RELATIVISTIC MODEL OF A SPHERICAL STAR EMITTING NEUTRINOS*

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ABSTRACT

We present a simple but complete relativistic model of a spherical star emitting neutrinos, with basis in the coupled Einstein-Dirac equations. The interior of the star is assumed to be a perfect fluid — described by its energy-matter density ρ, pressure p and barion number density n — bounded in space. Matter is considered transparent for neutrinos and the exterior region contains only neutrinos and gravitational field. The question of compatibility of neutrinos with spherically symmetric gravitational fields is discussed and a redefinition proposed for the physical energy-momentum tensor which enters the RHS of Einstein equations. Analytical solutions are obtained and are shown to correspond to a description of emission of neutrinos with cooling and contraction of the configuration. The local conservation laws and the junction and boundary conditions of the exterior and interior solutions in the surface of the fluid are studied and allow to characterize two classes of solutions. In one case the solution describes the stage of neutrinos emission with consequent contraction of the configuration of the star immediately before the fluid is totally contained inside its Schwarzschild radius, when the emission of neutrinos and the contraction of the star cease. The other possibility can correspond to a

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quasi-static configuration emitting neutrinos; the relativistic equation of radiative equilibrium for neutrinos is derived and permits to define the equivalent of a "radiation pressure" for neutrinos, which has an additive contribution to the gravitatio nal pressure and is not purely relativistic, with an eventual Newtonian limit.

1. INTRODUCTION

Although the study of the interaction of neutrinos and gravitational fields has begun with the works of Cartan[1] and Infeld and Van der Waerden[2] on spinors in a Riemannian manifold, we can consider the paper by Brill and Wheeler[3] as a basic refe rence. To the theoretical motivations these two authors have given of the importance of considering the physics of neutrinos in a curved space-time, many substantial arguments have been added in the last fifteen years. In cosmology, neutrinos are believed to play an important role in the question of the ener gy-density of the Universe[4,5]; also astrophysical processes connec ted to the emission and absorption of neutrinos have been exten sively discussed where, in certain cases (advanced stages of stellar evolution, etc)[6] the General Theory of Relativity beco mes important.

In this vein we present here a simple exact model of a localized source of neutrinos in the scheme of General Relativity. Anteriorly Misner[7] examined the gravitational colapse of a sphe rically symmetric perfect fluid with neutrino production, neutri nos being treated phenomenologically as a null fluid(*) and matter

(*) That is, with energy-momentum tensor of the form $T_{\alpha\beta} \sim k_\alpha k_\beta$, $k^\alpha k_\alpha = 0$.

transparent to neutrinos. The paper is limited to the formulation of the basic equations, describing "a simple heat transfer process in which internal energy is converted into an outward flux of neutrinos". Also Vaidya[8], with an analogous model, obtained some non-static solutions of Einstein equations for fluid spheres radiating electromagnetic energy. Since in a general spherically symmetric space-time the electromagnetic energy-momentum tensor can assume the form of a null fluid but with covariant divergence identically zero[9], Vaidya was led to consider the partial absorption of the radiation when traversing the medium, which is an effect of non-gravitational origin and demands further assumptions. Although both authors use a null fluid description for radiation (neutrinos or photons) the essential difference in the models is: (i) in Vaidya, the energy-momentum tensor of the null fluid satisfies $\underset{(n)}{T}{}_{\mu}{}^{\nu}{}_{\parallel\nu} = 0$ identically and one is then led to consider radiation is partially absorbed by the medium; (ii) in Misner, the energy-momentum tensor is by definition, one of a null fluid but $\underset{(n)}{T}{}_{\mu}{}^{\nu}{}_{\parallel\nu} \neq 0$ such that the sum of the energy-momentum tensors of the cooling matter and of the emitted neutrinos satisfies the local conservation law, with matter transparent for neutrinos.

We present here a class of analytical solutions corresponding to a model which has many similarities with the above two models: we consider a spherically symmetric bounded distribution of a perfect fluid(*) with the only assumptions; (i) neutrinos in interaction with gravitation are described by spinorial fields in the curved space-time; (ii) once emitted neutrinos have gravitational interaction only (matter is transparent for neutri

(*) for instance, a sphere of a degenerate neutral barion gas under self-gravity.

nos); (iii) the model is solution of the coupled Einstein-Dirac equations. We distinguish two regions: I. interior region, constituted of a perfect fluid distribution and neutrinos flowing radially outwards and II. exterior region, only neutrinos in interaction with gravitational field. Einstein equations and junction conditions of the exterior and interior solutions in the surface of the fluid sphere are sufficient to determine substantial properties of the model. The analytical solutions presented have their properties extensively discussed and some modifications – as the introduction of a Λ-term in the field equations – are made in order to attain to physically relevant situations. On examining the exterior region, we discuss the question of the compatibility of neutrinos as source with a spherically symmetric gravitational field.

For a general review of the formalism of spinors on a Riemannian space-time, see reference (3). Here we use four-component spinors from the point of view of tetrad formalism, with spin transformations generated by the local Lorentz rotations of the tetrads. It has the advantage of being more operational, allowing for a simple unification with Cartan calculus of differential forms[10], which we use in the calculations.

We choose a tetrad field $\left\{e^{\alpha}_{(A)}(x)\,;\,A=0,1,2,3\right\}$ such that locally the line element can be reduced to

$$ds^2 = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 \tag{1.1}$$

where(*)

$$\theta^A = e^{(A)}_{\alpha}\, dx^{\alpha} \tag{1.2}$$

(*) Capital Latin indices are tetrad indices and run 0 to 3; they are raised lowered with Minkowski metric η^{AB}, $\eta_{AB} = diag(+1,-1,-1,-1)$. Greek indices run 0 to 3 and are raised and lowered with $g^{\kappa\beta}$, $g_{\kappa\beta}$.

(1.1) is invariant under local Lorentz transformations

$$\tilde{\theta}^A = L^A{}_B(x)\,\theta^B \tag{1.3}$$

$$L^A{}_B(x)\,\eta_{AD}\,L^D{}_C(x) = \eta_{BC} \tag{1.4}$$

Such transformations correspond to a rotation in the tetrad basis

$$\tilde{e}^{(A)}_{\alpha} = L^A{}_B(x)\,e^{(B)}_{\alpha}$$

Relative to the local Lorentz structure (1.1), (1.4) which exists independently in each point of the manifold , we define Dirac spinors as four component objects which under the group (1.3), (1.4) transform like its correspondents in flat space,

$$\psi(x) \to \psi'(x) = S(L(x))\,\psi(x) \tag{1.5a}$$

and its conjugate correspondent

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\,S^{-1}(L(x)) \tag{1.5b}$$

where $S(x)$ is a 4 x 4 matrix representation of the Lorentz transformation $L^A{}_B(x)$ with the restriction det $S = 1$. Under general coordinate transformations $x^\alpha \to x'^\alpha = x'^\alpha(x)$ 4-spinors transform like scalars.

The constant Dirac matrices(*) γ^A satisfy

$$\gamma^A\gamma^B + \gamma^B\gamma^A = 2\eta^{AB}\,\mathbb{1} \tag{1.6}$$

and constitute a representation of the Clifford algebra associated to the local Minkowski metric η^{AB} . Using the tetrad basis we can define the field of Dirac matrices

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(*) We use a representation such that $(\gamma^A)^+ = \gamma^0\gamma^A\gamma^0$, with $(\gamma^0)^2 = -(\gamma^k)^2 = \mathbb{1}$, $k = 1,2,3.$

$$\gamma^{\mu}(x) = e^{\mu}_{(A)}(x)\gamma^{A} \tag{1.7}$$

which then satisfy

$$\gamma^{\mu}(x)\gamma^{\nu}(x) + \gamma^{\nu}(x)\gamma^{\mu}(x) = 2g^{\mu\nu}(x)\mathbb{1} \tag{1.8}$$

corresponding to a Clifford algebra associated to the metric $g_{\mu\nu}(x)$. Under (1.5) the matrices $\gamma^{\mu}(x)$ transform like

$$\tilde{\gamma}^{\mu}(x) = S(x)\gamma^{\mu}(x)\,S^{-1}(x)$$

and by (1.3), (1.4) and (1.7) we have

$$L^{A}{}_{B}(x)\gamma^{B} = S(x)\,\gamma^{A}\,S^{-1}(x) \tag{1.9}$$

In the above formalism

$$\bar{\psi} = \psi^{+}\gamma^{0} \tag{1.10}$$

where γ^{0} is the constant Dirac matrix. The covariant derivatives of ψ and $\bar{\psi}$ (defined such that under (1.5) they transform as spinors and $\bar{\psi}\psi$ as scalar) are given by

$$\nabla_{\alpha}\psi = (\partial_{\alpha}\psi - \Gamma_{\alpha}\psi), \quad \nabla_{\alpha}\bar{\psi} = \partial_{\alpha}\bar{\psi} + \bar{\psi}\Gamma_{\alpha} \tag{1.11}$$

with

$$\nabla_{\alpha}\gamma_{\beta} = \partial_{\alpha}\gamma_{\beta} - \left\{{\rho \atop \alpha\beta}\right\}\gamma_{\rho} - \Gamma_{\alpha}\gamma_{\beta} + \gamma_{\beta}\Gamma_{\alpha} = 0$$

which implies

$$\Gamma_{\alpha} = -\frac{1}{4}\gamma_{ABC}\,\gamma^{A}\gamma^{B}\,e^{(C)}_{\alpha} + A_{\alpha}\mathbb{1} \tag{1.12}$$

For neutrinos we take $A_{\alpha} = 0$. γ_{ABC} are the Ricci rotation coefficient defined by

$$\gamma_{ABC} = -\,e^{\alpha}_{(A)\|\beta}\,e_{\alpha(B)}\,e^{\beta}_{(C)} = -\,\gamma_{BAC} \tag{1.13}$$

The Lagrangean for neutrinos is given by

$$L = i\sqrt{-g}\left\{\bar{\psi}\gamma^{\alpha}(x)\nabla_{\alpha}\psi - \nabla_{\alpha}\bar{\psi}\gamma^{\alpha}(x)\psi\right\} \tag{1.14}$$

with an associated energy-momentum tensor

$$T_{\alpha\beta}(\psi) = i\left\{\bar{\psi}\,\gamma_{(\alpha}\nabla_{\beta)}\psi - \nabla_{(\alpha}\bar{\psi}\,\gamma_{\beta)}\psi\right\} \tag{1.15}$$

The coupled Einstein-Dirac equations for neutrinos as derived from a variational principle are given(*)

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -\kappa\left\{T_{\alpha\beta}(\psi) + T_{\alpha\beta}(\text{matter})\right\} \tag{1.16a}$$

$$\gamma^{\alpha}\nabla_{\alpha}\psi = 0 \tag{1.16b}$$

where $T_{\alpha\beta}(\psi)$ is expressed by (1.15) and $T_{\alpha\beta}$(matter) is to be specified for the particular model we are considering. We remark that in the tetrad basis the Fock-Ivanenko coefficients (1.12) for neutrinos can be expressed

$$\Gamma_{C} = e^{\alpha}_{(C)}\,\Gamma_{\alpha} = -\frac{1}{4}\gamma_{ABC}\,\gamma^{A}\gamma^{B} \tag{1.17}$$

and Dirac equation (1.16b)

$$\gamma^{A}e^{\alpha}_{(A)}\,\partial_{\alpha}\psi + \frac{1}{4}\gamma_{MNA}\,\gamma^{A}\gamma^{M}\gamma^{N}\psi = 0 \tag{1.18}$$

Equations (1.16) constitute the basis for the study of the interaction of neutrinos and gravitational fields, a solution of which – corresponding to a physical situation where this interaction should be dominant – is the object of the present paper.

