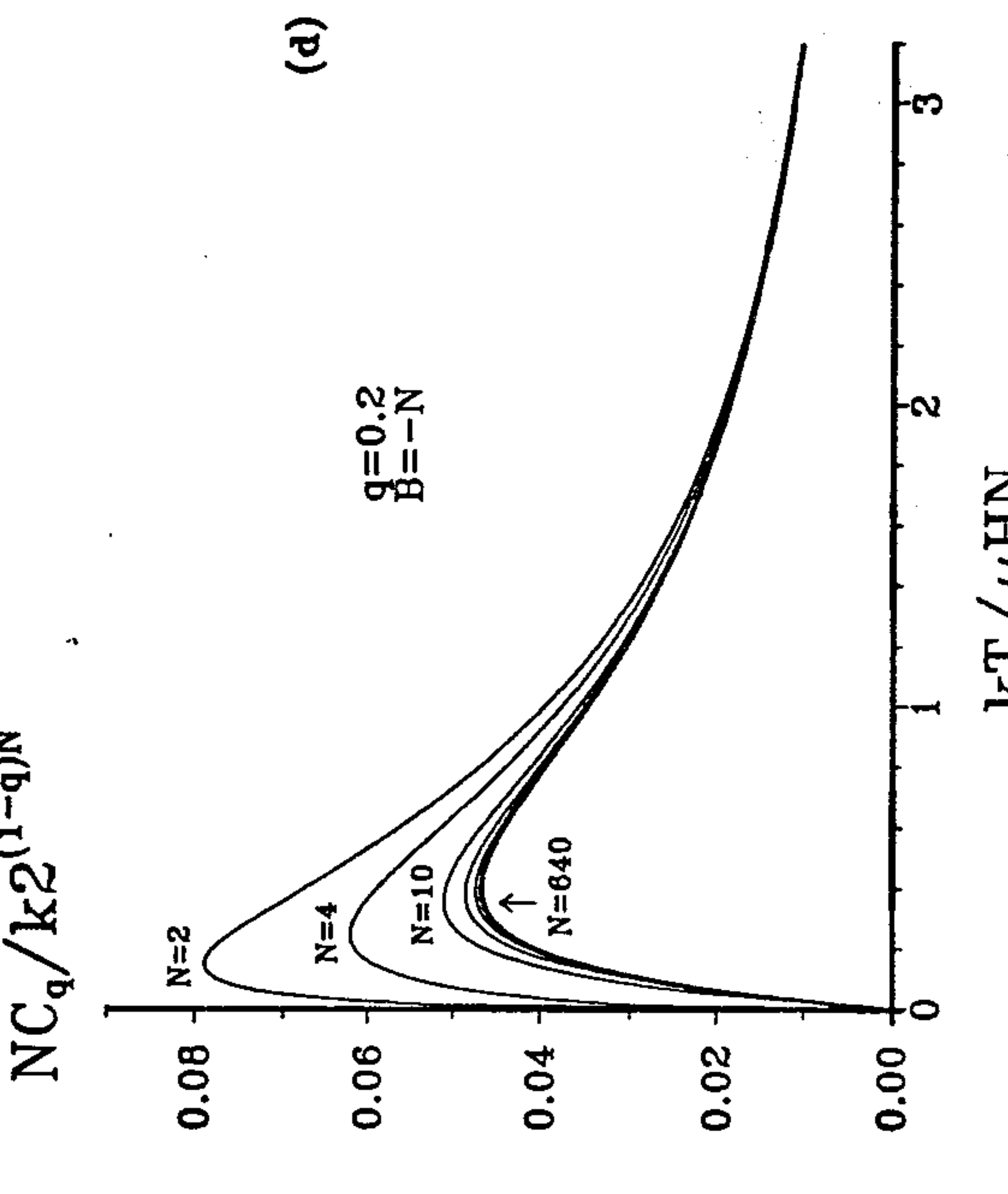

Fig. 6(d)
Nobre and Tsallis

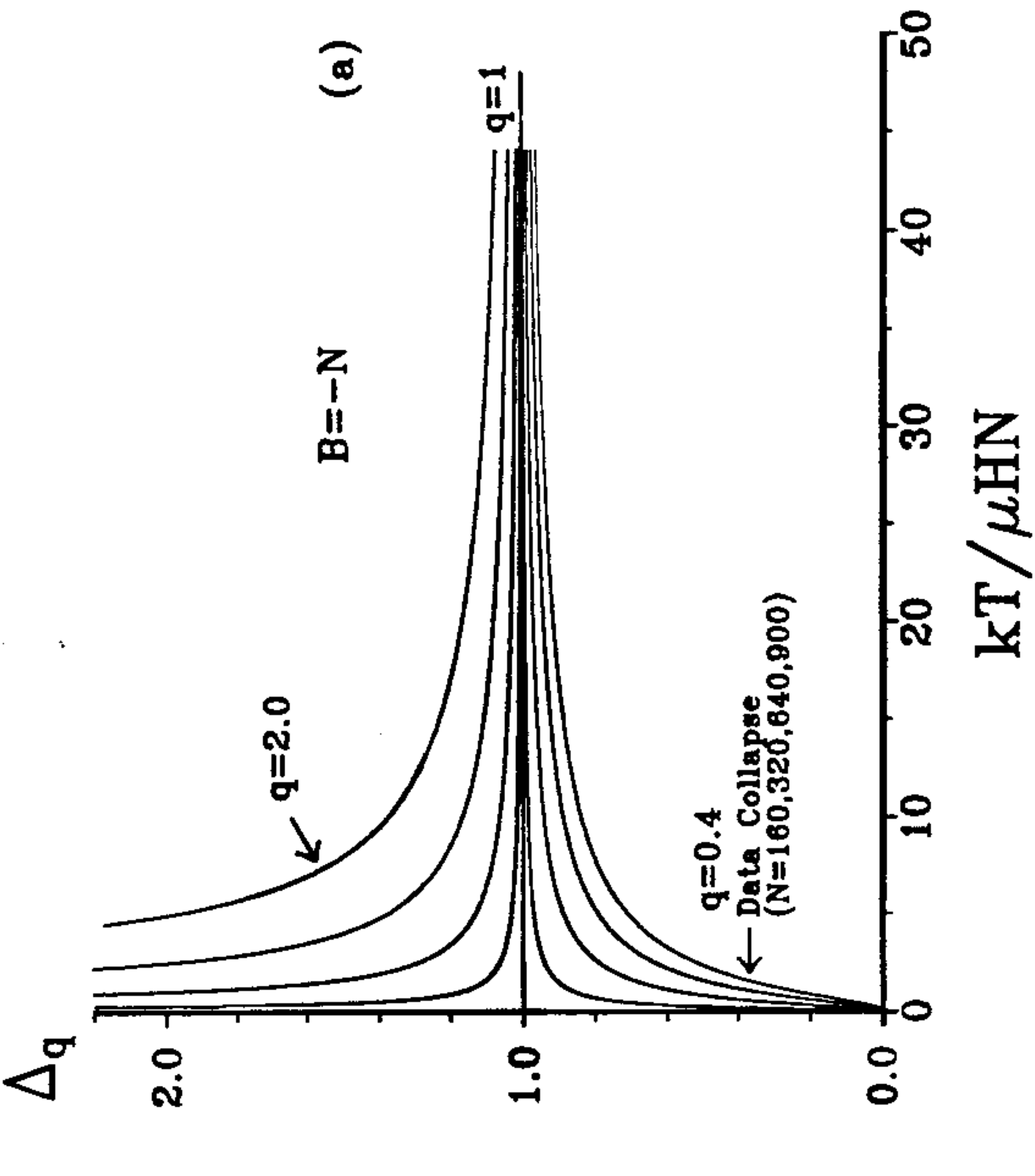

Fig. 7(a)
Nobre and Tsallis

