ANISOTROPIC GRAVITATIONAL WAVES PERTURBATIONS AND THE CREATION OF SCALAR PARTICLES IN THE QUASI-EUCLIDEAN FRIEDMANN'S COSMOLOGICAL MODEL

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ABSTRACT

We investigate the creation of scalar particles due to the propagation of anisotropic time dependent gravitational waves perturbations in the isotropic Friedmann's cosmological model. We consider a conformal invariant equation for the scalar field and for the case where the equation of state is $p = \rho$ we obtain the contribution for the energy-momentum tensor due to the created particles and show that the reaction of these particles on the metric does not tends to isotropize the perturbed background.

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I. THE INTERACTION BETWEEN CLASSICAL GRAVITATIONAL FIELDS AND REAL SCALAR FIELDS

Let $\phi(x)$ be a real scalar field and $g_{\mu\nu}(x)$ the metric tensor of a Riemannian space-time, with signature (+---). We assume that the sources of both the gravitational and scalar fields are specified independently of each other. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\ (g^{\alpha\beta}\ \phi_{|\alpha}\ \phi_{|\beta} - (m^2 + \tfrac{1}{6}R)\phi^2) \tag{1.1}$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar of curvature and a bar means the derivative with respect to the space-time coordinates. From (1.1) it follows that the equation of motion for the scalar field is the covariant generalization of the Klein-Gordon equation

$$g^{\alpha\beta}\ \nabla_\alpha\ \nabla_\beta\ \phi + (m^2 + \tfrac{1}{6}R)\ \phi = 0 \tag{1.2}$$

where ∇_μ denotes covariant derivative.

Equation (1.2) reduces to the ordinary Klein-Gordon equation in the absence of gravitational fields, i.e., if $g_{\alpha\beta}$ is $\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the Minkowski metric tensor. The reason for introducing the term proportional to R in (1.2) is that if m=0 this equation is invariant under the conformal transformation defined by

$$\begin{aligned} g_{\alpha\beta} &\longrightarrow \bar{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta} \\ \phi &\longrightarrow \bar{\phi} = \Omega^{-1}(x)\phi \end{aligned} \tag{1.3}$$

and as it is well known, conformal invariance of the massless scalar field equation has important consequences in the mechanism of creation of particles[1,2].

With the equation of motion written in the form (1.2, we say that the scalar field is conformally coupled to the gravitational field.

The energy-momentum tensor $T_{\alpha\beta}[\phi(x)]$ is obtained as usual, namely by the variation of the action integral with respect to $g_{\alpha\beta}$:

$$\delta I = \delta\int \mathcal{L}\, d^4x \equiv \int T_{\alpha\beta}\delta g^{\alpha\beta} \sqrt{-g}\, d^4x$$

The result is

$$T_{\alpha\beta} = \phi_{|\alpha}\phi_{|\beta} - g_{\alpha\beta}\mathcal{L} + \frac{1}{6}\left[R_{\alpha\beta}-g_{\alpha\beta}\Box+\nabla_\alpha\nabla_\beta\right]\phi^2 \tag{1.4}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. This tensor has the following properties :

$$\begin{aligned} T_{\alpha\beta} &= T_{\beta\alpha} \\ T \equiv T_\alpha{}^\alpha &= m^2\phi^2 \\ \nabla_\alpha T^{\alpha\beta} &= 0 \end{aligned} \tag{1.5}$$

In order to canonically quantize the scalar field in the classical Riemannian space-time manifold we define the momentum π conjugated to the scalar field as

$$\pi(\vec{x}\ ,t) = \frac{\partial\mathcal{L}}{\partial\phi_{|o}} \tag{1.6}$$

and introduce the equal time canonical commutation relations

$$\begin{aligned} \left[\phi(\vec{x},t),\phi(\vec{x}',t)\right] &= 0 \\ \left[\pi(\vec{x},t),\pi(\vec{x}',t)\right] &= 0 \\ \left[\phi(\vec{x},t),\pi(\vec{x}',t)\right] &= i\delta^{(3)}(\vec{x}-\vec{x}') \end{aligned} \tag{1.7}$$

where x and x' are space coordinates on the same hypersurface. It should be mentioned that the quantization procedure outlined above is generally covariant and consistent. See ref. 3 for details

We shall be concerned only with the isotropic quasi-Euclidean Friedmann's model and so, the space-time metric is

