THE UNIFICATION OF ELECTRODYNAMICS AND GRAVITY

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ABSTRACT : We present some arguments which favours a unification of Gravity and Electrodynamics in the realm of Cartan Restricted Geometry.

The idea of geometrization of gravity has its main support in the well-known universality of the constancy of the ratio of the inertial mass to the gravitational mass. Such universality becomes comprehensible and meaningful in the realm of a geometrical vision.

The program by means of which gravity is intimately connected to the geometry of space time has been carried on very succesfully by Einstein.

Analogous attemps to link the other classical long range field, e.g. electrodynamics, to the structure of space time, however, have failed .

It seems to me that the main reason of such insuccess is due to the lack of a explicit corresponding universality in electrodynamical interactions, which should support the geometrical idea.

The purpose of this letter is precisely to show where, in Electrodynamics, we can find such universality and what modifications on the geometric structure of the space-time the Electromagnetic field can provide.

Dirac, in the early thirties, in examining the dual symmetry of Electrodynamics suggested a modification of Maxwell's theory by the postulate of a magnetic current I^{μ}. The new set of equations have the form

(1a) $$F^{\mu\nu}{}_{;\nu} = e\ J^{\mu}$$

(1b) $$\overset{*}{F}{}^{\mu\nu}{}_{;\nu} = g\, I^{\mu}$$

We define the dual as usual $\overset{*}{F}{}^{\mu\nu} = \frac{1}{2}\, \eta^{\mu\nu\rho\sigma}\, F_{\rho\sigma}$. The symbol ; means co-variant derivative in a Riemannian space.

In this form, the invariance of the theory under the dual map

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \cos\Theta\, F_{\mu\nu} + \sin\Theta\, \overset{*}{F}_{\mu\nu}$$

is evident.

A geometrical approach to Electrodynamics should take such invariance as a guidance to the new structural relationship of space time properties.

Recently, Strazhev has shown that it is not necessary to postulate, as Dirac did, the existence of a magnetic monopole in order to have dual symmetry in the presence of currents and the reason is that one can treat any particle as being doubly charged, that is, any particle have both electric and magnetic charge.

If there is no magnetic current, equation (1b) implies that a vector potential ω^{μ} can be introduced and usually one sets

$$F_{\mu\nu} = \omega_{\mu;\nu} - \omega_{\nu;\mu} = \frac{\partial \omega_{\mu}}{\partial x^{\nu}} - \frac{\partial \omega_{\nu}}{\partial x^{\mu}}$$

However, as some authors (e.g., Cabibo, Ferrari , Strazhev and others) have notted, the most general form for $F_{\mu\nu}$ (even in the case $I^{\mu} \neq 0$) can be written in terms of two potentials ω^{μ} and z^{μ}:

$$(2) \qquad F_{\mu\nu} = \frac{\partial \omega_{\mu}}{\partial x^{\nu}} - \frac{\partial \omega_{\nu}}{\partial x^{\mu}} + \frac{1}{2} \eta_{\mu\nu}{}^{\rho\sigma} \frac{\partial z_{\rho}}{\partial x^{\sigma}}$$

There is a gauge group in the theory, which is characterized by the map

$$(3) \qquad \begin{aligned} \omega_{\mu} &\rightarrow \omega'_{\mu} = \omega_{\mu} + A_{\mu} \\ z_{\mu} &\rightarrow z'_{\mu} = z_{\mu} + B_{\mu} \end{aligned}$$

in which A^{μ} and B^{μ} must satisfy the null-field condition:

$$(4) \qquad \frac{\partial A_{\mu}}{\partial x^{\nu}} - \frac{\partial A_{\nu}}{\partial x^{\mu}} + \frac{1}{2} \eta_{\mu\nu}{}^{\rho\sigma} \frac{\partial B_{\rho}}{\partial x^{\sigma}} = 0$$

Thus dual invariance of the theory induces us to search for a geometry which accomodates naturaly two four-vectors connected to a gauge group.

For a geometer, this is not a difficult task once there is a special class of Cartan affine geometry which has precisely the required property. In [6] Cartan presented a generalization of Riemannian geometry in which the notion of paralel transport is generated by a non symmetric connection

$\Gamma^{\alpha}_{\mu\nu}$.

Some authors, erroneously, have tried to relate the anti-symmetric part of $\Gamma^{\alpha}_{\mu\nu}$, to the Electromagnetic field. These attempts have failed for two reasons: Firstly, they do not take care of dual invariance of Electrodynamics and secondly, thay have not explored the consequences of the existence of a class of equivalence of geometries related to the universal role of the dual charge, as we will see next.

We remark that although the torsion $T^{\alpha}_{\mu\nu} \equiv \frac{1}{2}(\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu})$ has 24 degrees of freedom, 16 of them will be frozen by setting

$$T^{\alpha}_{\mu\nu} = \frac{1}{3}(\delta^{\alpha}_{\mu}\,\omega_{\nu} - \delta^{\alpha}_{\nu}\,\omega_{\mu}) + \frac{1}{3}\eta^{\alpha}{}_{\mu\nu\lambda}\,z^{\lambda}$$

The reason for this is simple: we do not need the whole 24 independent numbers to describe the electromagnetic field and we do not want to introduce any new field into the structure of space-time for the time being. Further, the decomposition of torsion into its trace $T^{\alpha}{}_{\alpha\nu} \equiv \omega_{\nu}$ and pseudo-trace $\overset{*}{T}{}^{\alpha}{}_{\alpha\nu} \equiv z_{\nu}$ is irreducible. We will call such structure a Cartan Restricted Geometry (GCR).