2. THE EXTERIOR PROBLEM AND THE COMPATIBILITY OF NEUTRINOS WITH SPHERICALLY SYMMETRIC GRAVITATIONAL FIELDS

The exterior region is supposed to contain only neutrinos

(*) Einstein constant $\kappa = 8\pi G$ is positive; throughout the paper we use units such that $\hbar = c = 1$.

in interaction with the gravitational field. Hence in (1.16b) $T_{\alpha\beta}$ (matter) = 0. Since the problem is non-stationary, spherically symmetric and the trace of $T_{\alpha\beta}(\psi)$ is null, we take a metric of the class $[4N]_2$ of the Plebansky-Stachel classification[11], namely the Schwarzschild radiating solution[12]; in (u, r, θ, φ) coordinates, it assumes the form

$$ds^2 = \alpha^2 du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2.1}$$

where $\alpha^2 = 1 - 2m(u)r^{-1}$. Here u ($-\infty \leq u \leq +\infty$) is such that $u = const.$ defines null hypersurfaces. Light signals propagate along null lines of constant u, such that du is the proper time[16] (or Newtonian Time[17]) of an observer at rest at infinity. The vector normal to the hypersurfaces $u = const.$

$$k_\alpha = u_{,\alpha} \quad , \quad k^\alpha = \delta^\alpha_1 \tag{2.2}$$

satisfy

$$k_\alpha k^\alpha = 0 \quad , \quad k^\alpha{}_{\parallel\beta} k^\beta = 0 \tag{2.3}$$

The null lines with tangent k^α constitute a congruence of geodesics, parametrized with the affine parameter r (cf. (2.2)). The coordinate $x^1 = r$ can then be interpreted as a luminosity distance in the usual sense[13]. Along this congruence (u, θ, φ) are constant.

For the metric (2.1) we take the tetrad basis ,

$$\begin{aligned} &e^{(0)}{}_0 = 1 && e^{(0)}{}_1 = \alpha^{-1} && \\ &e^{(1)}{}_1 = \alpha^{-1} && e^{(2)}{}_2 = r && e^{(3)}{}_3 = r\sin\theta \end{aligned} \tag{2.4}$$

where (2.1) assumes the form (1.1). Due to the spherical symmetry of the problem we restrict ourselves to neutrino fields of the form

$$\psi = \begin{pmatrix} \varphi \\ \sigma^1 \varphi \end{pmatrix} \tag{2.5}$$

corresponding to a four current

$$j^\alpha = \bar{\psi} \gamma^\alpha(x) \psi = \alpha \rho \, \delta^\alpha_1 \tag{2.6}$$

radially along the light cones of (2.1)(*). φ is a 2-spinor to be determined, σ^1 is Pauli constant matrix and $\rho = 2\varphi^+\varphi$. The Ricci rotation coefficients γ_{MNA} defined in (1.13) can be calculated for (2.4) and the non-null components are

$$\begin{aligned} \gamma_{010} &= \alpha' + \frac{\dot\alpha}{\alpha^2} & \qquad \gamma_{133} &= \frac{\alpha}{r} \\ \gamma_{011} &= -\frac{\dot\alpha}{\alpha^2} & \qquad \gamma_{233} &= \frac{\cot g\,\theta}{r} \\ \gamma_{122} &= \frac{\alpha}{r} \end{aligned} \tag{2.7}$$

where $\alpha' = \partial\alpha/\partial r$ and $\dot\alpha = \partial\alpha/\partial u$. Using (2.7) in (1.17) and the standard property (1.6) of γ^A we obtain

$$\begin{aligned} \Gamma_0 &= -\frac{1}{2}\left(\alpha' + \frac{\dot\alpha}{\alpha^2}\right) \gamma^0 \gamma^1 \\ \Gamma_1 &= \frac{1}{2} \frac{\dot\alpha}{\alpha^2} \gamma^0 \gamma^1 \\ \Gamma_2 &= -\frac{1}{2} \frac{\alpha}{r} \gamma^1 \gamma^2 \\ \Gamma_3 &= -\frac{1}{2}\left(\frac{\alpha}{r} \gamma^1 \gamma^3 + \frac{\cot g\,\theta}{r} \gamma^2 \gamma^3\right) \end{aligned} \tag{2.8}$$

Noting that Γ_3 depends on θ , we are then led to take in (2.5), $\varphi = \varphi(u, r, \theta)$. In the present representation of the γ^A and with the property of the spinor (2.5),

$$\gamma^0 \psi = \gamma^1 \psi \tag{2.9}$$

Dirac equation (1.18) in the basis (2.4) reduces to

(*) This equivalent to assume that all neutrinos move radially when emitted, or that only neutrinos radially emitted contribute to the energy-momentum tensor.

$$\alpha \varphi' + \left(\frac{\alpha'}{2} + \frac{\alpha}{r} \right) \varphi = 0$$
$$\frac{\partial \varphi}{\partial \theta} + \frac{1}{2} \operatorname{cotg} \theta \, \varphi = 0 \tag{2.10}$$

Equations (2.10) can be immediately integrated, to give

$$\varphi(u,r,\theta) = \frac{1}{(\alpha \sin\theta)^{1/2} r} \lambda(u) \tag{2.11}$$

where $\lambda(u)$ is an arbitrary 2-spinor, corresponding to the most general spinor (2.5), solution of Dirac equation of the neutrino in the metric (2.1). The current (2.6) is calculated

$$j^{\alpha} = \frac{1}{\sin\theta \; r^2} \lambda^{+}(u) \lambda(u) \, \delta^{\alpha}_{1} \tag{2.12}$$

Analogously, using (2.8) and (2.10) the non-null components of the energy-momentum tensor of the neutrino in the basis (2.4) are calculated(*)

$$T_{00} = T_{11} = - T_{01} = \frac{4i}{\sin\theta} \frac{1}{r^2 \alpha^2} \left(\lambda^{+} \dot{\lambda} - \dot{\lambda}^{+} \lambda \right) \tag{2.13a}$$

$$T_{03} = - T_{13} = \frac{2}{\sin\theta} \frac{\operatorname{cotg}\theta}{r^3 \alpha} \lambda^{+} \sigma^{1} \lambda \tag{2.13b}$$

Expressions (2.12) and (2.13) show clearly why has been widely stated in the literature[14] that neutrinos cannot generate a curvature compatible with spherical symmetry, or equivalently neutrinos are not compatible (as source) with spherically symmetric gravitational fields: the components of the energy-momentum tensor (2.13) depend also on θ while R_{AB} depends only on (u, r). We now discuss why the angle dependence of (2.12) and (2.13) is

(*) Due to the form (2.5) of ψ , the expressions $\bar{\psi} \gamma^0 \gamma^1 \gamma^2 \psi$ and $\bar{\psi} \gamma^0 \gamma^1 \gamma^3 \psi$ are identically null.

in fact coordinate dependent and not physically significant, and can be suitably eliminated.

The θ-dependence of (2.13b) is not so drastic because we could assume

$$\lambda^{+}\sigma^{1}\lambda = 0 \tag{2.14}$$

which corresponds to neutrino fields which are not eigenstates of γ^{r}. Actually these components (2.13b) should vanish by Einstein equations for (2.1). The crucial factor is $1/\sin\theta$ which appears not only in the current (2.12) but also in the relevant components (2.13a). This dependence is suggestive because it is exactly the factor $1/\sin\theta$ that corrects areas in a spherical co ordinate system.

Indeed if we consider the measurement of a radial flux of neutrinos, we can easily see that the number of particles by unit of time and area meassured in the direction θ , for r fixed, is proportional to

$$\sqrt{-g}\; j^{r}\;(\theta) \tag{2.15}$$

where j^{r} is given by $\frac{1}{r^{2}\sin\theta}\lambda^{+}\lambda(u)$ (cf. (2.12)). Hence the observed flux is independent of the direction of measurement whether observed locally or globally. Since the current associated to the neutrino field (2.5), (2.11) corresponds to an isotropic (or spherically symmetric) emission, the metric (2.1) should have curvature compatible – through Einstein equations – with this flux of neutrinos. Analogously we can interpret $T_{0}^{\;\mu}$ as a current density of energy-momentum which depends on θ as $1/\sin\theta$ and so corresponding to an isotropic (or spherically symmetric) flux of energy. Therefore in a spherically symmetric space-time we are led to redefine the energy-momentum tensor of the neutrino – which shall enter the right hand side of Einstein equations — as

$$\tilde{T}_{\mu\nu} = \frac{1}{4\pi} \int T_{\mu\nu} \, d\Omega \tag{2.16}$$

We consider that the spherically symmetric metric (2.1), solution of Einstein equations

$$R_{\mu\nu} = -\kappa \, \tilde{T}_{\mu\nu} \tag{2.17}$$

describes the exterior geometry of a spherically symmetric distribution of fluid, emitting neutrinos isotropically.

Condition (2.14) can be discarded as artificial — because in (2.16) we could take the θ-integral as the principal value

$$\tilde{T}_{\alpha\beta} = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \;\; PV \int_0^{\pi} d\theta \sin\theta \; T_{\alpha\beta} \tag{2.18}$$

what implies $\tilde{T}_{03} = \tilde{T}_{13} = 0$ — and we can eventually have emission of neutrinos of only one type $\psi_{(\pm)} = \pm \gamma^5 \psi_{(\pm)}$.

To summarize, once stablished that the factor $1/\sin\theta$ is merely a correction due to the coordinate system used and has no physical significance, that all observables quantities constructed with (2.5), (2.11) are independent of the direction of measurement, and so isotropic, the average over angles (2.16) is legitimate and the redefined energy-momentum tensor (2.16) (or(2.18)) is the physical energy-momentum tensor of neutrinos, which shall enter the RHS of Einstein equations in a spherically symmetric space-time$^{(*)}$.

Redefinition (2.16) (or (2.18)) has two important properties:

(*) In an analogous context, Griffiths[15] proposed an averaged energy-momentum tensor for neutrinos in the spherically symmetric metric (2.1).

1) The energy-momentum tensor $\tilde{T}_{\alpha\beta}$ is still conserved locally in the metric (2.1), $\tilde{T}_{\alpha}{}^{\beta}{}_{\|\beta}=0$.

2) $\tilde{T}_{\alpha\beta}$ has the form of the energy-momentum tensor of a null fluid which is the usual phenomenological description of neutrinos[7,8].

Substituting (2.13) in (2.18) we obtain the non-null components

$$\tilde{T}_{00} = \tilde{T}_{11} = - \tilde{T}_{01} = 2\pi i \frac{1}{\kappa^2 r^2} \left(\lambda^+ \dot{\lambda} - \dot{\lambda}^+ \lambda \right) \tag{2.19}$$

Expressing the energy-momentum tensor (2.19) in the coordinate basis, $\tilde{T}_{\alpha\beta} = e_{\alpha}^{(A)} e_{\beta}^{(B)} \tilde{T}_{AB}$, with $e_{\alpha}^{(A)}$ given by (2.4) we obtain the only non-null component $\tilde{T}_{00} = \frac{2\pi i}{r^2}(\lambda^+\dot{\lambda}-\dot{\lambda}^+\lambda)$ or

$$\tilde{T}_{\alpha\beta} = \frac{2\pi i}{r^2} \left(\lambda^+ \dot{\lambda} - \dot{\lambda}^+ \lambda \right) u_{1\alpha} u_{1\beta} \tag{2.20}$$

which shows property (2).