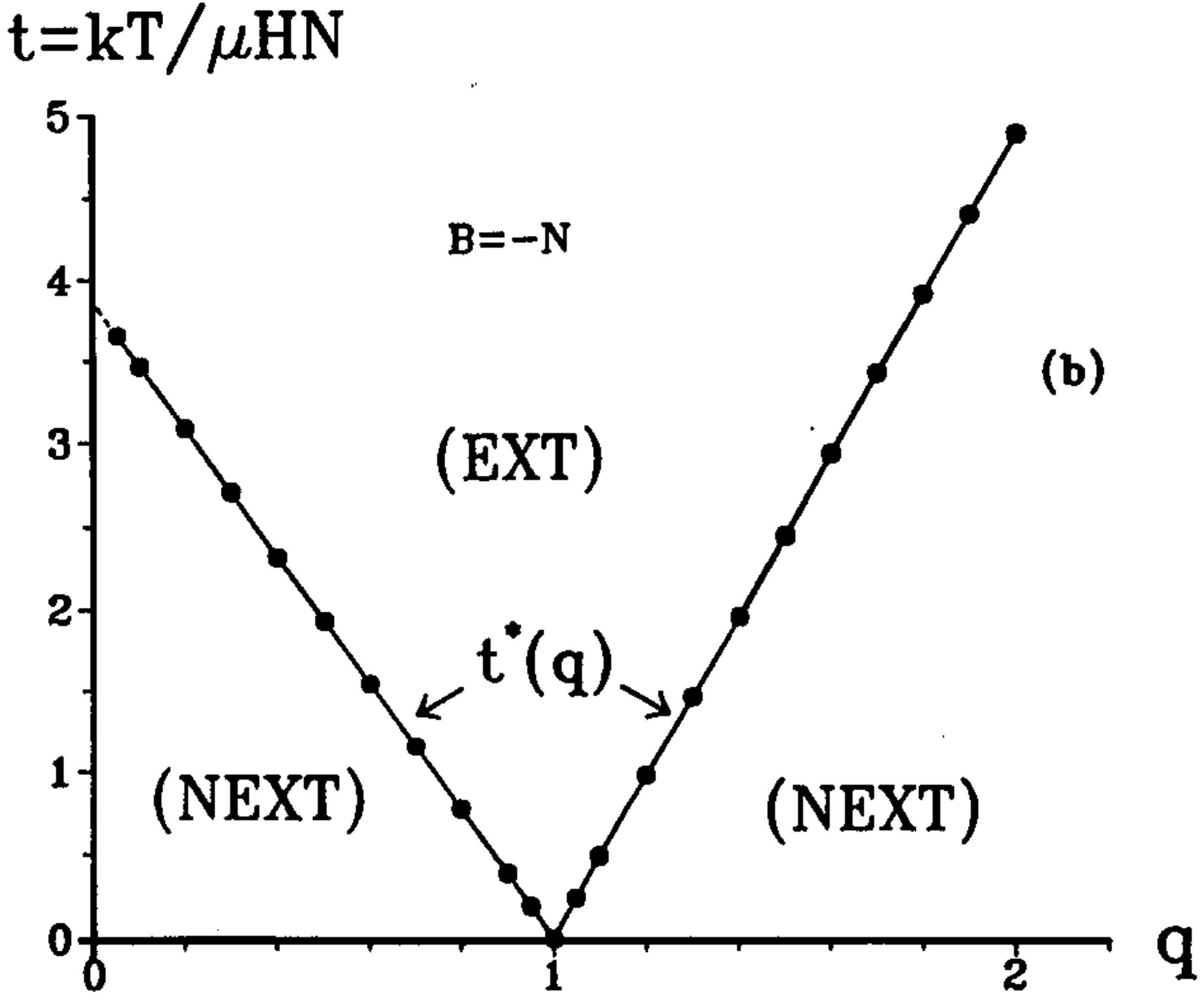

Fig. 7(b)
Nobre and Tsallis

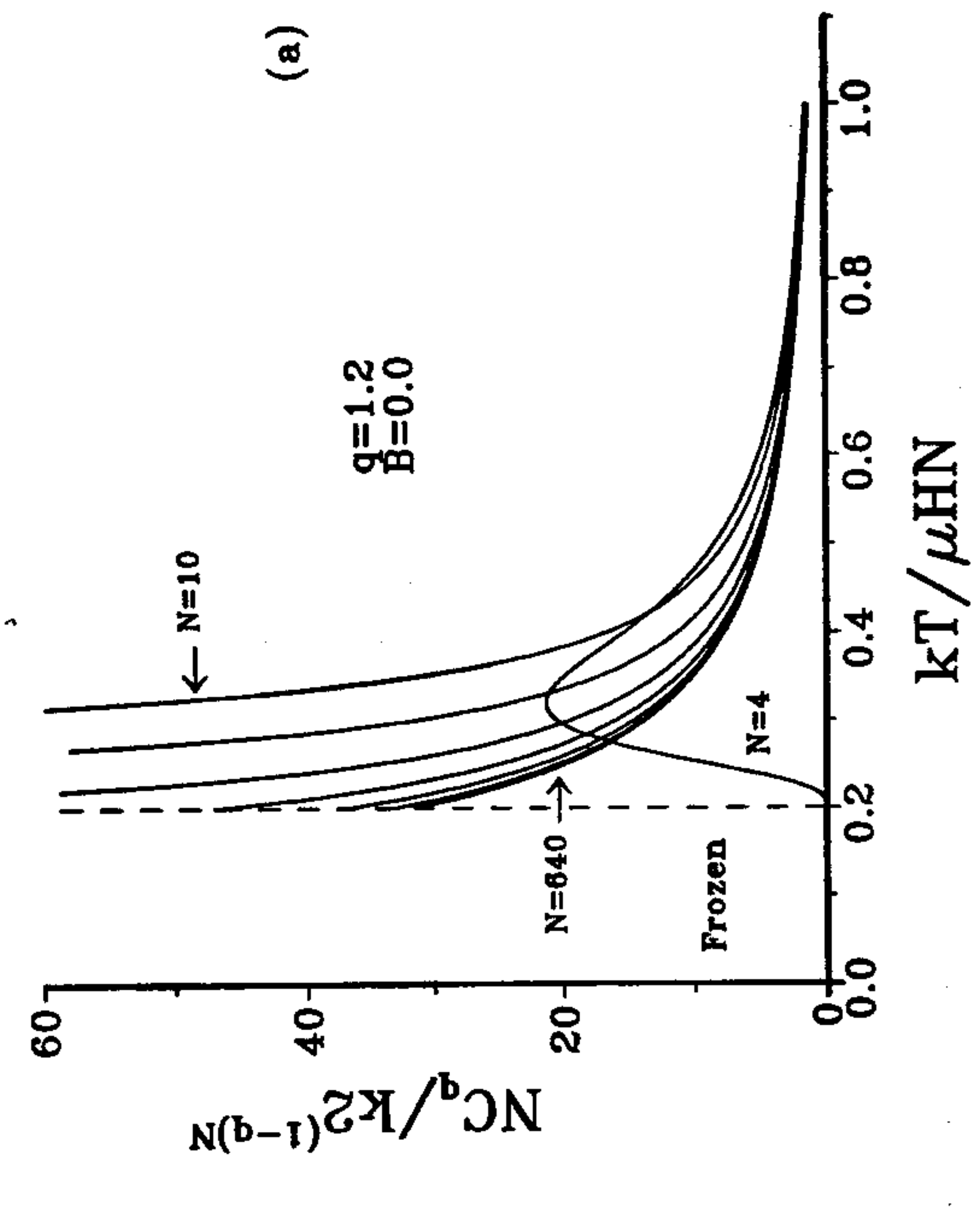

Fig. 8(a)
Nobre and Tsallis

