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by

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ABSTRACT

It is shown that Einstein equations allow a special class of stationary solutions which correspond to spherically symmetric clusters of particles in circular motions, the total angular momentum of the cluster being zero, and all orbits being performed with the same period. The mass density of such clusters is everywhere regular and positive, decreasing with increasing radius. No restriction is found either on the radii of the clusters, or on the value of the period of the orbits.

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INTRODUCTION

To investigate the mathematical and physical significance of the Schwarzschild singularity, Einstein [1] in an ingenious way introduced rotation without angular momentum in a system with spherical symmetry. He considered a stationary cluster of particles moving in circular orbits about the centre of symmetry under the influence of the gravitational field produced by all of them together. To have spherical symmetry it was assumed that the phases of motion and the orientation of orbits were perfectly at random. For such a distribution Schwarzschild singularities do not exist in physical reality, because if a cluster of given mass shrinked to Schwarzschild radius its outermost particles would attain velocities greater than that of light.

The aim of the present work is to investigate a similar stationary spherically symmetric cluster of particles under a constraint of motion. If one assumes that all orbits are performed with the same period, the size of the orbit may tend to any arbitrarily small value in agreement with the result of Einstein's spherical clusters. The mass density of such a cluster is everywhere regular and positive, decreasing with increasing radius. For a given gravitational mass the radius of cluster depends only on the period, but has a minimum which is three halves the Schwarzschild radius.

FIELD EQUATIONS

With a suitable choice of spherical coordinates, $x^{\mu} = (x^0, r, \Theta, \phi)$, it is possible to obtain stationary spherically symmetric line elements in the form (Anderson (2))

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}\, dr^2 - r^2(d\Theta^2 + \sin^2\Theta\, d\phi^2) \qquad (1)$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. The Einstein's field equations are given by

$$G^{\mu}_{\nu} \equiv R^{\mu}_{\nu} - \frac{1}{2} R\, \delta^{\mu}_{\nu} = - \frac{8\pi G}{c^4} T^{\mu}_{\nu} \quad , \tag{2}$$

where T^{μ}_{ν} is energy-stress tensor. For a stationary spherically symmetric cluster of particles one can consider this tensor in the form

$$T^{\mu}_{\nu} = c^2 \rho \text{ diag.} \left[1 + \alpha^2,\ 0,\ -\alpha^2/2,\ -\alpha^2/2\right] , \tag{4}$$

where $\alpha(r)$ is some function related to the components of the velocity vectors of the particles, and $\rho(r)$ is a continuous mass distribution corresponding to the whole of the particles.

Inserting the expressions for $g_{\mu\nu}$ and T^{μ}_{ν} from (1) and (4) respectively into the field equations (2) we obtain

$$G^{0}_{0} \equiv e^{-\lambda}(r^{-2} - r^{-1} \lambda_1) - r^{-2} = -8\pi G\rho(1+\alpha^2)/c^2 , \tag{5}$$

$$G^{1}_{1} \equiv e^{-\lambda}(r^{-2} + r^{-1} \nu_1) - r^{-2} = 0 \qquad \text{and} \tag{6}$$

$$G^{2}_{2} = G^{3}_{3} \equiv e^{-\lambda}\left[2r\nu_{11} + r\nu_1^2 + 2\nu_1 - 2\lambda_1 - r\nu_1\lambda_1\right] /(4r) = 4\pi G\rho\alpha^2/c^2 . \tag{7}$$

We can simplify the task of obtaining the solutions of this set of equations. Indeed, Bianchi identities $G_{\nu;\mu}^{\ \mu} \equiv 0$ connected to Einstein equations (2) impose the equations of motion $T^{\mu}_{\nu;\mu} = 0$; from these, equation $\nu = 1$ is the only one which does not vanish identically, and gives

$$r\nu_1 = 2\alpha^2/(1+\alpha^2) . \tag{8}$$

The four equations (5) to (8) are not independent, however. We shall conveniently consider (5), (6) and (8) as the equations of our problem. We have thus three equations to be satisfied by four unknown functions

(ν, λ, ρ and α).

Our purpose is to investigate the distribution under a particular constraint of motion, so we choose α arbitrarily. Equation (8) determines ν. Substitution of ν in equation (6) determines λ. One then easily can find the mass density ρ from equation (5).