In basis (2.4) the non-null components of Ricci tensor are given by

$$R_{00} = R_{11} = - R_{01} = 4\dot{m} / \kappa^2 r^2 \tag{2.21}$$

and Einstein equations (2.17) imply

$$\dot{m} = \frac{k\pi i}{2} \left(\dot{\lambda}^+ \lambda - \lambda^+ \dot{\lambda} \right) \tag{2.22}$$

The geometric properties of the Schwarzschild radiating space-time have been extensively studied by Lindquist, Schwarz and Misner[16] but we have some comments to do here. The total power emission of the system described by the metric (2.1) was calculated in reference (16) to be

$$L_{Tot} = - \frac{16\pi}{k} \dot{m}$$

using the Landau and Lifshitz's energy-momentum pseudo-tensor for the gravitational field. Equating with (2.22) we have

$$L_{tot} = 8\pi^2 i\,(\lambda^+\dot{\lambda} - \dot{\lambda}^+\lambda) \tag{2.23}$$

We call L_{tot} the luminosity of neutrinos emitted, as measured by an observer at rest at infinity ($r \to \infty$). Result (2.23) could alternatively be derived by using the following arguments. The Weyl tensor for the exterior metric (2.1) with neutrinos is formally identical to the Weyl tensor of the Schwarzschild metric. The null vector field $u_{|\alpha}$ which is a degenerate principal null direction[18] of the space-time defines the null direction along which neutrinos propagate (cf. (2.12) and (2.20)). Since for m = const. $u_{|\alpha}$ is still a degenerate principal null direction of the space-time we can conclude that principal null directions in the Schwarzschild space-time are null directions of propagations of neutrinos in case of an eventual spherical emission. This suggests that in the emission of neutrinos as described by (2.1) and (2.22) there is no simultaneous emission of gravitational radiation. Indeed,if we examine the curvature tensor R_{ABCD} of the metric (2.1) for r sufficiently large we see that the leading term (corresponding to the lowest power of r^{-1}) is proportional to

$$R_{ABCD} \sim \frac{\dot{m}}{r^2}$$

The radiative part of the space-time thus comes from $\dot{m} \neq 0$ which by (2.22) is due to neutrinos only. We then have no gravitational radiation emission together with neutrinos and the total radiated power (which is associated to neutrinos only) can be calculated from the energy-momentum tensor (2.20). If we denote $v^\mu = \alpha^{-1}\delta^\mu_0$ (corresponding to a local inertial observer) the total luminosity (neutrino luminosity) as measured by an asymptotic observer at rest is then given by

$$L_{tot} = \lim_{r\to\infty} 4\pi r^2 \tilde{T}_{\mu\nu}(\text{neutrino})\, v^\mu v^\nu = 8\pi^2 i(\lambda^+\dot{\lambda} - \dot{\lambda}^+\lambda)$$

which is the result (2.23).

The hypersurface $r=2m(u)$ for $\dot{m}<0$ is space-like and a light signal emitted in the region $r>2m(u)$ or any material particle following a time-like trajectory starting from a point in $r>2m(u)$ cannot reach the hypersurface $r=2m(u)$. This led LSM[16] to consider the region $r\leq 2m(u)$ unphysical. Nevertheless although this region is not accessible for objects in the exterior region of a Vaidya metric, it is probable that when matter was assembled to form the source in this region the metric of the space-time was not Vaidya. Only after assembled in a spherically symmetric configuration and eventually radiating, can the space-time be characterized by the radiating metric (2.1). We here disconsider the region $r\lesssim 2m(u)$ by assuming that the emission takes place before the fluid source is inside its Schwarzschild radius, the fluid having a boundary $r_s(u)>2m(u)$ (the static limit corresponds to the Schwarzschild configuration for $r>2m=\text{const.}$ (cf. section 3)). We exclude the case of the whole mass of the object being emitted as neutrino before the mass reaches its Schwarzschild radius (i.e., $\dot{r}_s<\dot{m}$); the conditions which eliminate this possibility for our solution are examined later (cf. equation (3.103)).

3. THE INTERIOR PROBLEM: A CLASS OF SOLUTIONS

The interior region is constituted of a distribution of matter and neutrinos flowing outwards. The matter distribution is a perfect fluid sphere characterized by a total density ρ , pressure p and radius r_s and which emits neutrinos. Neutrinos are supposed to move radially when emitted, or only radial neutrinos contribute to the energy-momentum tensor. Matter is transparent to neutrinos, i.e., once emitted neutrinos have interaction with gravitation only, being not scattered or absorbed by the adjacent matter. The above model is to be solution of the coupled Einstein-Dirac equations, joined to the exterior solution of section 2 on the surface of the fluid sphere.

For the interior problem our choice of coordinates is $x^\alpha=(u,r,\theta,\varphi)$, $-\infty \leq u \leq \infty$, $0<r<\infty$, with the following properties: (i) the hypersurfaces $u = \text{const.}$ are null hypersurfaces tangent in each point to the local light cone; (ii) r is an affine parameter along the null curves with tangent $k^\alpha = g^{\alpha\beta} u_{,\beta}$ defining a luminosity distance; (iii) the scalars θ and φ are constant along the null curves in (ii). In this coordinate system, a spherically symmetric line element can always be expressed

$$ds^2 = \alpha^2 du^2 + 2\,du\,dr - \beta^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{3.1}$$

where α and β are functions of u and r. We choose a tetrad basis $e^{(A)}_{\alpha}$ with non-null components

$$e^{(0)}_{0} = \alpha \qquad e^{(0)}_{1} = \alpha^{-1}$$
$$e^{(1)}_{1} = \alpha^{-1} \qquad e^{(2)}_{2} = \beta \qquad e^{(3)}_{3} = \beta \sin\theta \tag{3.2}$$

such that according to (1.2) the metric (3.1) assumes the form (1.1). The Ricci coefficients γ_{ABC} calculated for (3.2) have non-null components

$$\begin{aligned}
\gamma_{010} &= \alpha' + \dot{\alpha}/\alpha^2 & \gamma_{122} &= -\dot{\beta}/\beta\alpha + \beta'\alpha/\beta \\
\gamma_{011} &= -\dot{\alpha}/\alpha^2 & \gamma_{033} &= \dot{\beta}/\beta\alpha \\
\gamma_{022} &= \dot{\beta}/\beta\alpha & \gamma_{133} &= -\dot{\beta}/\beta\alpha + \beta'\alpha/\beta \\
\gamma_{233} &= \cot\theta/\beta & &
\end{aligned} \tag{3.3}$$

With (3.3) the Fock-Ivanenko coefficients (1.17) are calculated

$$
\begin{aligned}
\Gamma_0 &= -\frac{1}{2}\left(\alpha' + \dot{\alpha}/\alpha^2\right)\gamma^0\gamma^1 \\
\Gamma_1 &= \frac{1}{2}\,\frac{\dot{\alpha}}{\alpha^2}\,\gamma^0\gamma^1 \\
\Gamma_2 &= -\frac{1}{2}\,\frac{\dot{\beta}}{\beta\alpha}\,\gamma^0\gamma^2 - \frac{1}{2}\left(\frac{\beta'\alpha}{\beta} - \frac{\dot{\beta}}{\beta\alpha}\right)\gamma^1\gamma^2 \\
\Gamma_3 &= -\frac{1}{2}\,\frac{\dot{\beta}}{\beta\alpha}\,\gamma^0\gamma^3 - \frac{1}{2}\left(\frac{\beta'\alpha}{\beta} - \frac{\dot{\beta}}{\beta\alpha}\right)\gamma^1\gamma^3 - \frac{\cot\theta}{2\beta}\,\gamma^2\gamma^3
\end{aligned}
\tag{3.4}
$$

Since only radial neutrinos are considered, we are restricted to spinorial fields of the form

$$
\psi = \begin{pmatrix} \varphi \\ \sigma^1\varphi \end{pmatrix}
\tag{3.5}
$$

where φ is a 2-spinor and σ^1 the constant Pauli matrix, with the associated current

$$
j^A = (\rho, \rho, 0, 0)
\tag{3.6}
$$

radially along the local light cones. Because Γ_3 depends on θ we take in (3.5), $\varphi = \varphi(u, r, \theta)$. With the properties of the constant Dirac matrices γ^A and of the spinor (3.5), Dirac equation (1.18) reduces to

$$
\begin{aligned}
&\alpha\varphi' + \left(\frac{\alpha'}{2} + \frac{\beta'\alpha}{\beta}\right)\varphi = 0 \\
&\frac{\partial\varphi}{\partial\theta} + \frac{1}{2}\cot\theta\,\varphi = 0
\end{aligned}
\tag{3.7}
$$

which can be immediately integrated to give

$$
\varphi(u, r, \theta) = \frac{1}{(\alpha\sin\theta)^{1/2}\beta}\,\Lambda(u)
\tag{3.8}
$$

where $\Lambda(u)$ is an arbitrary 2-spinor, which corresponds to the most general solution (3.5) of Dirac equation in the metric (3.1). We remark that in the present coordinate system Dirac equation is immediately integrated as (3.8) and whose functional dependence in (u, r, θ) reflects in the current (3.6) and in the energy-momentum tensor, allowing a simple interpretation as we shall see. This is the strongest reason for the choice of the present coordinates. In the current (3.6) the solution (3.8) yields

$$\rho = \frac{1}{\sin\theta} \frac{2}{\alpha\beta^2} \Lambda^+(u)\Lambda(u) \tag{3.9}$$

whose θ-dependence is identical as in (2.12). Using (3.4), (3.5) and (3.7) we obtain the non-null components T_{AB} of the energy-momentum tensor of the neutrino field (3.5) in the basis (3.2):

$$T_{00} = T_{11} = -T_{01} = \frac{4i}{\sin\theta} \frac{1}{\alpha^2\beta^2} (\Lambda^+\dot{\Lambda} - \dot{\Lambda}^+\Lambda) \tag{3.10a}$$

$$T_{03} = -T_{13} = \frac{2}{\sin\theta} \frac{\cot\theta}{\beta^3\alpha} \Lambda^+\sigma^1\Lambda \tag{3.10b}$$

with a θ-dependence as in (2.13). As for the exterior solution, the angle dependence of (3.10b) is not so drastic because we could consider neutrino fields satisfying

$$\Lambda^+(u)\ \sigma^1 \Lambda(u) = 0 \tag{3.11}$$

which are not eigenstates of γ^5. In an analysis analogous to the exterior case, we can show that the factor $1/\sin\theta$ in (3.9) and (3.10a) is simply a correction due to the spherical coordinate system used, such that all observable quantities constructed with (3.9) and (3.10a) are independent of the direction θ of measurement, corresponding to an isotropic (or spherically symmetric) emission of neutrinos. Hence as in (2.16) we redefine the energy-momentum tensor of the neutrino, which shall enter the RHS of Einstein equations, as

$$\tilde{T}_{\mu\nu} = \frac{1}{4\pi} \int T_{\mu\nu} \, d\Omega \tag{3.12}$$

The spherically symmetric metric (3.1), solution of Einstein equa tions

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left\{ T_{\mu\nu}(\rho, P) + \tilde{T}_{\mu\nu}(\text{neutrino}) \right\} \tag{3.13}$$

is then considered to describe the interior geometry of a fluid sphere emitting neutrinos isotropically, eigenstates or not of γ^5 (*). The redefinition (3.12) has two important properties: (i) the energy-momentum tensor $\tilde{T}_{\alpha\beta}$ is still conserved locally, that is

$$\tilde{T}_{\alpha}{}^{\beta}{}_{\|\beta} = 0$$

in the metric (3.1); (ii) $\tilde{T}_{\alpha\beta}$ has the form of the energy-momentum tensor of a null fluid, which is the usual phenomenological des cription of neutrinos in General Relativity[7,8].

Substituting (3.10) in (3.12) we have

$$\begin{aligned} &\tilde{T}_{00} = \tilde{T}_{11} = -\tilde{T}_{01} = 2\pi i \, \frac{1}{\alpha^2\beta^2} \left(\Lambda^+ \dot{\Lambda} - \dot{\Lambda}^+ \Lambda \right) \\ &\tilde{T}_{03} = \tilde{T}_{13} = 0 \end{aligned} \tag{3.14}$$

In the coordinate basis, the energy-momentum tensor $\tilde{T}_{\mu\nu} = e^{(A)}_{\mu} e^{(B)}_{\nu} \tilde{T}_{AB}$ has only one non-null component $\tilde{T}_{00} = \frac{2\pi i}{\beta^2}(\Lambda^+\dot{\Lambda} - \dot{\Lambda}^+\Lambda)$ and we can write

$$\tilde{T}_{\mu\nu} = \frac{2\pi i}{\beta^2} \left(\Lambda^+ \dot{\Lambda} - \dot{\Lambda}^+ \Lambda \right) u_{1\mu} u_{1\nu} \tag{3.15}$$

with $u_{1\mu} = \delta^0_\mu$ a radial null vector tangent to the local light cone, what shows property (ii). The factor $2\pi i \frac{1}{\alpha^2\beta^2}(\Lambda^+\dot{\Lambda} - \dot{\Lambda}^+\Lambda)$ can be conveniently interpreted as the energy-density of neutrinos as

---

(*) in (3.12) the θ-integral is also taken as the principal value, that implies $\tilde{T}_{03} = \tilde{T}_{13} = 0$.

measured locally by the observer with four-velocity $\alpha^{-1}\delta^{\mu}_{0}$.