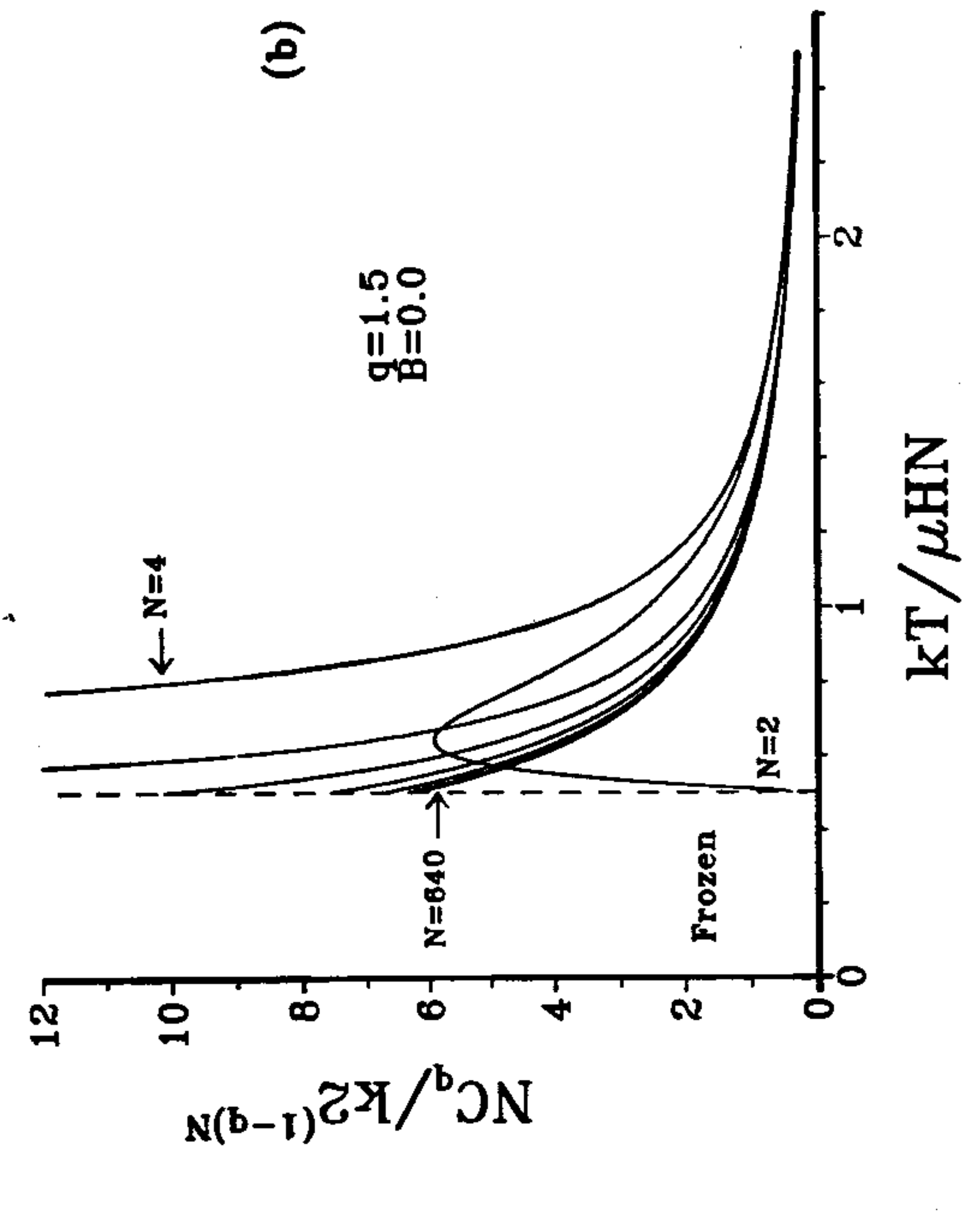

Fig. 8(b)
Nobre and Tsallis

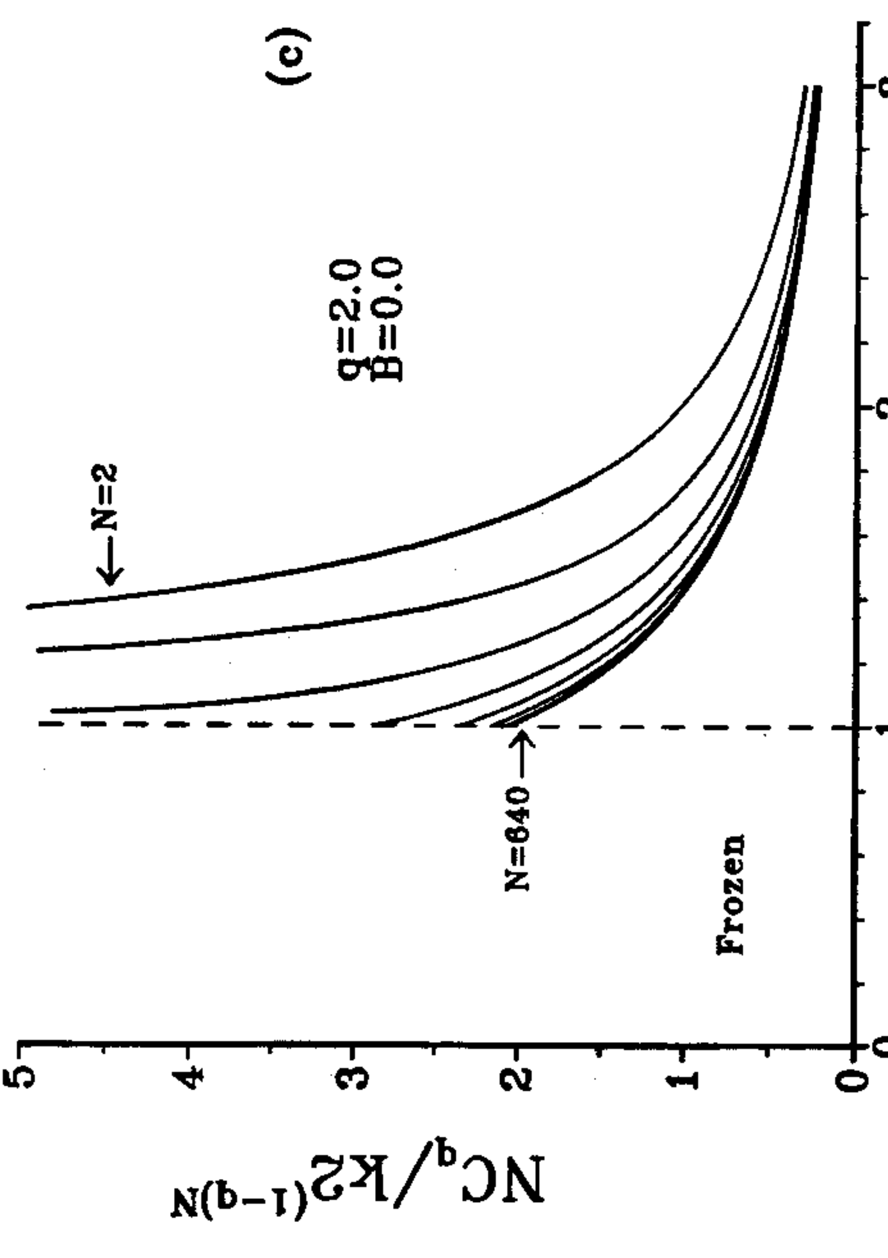

Fig. 8(c)
Nobre and Tsallis

