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GAUGE TRANSFORMATIONS AND GENERALIZED MULTIPOLE MOMENT OPERATORS *

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ABSTRACT

A space-time dependent gauge transformation of a non-hermitian field is assumed to induce a unitary transformation in Hilbert space. This is determined by a linear functional of the gauge function which is regarded as a test function in the sense of Laurent Schwartz. It is shown that the generators of the corresponding infinitesimal transformations are multipole moment operators; the commutation rules of these operators with the field are automatically obtained and are valid for any non-hermitian field. The commutators of these moments with one another are also written down.

These operators might have a meaning in the quark theory of elementary particles where a meson, which has vanishing baryonic number, might have a baryonic multipole moment since it is a set of quarks with opposite baryonic charges.

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1. UNITARY IMPLEMENTATION IN HILBERT SPACE OF GAUGE TRANSFORMATIONS

It is well known that any non-hermitian field operator $\psi(x)$ may undergo a phase transformation of the type

$$\psi'(x) = e^{i\alpha}\psi(x) \tag{1}$$

where α is a real number, all physical theories which involve this field being invariant under this group (1).

The unitary implementation in Hilbert space corresponding to equation (1) is

$$\psi'(x) = U^{+}(\alpha)\,\psi(x)\,U(\alpha)$$

where

$$U(\alpha) = e^{i\alpha Q}$$

and Q is the charge operator, $Q^{+} = Q$.

In the case of infinitesimal transformations, the equation:

$$e^{-i\alpha Q}\,\psi(x)\,e^{i\alpha Q} = e^{i\alpha}\,\psi(x) \tag{2}$$

reduces to

$$(I - i\alpha Q)\,\psi(x)(I + i\alpha Q) = (1 + i\alpha)\,\psi(x)\ .$$

Whence the well known commutation relations which characterise the charge operator:

$$\begin{aligned} [\psi(x), Q] &= \psi(x) \\ [\psi^{+}(x), Q] &= -\psi^{+}(x)\ . \end{aligned} \tag{3}$$

In this note we examine the question if it is possible to define an analogous unitary implementation in Hilbert space when the gauge transformation is of the type:

$$\psi''(x) = e^{i\Lambda(x)}\,\psi(x) \tag{4}$$

that is, when the constant α in equation (1) is replaced by a real function $\Lambda(x)$.

We shall make a few assumptions: 1) The function $\Lambda(x)$ (which belongs to the class C^{∞}) is a test function in the sense of L. Schwartz. 2) We assume that it is possible to define a current vector $j^{\mu}(x)$. 3) We define a distribution:

$$J[x^o; \Lambda] = \int \Lambda(x)\, j^o(x)\, d^3x \tag{5}$$

which is a linear functional of Λ and depends on the time x^o.

Let us further assume that $\Lambda(x)$ vanishes outside of a small set around the origin in three-dimensional space.

$$\Lambda(\vec{x}, x^o) = 0, \quad |\underset{\sim}{x}| > a \tag{6}$$

where $\underline{a}$ is a small radius. We can thus develop $\Lambda(x)$ around the origin:

$$\Lambda(\vec{x},x^o) = \Lambda(0,x^o) + x^k\Lambda_{,k}(0,x^o) + \frac{x^k x^\ell}{2!}\Lambda_{,k,\ell}(0, x^o) + \ldots \tag{7}$$

An operator is now defined

$$\mathcal{U}[x^o; \Lambda] = e^{iJ[x^o;\Lambda]} \tag{8}$$

and we assume it to determine the unitary implementation in Hilbert space of the symmetry corresponding to the group of transformations (4). That is, we assume that the generalization of equation (2) for the group of transformation (4) is the following:

$$e^{-iJ}\,\psi(x)\,e^{iJ} = e^{i\Lambda(x)}\,\psi(x) \tag{9}$$

where J is defined in equation (5).

In view of the conditions (6) imposed on the test function $\Lambda(x)$ we suppose that the charge density $j^o(x)$ makes the integral (5) such that in the exponential (8) we may neglect terms of higher order. This means that $\Lambda(x)$ contains a factor ϵ and that one is allowed to neglect second and higher powers of ϵ in equation (9). Therefore, for an infinitesimal transformation we approximate equation (9) by the following one:

$$\left\{1 - iJ[x^o;\Lambda]\right\} \psi(x) \left\{1 + iJ[x^o;\Lambda]\right\} = (1 + i\Lambda(x)\,\psi(x) \ .$$

The series (7) in this equation gives then rise to the following commutation rules:

$$[\psi(x),\, Q] = \psi(x)$$
$$[\psi(x),\, Q^k] = x^k\psi(x)$$
$$[\psi(x),\, Q^{k\ell}] = x^k x^\ell \psi(x)$$

where the Q's are:

a) the charge operator of the field:

$$Q = \int j^o(x)\, d^3x \qquad (10)$$

b) the dipole moment:

$$Q^k = \int x^k\, j^o(x)\, d^3x \qquad (10a)$$

c) the quadrupole moment $Q^{k\ell}$ operator (with non-vanishing trace):

$$Q^{k\ell} = \int x^k x^\ell\, j^o(x)\, d^3x \qquad (10b)$$

etc.