For the energy-momentum tensor of the perfect fluid we assume that an observer comoving with matter has four velocity

$$V^{A}=\delta^{A}_{0}\,,\qquad V_{A}=\delta^{0}_{A} \tag{3.16}$$

in the local inertial frame determined by (3.2). This corresponds to a matter velocity field given by

$$V^{\mu}=e^{\mu}_{(A)}\,V^{A}=e^{\mu}_{(0)}$$

We denote respectively by ρ and p the density of mass-energy and the pressure of the fluid, as measured locally by the observer (3.16). Thus in the local inertial frame defined by (3.2) the energy-momentum tensor of the fluid has the form

$$T_{AB}=(\rho+p)\,\delta^{0}_{A}\,\delta^{0}_{B}-p\,\eta_{AB} \tag{3.17}$$

which is the expression of the energy-momentum tensor of a perfect fluid in Special Relativity[19].

The total energy-momentum tensor for the interior problem shall be the sum of the energy-momentum tensors of the neutrino and of the fluid, which in the local basis (3.2) is expressed

$$T_{AB}(\text{total})=(\rho+p)\,\delta^{0}_{A}\,\delta^{0}_{B}-p\,\eta_{AB}+\frac{2\mathcal{L}(u)}{\alpha^{2}\beta^{2}}\,k_{A}\,k_{B} \tag{3.18}$$

where $k_{A}=(1,-1,0,0)$ and

$$\mathcal{L}(u)=\pi i\left(\Lambda^{+}\dot{\Lambda}-\dot{\Lambda}^{+}\Lambda\right) \tag{3.19}$$

The non-null components R_{AB} of the Ricci tensor in the local basis (3.2) are calculated to be

$$R_{00} = -2\alpha\alpha'' - 2\alpha'^2 + 4\frac{\ddot{\beta}}{\beta\alpha^2} + 4\frac{\dot{\beta}\alpha'}{\beta\alpha} - 4\frac{\alpha\alpha'\beta'}{\beta} - 4\frac{\dot{\alpha}\beta'}{\alpha\beta}$$

$$R_{01} = -4\frac{\ddot{\beta}}{\beta\alpha^2} + \frac{4\dot{\beta}'}{\beta} - 4\frac{\dot{\beta}\alpha'}{\beta\alpha} + \frac{4\beta'\dot{\alpha}}{\beta\alpha}$$

$$R_{11} = 2\alpha\alpha'' + 2\alpha'^2 + 4\frac{\ddot{\beta}}{\beta\alpha^2} - 8\frac{\dot{\beta}'}{\beta} - 4\frac{\beta'\dot{\alpha}}{\beta\alpha} + 4\frac{\dot{\beta}\alpha'}{\beta\alpha} + $$
$$+ 4\frac{\beta''\alpha^2}{\beta} + 4\frac{\alpha\alpha'\beta'}{\beta} \tag{3.20}$$

$$R_{22} = R_{33} = -\frac{4\dot{\beta}'}{\beta} + 2\frac{\beta''\alpha^2}{\beta} + 4\frac{\alpha\alpha'\beta'}{\beta} - \frac{2}{\beta^2} - 4\frac{\dot{\beta}\beta'}{\beta^2} + 2\frac{\beta'^2\alpha^2}{\beta^2}$$

Einstein equations in the local basis are given by

$$R_{AB} - \frac{1}{2}\eta_{AB}\, R = k\, T_{AB}\,(\text{Total})$$

Here we take $k<0$. Using (3.18) we can write the independent equations

$$R_{00} - \frac{1}{2}R = 2k\frac{\mathcal{L}(u)}{\alpha^2\beta^2} + k\rho$$

$$R_{01} = -2k\frac{\mathcal{L}(u)}{\alpha^2\beta^2}$$

$$R_{11} + \frac{1}{2}R = kp + 2k\frac{\mathcal{L}(u)}{\alpha^2\beta^2} \tag{3.21}$$

$$R_{22} + \frac{1}{2}R = kp$$

The Ricci scalar R is given by $R = R_{00} - R_{11} - R_{22} - R_{33}$ and from (3.20) we obtain

$$R = -4\alpha\alpha'' - 4\alpha'^2 - 16\,\alpha\alpha'\beta'/\beta + 16\,\dot{\beta}'/\beta - 8\beta''\alpha^2/\beta \tag{3.22}$$
$$+ 4/\beta^2 + 8\dot{\beta}\beta'/\beta^2 - 4\beta'^2\alpha^2/\beta^2$$

The field equations (3.21) can be rewritten

$$R_{00} = 2\kappa \frac{\mathcal{L}(u)}{\alpha^2\beta^2} + \kappa\rho + \frac{1}{2} R \tag{3.23a}$$

$$R_{01} = -2\kappa \frac{\mathcal{L}(u)}{\alpha^2\beta^2} \tag{3.23b}$$

$$R_{22} = \kappa p - \frac{1}{2} R \tag{3.23c}$$

$$R_{11} + R_{01} - R_{22} = 0 \tag{3.23d}$$

Equation (3.23d) is obtained from the relation $T_{11} - T_{22} + T_{01} = 0$, which holds for the total energy-momentum tensor (3.18) and it is equivalent to a linear combination of the three last equations (3.21). Also from (3.21) we can see

$$R_{00} + R_{11} + 2R_{01} = \kappa(\rho + p) \tag{3.24}$$

and an explicit calculation of the LHS of (3.24) yields the important relation

$$4\frac{\beta''}{\beta}\alpha^2 = \kappa(\rho + p) \tag{3.25}$$

Since for physically reasonable equations of state, $\rho + p > 0$ and since $\kappa < 0$ we must have $\beta''/\beta < 0$ in all points of the interior region. Also the existence of matter in the interior region described by the metric (3.1) depends essentially on β''/β being non-zero.

A inspection of (3.20) and (3.23) can convince us of the difficulty of finding on explicit solution of the field equations (3.23). We try a separation of variables solution for the present case of matter without opacity for neutrinos. We take

$$\alpha = R_1(r)\, T_1(u)$$
$$\beta = R_2(r)\, T_2(u) \tag{3.26}$$

Using (3.20) and (3.26) in the field equation (3.23d) we have

$$\left\{ R_1 R_1'' + R_1'^2 + \frac{R_2''}{R_2} R_1^2 - \left(\frac{R_2' R_1}{R_2}\right)^2 \right\} T_1^2 + \frac{1}{R_2^2} T_2^{-2} + 2\,\frac{R_2'}{R_2}\;\frac{\dot{T}_2}{T_2} = 0 \tag{3.27}$$

We make the choice

$$\frac{\dot{T}_2}{T_2} = \xi\, T_2^{-2} \tag{3.28}$$

that implies

$$\frac{R_1 R_1'' + R_1'^2 + \frac{R_2''}{R_2}\; R_1^2 - \left(\frac{R_2' R_1}{R_2}\right)^2}{\frac{1}{R_2^2} + 2\xi\; R_2'/R_2} = -\eta^2 = -\,\frac{1}{T_1^2 T_2^2} \tag{3.29,30}$$

where ξ and η are arbitrary separation constants. Equations (3.28) to (3.30) can be rewritten

$$\dot{T}_2 = \xi\; T_2^{-1} \tag{3.28}$$

$$T_1^2 T_2^2 = \frac{1}{\eta^2} \tag{3.29}$$

$$R_1 R_1'' + R_1'^2 + \frac{R_2''}{R_2} R_1^2 - \left(\frac{R_2' R_1}{R_2}\right)^2 = -\eta^2 \left(\frac{1}{R_2^2} + 2\xi\; \frac{R_2'}{R_2}\right) \tag{3.30}$$

Let us examine now equation (3.23b). Using (3.26) and (3.28), it can be reduced to

$$2R_2^2\, T_2\, \ddot{T}_2 + 2R_1 R_1'\, R_2^2\, T_1^2\, T_2\, \dot{T}_2 = \kappa \mathcal{L} \tag{3.31}$$

By (3.28) $\ddot{T}_2 = -\xi^2\, T_2^{-3}$ which together with (3.28) and (3.29) reduces (3.31) to

$$-2R_2^2 \xi^2 + 2R_1 R_1'\, R_2^2\, \frac{\xi}{\eta^2} = \kappa \mathcal{L}\, T_2^2 = \chi \tag{3.32}$$

where χ is a separation constant. Equation (3.32) yields

$$\mathcal{L} = \frac{\chi}{k} \; T_2^{-2} \tag{3.33}$$

$$R_1' = \frac{\eta^2}{\xi}\left(\chi + 2\xi^2 R_2^2\right) \frac{1}{2R_1 R_2^2} \tag{3.34}$$

From (3.33) we see that the functional (3.19) of the arbitrary spinorial field $\Lambda(u)$ of the neutrino can be described by the metric function T_2 or vice-versa. Also, for a class of junction conditions of the exterior and interior solutions, the relation (3.33) will be useful in relating the sign of $\dot{m}$ and the sign of χ .

Taking the derivative of (3.34) results

$$\begin{aligned} R_1'' = & -2\eta^2\xi \frac{R_2'}{R_1 R_2} - \frac{\eta^4}{4\xi^2}\left(\chi + 2\xi^2 R_2^2\right)^2 \frac{1}{R_1^3 R_2^4} \\ & - \frac{\eta^2}{\xi}\left(\chi + 2\xi^2 R_2^2\right) \frac{R_2'}{R_2^3 R_1} \end{aligned} \tag{3.35}$$

and substituting (3.34) and (3.35) in (3.30) we obtain after a long calculation

$$\frac{R_2''}{R_2} = \frac{R_2'}{R_2}\left(\frac{R_2'}{R_2} + 2\frac{R_1'}{R_1}\right) - \frac{1}{R_1^2}\left(\frac{\eta^2}{R_2^2} + 4\xi\eta^2 \frac{R_2'}{R_2}\right) \tag{3.36}$$

(3.34) and (3.36) constitute a pair of coupled differential equations for the two metric functions R_1 and R_2 . Once we have a solution (R_1, R_2), $\rho + p$ is determined by (3.25), $\rho + p = \frac{4}{k} R_1^2 R_2''/R_2 \; T_1^2$ and assuming an equation of state for the fluid, $p = \lambda \rho$, the total matter density ρ is determined

$$\rho = \frac{4}{k(1+\lambda)}\left\{R_1^2 \; \frac{R_2''}{R_2}\right\} T_1^2 = \frac{4}{k(1+\lambda)\eta^2}\left\{R_1 \; \frac{R_2''}{R_2}\right\} T_2^{-2} \tag{3.37}$$

In (3.37) T_2 is a solution of (3.28).