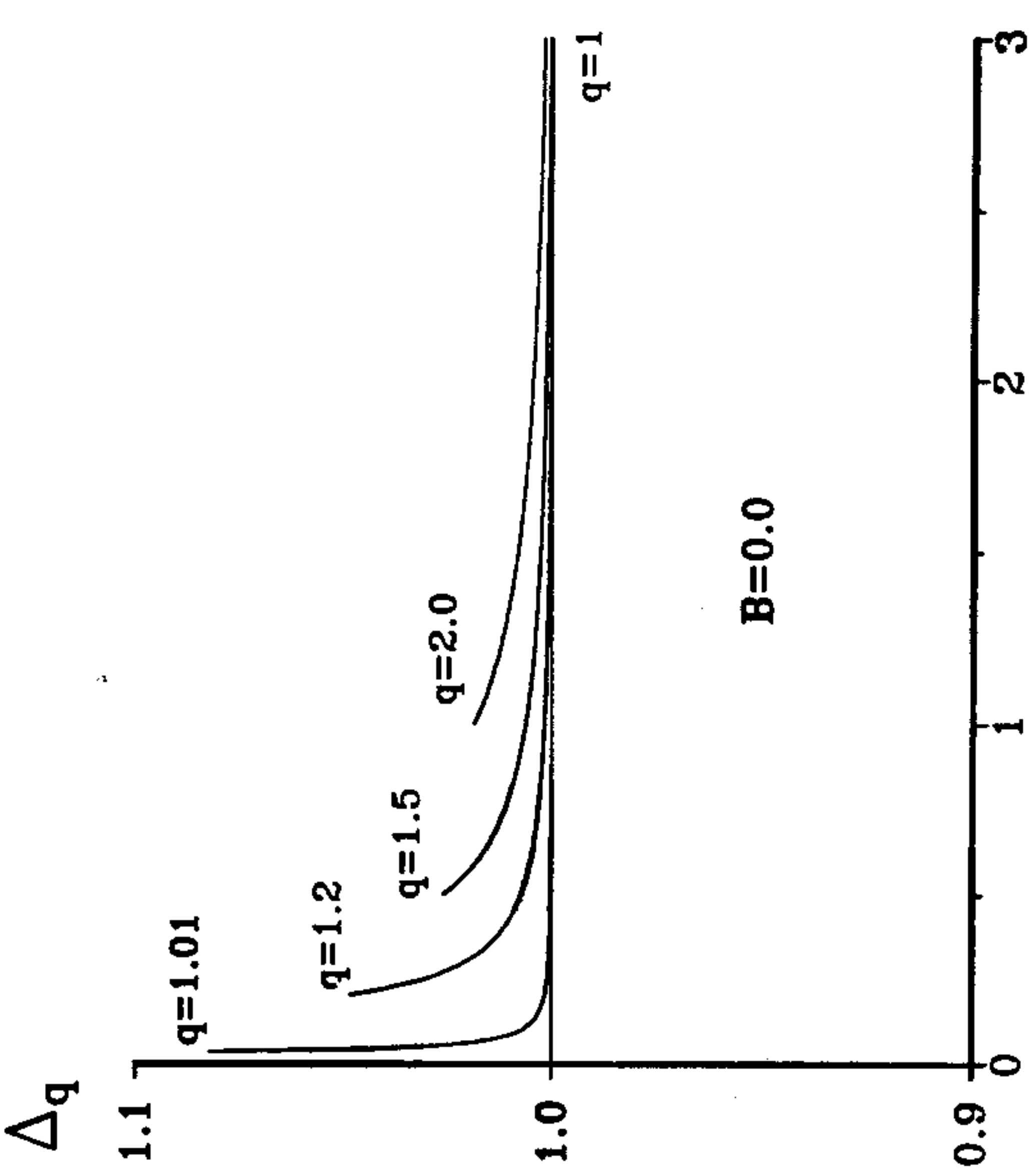

Fig. 9
Nobre and Tsallis

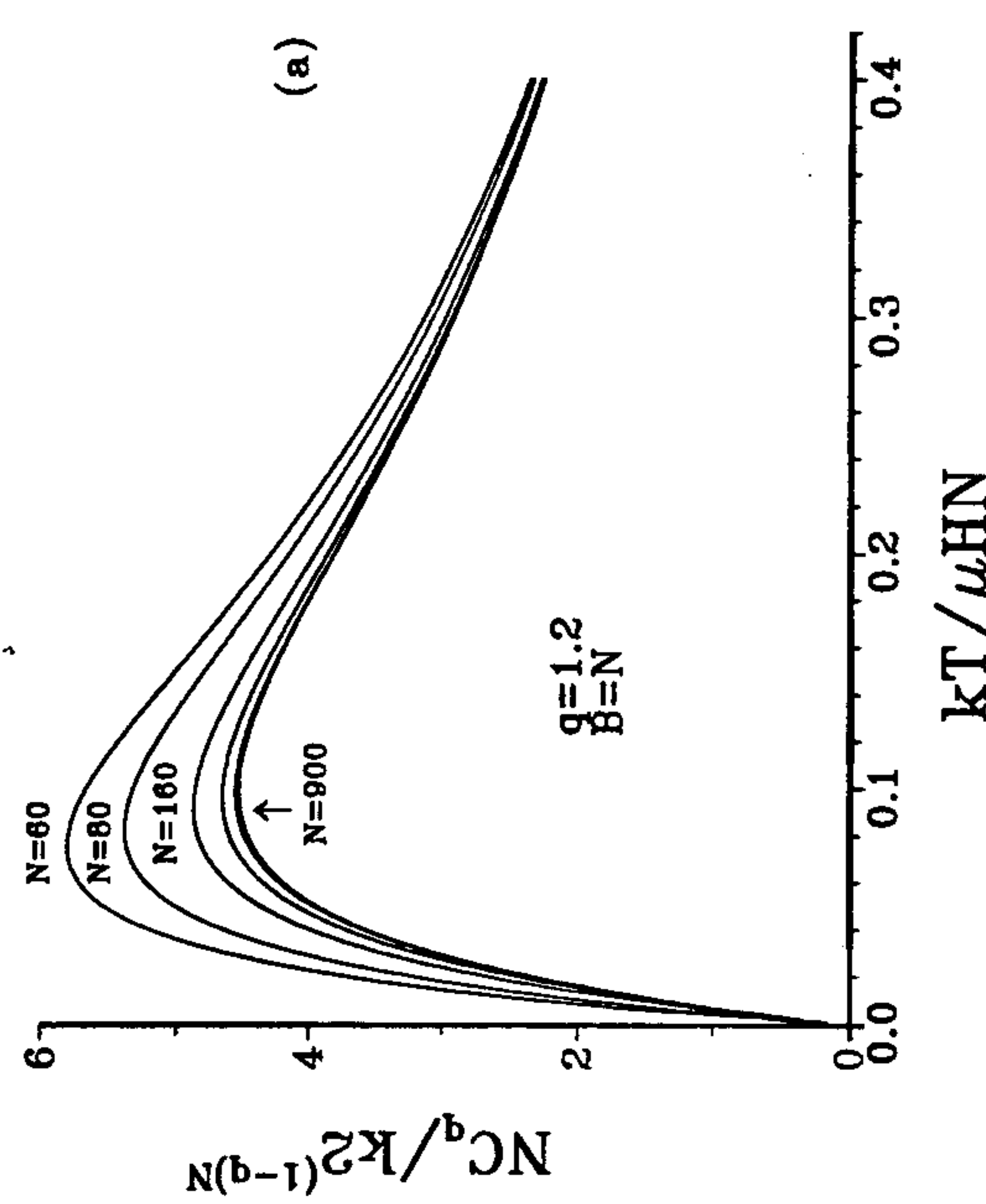

Fig. 10(a)
Nobre and Tsallis