Let us put $\alpha^2 = \omega^2 r^2/c^2$ where ω is an arbitrary positive constant. Then from equations (5), (6) and (8) we have

$$\rho = 3\,\omega^2(4\pi G)^{-1}\,(1+3\,\omega^2 r^2/c^2)^{-2}\ , \tag{9}$$

$$e^{\lambda} = (1 + 3\,\omega^2\, r^2/c^2)(1 + \omega^2\, r^2/c^2)^{-1} \quad \text{and} \tag{10}$$

$$e^{\nu} = A(1 + \omega^2\, r^2/c^2)\ , \tag{11}$$

where A is a constant of integration. The constants A and ω will be interpreted from the boundary condition and geodesic equations.

MOTION OF PARTICLES IN THE GRAVITATIONAL FIELD

The motion of any particle in the field of others is governed by the geodesic equation

$$\frac{d^2x^{\mu}}{ds^2} + \left\{{\mu \atop \nu\rho}\right\} \frac{dx^{\nu}}{ds}\,\frac{dx^{\rho}}{ds} = 0 \tag{12}$$

with the supplementary condition

$$g_{\mu\nu}\,\frac{dx^{\mu}}{ds}\,\frac{dx^{\nu}}{ds} = 1\ . \tag{13}$$

Now we define a time-like Killing vector field τ^{μ} associated to our spherically symmetric stationary field, that is,

$$\tau^2 \equiv g_{\mu\nu}\, \tau^\mu \tau^\nu > 0 \qquad \text{and} \qquad \tau_{\mu;\nu} + \tau_{\nu;\mu} = 0 \quad ; \tag{14}$$

with its help we construct the projection tensor

$$\mathcal{B}^\mu_\nu = \delta^\mu_\nu - \tau^\mu \tau_\nu / \tau^2 \quad , \tag{15}$$

and the covariant *normal* velocity (of an object of velocity $u^\mu = dx^\mu/ds$)

$$v^\mu = \mathcal{B}^\mu_\nu\, u^\nu (\tau^2)^{1/2}\, (\tau_\rho\, u^\rho)^{-1} \quad . \tag{16}$$

The norm of this vector corresponds to the norm of the classical velocity of the object,

$$v_c^2 = -\, c^2\, g_{\mu\nu}\, v^\mu v^\nu \; . \tag{17}$$

One can easily verify that $\tau^\mu = (1, 0, 0, 0)$ satisfies (14), and that the corresponding $\mathcal{B}^\mu_\nu = \text{diag}\ (0, 1, 1, 1)$. In order to simplify calculation of v_c we consider a particle of the cluster in equatorial motion: then its

$$u^\mu = (u^0, 0, 0, u^3) \; ,$$

where due to the restriction imposed by (13)

$$(u^0)^2 = A^{-1} \left[1 + (r\, u^3)^2 \right] (1 + \omega^2 r^2/c^2)^{-1} \; .$$

Then the geodesic equation (12) gives for $\mu = 1$ after straightforward calculation $|u^3| = \omega/c$ and thus $u^0 = A^{-1/2}$; finally substituting $u^\mu = (A^{-1/2}, 0, 0, \omega/c)$ into (16) and (17) we get

$$v_c = r\omega (1 + \omega^2\, r^2/c^2)^{-1/2} \; .$$

Thus we see that for $\omega r/c \gg 1$ we have $v_c \to c$ and for $\omega r/c \ll 1$, $v_c \to r\omega$, so that ω is angular velocity in that limit.

We have shown already that for any equatorial particle $|u^3| \equiv |d\phi/ds| = \omega/c$, irrespective of radial distance from the center of symmetry. Since the radius

$$a_{min} = 3\ r_s/2$$

where $r_s = 2\ G\ m/c^2$ is the Schwarzschild radius associated to m.

DISCUSSION

We see that near the origin ρ and g_{11} tend to their classical values, the same happening to g_{00} in the case of "slow" ($\omega a/c << 1$) clusters. With increasing distance inside the cluster we have a decreasing ρ and increasing g_{00} and $|g_{11}|$; on the boundary we have the values

$$g_{00} = |g_{11}|^{-1} = (1+\omega^2 a^2/c^2)(1+3\ \omega^2 a^2/c^2)\ .$$

Outside the cluster we have $g_{00} = |g_{11}|^{-1}$ tending monotonically to their Minkowski value at infinity.

Slow clusters ($\omega a/c << 1$) have a total mass which is approximately proportional to the cube of the radius and to the square of the frequency of its particles: and "fast" clusters have total mass nearly independent of ω and linearly proportional to the radius.

From equation (21) it is evident that Schwarzschild singularity does not appear in any region of the bounded cluster in striking constrast to the incompressible fluid sphere of Schwarzschild [3].

Very shrinked clusters ($a \to 3Gm/c^2$) have finite mass m, with the density of mass ρ showing strong concentration near the center, and with particles having classical velocity near c, except those near the center.

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