Nevertheless it is to be verified if the solutions given by (3.28), (3.29), (3.34) and (3.36) are compatible with the re maining field equations (3.23 a,c). To this end we initially show that equation (3.23c), $R_{22} = \kappa p - \frac{1}{2}R$, can be obtained by a convenient linear combination of (3.23a), (3.23d), (3.23b) and (3.24): substituting (3.23d) in (3.24) results $R_{00} + R_{22} + R_{01} = \kappa(\rho + p)$ and using (3.23b) we obtain $R_{00} + R_{22} = \kappa\rho + \frac{2\kappa\mathcal{L}}{\alpha^2\beta^2} + \kappa p$, which by (3.23a) reproduces (3.23c). We note that equation (3.24) does not impose further restrictions on the solutions R_1, T_1, R_2, T_2 but defines only the additional variable $\rho + p$ in terms of the metric functions. Thus the only remaining condition to be satisfied by solutions (3.28), (3.29), (3.34) and (3.36) is the equation

$$R_{00} = \frac{2\kappa\mathcal{L}}{\alpha^2\beta^2} + \kappa\rho + \frac{1}{2}R \tag{3.23a}$$

Using anterior expressions we can see that (3.23a) is equivalent to

$$R_{22} = \frac{\kappa}{2}(\rho - p) \tag{3.38}$$

or, from (3.20), (3.26), (3.28), (3.29)

$$\frac{\kappa}{2}(\rho - p) = \Big[-8\xi\eta^2\frac{R_2'}{R_2} + 2\frac{R_2''}{R_2}R_1^2 + 4\frac{R_2'}{R_2}R_1'R_1 - \frac{2}{R_2^2}\eta^2 + 2\Big(\frac{R_2'R_1}{R_2}\Big)^2\Big]T_1^2 \tag{3.39}$$

Now (3.39) together with

$$\kappa(\rho + p) = 4\frac{R_2''}{R_2}R_1^2T_1^2 \tag{3.25}$$

determine uniquely the equation of state $p = p(\rho)$. Substituting (3.25) above in (3.39) and using expression (3.36) for $\frac{R_2''}{R_2}$, the matter density ρ is given

$$\kappa\rho = 3\Big[-8\xi\eta^2\frac{R_2'}{R_2} + 4\frac{R_2'}{R_2}R_1'R_1 + 2\Big(\frac{R_2'R_1}{R_2}\Big)^2 - \frac{2\eta^2}{R_2^2}\Big]T_1^2 \tag{3.40}$$

Similarly using (3.40) and (3.36) in (3.25) above, we have

$$\kappa p = \left[8 z \gamma^2 \frac{R_2'}{R_2} - 2\left(\frac{R_2' R_1}{R_2}\right)^2 + \frac{2}{R_2^2}\gamma^2 - \frac{4R_2'}{R_2} R_1' R_1 \right] T_1^2 \qquad (3.41)$$

From (3.40) and (3.41) we have the relation $p/\rho = -\frac{1}{3}$ or

$$p = -\frac{1}{3}\rho \qquad (3.42)$$

The equation of state (3.42), though satisfying energy-conditions[20] implies the existence of negative scalar pressures, what is not physically satisfactory. To circunvent this, we later introduce in the field equations (or, equivalently in the total energy-momentum tensor) a term which describes the cooling of the fluid by emission of neutrinos.

Let us examine now the local conservation law

$$T^{\mu\nu}{}_{\|\nu}\,(\text{total}) = 0 \qquad (3.43)$$

of the total energy-momentum tensor (3.18). An explicit calculation of (3.43) gives

$$\dot{\rho} - (1+\lambda)\,\alpha\rho\left(\frac{\dot{\alpha}}{\alpha^2} - \frac{2\dot{\beta}}{\beta\alpha}\right) = 0 \qquad (3.44a)$$

$$\dot{\rho} - \alpha^2\rho' - \left(\frac{1+\lambda}{\lambda}\right)\alpha\rho\left(\alpha' + \frac{\dot{\alpha}}{\alpha^2}\right) = 0 \qquad (3.44b)$$

where we used $p = \lambda\rho$, $\lambda = \text{const}$. We note that the components of the energy-momentum tensor of neutrino do not contribute in (3.44) since by construction (1.15) has null covariant divergence and redefinition (3.12) does not alter this property. Using expression (3.37) for $\rho(u,r)$, equation (3.44a) reads

$$\left\{ 2T_1\dot{T}_1 - (1+\lambda)\,R_1 T_1^3\left(\frac{\dot{T}_1}{R_1 T_1^2} - 2\,\frac{\dot{T}_2}{R_1 T_1 T_2}\right)\right\} \frac{4}{k(1+\lambda)} R_1^2\,\frac{R_2''}{R_2} = 0$$

which implies

$$2T_1\dot{T}_1 - (1+\lambda)\left(T_1\dot{T}_1 - 2T_1^2\frac{\dot{T}_2}{T_2}\right) = 0 \qquad (3.45)$$

From (3.28) $\frac{\dot{T}_2}{T_2} = -\frac{\dot{T}_1}{T_1}$ and (3.45) results

$$\{2 - 3(1+\lambda)\}\, T_1\dot{T}_1 = 0$$

or

$$\lambda = -\frac{1}{3}$$

what is consistent with (3.42), without no new restriction on the solution. We now examine equation (3.44b). Using $\lambda = -\frac{1}{3}$ and the expression for ρ ,

$$\rho = \frac{4}{k(1+\lambda)}\left(R_1^2\frac{R_2''}{R_2}\right)T_1^2$$

we can rewrite (3.44b) as

$$4\left(R_1^2\frac{R_2''}{R_2}\right)T_1\dot{T}_1 - R_1^2\left(R_1^2\frac{R_2''}{R_2}\right)'T_1^4 + 2R_1R_1'\left(R_1^2\frac{R_2''}{R_2}\right)T_1^4 = 0 \qquad (3.46)$$

Using (3.28) and (3.29), (3.46) yields after some calculations

$$-4\xi\eta^2\left(R_1\frac{R_2''}{R_2}\right) - R_1^2\left(R_1^2\frac{R_2''}{R_2}\right)' + 2R_1R_1'\left(R_1^2\frac{R_2''}{R_2}\right) = 0 \qquad (3.47)$$

and by a further simplification in (3.47)

$$\left(\frac{R_2''}{R_2}\right)' + 4\xi\eta^2R_1^{-2}\left(\frac{R_2''}{R_2}\right) = 0 \qquad (3.48)$$

Hence the interior solution given by (3.26), (3.28), (3.29),(3.34) and (3.36) satisfies Einstein field equations and the local conservation law of the total energy-momentum tensor for an equation of state $p = -\frac{1}{3}\rho$ and provided (3.48) holds. That is, the functions

$R_1(r)$ and $R_2(r)$ must satisfy simultaneously the three equations

$$R_1' = \frac{\eta^2}{\xi}\left(\chi + 2\xi^2 R_2^2\right)\frac{1}{2R_1 R_2^2} \qquad (3.34)$$

$$\frac{R_2''}{R_2} = \left(\frac{R_2'}{R_2}\right)^2 - \frac{\eta^2}{R_1^2 R_2^2} - 4\xi\eta^2\frac{R_2'}{R_2}\frac{1}{R_1^2} + 2\frac{R_1'}{R_1}\frac{R_2'}{R_2} \qquad (3.36)$$

$$\left(\frac{R_2''}{R_2}\right)' + 4\xi\eta^2 R_1^{-2}\left(\frac{R_2''}{R_2}\right) = 0 \qquad (3.48)$$

A straightforward but very long calculation shows us that (3.36) is a first integral of (3.48) provided (3.34) holds. Thus the above equations for R_1 and R_2 are consistent and indeed it remains only two independent coupled equations which determine R_1 and R_2. We can interpret this solution as describing the interior metric (gravitational field) of a perfect fluid, in interaction with neutrinos as described by the energy-momentum tensor (3.15) but completely decoupled for the equation of state $p = -\frac{1}{3}\rho$.

From equation (3.48) we can study the behaviour of ρ with r. By (3.37) we have $\rho(r,u) = \rho(r)\, T_1^2$ where

$$\rho(r) = \frac{6}{k}\,\frac{R_2''}{R_2}\,R_1^2$$

Using (3.34) and (3.48) we obtain

$$\frac{d\rho(r)}{dr} = \rho(r)\,\frac{\eta^2}{R_1^2}\left\{\frac{\chi}{\xi}\,\frac{1}{R_2^2} - 2\xi\right\} \qquad (3.49)$$

and since $\rho(r) > 0$, the sign of $d\rho/dr$ is given by $\left(\frac{\chi}{\xi}\frac{1}{R_2^2} - 2\xi\right)$. For instance, for the density ρ decreasing with r we have the inequality

$$\frac{\chi}{\xi} - 2\xi R_2^2 < 0 \qquad (3.50)$$

which is verified for the following cases: (i) $\chi > 0$, $\xi > 0$, $-\sqrt{\frac{\chi}{2\xi^2}} > R_2(r) > \sqrt{\frac{\chi}{2\xi^2}}$; (ii) $\chi > 0$, $\xi < 0$, $-\sqrt{\frac{\chi}{2\xi^2}} < R_2(r) < \sqrt{\frac{\chi}{2\xi^2}}$; (iii)

$\chi<0,\ \xi>0$ for any value of $R_2(r)$; $\chi<0,\ \xi<0$ is incompatible with (3.50). Each possibility for ρ decreasing with r is restricted by the signs of χ and ξ, which shall describe emission or absorption of neutrinos, contraction or expansion of the fluid configuration, as discussed in the junction of the present interior solution with the exterior solution. For the density increasing with r we must have

$$\frac{\chi}{\xi} - 2\xi R_2^2 > 0 \tag{3.51}$$

which is verified for the following cases: (i') $\chi>0,\ \xi>0$, $-\sqrt{\frac{\chi}{2\xi^2}} < R_2(r) < \sqrt{\frac{\chi}{2\xi^2}}$;(ii') $\chi>0,\ \xi<0$, $-\sqrt{\frac{\chi}{2\xi^2}} > R_2(r) > \sqrt{\frac{\chi}{2\xi^2}}$ (iii') $\chi<0,\ \xi<0$, for any value of $R_2(r)$; $\chi<0,\ \xi>0$ is incompatible with (3.51). If in some point $r=r_m$ corresponding to the interior of the star, the function $R_2(r)$ assumes the value $R_2(r_m)=\sqrt{\chi/2\xi^2}$ and the density ρ of star has an extremum on the 2-spheres with radius $R_2(r_m)$. For the two regions $0<r<r_m$ and $r_m<r<r_s$, where r_s is the value of r corresponding to the radius of the star, compatible choices of (i) - (iii) and (i') - (iii') can be made. As the junction of the choices at $r=r_m$ must hold for any u, both choices must have ξ with the same sign — this excludes the possibility of $\chi<0$ with the density ρ having an extremum in the interior of the star; for $\chi<0$ the density ρ must be a monotonous function of r.

We now interpret the parameter ξ. The congruence of observers comoving with the fluid is defined by the velocity field (cf. (3.16))

$$V^{\mu} = e^{\mu}_{(0)} = \alpha^{-1}\delta^{\mu}_{0} \tag{3.52}$$

For our choice of observers in the interior of the future light cones we have the condition

$$\alpha > 0 \tag{3.53}$$

For the congruence determined by (3.52), the expansion parameter(*) is defined by $\theta = V^{\mu}{}_{\parallel\mu} = e^{\mu}_{(0)\parallel\mu}$ or, using (1.13),

$$\theta = -\gamma_0{}^{A}{}_{A} \tag{3.54}$$

The volume of the fluid expands if $\theta>0$ or contracts if $\theta<0$, the contraction or expansion due to the physical processes that occur along its world-line. Using (3.3) in (3.54) results

$$\theta = 2\dot{\beta}/\beta\alpha - \dot{\alpha}/\alpha^2$$

or, by (3.26), (3.28) and (3.29),

$$\theta = \xi \frac{3}{R_1 T_1 T_2^2} \tag{3.55}$$

Since $\alpha>0$ the sign of θ is determined by the sign of ξ, i.e., the fluid is contracting for $\xi<0$ or expanding for $\xi>0$.

Reinterpretation of the Equation of State

We have presented a complete analytic solution for the interior problem but the field equations and the local conservation of the energy-momentum tensor imply the equation of state $p=-\frac{1}{3}\rho$, involving thus negative scalar pressure. We can consider this solution, with the above equation of state, as an effective description of a more complicated solution describing the cooling of the fluid by emission of neutrinos.

(*) In the local inertial frame of an observer determined by (3.52), θ measures the relative variation of an infinitesimal spherical volume of the fluid, centered in the origin of this inertial frame, along its world-line defined by (3.52).