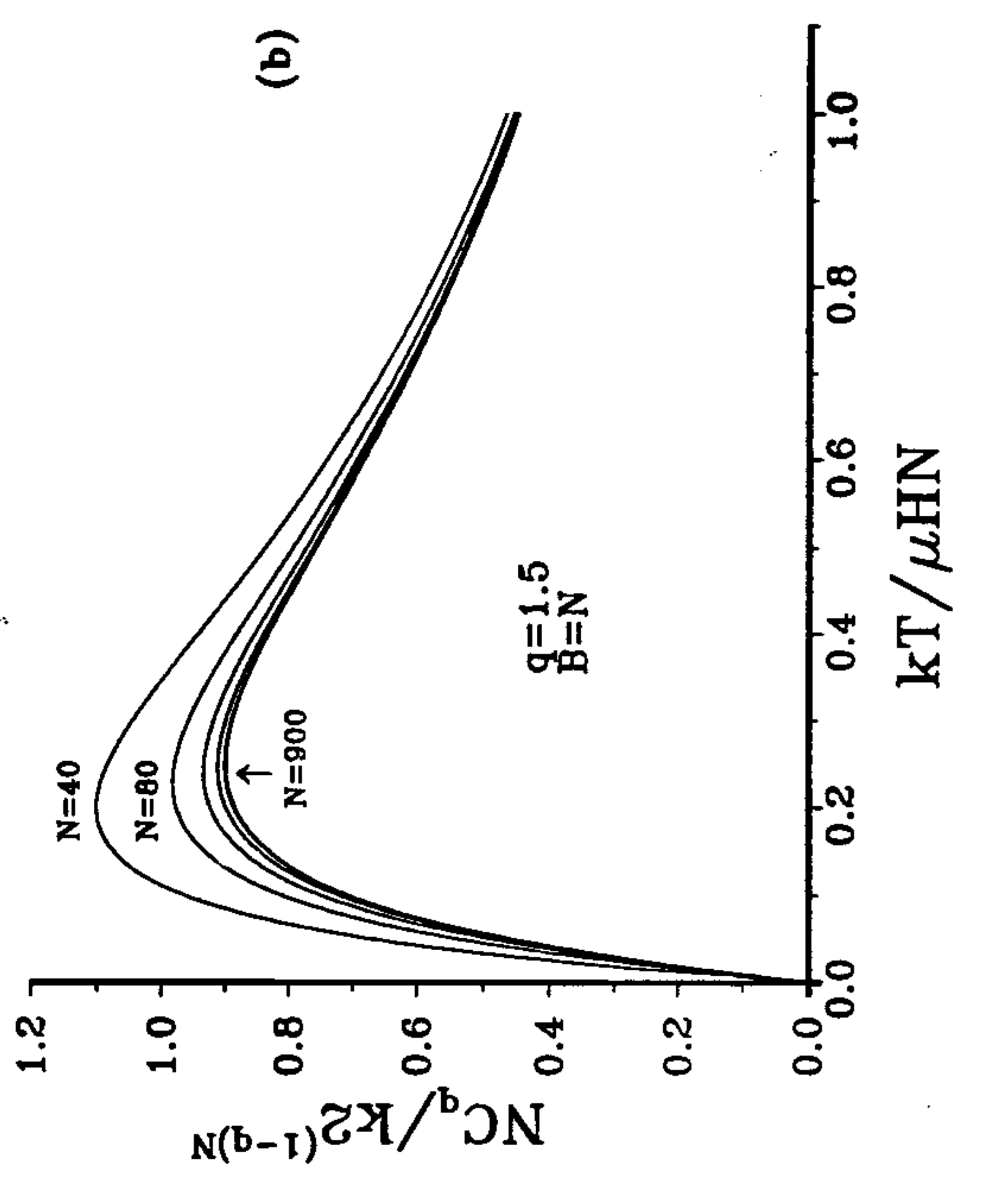

Fig. 10(b)
Nobre and Tsallis

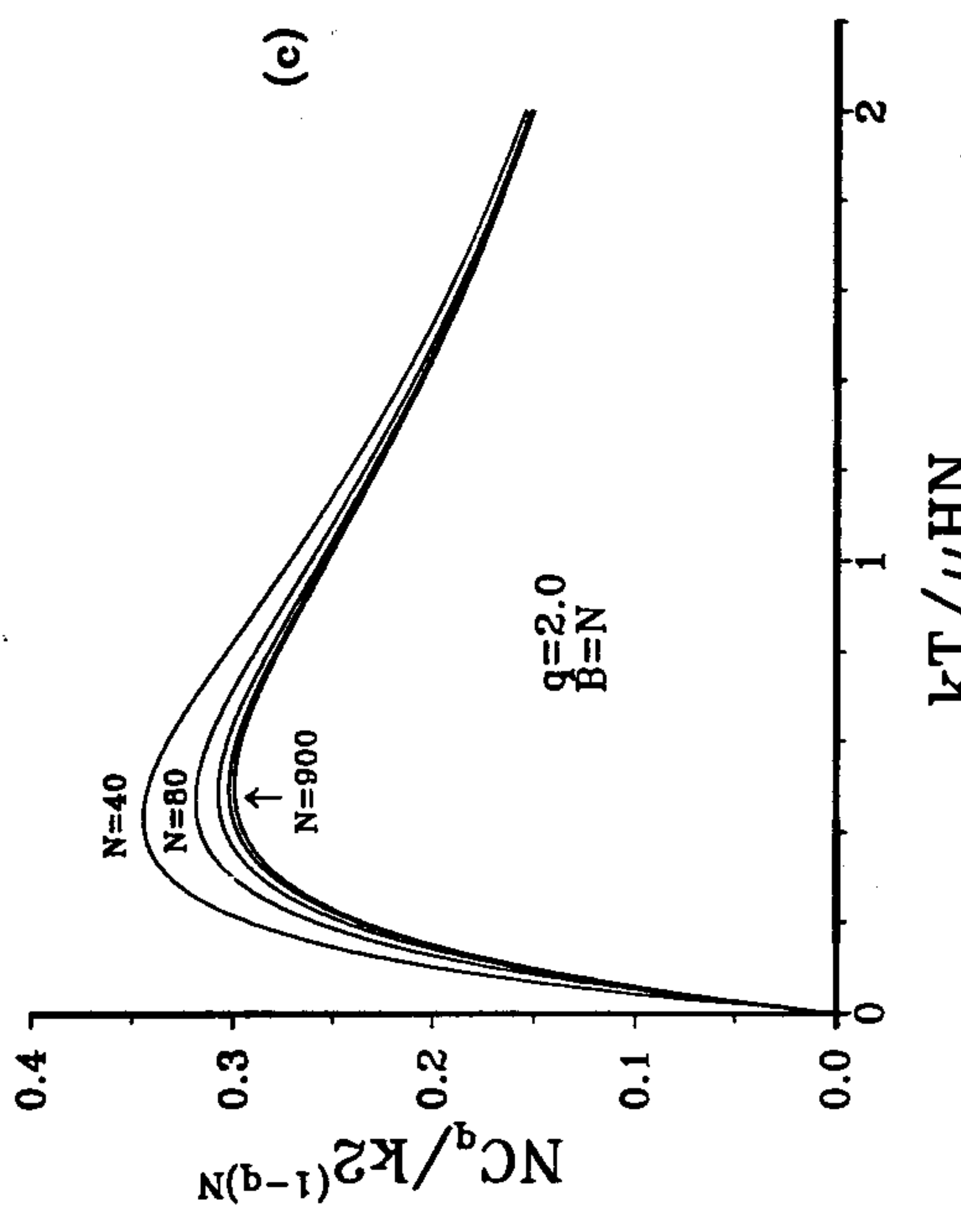

Fig. 10(c)