Now, if we want to identify these two traces with the electromagnetic potentials we should ask: What is the relation of the Maxwell tensor $F_{\mu\nu}$ with the geometry?

The answer to this question comes from an examination of the curvature tensor $R_{\alpha\beta\mu\nu}$. By definition we have

$$R^{\mu}{}_{\varepsilon\nu\lambda} = \frac{\partial\Gamma^{\mu}_{\nu\varepsilon}}{\partial x^{\lambda}} - \frac{\partial\Gamma^{\mu}_{\lambda\varepsilon}}{\partial x^{\nu}} + \Gamma^{\mu}_{\lambda\sigma}\,\Gamma^{\sigma}_{\nu\varepsilon} - \Gamma^{\mu}_{\nu\sigma}\,\Gamma^{\sigma}_{\lambda\varepsilon}$$

in which, in the GCR, the connection $\Gamma^{\mu}{}_{\nu\rho}$ is given by

$$\Gamma^{\mu}{}_{\varepsilon\lambda} = \left\{ {\mu \atop \varepsilon\lambda} \right\} + \frac{2}{3}(\delta^{\mu}{}_{\varepsilon}\,\omega_{\lambda} - g_{\varepsilon\lambda}\,\omega^{\mu}) - \frac{1}{3}\,\eta^{\mu}{}_{\varepsilon\lambda\alpha}\,z^{\alpha}$$

and $\left\{ {\mu \atop \varepsilon\lambda} \right\}$ is the Christoffel symbol which depends only on $g_{\mu\nu}$ and its derivatives.

Then, a straight forward calculation gives for the symmetric and anti-symmetric parts of the contracted tensor $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$:

(5a) $$\frac{1}{2}(R_{\mu\nu} + R_{\nu\mu}) = \dot{R}_{\mu\nu} + \frac{2}{3}(\omega_{\mu;\nu} + \omega_{\nu;\mu} + \omega^{\alpha}{}_{;\alpha}\,g_{\mu\nu}) - \frac{8}{9}(\omega_{\mu}\,\omega_{\nu} - \omega^{2}\,g_{\nu\mu}) + \frac{2}{9}(z_{\mu}\,z_{\nu} - z^{2}g_{\mu\nu})$$

(5b) $$\frac{1}{2}(R_{\mu\nu} - R_{\nu\mu}) = \frac{2}{3}\left\{ \frac{\partial\omega_{\mu}}{\partial x^{\nu}} - \frac{\partial\omega_{\nu}}{\partial x^{\mu}} + \frac{1}{2}\,\eta_{\mu\nu}{}^{\rho\sigma}\frac{\partial z_{\rho}}{\partial x^{\sigma}} \right\}$$

in which $\dot{R}_{\mu\nu}$ is constructed only with the Christoffel symbols $\left\{ {\alpha \atop \nu\mu} \right\}$

Thus, it is natural to identify the Maxwell tensor $F_{\mu\nu}$ with the anti-symmetric part of $R_{\mu\nu}$. We set

(6) $$F_{\mu\nu} = \frac{3}{4}(R_{\mu\nu} - R_{\nu\mu})$$

With such identification we recognize that $F_{\mu\nu}$ does not fix completely the structure of the Cartan restricted geometry.

We conclude then that each electromagnetic field is related to a set of geometries of the type GCR, for $N \equiv \{T^{\alpha(1)}_{\mu\nu}, T^{\alpha(2)}_{\mu\nu}, T^{\alpha(3)}_{\mu\nu}, \ldots\}$. The elements of N are related by a dual gauge giving by

$$\omega'_{\mu} = \omega_{\mu} \cos \Theta + z_{\mu} \operatorname{sen} \Theta$$
$$z'_{\mu} = -\omega_{\mu} \operatorname{sen} \Theta + z_{\mu} \cos \Theta \ .$$

Thus we can state that Electrodynamics generates a class of equivalence of Cartan Restricted Geometries. We note that the impossibility to fix a unique geometry of N is due to the dual symmetry of Electrodynamics.

The specification of a given element of N corresponds to a definite choice of the dual angle.

Now comes the second big question: if there is any sense in geometrizing electrodynamics, the set N should be constructed independently of a particular electromagnetic configuration. In other words, should all charged particles perceive the same set N ?

At this point we make an appeal to the universal dual behaviour of charged particles. In (1) Strazhev has shown that any charged particle can be regarded as an admixture of electric charge e and magnetic charge g different for each particle, thus giving origin to an effective electric charge $q^2 = e^2 + g^2$.

The condition of compatibility of the doubly charged hypothesis is that the ratio $\frac{g}{e}$ must be the same constant for any real particle. Such universality of the ratio $\frac{g}{e}$, which preserves the dual invariance of Electrodynamics even in the presence of charged particles corresponds to the dual symmetry of Cartan geometries of N.

If a particle, call it I, fix the dual angle through the value $\left(\frac{g}{e}\right)_I$, it implies that for any other particle, say

II, the value of $\left(\frac{g}{e}\right)_{II}$ is fixed and they are equals, $\left(\frac{g}{e}\right)_{I} = \left(\frac{g}{e}\right)_{II}$. This is guaranteed by the universality.

Fixing the dual angle $\left(\frac{g}{e}\right)_{I}$ implies a selection of a given geometry of N.

Thus, universality implies that fixing the geometry of a given particle, say fixing $\left(\frac{g}{e}\right)_{I}$, then the structure of the geometry (which is given by selecting an element of N) is established for all particles.

It is precisely this fact that enable us to pursue the idea to relate electrodynamics to the structure of space time within the context of Cartan restricted geometries.

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