To this end we write Einstein equation with a Λ-term, namely,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda(x)\, g_{\mu\nu} = \kappa\, T_{\mu\nu}\,(\text{total}) \tag{3.56}$$

or, in the local basis (3.2),

$$R_{AB} - \frac{1}{2}\eta_{AB}\, R = \kappa\, T_{AB}\,(\text{total}) + \Lambda\,\eta_{AB}$$

Using the form (3.18) for T_{AB}(total) we have

$$\kappa\, T_{AB}(\text{total}) + \Lambda\eta_{AB} = \kappa\left[(\tilde{\rho}+\tilde{p})\,\delta^0_A\delta^0_B - \tilde{p}\,\eta_{AB}\right] + \Lambda\,\eta_{AB} + T_{AB}\,(\text{neutrino}) \tag{3.57}$$

where $\tilde{\rho}$ and $\tilde{p}$ are the real density of energy and pressure of the fluid, as measured in the local inertial frame of an observer comoving with the fluid. The RHS of (3.57) can be rewritten

$$\kappa T_{AB}(\text{total}) + \Lambda\,\eta_{AB} = \kappa\left[\left(\tilde{\rho} + \frac{\Lambda}{\kappa} + \tilde{p} - \frac{\Lambda}{\kappa}\right)\delta^0_A\,\delta^0_B - \left(\tilde{p} - \frac{\Lambda}{\kappa}\right)\eta_{AB}\right] + \kappa\, T_{AB}\,(\text{neutrino}) \tag{3.58}$$

and has the form (3.18) for density ρ and pressure p defined by

$$\rho = \tilde{\rho} + \Lambda/\kappa \tag{3.59a}$$

$$p = \tilde{p} - \Lambda/\kappa \tag{3.59b}$$

If we take (ρ, p) as (3.40) and (3.41) corresponding to the analytical solution (3.26), (3.28), (3.29), (3.34) and (3.36), and which satisfy $p = -\frac{1}{3}\rho$, it results that $\tilde{\rho}$ and $\tilde{p}$ correspond to the same metric solution for Einstein equations (3.56), with the Λ-term satisfying

$$\tilde{p} + \frac{1}{3}\tilde{\rho} = \frac{2}{3}\,\frac{\Lambda}{k} \tag{3.60}$$

From (3.56) Bianchi's identities imply the conservation law

$$\kappa \, T_{\mu}{}^{\nu}{}_{\parallel\nu}(\text{total}) = -\Lambda_{1\mu} \tag{3.61}$$

The RHS of (3.61) permit to describe the heat output (input) rate by specific volume of the system and which can be interpreted as the rate of cooling (heating) of the fluid due to the emission (absorption) of neutrinos. To this let us write (3.61) in the local basis (3.2). A straightforward calculation gives

$$\begin{aligned} \kappa \, e^{\mu}_{(A)} \Big\{ (\tilde{\rho}+\tilde{p})^{\cdot} \alpha^{-1} e^{\mu}_{(0)} - (\tilde{\rho}+\tilde{p}) \, e_{\mu(B)} \, \gamma_0{}^{B}{}_{0} - (\tilde{\rho}+\tilde{p}) \, e_{\mu(0)} \, \gamma_0{}^{B}{}_{B} \\ - \dot{\tilde{p}} \, \delta^{0}_{\mu} - \tilde{p}' \, \delta^{1}_{\mu} \Big\} = - e^{\mu}_{(A)} \, \Lambda_{1\mu} \end{aligned} \tag{3.62}$$

We note that for A=0 we have equation (3.61) described in the rest frame of the fluid since by (3.16) $V^{\mu} = e^{\mu}_{(0)}$:

$$\kappa \Big\{ \dot{\tilde{\rho}} - (\tilde{\rho}+\tilde{p}) \Big(\frac{\dot{\alpha}}{\alpha} - 2 \frac{\dot{\beta}}{\beta} \Big) \Big\} = - \dot{\Lambda} \tag{3.63}$$

Denoting by n the barion number density as measured locally by observer (3.16), matter conservation is expressed

$$\big(n \, e^{\mu}_{(0)} \big)_{\parallel\mu} = 0 \tag{3.64}$$

or, by (3.2) and (3.3),

$$\dot{n} - n \Big(\frac{\dot{\alpha}}{\alpha} - 2 \frac{\dot{\beta}}{\beta} \Big) = 0 \tag{3.64'}$$

Defining an specific internal energy ε by

$$\tilde{\rho} = n(\mu_0 + \varepsilon) \tag{3.65}$$

where μ_0 is the rest mass of the barion, and using (3.64') and (3.65) in (3.63) results

$$\dot{\varepsilon} + \tilde{p} \Big(\frac{1}{n} \Big)^{\cdot} = - \frac{1}{\kappa} \, \frac{\dot{\Lambda}}{n} \tag{3.66}$$

Equation (3.66) is the expression in the rest frame of the fluid(*) of the first law of thermodynamics, where $\dot{\Lambda}/m$ is proportional to the heat output (input) rate per barion of the fluid.

For A = 1, (3.62) yields

$$K\left\{(\tilde{\rho}+\tilde{p})\gamma_0{}'_0 + \dot{\tilde{p}}\,\alpha^{-1} - \tilde{p}'\alpha\right\} = \dot{\alpha}'\dot{\Lambda} - \alpha\Lambda' \tag{3.67}$$

Using (3.63) in the RHS of (3.67), and also (3.3) we obtain

$$\tilde{p}' + (\tilde{\rho}+\tilde{p})\left(\alpha' + \frac{\dot{\alpha}}{\alpha^2}\right)\frac{1}{\alpha} = \frac{1}{\alpha^2}\left(\dot{\tilde{\rho}}+\dot{\tilde{p}}\right) + \frac{1}{\alpha}(\tilde{\rho}+\tilde{p})\left(\frac{2\dot{\beta}}{\beta\alpha} - \frac{\dot{\alpha}}{\alpha^2}\right) + \frac{\Lambda'}{K} \tag{3.68}$$

which substitutes the usual equation for static distributions $\tilde{\rho}' + (\tilde{\rho}+\tilde{p})\frac{\alpha'}{\alpha} = 0$. Equation (3.68) is the equation of hydrodynamic equilibrium for the star configuration. We now consider quasi-static distributions. Since by (3.55) the expansion or contraction is determined by ξ , we define quasi-static configurations for values of ξ such that

$$\xi^2 \ll |\xi| \tag{3.69}$$

An immediate integration of (3.28) and (3.29) gives

$$T_2^2 = 2\xi u + \xi_0 \tag{3.70}$$

$$T_1{}^2 = \frac{1}{\eta^2(2\xi u + \xi_0)} \tag{3.71}$$

where ξ_0 is a constant of integration. For (3.69) and finite values of u the functions (3.70) and (3.71) have approximately

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(*) We note, for instance, $\dot{\mathcal{E}} = \alpha\, \mathcal{E}_{|\mu} V^{\mu}$.

An explicit determination of the form of $\Lambda(r,u)$ will be done for a class of junction (and boundary) conditions with the exterior solution, and consequently the determination of $\tilde{\rho}$ and $\tilde{p}$.

Junction (and Boundary) Conditions of the Interior and Exterior Solutions.

Here we denote the coordinate system of the interior solution (3.1) and the exterior solution (2.1) respectively by $x^\alpha_I=(\mathcal{V}, R, \theta, \varphi)$ and $x^\alpha_{II}=(u, r, \theta, \varphi)$. For the exterior solution the coordinates x^α_{II} are admissible in the region of the space-time restricted by $r>2m(u)$. For the interior solution we see by (3.34), (3.36) and (3.48) that the coordinate R is admissible in all its domain, excluded the points such that $R_2(R)=0$ (which are improbable since R_2^{-2} is proportional to the curvature of the $\mathcal{V}, R$ = const. spheres, and which is always assumed to be finite). The coordinate $\mathcal{V}$ must be restricted by (cf. (3.70))

$$\mathcal{V}< -\frac{\xi_0}{2\xi} \quad , \quad \xi<0$$

consequence of $T_2(\mathcal{V})>0$ and $T_1(\mathcal{V})$ finite in the admissible domain of $\mathcal{V}$. In the following analysis we are considering only the region of the space-time of the solutions covered by the above coordinates.

Let us now consider the 3-dim hypersurface Σ of junction of the two solutions and D a finite neighbourhood of Σ . D and Σ are chosen such that x^α_I and x^α_{II} are simultaneously admissible in D . Following Lichnerowicz[22] we assume reasonable conditions on the continuity of the metric and its partial derivatives on D such that in the present admissible coordinate systems we have the continuity of the metric through Σ and the junction conditions[22,23]

$$G_\mu{}^\nu \phi_{,\nu} = \text{continuous through } \Sigma \tag{3.74}$$

where the equation of Σ is given by $\phi = 0$. We also suppose that on D the transformation functions $x^\alpha_I \rightleftarrows x^\alpha_{II}$ have continuous first order partial derivatives and piecewise continuous second--order partial derivatives (that is, the second order partial derivatives may have finite different limits on each side of Σ) and thus the junction conditions (3.74) are preserved under these transformations. Using Einstein equations in (3.74) we have the O'Brien-Synge junction conditions[24]

$$T_\mu{}^\nu \phi_{,\nu} = \text{continuous through } \Sigma \tag{3.75}$$

which express the continuity of the flux of four-momentum through Σ. Although the metric shall be continuous through Σ, (3.75) implies that the first order partial derivatives of the metric are discontinuous through Σ if the energy-momentum tensor is discontinuous through Σ.

We take as junction hypersurface a sphere with u-dependent radius, described in exterior coordinates by

$$\Sigma : \quad r = r_s(u) \tag{3.76}$$

In interior coordinates x^α_I, we note that R is a comoving coordinate because

$$\underset{I}{e}{}^\mu_{(0)} \frac{\partial R}{\partial x^\mu_I} = 0$$

and then Σ is described by

$$R = R_s = \text{const.} \tag{3.77}$$

The metric induced on Σ by the exterior metric can be calculated:

on Σ we take the coordinates $\lambda^i=(u,\theta,\varphi)$ and the extrinsec co ordinates of points on Σ parametrized by λ^i are given by $x^\alpha_{II}(\lambda^i)=(u, r_s(u), \theta, \varphi)$. The metric induced on Σ by the exterior metric (2.1) results

$$(ds^2)_\Sigma = \left(1-\frac{2m(u)}{r_s(u)}+2\dot{r}_s\right)du^2 - r_s^2\, d\Omega^2 \tag{3.78}$$

Analogously, the metric induced on Σ by the interior metric is calculated: on Σ we take the coordinates $\tilde{\lambda}^i=(\mathcal{V},\theta,\varphi)$ and the extrinsec coordinates of points on Σ parametrized by $\tilde{\lambda}^i$ are given by $x^\alpha_I(\tilde{\lambda}^i)=(\mathcal{V}, R_s=\text{const.}, \theta, \varphi)$. It results

$$(ds^2)_\Sigma = \alpha^2(R_s,\mathcal{V})\,d\mathcal{V}^2 - \beta^2(R_s,\mathcal{V})\,d\Omega^2 \tag{3.79}$$

The first junction condition can be expressed by the equality of (3.78) and (3.79), and results

$$\beta^2(R_s,\mathcal{V}) = r_s^2(u) \tag{3.80}$$

$$\alpha^2(R_s,\mathcal{V})\,d\mathcal{V}^2 = \left(1-\frac{2m(u)}{r_s(u)}+2\dot{r}_s\right)du^2 \tag{3.81}$$