Nobre and Tsallis

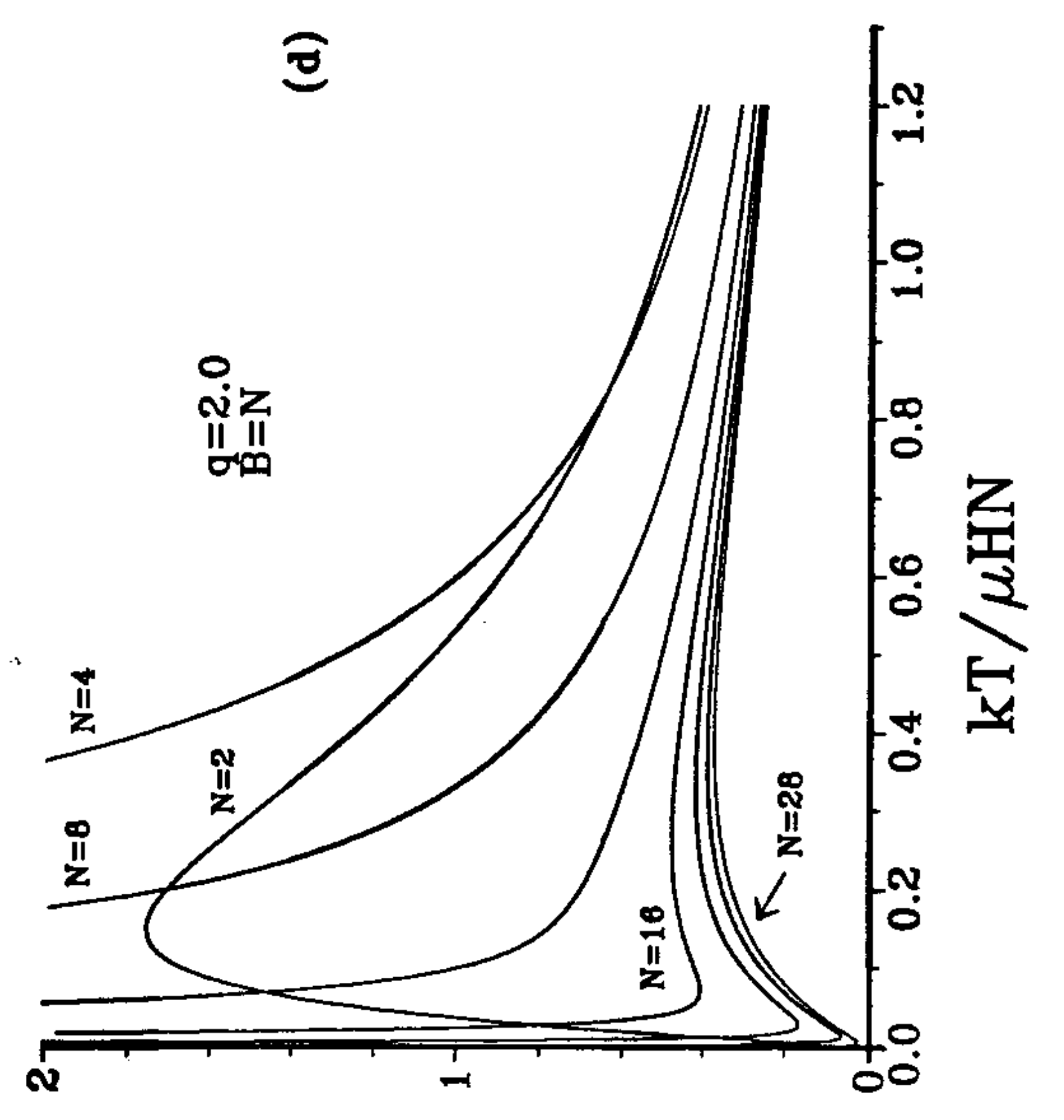

Fig. 10(d)
Nobre and Tsallis

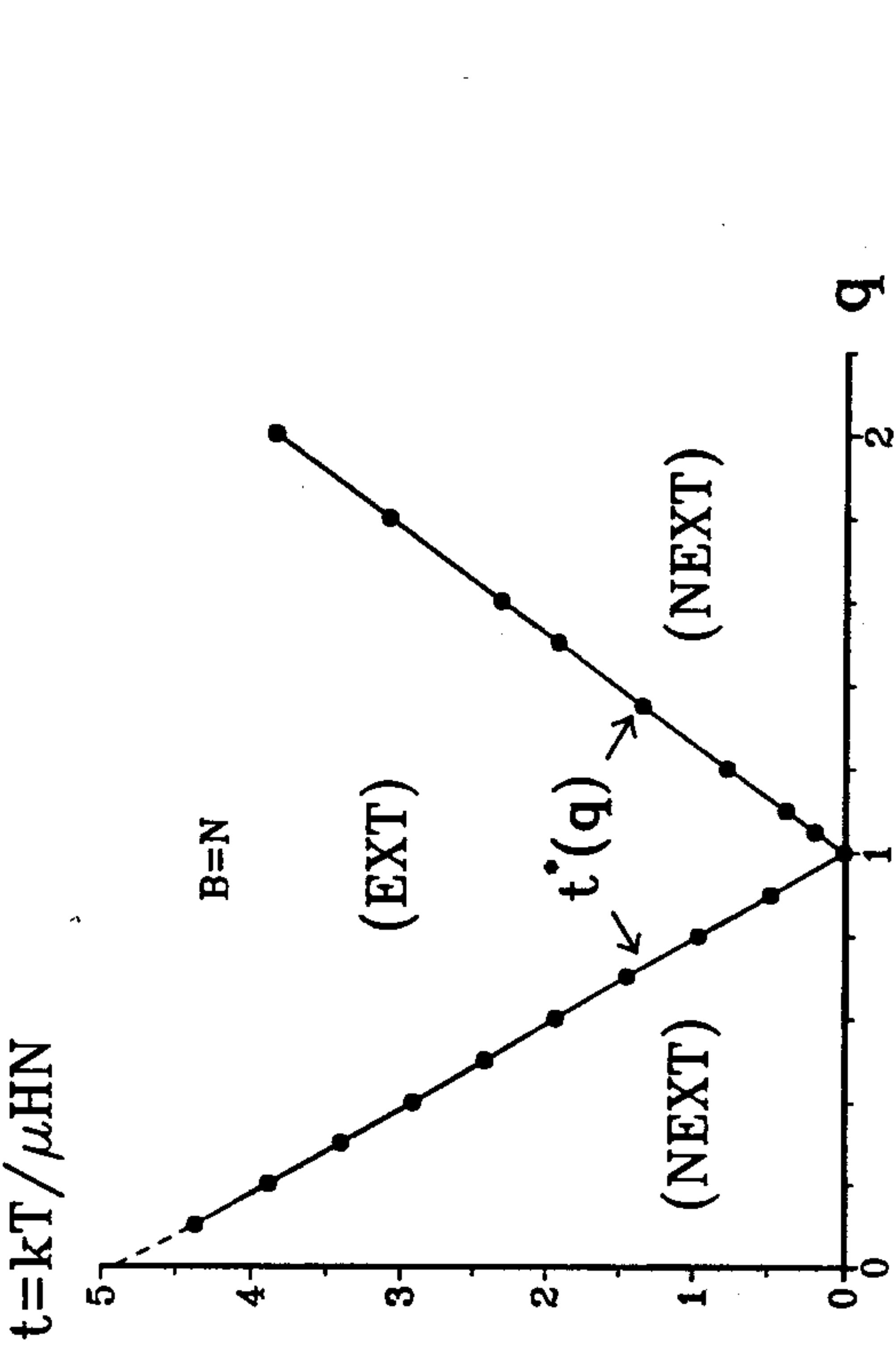

Fig. 11