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j \tag{1.8}$$

$$(i,j = 1,2,3)$$

The scalar field may be expanded as

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[A_k \Phi_k(t)\psi_k(\vec{x}) + A_k^\dagger \Phi_k^*(t)\psi_k^*(\vec{x})\right] \tag{1.9}$$

In the above expression A_k and $A_k^\dagger$ are time independent operators which due to (1.7) satisfies the commutation relations

$$\begin{aligned} \left[A_k, A_{k'}\right] &= 0 \\ \left[A_k^\dagger, A_{k'}^\dagger\right] &= 0 \\ \left[A_k, A_{k'}^\dagger\right] &= \delta(\vec{k}-\vec{k}') \end{aligned} \tag{1.10}$$

The functions $\psi_k(\vec{x})$ are eigen-functions of the three-dimensional Laplacian operator, $\nabla^2\psi_k(\vec{x}) - k^2\psi_k(\vec{x}) = 0$. Clearly, we have $\psi_k(\vec{x}) \sim e^{\pm i\vec{k}.\vec{x}}$. The time-dependent function $\Phi_k(t)$ satisfies the differential equation

$$\ddot{\Phi}_k + \frac{\dot{V}}{V}\dot{\Phi}_k + \left[\omega_k^2(t) + \frac{1}{6}R\right]\Phi_k = 0 \tag{1.11}$$

where the dot means the derivative with respect to time, $V = a^3$ and

$$\omega_k^2(t) = m^2 + \frac{1}{a^2}(k_1^2+k_2^2+k_3^2) \qquad (1.12)$$

Consistence of (1.7), (1.10) and equation (1.11) requires that

$$\Phi_k\dot{\Phi}_k^* - \dot{\Phi}_k\Phi_k^* = \frac{i}{V} \qquad (1.13)$$

Let us introduce the new variables defined by

$$dt = a d\eta \quad , \quad \Phi_k = a^{-1}(\eta)\chi_k(\eta) \qquad (1.14)$$

It follows that (1.10) and (1.12), respectively, become

$$\chi_k''(\eta) + \Omega_k^2(\eta)\chi_k(\eta) = 0 \qquad (1.15)$$

and

$$\Omega_k^2(\eta) = a^3(\eta)\omega_k^2(\eta) \qquad (1.16)$$

We mention that the conformal transformation (1.3) preserves the canonical commutation relations so that if $\bar{\phi}$ and $\bar{\pi}$ are the transformed quantities, they will satisfy commutation relations analogous to (1.7). The inverse is also true. It is important to observe that if the space-time is conformally flat, them $g_{\mu\nu}=\Omega^2\eta_{\mu\nu}$ and equation (1.2) can be transformed in the ordinary Klein-Gordon equation with "variable mass" $\eta^{\mu\nu}\,\bar{\phi}_{|\mu|\nu} + \bar{\Omega}^{4}(x)m^2\bar{\phi}=0$. In this case, if m=0 the gravitational field does not influence the field $\bar{\phi}$ [4,5].

II. THE PARTICLE CREATION MECHANISM

With the Lagrangian density (1.1) and the momentum conjugated to the scalar field, we can construct the Hamiltonian den sity

$$\mathcal{H} = \pi(x)\phi(x)|_o - \mathcal{L} \tag{2.1}$$

$$= \frac{1}{2}\sqrt{-g}\left[(\phi|_o)^2 - g^{ij}\phi|_i\phi|_j + (m^2+\frac{1}{6}R)\phi^2\right] \tag{2.2}$$

and the Hamiltonian is

$$H(t) = \frac{1}{2}\int d^3x \left[\frac{\pi^2}{\sqrt{-g}} - \sqrt{-g}\, g^{ij}\phi|_i\phi|_j + (m^2+\frac{1}{6}R)\phi^2\right] \tag{2.3}$$

This expression is valid for the general case where the gravitational field is homogeneous and anisotropic. Particularizing for the metric given by (1.8) and using the expansion (1.9) for the scalar field together with the commutation relations (1.10), we get for the second quantized Hamiltonian operator

$$H(\eta) = \frac{1}{2}\int d^3k \Big[U(\vec{k},\eta)(A_k^\dagger A_k + A_{-k}^\dagger A_{-k}) + V(\vec{k},\eta)A_k^\dagger A_{-k}^\dagger + V^*(\vec{k},\eta)A_{-k}A_k\Big] \tag{2.4}$$

where $U(\vec{k},\eta) = |\chi_k'(\eta)|^2 + \Omega_k^2(\eta)|\chi_k(\eta)|^2$ and $V(\vec{k},\eta) = \chi_k'(\eta)\chi_k^*(\eta) + \Omega_k^2(\eta)|\chi_k(\eta)|^2$.