2. GENERALIZED MULTIPOLE MOMENT OPERATORS

This association of multipole moment operators and a space-time dependent gauge transformation may be easily generalized to a set of non-hermitian fields $\psi_A(x)$, where the label A refers to internal symmetry as well as to space-time components of tensor or spinor fields. This is accomplished by considering the following infinitesimal gauge transformation [1].

$$\psi_A(x) \rightarrow \psi'_A(x) = (\delta_{AC} + i\, C_{ABC}\, \Lambda_B(x))\, \psi_C(x) \tag{11}$$

The corresponding unitary implementation in Hilbert space will be assumed to be of the form:

$$\left\{I - i\int \Lambda_B(x)\, j^o_B(x)\, d^3x\right\} \psi_A(x) \left\{I + i\int \Lambda_B(x)\, j^o_B(x)\, d^3x\right\} = (\delta_{AC} + iC_{ABC}\, \Lambda_B(x))\, \psi_C(x) \tag{12}$$

The development of $\Lambda_B(x)$ around the origin in 3-dimensional space for all times gives us:

$$\left\{I - i\Lambda_B(0,x^o)Q_B - i\Lambda_{B,k}(0,x^o)Q^k_B - \tfrac{1}{2}\Lambda_{B,k,\ell}(0,x^o)Q^{k\ell}_B \ldots\right\} \psi_A(x) \left\{I + i\Lambda_B(0,x^o)Q_B + i\Lambda_{B,k}(0,x^o)Q^k_B + \tfrac{1}{2}\Lambda_{B,k,\ell}(0,x^o)Q^{k\ell}_B + \ldots\right\} =$$

$$\left\{\delta_{AC} + i\, C_{ABC}(\Lambda_B(0,x^o) + x^k \Lambda_{B,k}(0,x^o) + \tfrac{1}{2} x^k x^\ell \Lambda_{B,k,\ell}(0,x^o) + \ldots)\right\} \psi_C(x)$$

whence the commutation rules:

$$\begin{aligned} \left[\psi_A(x),\, Q_B\right] &= C_{ABC}\, \psi_C(x) \\ \left[\psi_A(x),\, Q^k_B\right] &= C_{ABC}\, x^k\, \psi_C(x) \\ \left[\psi_A(x),\, Q^{k\ell}_B\right] &= C_{ABC}\, x^k\, x^\ell\, \psi_C(x) \end{aligned} \tag{13}$$

etc....

The operators

$$Q_B = \int j_B^o(x)\ d^3x$$
$$Q_B^k = \int x^k\ j_B^o(x)\ d^3x \qquad (14)$$
$$Q_B^{k\ell} = \int x^k x^\ell\ j_B^o(x)\ d^3x$$

are the generalizations of the multipole moment operators (10). They may be associated not only to a system of electric charges but also to systems with baryon number, lepton number, etc. Thus, a positive pion is formed, according to the quark model, of a p_o - quark and a $\bar{n}_o$-quark:

$$\pi^+ \sim \bar{n}_o\ p_o\ .$$

The baryonic num[illegible] of the corresponding field is zero. However, this meson, regarded as a set of two entities with baryonic numbers 1/3 and - 1/3 respectively could well have a baryonic dipole moment or a higher order baryonic multipole moment. Clearly, these are operators which refer to static multipole moments. If the current $j_B^\mu(x)$ is conserved then the charge Q_B is time-independent (provided of course that $j_B^k(x)$ vanishes at infinity, k = 1, 2, 3). In this case, the charge Q_B may be written

$$Q_B = \int_\sigma j_B^\mu(x)\ d\,\sigma_\mu \qquad (15)$$

as an integral over a space-like surface σ and Q_B is σ-independent because:

$$\frac{\delta Q_B}{\delta\sigma(x)} = \frac{\partial\, j_B^\mu(x)}{\partial\, x^\mu} = 0\ . \qquad (16)$$

Hence Q_B is a Poincaré scalar.

The higher order multipole moments are tensors in three-dimensional space. This comes from the fact that the integral (5) refers to an instant X^o.

A relativistic generalization of equation (5) which suggests itself would be an integral of the type:

$$J\,[\sigma; \Lambda] = \int_{\sigma} \Lambda_B(x)\; j_B^{\alpha}(x)\; d\sigma_{\alpha} \tag{17}$$

where σ is a space-like surface.