Nobre and Tsallis

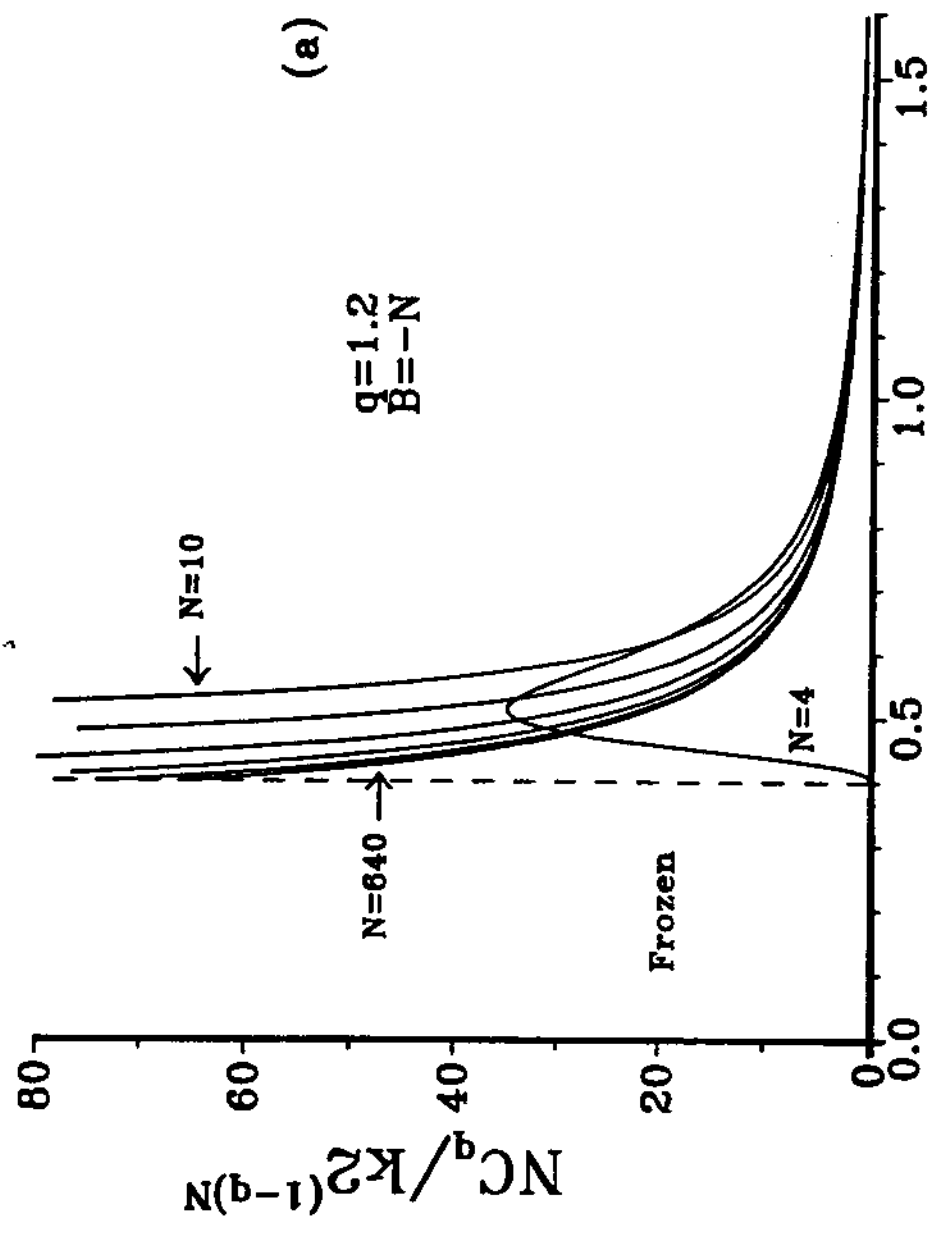

Fig. 12(a)
Nobre and Tsallis

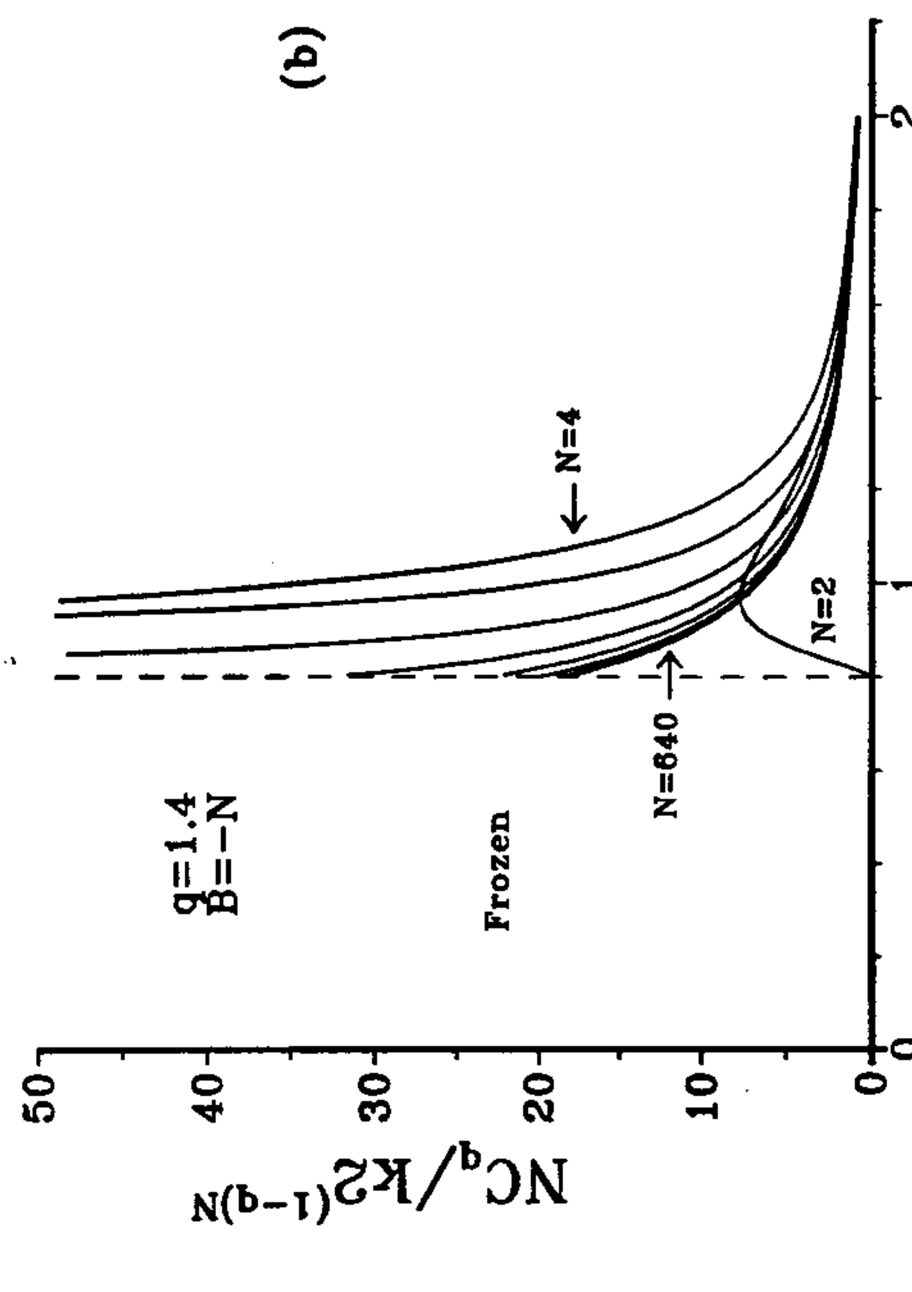

Fig. 12(b)