From the above expression we can conclude that $H(\eta)$ is not diagonal in the operators A_k and $A_k^\dagger$ and so it does not define a self-adjoint operator in a Hilbert space. Then, we must consider at different times non equivalent representations of the commutation relations. This follows from the fact that the solutions of equation (1.11) are not uniquely determined by the condition (1.13) and this implies that the operators A_k and $A_k^\dagger$ are not also uniquely determined. The Hamiltonian (2.4) may be diagonalized by a time dependent transformation on the operators A_k and $A_k^\dagger$. We write

$$A_k(t) = \alpha_k^*(t)A_k(t_o)+\beta_k(t)A_k^+(t_o) \qquad (2.5)$$

$\alpha_k(t)$ and $\beta_k(t)$ are complex functions which due to (1.13) must satisfy the condition

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1 \qquad (2.6)$$

The transformation (2.5) is then a time dependent Bogoliubov transformation[6,7].

Now, if we define the vacuum state at the time t_o by the relation $A_k(t_o)|0>_{t_o} = 0$, at latter times we will have $A_k(t)|0>_{t_o} \neq 0$ which means that the vacuum for $H(t)$ is not the same for $H(t_o)$. Under the hypothesis that at the time t_o there were no particles present, $<N_k(t_o)>_{t_o} = 0$, we have for later times $<N_k(t)>_{t_o} =|\beta_k(t)|^2$. In other circunstances we will have $<N_k(t)>_{t_o} =<N_k(t_o)>_{t_o}+ |\beta_k(t)|^2(1+2<N_k(t_o)>_{t_o})$. Basically, this is the phenomena of creation of particles from vacuum due to the presence of an external non stationary gravitational field. Then $|\beta_k(t)|^2$ is a measure of the mean number of particles created by the gravitational field.

Let us return to the case of massless scalar particles and conformally flat space-time. We saw that if the scalar field is conformally coupled to the gravitational field then the transformed field $\bar{\phi}$ satisfies the equation $\eta^{\mu\nu}\bar{\phi}_{|\mu|\nu} = 0$. It is clear that in this case we have $\Phi_k(t)\sim e^{\pm i\omega_k t/a(t)}$, with ω_k independent of t. It follows that $<N_k>$ does not depend on the time and so it is a constant of motion, and no particles are created by the gravitational field. This result is valid for particles with any spin and was shown independently by Parker[8] and Zel'dovich[1].

III. QUANTIZED SCALAR FIELDS AS SOURCES OF CLASSICAL GRAVITATIONAL FIELDS

The gravitational field produced by a quantized scalar field may be computed using a "semi-classical" theory of gravitation based on Einstein's equations written as

$$R_\mu{}^\nu - \frac{1}{2}R\, g_\mu{}^\nu = - \langle T_\mu{}^\nu\left[\phi(x)\right]\rangle \tag{3.1}$$

where $\langle T_\mu{}^\nu\left[\phi(x)\right]\rangle \equiv \langle 0| :T_\mu{}^\nu\left[\phi(x)\right]: |0\rangle$ is the mean value of the energy-momentum tensor operator and: : indicates normal ordering of the operators A_k and $A_k^\dagger$.

Besides the conceptual problems that arises in such theory[1,9] there is another source of difficulties. When the equation of motion for the field $\phi(x)$ can be solved by separating variables, the mean value of the energy-momentum tensor operator is formally expressed in terms of divergent integrals. Using the expansion (1.9) for the scalar field and the commutation relations (1.10) in (1.4) one find

$$\langle 0|:T_o{}^o:|0\rangle = \frac{1}{(2\pi)^3}\int d^3k\left\{\frac{1}{2}\left(|\dot{\Phi}_k|^2+\omega_k^2|\Phi_k|^2\right) - \frac{1}{6}\left(R_o{}^o-\frac{1}{2}R\right)|\Phi_k|^2 - \frac{1}{6}\frac{\dot{V}}{V}\frac{d}{dt}|\Phi_k|^2\right\} \tag{3.2}$$

$$\langle 0|:T_1{}^1:|0\rangle = \frac{1}{(2\pi)^3}\int d^3k\left\{\frac{k_1^2}{a^2}|\Phi_k|^2 + \frac{1}{2}\left(|\dot{\Phi}_k|^2-\omega_k^2|\Phi_k|^2\right)+\frac{1}{6}\left(R_1{}^1-\frac{1}{2}R\right)|\Phi_k|^2 + \frac{1}{6}\frac{1}{V}\frac{d}{dt}\left(V\frac{d}{dt}|\Phi_k|^2\right)+\frac{1}{6}\frac{\dot{a}}{a}\frac{d}{dt}|\Phi_k|^2\right\} \tag{3.3}$$

and analogous expressions for $\langle 0|:T_2{}^2:|0\rangle$ and $\langle 0|:T_3{}^3:|0\rangle$.