Equation (5) would be a special case of equation (17) for a surface σ perpendicular to the time-axis if $J[\sigma; \Lambda]$ were σ-independent:

$$\frac{\delta J[\sigma;\Lambda]}{\delta\sigma(y)} = \frac{\partial}{\partial y^{\alpha}}\left(\Lambda_B(y)\; j_B^{\alpha}(y)\right) = 0\ . \tag{18}$$

3. COMPARISON WITH THE LAGRANGEAN FORMALISM

Let us now assume the existence of an effective lagrangean density:

$$\mathcal{L} = \mathcal{L}\left(\psi_A(x), \frac{\partial\psi_A(x)}{\partial x^{\mu}}\right)\ . \tag{19}$$

The transformation (11) and its associate:

$$\frac{\partial\psi_A(x)}{\partial x^{\mu}} \longrightarrow \frac{\partial\psi'_A(x)}{\partial x^{\mu}} = \frac{\partial\psi_A(x)}{\partial x^{\mu}} + i\; C_{ABC}\,\frac{\partial}{\partial x^{\mu}}\left(\Lambda_B(x)\,\psi_C(x)\right) \tag{20}$$

induces a change in the lagrangean density by the amount:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi_A(x)}\,\delta\psi_A(x) + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi_A(x)}{\partial x^\mu}\right)}\ \frac{\partial}{\partial x^\mu}\ \left(\delta\psi_A(x)\right)$$

which, in view of Lagrange's equations:

$$\frac{\partial\mathcal{L}}{\partial\psi_A(x)} = \frac{\partial}{\partial x^\mu}\ \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi_A}{\partial x^\mu}\right)}$$

may be written:

$$\delta\mathcal{L} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi_A}{\partial x^\mu}\right)}\ \delta\psi_A\right) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi_A}{\partial x^\mu}\right)}\ i\,C_{ABC}\,\Lambda_B(x)\,\psi_C(x)\right)$$

If we define the vector-current by the relation:

$$j_B^\mu(x) = i\,\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi_A}{\partial x^\mu}\right)}\ \ C_{ABC}\ \ \psi_C(x)$$

we obtain for the change in the lagrangean density:

$$\delta\mathcal{L} = \frac{\partial}{\partial x^\mu}\ \left(\Lambda_B(x)\ j_B^\mu(x)\right) \tag{21}$$

The invariance of the lagrangean by the transformation (11) and (20) would thus be equivalent to assuming that the integral (17) be σ-independent, equation (18).

We thus see that:

$$\delta\mathcal{L} = \frac{\delta J[\sigma;\Lambda_B]}{\delta\sigma(x)}$$

and that, in general:

$$j^{\mu}_{B}(x) = \frac{\partial}{\partial\left(\frac{\partial \Lambda_B}{\partial x^{\mu}}\right)} \frac{\delta J[\sigma; \Lambda_B]}{\delta\sigma(x)}$$

$$\frac{\partial j^{\mu}_{B}(x)}{\partial x^{\mu}} = \frac{\partial}{\partial \Lambda_B(x)} \frac{\delta J[\sigma; \Lambda_B]}{\delta\,\sigma(x)} \tag{22}$$

These are the well-known Gell-Mann-Lévy equations.

Equation (18) is therefore not generally valid.

If we develop $\Lambda_B(x)$ in a power series around the origin we obtain:

$$\delta \mathcal{L} = \Lambda_B(0) \frac{\partial j^{\mu}_{B}(x}{\partial x^{\mu}} + \Lambda_{B,\alpha}(o) \frac{\partial}{\partial x^{\mu}} \left(x^{\alpha} j^{\mu}_{B}(x)\right) +$$

$$+ \frac{1}{2!} \Lambda_{B,\alpha,\lambda}(o) \frac{\partial}{\partial x^{\mu}} \left(x^{\alpha} x^{\lambda} j^{\mu}_{B}(x)\right) + \dots \tag{23}$$

The invariance of the lagrangean density is assured in the case in which $\Lambda_B(x)$ is a constant (which is not a test function) and is equivalent to the existence of a conserved vector current.

The non-invariance of the lagrangean density for a transformation of the type (4) or of the type (11) may be interpreted by the statement that non-vanishing divergences of currents of the form $x^{\alpha} x^{\beta} \dots j^{\mu}_{B}(x)$ are added to the first, usual term $\Lambda_B(o) \frac{\partial j^{\mu}_{B}}{\partial x^{\mu}}$ as shown in equation (23).

4. COMMUTATION RULES OF THE GENERALIZED MULTIPOLE MOMENT OPERATORS

Let us require that the generalized charges Q_A, as defined in equation (14), form a Lie Algebra:

$$[Q_A, Q_B] = i f_{ABC} Q_C \qquad (24)$$

where f_{ABC} are the structure constants of the group. This will be satisfied by the well-known equal-time commutation relation of the charge densities:

$$\left[j_A^o(x), j_B^o(y)\right]_{x^o=y^o} = i f_{ABC} j_C^o(x) \, \delta(\underset{\sim}{x} - \underset{\sim}{y}) \, . \qquad (25)$$

If then follows from equation (14) and (25) that the multipole moment operators obey the following commutation rules:

$$[Q_A, Q_B^k] = i f_{ABC} Q_C^k$$

$$[Q_A, Q_B^{k\ell}] = i f_{ABC} Q_C^{k\ell}$$

$$[Q_A, Q_B^{k\ell\ldots p}] = i f_{ABC} Q_C^{k\ell\ldots p}$$

$$[Q_A^k, Q_B^\ell] = i f_{ABC} Q_C^{k\ell}$$

$$[Q_A^k, Q_B^{\ell m\ldots p}] = i f_{ABC} Q_C^{k\ell m\ldots p}$$

and so on. This set of operators is closed under the commutation rule.

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REFERENCES

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