Nobre and Tsallis

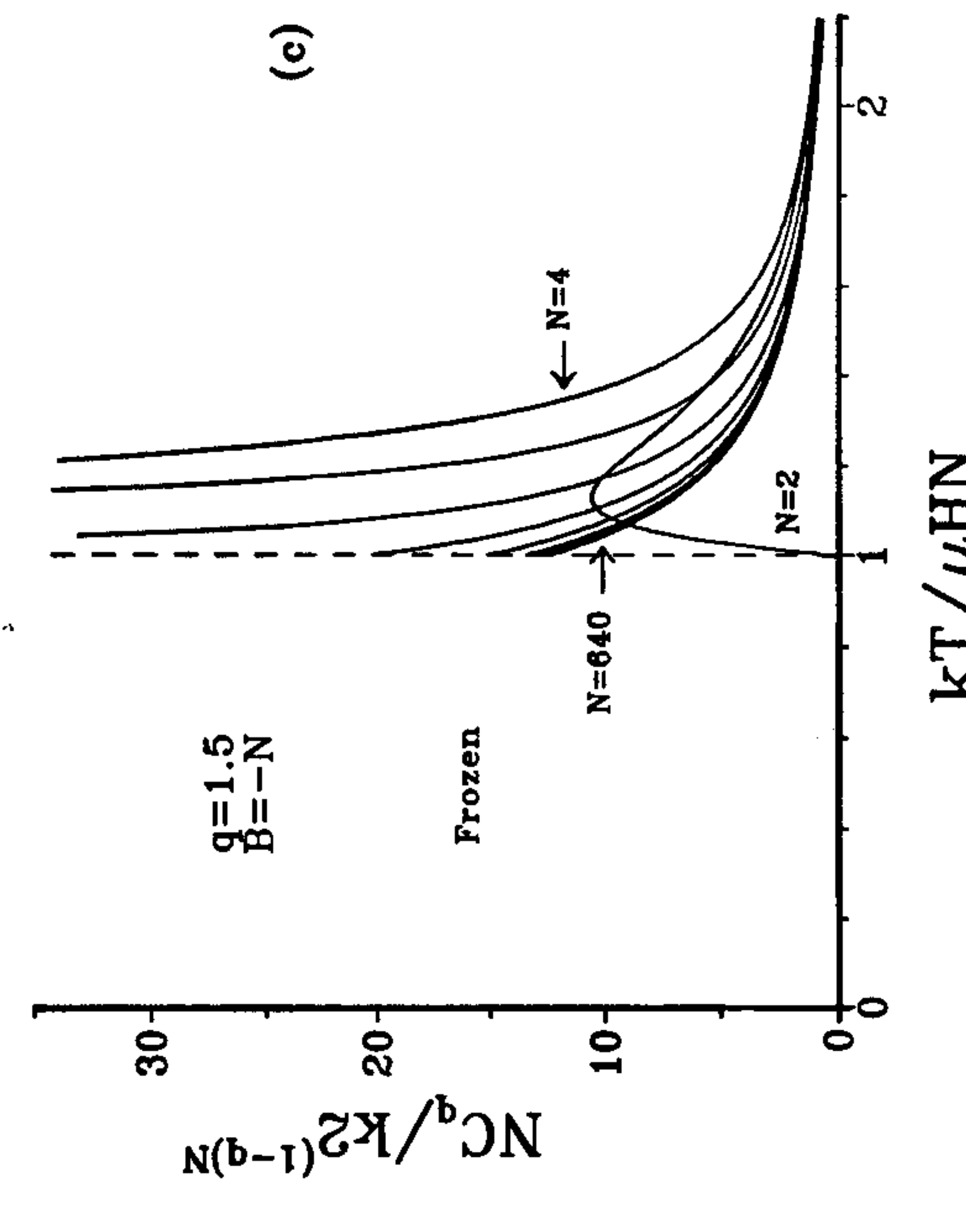

Fig. 12(c)
Nobre and Tsallis

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