Some methods of regularization and renormalization for the mean values $<T_{\mu}^{\nu}>$ has recently been proposed[1,10,11]. We shall not be concerned with this problem here and in what follows we suppose that these expressions can be regularized (or renormalized) in some way. The results of our calculations will not be affected by the regularization or renormalization process.

IV. CREATION OF PARTICLES DUE TO ANISOTROPIC PERTURBATIONS IN FRIEDMANN'S QUASI-EUCLIDEAN MODEL

Let us consider a time dependent anisotropic perturbation tensor defined by

$$h_{ij}(t)=a(t)\left[\alpha(t)\delta_{1i}\delta_{1j}+\beta(t)\delta_{2i}\delta_{2j}+\gamma(t)\delta_{3i}\delta_{3j}\right] \qquad (4.1)$$

$$h_{o\mu} = 0 \qquad (\mu=0,1,2,3,; \quad i,j = 1,2,3)$$

so that the perturbed metric is written as

$$d\bar{s}^2=dt^2-a^2(t) \left[(1+ \frac{\alpha(t)}{a(t)})dx^2+(1+ \frac{\beta(t)}{a(t)})dy^2+(1+ \frac{\gamma(t)}{a(t)})dz^2\right] \qquad (4.2)$$

Using the results obtained in[12], the corrections for the components of the Ricci tensor can be calculated and the result is

$$\delta R_{oo} \simeq - \frac{\ddot{\theta}}{a} + \frac{\ddot{a}}{a}\theta$$

$$\delta R_{ij} \simeq \frac{\ddot{\lambda}_{ij}}{a} + \frac{\dot{a}}{a^2}\dot{\lambda}_{ij} - (\frac{\ddot{a}}{a^2} + \frac{\dot{a}^2}{a^3})\lambda_{ij}+(\frac{\dot{a}}{a^2}\dot{\theta} - \frac{\dot{a}^2}{a^3}\theta) \qquad (4.3)$$

where

$$\lambda_{ij} = \frac{h_{ij}}{a} \quad , \quad \theta(t) = \alpha(t)+\beta(t)+\gamma(t) \qquad (4.4)$$

With these values we can construct the corrections $\overset{(1)}{\delta} G_\mu{}^\nu$ to the tensor $G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2} R g_\mu{}^\nu$ due to the perturbations defined in (4.1).

Now, suppose that there were no massless scalar particles before the perturbation. After the perturbation the metric becomes anisotropic and massless particles can be created by the new gravitational field. Let us call $\overset{(2)}{\delta} \langle T_\mu{}^\nu \rangle$ the contribution to the energy-momentum tensor due to the newly created particles and $\overset{(2)}{\delta} G_\mu{}^\nu$ the reaction back on the tensor $G_\mu{}^\nu$ induced by this term. If we call $\overset{(o)}{T}_\mu{}^\nu$ the energy-momentum tensor which determines the unperturbed metric through Einstein's equations $\overset{(o)}{G}_\mu{}^\nu = - \overset{(o)}{T}_\mu{}^\nu$, then we can write the perturbed Einstein's equations as

$$\overset{(o)}{G}_\mu{}^\nu + \varepsilon \overset{(1)}{\delta} G_\mu{}^\nu + \varepsilon^2 \overset{(2)}{\delta} G_\mu{}^\nu + \ldots = - \overset{(o)}{T}_\mu{}^\nu - \varepsilon \overset{(1)}{\delta} T_\mu{}^\nu - \varepsilon^2 \overset{(2)}{\delta} \langle T_\mu{}^\nu \rangle + \ldots \qquad (4.5)$$

where $|\varepsilon|^2 \ll \varepsilon$.

We choose the equation of state for the background gravitational field as being $p=\rho$ which through the equations $\overset{(o)}{G}_\mu{}^\nu = - \overset{(o)}{T}_\mu{}^\nu$ leads to $a(t) \sim t^{1/3}$.