This equality of the first fundamental forms of Σ, (3.78) and (3.79), guarantee the continuity of the components of the metric through Σ . Equation (3.81) relates the proper time (or Newtonian time) interval du of an observer at rest at infinity and the interval $d\mathcal{V}$ of an observer on the surface Σ , and (3.8) is an expression for the boundary radius of the surface Σ of the fluid. We remark that if $\dot{r}_s=0$ by equation (3.81) we could take $\mathcal{V}\equiv u$ and prolongate the exterior coordinate naturally to the interior $R<r_s$, corresponding to a complete solution of the Schwarzschild problem for a static fluid sphere. For $\dot{r}_s \neq 0$ the identification $x^\alpha_{I} \equiv x^\alpha_{I}$ makes (3.80) and (3.81) incompatible with the continuity of the metric components through Σ . For the two metrics (2.1) and (3.1) let us examine the coordinate transformations $x^\alpha_I \rightarrow x^\alpha_{II}$ given by

$$u = F(\mathcal{V}, R) \tag{3.82}$$

$$r = G(\mathcal{V}, R) \tag{3.83}$$

Differentiating (3.82) ($du = F' dR + \dot{F}\, d\mathcal{V}$) and calculating in $R = R_S$, and comparing with (3.81), we have the condition

$$F'(R_S, \mathcal{V}) = 0 \tag{3.84}$$

Using (3.82) and (3.83) in (2.1), comparing for $R = R_S$ with (3.1) and noting (3.84) we get

$$G^2(R_S, \mathcal{V}) = R_2^2(R_S)\, T_2^2(\mathcal{V}) \tag{3.85a}$$

$$(\dot{F} G')(R_s) = 1 \tag{3.85b}$$

$$\left\{(1 - 2m(F)\, G^{-1})(\dot{F}^2 + \dot{F}\dot{G})\right\}_{R=R_S} = R_1^2(R_s)\, T_1^2(\mathcal{V}) \tag{3.85c}$$

Equations (3.85) define a class of coordinate transformations in a finite neighbourhood of Σ, which permit to accomplish the junction conditions discussed above and with suitable continuity properties. Indeed (3.85) express the continuity of the metric through Σ . From (3.82), (3.84) and (3.81) we have

$$R_1^2(R_s)\, T_1^2(\mathcal{V}) = \left\{1 - 2m(F(R_s,\mathcal{V}))\, G^{-1}(R_s,\mathcal{V}) + 2\dot{G}(R_s,\mathcal{V})\right\} \dot{F}^2(R_s,\mathcal{V}) \tag{3.86}$$

We can also calculate dr_s/du by using (3.80), (3.81), (3.28), (3.29) and it results

$$\frac{dr_s(u)}{du} = \xi\eta \frac{R_2(R_s)}{R_1(R_s)} \left\{ 1- \frac{2m(u)}{r_s(u)} + 2\dot{r}_s(u) \right\}^{1/2} \tag{3.87}$$

where the variable u is related to $\mathcal{V}$ by $u = F(R_s, \mathcal{V})$. Equation (3.87) is a equation for the evolution of the surface Σ . We remark that whenever root square is taken in obtaining the above expressions, it is geometrically reasonable to consider the positive root only. Since by (3.81) $\{1-2m(u)r_s^{-1}(u) + 2\dot{r}_s(u)\}^{1/2}$ is always positive, we can see that the sign of $\dot{r}_s$ is given by ξ, such that if the fluid is contracting $\dot{r}_s<0$ and if the fluid is expanding $\dot{r}_s>0$, corresponding to a decreasing or increasing of the area of Σ .

Let us examine now the junction conditions (3.75) in the junction surface (3.76), $\Sigma: r = r_s(u)$ in exterior coordinates. Denoting the normal to the surface Σ by $\Sigma_{|\mu}$ we have

$$\Sigma_{|\mu} = (-\dot{r}_s, 1, 0, 0) \quad , \quad \Sigma^{\mu} = g^{\mu\alpha}\Sigma_{|\alpha} = (1, -\dot{r}_s + \alpha^2, 0, 0) \tag{3.88}$$

Since the metric is continuous through Σ the junction conditions (3.75) can be expressed in the tetrad basis (2.4) as

$$\underset{II}{e}{}^{\mu}_{(A)}\, T_{\mu}{}^{\nu}\, \underset{II}{e}{}_{\nu}{}^{(B)}\, \underset{II}{e}{}^{\rho}_{(B)}\, \Sigma_{|\rho} \quad = \text{continuous through } \Sigma$$

or

$$T_A{}^B\, \Sigma_{|B} \qquad = \text{continuous through } \Sigma \tag{3.89}$$

where $\Sigma_{|B} = (-\alpha_{II}^{-1}\dot{r}_s, \alpha_{II}^{-1}(\dot{r}_s + \alpha_{II}^2), 0, 0)$ and α_{II} is given by $\alpha_{II}^2 = 1-2m(u)r^{-1}$ since the normal is expressed in terms of the exterior metric. From (3.89) we have

$$(\Delta T_A{}^B)\, \Sigma_{1B} = 0 \tag{3.90}$$

where $\Delta T_A{}^B$ denotes the discontinuity in $T_A{}^B$ on crossing Σ, explicitly $\Delta T_A{}^B = T_{I A}{}^B(\Sigma) - T_{II A}{}^B(\Sigma)$. In (3.90) α^2_{II} is calculated on Σ, which we denote $\alpha^2_{II}(\Sigma) = 1 - 2m(u)\, r_s^{-1}(u)$, where the coordinate u is given by $u = F(R_s, \mathcal{V})$ in terms of the coordinates x^α_I of the interior solution. We follow an analogous notation in all cases.

Noting that the quantities in (3.90) are expressed in a tetrad basis and are invariant under the coordinate transformations (3.82), we can use directly the components (2.19) and (3.18) of the exterior and interior energy-momentum tensor in the calculation of $\Delta T_A{}^B$. With the notation

$$\ell(u) = 2\pi i(\lambda^+ \dot\lambda - \dot\lambda^+ \lambda)$$

in (2.19) and using (3.80) we have

$$\begin{aligned}
\Delta T_0{}^1 &= \frac{2}{r_s^2}\left(\frac{\mathcal{L}}{\alpha^2_I(\Sigma)} - \frac{\ell}{\alpha^2_{II}(\Sigma)}\right)\\
\Delta T_0{}^0 &= \Delta \rho + \frac{2}{r_s^2}\left(\frac{\mathcal{L}}{\alpha^2_I(\Sigma)} - \frac{\ell}{\alpha^2_{II}(\Sigma)}\right)\\
\Delta T_1{}^1 &= -\Delta p - \frac{2}{r_s^2}\left(\frac{\mathcal{L}}{\alpha^2_I(\Sigma)} - \frac{\ell}{\alpha^2_{II}(\Sigma)}\right)
\end{aligned} \tag{3.91}$$

where $\alpha^2_I(\Sigma) = R_1^2(R_s)\, T_1^2(\mathcal{V})$. With the expression of Σ_{1B}, (3.90) can be written

$$\begin{aligned}
&\Delta T_0{}^1(\dot r_s + \alpha^2_{II}(\Sigma)) - \Delta T_0{}^0\, \dot r_s = 0\\
&\Delta T_1{}^1(\dot r_s + \alpha^2_{II}(\Sigma)) - \Delta T_1{}^0\, \dot r_s = 0
\end{aligned} \tag{3.92}$$

and substituting (3.91) in (3.92),

$$\Delta \rho \; \dot{r}_s = \frac{2}{r_s^2} \left(\frac{\mathcal{L}}{\alpha_I^2(\Sigma)} - \frac{\ell}{\alpha_{II}^2(\Sigma)} \right) \alpha_{II}^2(\Sigma) \tag{3.93}$$

$$-\Delta p \left(\dot{r}_s + \alpha_{II}^2(\Sigma) \right) - \frac{2}{r_s^2} \left(\frac{\mathcal{L}}{\alpha_I^2(\Sigma)} - \frac{\ell}{\alpha_{II}^2(\Sigma)} \right) \alpha_{II}^2(\Sigma) = 0 \tag{3.94}$$

Equations (3.93) and (3.94) express the junction conditions on Σ of the exterior and interior solutions for a spherically symmetric star emitting neutrinos, continuously through Σ. Here ρ and p are the quantities (3.40) and (3.41) of the interior solution which must satisfy $p = -\frac{1}{3}\rho$. This implies their discontinuities through Σ must satisfy

$$\Delta p = -\frac{1}{3} \Delta \rho \tag{3.95}$$

Condition (3.95) determines how the real solution ($\tilde{\rho}, \tilde{p}, \Lambda$) must be discontinuous through Σ. From (3.59a,b)

$$\Delta \rho = \Delta \tilde{\rho} + \frac{1}{\kappa} \Delta \Lambda \tag{3.96a}$$

$$\Delta p = \Delta \tilde{p} - \frac{1}{\kappa} \Delta \Lambda \tag{3.96b}$$

where $\Delta\Lambda = \Lambda(R_s, \mathcal{V})$, with the additional condition (3.95) yielding

$$\frac{1}{3} \Delta \tilde{\rho} + \Delta \tilde{p} = \frac{2}{3\kappa} \Delta \Lambda \tag{3.97}$$

It is physically reasonable to have

$$\Delta \tilde{p} = 0 \tag{3.98}$$

that reduces (3.96a,b) and (3.97) to

$$\Delta\varrho = \Delta\tilde{\varrho} + \frac{1}{k}\Delta\Lambda$$
$$\Delta p = -\frac{1}{k}\Delta\Lambda \qquad\qquad \Delta\tilde{\varrho} = \frac{2}{k}\Delta\Lambda \qquad (3.99)$$

On examining the junction conditions (3.93) and (3.94) we can distinguish two relevant situations:

(a) $\Delta\varrho,\ \Delta p \neq 0$

Substituting (3.93) and (3.94) in (3.95) we obtain

$$2\,\frac{\alpha_{II}^2(\Sigma)}{r_S^2}\left(\frac{\mathcal{L}}{\alpha_I^2(\Sigma)} - \frac{\ell}{\alpha_{II}^2(\Sigma)}\right)\left(\frac{1}{3}\;\frac{\dot{r}_S + \alpha_{II}^2(\Sigma)}{\dot{r}_S} - 1\right) = 0$$

that implies

$$\frac{\mathcal{L}}{\alpha_I^2(\Sigma)} - \frac{\ell}{\alpha_{II}^2(\Sigma)} = 0 \qquad (3.100)$$

or

$$\alpha_{II}^2(\Sigma) = 2\,\dot{r}_S \qquad (3.101)$$

The case (3.100) will be examined later. Equation (3.101) can be written

$$2\,\dot{r}_S = 1 - \frac{2m(u)}{r_S(u)} \quad , \quad u = F(R_S, \mathcal{V}) \qquad (3.102)$$

Substituting (3.102) in (3.8) we have the expression for $m(u)$,

$$m(u) = \left\{1 - 16\,\xi^2\eta^2\,\frac{R_2^2(R_S)}{R_1^2(R_S)}\right\}\;\frac{r_S(u)}{2} \qquad (3.103)$$

corresponding to the region of space-time where the coordinate system is admissible ($r_S > 2m(u)$) and imposing the parameters shall be restricted by

$$16\,\xi^2\eta^2\;\frac{R_2^2(R_S)}{R_1^2(R_S)} < 1$$

Equation (3.103) which basically results of the choice (3.101) contains the important information that in the static limit $\xi \to 0$, the surface Σ coincides with the Schwarzschild surface of the star, that is, for $\xi \to 0$ — when the contraction and the emission of neutrinos cease — the fluid is entirely contained inside its Schwarzschild radius. Using (3.80) and (3.70) in (3.103) the Schwarzschild mass for the static limit ($\xi \to 0$) of a star emitting neutrinos immediately before passing its Schwarzschild radius, can be evaluated

$$m = \xi_0 \frac{R_2(R_S)}{2} \tag{3.104}$$

From the above discussion, the choice (3.101) which results in (3.103) is not satisfatory for quasi-static distributions in radiative equilibrium (cf. (3.72)) with an eventual Newtonian limit. For this we consider

(b) distributions ($\tilde{\rho}, \tilde{p}, \Lambda$) which vanish smoothly on Σ , without discontinuity: $\Delta\tilde{\rho}, \Delta\tilde{p}, \Delta\Lambda = 0$. From (3.93) and (3.94) we then have

$$\frac{\mathcal{L}}{\alpha_I^2(\Sigma)} - \frac{\ell}{\alpha_{II}^2(\Sigma)} = 0 \tag{3.100}$$

which is the first case of (a). Using (3.33) in (3.100) results

$$\ell(u) = \frac{\alpha_{II}^2(\Sigma)}{\alpha_I^2(\Sigma)} \frac{\chi}{k} T_2^{-2}(\mathcal{V}) \tag{3.105}$$

or, by (3.29) and $\alpha_I^2(\Sigma) = R_1^2(R_S)\, T_1^2(\mathcal{V})$,

$$\ell(u) = \frac{1}{R_1^2(R_S)} \left(1 - \frac{2m(u)}{r_s(u)}\right) \eta^2 \chi/k \tag{3.106}$$

From (2.22) (here $k < 0$)

$$\ell(u) = 4\dot{m}/k ,$$

which in (3.106) gives an ordinary differential equation for $m(u)$,

$$4\dot{m} = \frac{\eta^2 \chi}{R_1^2(R_s)} \left(1 - \frac{2m(u)}{r_s(u)}\right) \qquad (3.107)$$

With equation (3.107) we can discuss the sign of χ. We know that for neutrino emission $\dot{m}<0$ and neutrino absorption $\dot{m}>0$. Since in the RHS of (3.107) the factor in parentesis is always positive, we have

(b.1) $\chi<0$ emission ($\dot{m}<0$)

(b.2) $\chi>0$ absorption ($\dot{m}>0$)

without relation with the sign of ξ (*). Contrary to the case (3.101), (3.103) where the sign of $\dot{m}$ is defined by ξ , in the present case (b.1) and (b.2) we can eventually have emission with expansion or aborption with contraction of the fluid, though these situations seem physically improbable.

Finally we now determine the function $\Lambda(R,\mathcal{V})$, which contains information on the dynamics of the fluid configuration, and the real density $\tilde{\rho}$ defined in (3.59a). For instance we consider the case of the junction condition (3.101). The density $\tilde{\rho}$ is measured in the local inertial frame of an observer comoving with the fluid. The proper volume of this observer is expressed by $dV = e^{(1)}_{\alpha} e^{(2)}_{\beta} e^{(3)}_{\gamma} dx^{\alpha} dx^{\beta} dx^{\gamma}$ with $e^{(A)}_{\alpha}$ given by (3.2). We could tentatively define the mass-energy distribution of the fluid inside a sphere of radius corresponding to the coordinate R , and for a given $\mathcal{V}$, as

$$M(R,\mathcal{V}) = \int_0^R \tilde{\rho}(R,\mathcal{V})\, dV \qquad (3.108)$$

(*) From (3.32) we note that in order to have $R_1'(R)$ finite, χ must be of the same order of ξ , that is, $\chi\to 0$ whenever $\xi\to 0$ and vice versa, independent of the sign of both.

Nevertheless the density $\tilde{\rho}$ — locally measured — is associated to the rest, and the internal, and barion interaction energies only but does not include the energy of gravitational interaction which is not locally defined. So the definition (3.108) for $R=R_s$ differs from $m(u)$ (cf. exterior metric (2.1)) by the energy of gravitational interaction. But since our interior solution has a natural static limit defined by $\xi \to 0$, we can define the total energy of a sphere of fluid, with radius corresponding to the co ordinate R , in an analogous way to the static case[25,26]

$$m(R,\mathcal{V}) = 4\pi \int_0^R \tilde{\rho}(R,\mathcal{V})\,\beta^2(R,\mathcal{V})\,\beta'(R,\mathcal{V})\,dR \qquad (3.109)$$

correctly including the energy of gravitational interaction. As discussed anteriorly no gravitational radiation is simultaneously emitted with neutrinos. Therefore, the problem of localization of the energy of gravitational waves is not present here and the total energy $m(R,\mathcal{V})$ defined by (3.109) is localized since other fluxes of energy (neutrinos, for instance) are always locally measurable (cf. (3.66) and (3.68)). For $R=R_s$ the total mass-energy of the fluid is given

$$m(R_s,\mathcal{V}) = 4\pi \int_0^{R_s} \left[\rho(R,\mathcal{V}) - \frac{\Delta}{K}(R,\mathcal{V})\right] R_2^2(R)\,R_2'(R)\,T_2^{\,3}(\mathcal{V})\,dR \qquad (3.110)$$

Using (3.80) the equality of (3.110) to (3.103) results

$$\gamma^2 \int_0^{R_s} R_2'(R)\,T_2(\mathcal{V})\,dR = 4\pi \int_0^{R_s} \left[\rho(R,\mathcal{V}) - \frac{\Delta}{K}(R,\mathcal{V})\right] R_2^2(R)\,R_2'(R)\,T_2^{\,3}\,dR \qquad (3.111)$$

where $\gamma^2 = \frac{1}{2}\left(1 - 16\,\xi^2\eta^2 R_2^2(R_s)/R_1^2(R_s)\right)$. From (3.37), writing $\rho(R,\mathcal{V}) = \rho(R)\,T_2^{-2}(\mathcal{V})$, we take from (3.111)

$$\gamma^2 R_2'(R)\,T_2(\mathcal{V}) = 4\pi\rho(R)\,R_2^2(R)\,R_2'(R)\,T_2(\mathcal{V}) - 4\pi\,\frac{\Delta}{K}(R,\mathcal{V}) \times R_2^2(R)\,R_2'(R)\,T_2^{\,3}(\mathcal{V})$$

up to a function of $(R, \mathcal{V})$ whose integral in R between 0 and R_s is identically zero. We then have for this case the expression for $\Lambda(R,\mathcal{V})$

$$\frac{\Lambda}{\kappa}(R,\mathcal{V}) = \rho(R,\mathcal{V}) - \frac{1}{4\pi R_2^2(R)} \gamma^2 \; T_2^{-2} \tag{3.112}$$

where $\rho(R,\mathcal{V})$ is given by (3.40). Substituting (3.112) in (3.59a) we obtain the expression for the real density

$$\tilde{\rho}(R,\mathcal{V}) = \frac{1}{4\pi R_2^2(R)} \gamma^2 \; T_2^{-2} \tag{3.113}$$

We can verify directly that $\tilde{\rho}(R) = \gamma^2/4\pi R_2^2(R)$ increases with R if $R_2'/R_2 < 0$ and decreases if $R_2'/R_2 > 0$. In the present case where ΔT_0^1, $\Delta\tilde{\rho}$, etc. are non-zero, the value R_s of the coordinate R corresponding to the radius of the configuration can be chosen with some arbitrariness subject for instance to the condition $R_2''/R_2 < 0$ for $0 < R < R_s$ (cf. (3.25)) etc. In case of the energy-momentum tensor has no discontinuity through Σ the value R_s can be taken as the first root of the equation $\tilde{\rho}(R) = 0$.

4. CONCLUSIONS

We have presented a simple but complete relativistic model of a spherically symmetric star emitting neutrinos. The interior of the star is assumed to be a perfect fluid — described by its total density ρ , pressure p and the barion number density n — and bounded in space by a spherical surface of radius r_s . Once emitted neutrinos are supposed to have only gravitational interaction, that is, the matter of the star is transparent for neutrinos, and the total energy momentum-tensor for the interior solution is the sum of the energy-momentum tensor of the fluid and the energy-momentum tensor of the neutrino. The latter is constructed with the classical spinorial fields ψ, solution of Dirac equation for neutrino in the metric of the space-time considered. In the coordinate systems used, and for the choice of a current $j^{\alpha}=\bar{\psi}\gamma^{\alpha}(x)\psi$ radially along the local light cones, Dirac equation is directly integrable and the spinor-solution depends on an arbitrary spinor function of the coordinate $x^{0}=u$, and also depends on θ as $(\sin\theta)^{-\frac{1}{2}}$. This θ-dependence is shown to be a correction due to the coordinate system and has no physical significance, with all observable quantities (constructed with the neutrino field) corresponding to an isotropic emission of neutrinos. We then propose a legitimate redefinition of the energy momentum tensor of neutrino which shall enter the RHS of Einstein equations. This redefined energy momentum tensor has two important porperties: (i) it has null covariant divergence; (ii) it has the form of the energy-momentum tensor of a null fluid, which is the usual phenomenological description for neutrinos in General Relativity. The exterior metric is the Schwarzschild radiating metric (the region contains neutrinos and gravitational field only) and we determine the variation of the mass parameter of the star as a functional of the spinor-solution of the neutrinos, and which is interpreted as the total luminosity of the star relative to an asymptotic observer at rest. The interior solution is obtained by separation of variables — for the solution obtained, field equations imply an equation of state $p=-\frac{1}{3}\rho$ which corresponds to neutrinos completely decoupled of matter, in the sense that for $p=-\frac{1}{3}\rho$ the energy-momentum tensor of the fluid has identically null covariant divergence, together with the null divergence of the neutrinos energy-momentum tensor. To eliminate negative scalar pressures (which are not phy sically satisfactory) we introduce a Λ-term in Einstein equa tions (or equivalently, in the total energy-momentum tensor). To the above solution, it corresponds a solution for energy-matter density and pressure, $\tilde{\rho} = \rho - \Lambda/k$ and $\tilde{p} = p + \Lambda/k$, respectively, with $\tilde{\rho} + \tilde{p}/3 = \Lambda/k$. Bianchi's identities imply the local conservation law $\kappa T_{\mu}{}^{\nu}{}_{\|\nu} = -\Lambda_{|\mu}$ and two conservation equation follows. For $\mu = 0$, we have the analogue of the first law of thermo-dyna mics for the system, in the frame of an observer comoving with the fluid, where $\dot{\Lambda}$ is seen to be proportional to the rate of cooling of the fluid. For quasi-static distributions these equa tions provide us the relativistic analogue for neutrinos of the radiative equilibrium equation of Chandrasekhar, and we interpret Λ/k as a radiation pressure for neutrinos (perhaps a better denomination would be gravitational pressure due to neutrinos). Contrary to the photons radiation pressure in Chandrasekhar's Newtonian equation, the gradient of Λ/k has negative sign and so has additive effect to the gravitational compression – in fact the effect of neutrinos is to cool the configuration (they do not interact with the matter of the star), this cooling being equivalent to pressure in the inverse direction of photons pressure, additive to the gravitational pressure. This Λ function is completely determined by a careful examination of the junction and boundary conditions of the interior and exterior solutions. We have used Israel-O'Brien-Synge junction conditions. Two possi bilities arise, corresponding to physically distinct situations. In one case, the solution describes a stage of emission of neutrinos, with consequent contraction of the configuration immedia tely before the surface of the fluid coincide with its Schwarzs child surface, i.e., imediately before the fluid is entirely inside its Schwarzschild radius – this occurs when the emission of neutrinos and the contraction of the star cease ($\xi \to 0$). The Λ function is determined completely for this case. The other possibility can, for instance, correspond to a quasi-static configuration where $\Lambda_{/k}$ has the interpretation of a radiation pressure for neutrinos.

Our detailed geometrical treatment of the problem has nevertheless been unilateral because of our description of neutrinos as classical spinorial fields. An improvement of it would be the quantization of the neutrino field on the classical curved space-time. At present we do not know a general procedure to construct a quantum field theory on an arbitrary Riemannian space-time. In case the space-time admits a time-like Killing vector and some other symmetries we can construct a basis system for an invariant definition of particel and anti-particle states and vacuum state[27]. In the present model, the only Killing vectors are the generators of the spatial rotation group. These are not sufficient to construct a basis for an invariant decomposition into excitation modes of the field. Alternatively, since the exterior solution is asymptotically flat we could essay an asymptotic quantization of the neutrino field, choosing the asymptotic basis functions as the usual basis functions in flat space[28] and for the junction condition (3.100) the form (3.33) would be an additional guide in the choice of the asymptotic basis to express $\ell(u)$. From the classical form of $\ell(u)$ and following an usual scheme of quantization it would in principle be possible (through (3.33) and (3.100)) to characterize the parameter χ as function of the generalized momenta corresponding to the basis chosen. At present we do not know if a complete quantization of the radiated neutrino field in the background metric should provide a drastic change in the problem of these particles contributing to the curvature of the background and if our solution as a first approximation would be significant at all.

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