With the additional hypothesis that the perturbations are produced by the propagation of gravitational waves in their lowest vibration mode, in the transverse traceless gauge, the functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ can be obtained from the equations

$$\overset{(1)}{\delta} G_\mu{}^\nu \simeq - \overset{(1)}{\delta} T_\mu{}^\nu = 0 \qquad (4.6)$$

with the condition

$$\theta(t) = 0 \qquad (4.7)$$

The differential equations that result for the functions

$\alpha(t)$, $\beta(t)$, and $\gamma(t)$ can be easily solved and leads to the following values for the components of the perturbed metric tensor

$$\bar{g}_{11} \sim t^{2/3}$$
$$\bar{g}_{22} \sim t^{2/3}(1+A\log t) \qquad (4.8)$$
$$\bar{g}_{33} \sim t^{2/3}(1+A\log t)$$

where A is a constant.

In the perturbed background the massless scalar field equation is

$$\overset{(o)}{g}{}^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \frac{1}{6}\overset{(o)}{R}\phi - h^{\mu\nu}\phi_{|\mu|\nu} = 0 \qquad (4.9)$$

and the equation (1.15) now read

$$\bar{\chi}_k'' + \Delta_k^2(\eta)\bar{\chi}_k = 0 \qquad (4.10)$$

where

$$\Delta_k^2(\eta) = (1+\frac{\alpha}{a})k_1^2 + (1+\frac{\beta}{a})k_2^2 + (1+\frac{\gamma}{a})k_3^2 \qquad (4.11)$$

With the initial conditions $\bar{\chi}_k(o) = 0$ and $\bar{\chi}_k'(o) = \sqrt{\omega_k(o)}$, the equation (4.10) can be transformed into the integral equation[6]

$$\bar{\chi}_k(\eta) = \frac{\sin \omega_k(o)\eta}{\omega_k(o)} - \int_o^\eta \frac{\Delta_k^2(\eta')}{\omega_k(o)} \sin \omega_k(o)(\eta-\eta')\bar{\chi}_k(\eta')d\eta' \qquad (4.12)$$

Considering only small values of t and using the first order approximation to (4.12) we obtained

$$\overset{(2)}{\delta}\langle T_o{}^o\rangle - \overset{(2)}{\delta}\langle T_1{}^1\rangle = F(k)t^{2/3}\log t$$
$$\overset{(2)}{\delta}\langle T_2{}^2\rangle = (F(k)t^{2/3}\log t - H(k)t^{-2/3}) \qquad (4.13)$$

$$\overset{(2)}{\delta} \langle T_3{}^3 \rangle \simeq (F(k)t^{2/3} \log t + H(k)\ t^{-2/3})$$

where F(k) and H(k) are time independent integrals in the components of the momenta $\vec{k}$ of the particles.

With these results and the equation $\overset{(2)}{\delta} G_\mu{}^\nu = - \overset{(2)}{\delta} \langle T_\mu{}^\nu \rangle$, the reaction of the created particles back on the perturbed metric, which we call $\bar{\alpha}(t)$, $\bar{\beta}(t)$, and $\bar{\gamma}(t)$, can be evaluated. After a lengthy calculation we obtained

$$\begin{aligned} \bar{\alpha}(t) &\sim t^{1/3} \\ \bar{\beta}(t) &\sim - H(k)t^{5/3} \\ \bar{\gamma}(t) &\sim H(k)t^{5/3} \end{aligned} \qquad (4.14)$$

V. CONCLUSIONS

We studied the evolution of anisotropic time dependent gravitational waves perturbations in a quasi-Euclidean Friedmann's cosmological model. The solutions to Einstein's perturbed equations have been obtained for the case where the equation of state for the background is $p=\rho$. The perturbations generates a plane of anisotropy and the amplitudes increases at increasing t. The solution to the time dependent part of the scalar field equation has been obtained as a first approximation of a Volterra type integral equation, valid for small values of the time t. With the help of this result we calculated the reaction of the created particles back on the perturbed metric. Such reaction clearly does not tend to isotropize the perturbed background.

It should be mentioned that recently Novello and Galvão[2] analized the influence of primeval homogeneous electromagnetic fields on the mechanism of creation of particles in an anisotropic cosmological model and showed that the existence of the electromagnetic field inhibits the creation of particles due to the expansion of the Universe, and so tends to difficult the isotropization process. What we can conclude from these results is that the creation of particles due to the expansion of the Universe does not acts in all situations as a mechanism of isotropization but only under special situations.

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