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ON THE INTERACTION OF THE GRAVITATIONAL AND SPINOR FIELDS

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INTRODUCTION

The problem of the interaction between spinor fields and gravitation has been treated by some authors. [1] Presently we use the simplified version of this problem given by Dirac [2] which uses the Hamiltonian formulation and a conveniently oriented set of tetrads.

The dynamical variables for the interacting system are not all independent, but they are related by a set of four secondary constraints [3] and by three primary constraints. As it is known these constraints are the generators of infinitesimal transformations under which the Lagrangian of the system is invariant. The primary constraint represents the generator of local Lorentz transformations, whereas the four secondary constraints, the so called Hamiltonian constraints, are the generators of infinitesimal coordinate transformations.

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The coupled system is studied from the point of view of the three-dimensional geometry and under the condition that the gravitational field is a linear field (weak field approximation); the transformation law for the spinor field is then interpreted as a gauge transformation in the three-dimensional hypersurface of constant time where the state of the system is given. An expression for a gauge invariant quantity involving the spinor field is derived, this expression is a linear functional of the spinor field.

The determination of a localized spinor quantity which is invariant under these gauge transformations calls first for the determination of the explicit form of the spinor wave function of the spin 1/2 particle in interaction with the gravitational field, that is, is a property which depends on the behaviour of the gravitational potentials.

The notation which will be used is the following: coordinate indices will be denoted by greek letters, they run from 0 to 3. Local or tetrad indices will be denoted by greek letters inside a bracket, these also run from 0 to 3. In the three-dimensional formulation of the system we will use small latin letters to denote the coordinate indices going from 1 to 3, and latin letters inside a bracket to denote the three-dimensional tetrad indices. Spinor indices will be denoted with capital latin letters, we will use only the four component spinor formalism.

Finally, the metric tensor will be used with the oo component negative, that is g_{oo} and the local component $\overset{\circ}{g}_{oo}$ are chosen to be negative, the last one being simply equal to -1.

1. THE LAGRANGIAN FOR THE COUPLED SYSTEM

It is known that in order to take into account the gravitational interaction of the spinor field it is necessary to associate to each point of the four dimensional Riemann space a system of tetrads [4] (or local Lorentz frames) to which the spinor is referred. From this consideration it is possible to derive the equation for the spin 1/2 field in interaction with gravitation, the so called generalized Dirac equation [5].

The tetrad vectors satisfy the following relations

$$h^{\mu}_{(\alpha)} h_{\mu(\beta)} = \overset{\circ}{g}_{\alpha\beta}, \qquad \overset{\circ}{g}^{\alpha\beta} h_{\mu(\alpha)} h_{\nu(\beta)} = g_{\mu\nu} \qquad (1)$$

where $\overset{\circ}{g}_{\alpha\beta}$ is the local (Galilean) metric tensor with signature +2.

They also satisfy

$$h_{\mu}{}^{(\alpha)} = \overset{\circ}{g}^{\alpha\beta} h_{\mu(\beta)}, \qquad h^{\mu}_{(\alpha)} = g^{\mu\lambda} h_{\lambda(\alpha)}$$

The Lagrangian which gives rise to the equation for the spin 1/2 particle in interaction with the gravitational field may be written in the form [2]

$$L_S = \int \mathcal{L}_S \, d_3x$$

$$\mathcal{L}_S = J h^{\mu}_{(\alpha)} \varphi \alpha^{(\alpha)} \psi_{,\mu} + \frac{1}{2}\left(J h^{\mu}_{(\alpha)}\right)_{,\mu} \varphi \alpha^{(\alpha)} \psi -$$

$$- \frac{1}{4} J h^{\mu}_{(\alpha)} h^{\nu}_{(\beta)} h_{\nu(\gamma),\mu} \varphi A\left(\alpha^{(\alpha)}\beta \alpha^{(\beta)} \beta \alpha^{(\gamma)}\right) \psi - imJ\varphi\beta\psi \qquad (2)$$

where the subscript S stands for "Spinor part", and the comma denote the usual partial derivative. The meaning of some symbols introduced in Eq. (2) is as follows

$$J^2 = -|g_{\mu\nu}| = |h_{\mu(\alpha)}|^2$$

$$A\left(\alpha^{(\alpha)}{}_{\beta}\,\alpha^{(\beta)}{}_{\beta}\,\alpha^{(\gamma)}\right) = \frac{1}{2}\left(\alpha^{(\alpha)}{}_{\beta}\alpha^{(\beta)} - \alpha^{(\beta)}{}_{\beta}\,\alpha^{(\alpha)}\right)_{\beta}\alpha^{(\gamma)} -$$

$$- \overset{\circ}{g}{}^{\alpha\gamma}\,\alpha^{(\beta)} + \overset{\circ}{g}{}^{\beta\gamma}\,\alpha^{(\alpha)}\,,$$

$$\varphi = i\psi^{+},\ \alpha^{(\alpha)} = \beta\gamma^{(\alpha)},\ \ \beta = \gamma^{(0)}\ .$$

The Lagrangian density for the gravitational field is the Einstein's Lagrangian density, where the surface term has been dropped

$$\mathcal{L}_G = Jg^{\mu\nu}\left(\begin{Bmatrix}\sigma\\ \mu\rho\end{Bmatrix}\begin{Bmatrix}\rho\\ \nu\sigma\end{Bmatrix} - \begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \rho\sigma\end{Bmatrix}\right)\ . \qquad (3)$$

$\begin{Bmatrix}\sigma\\ \mu\rho\end{Bmatrix}$ is the Christoffel symbol build up with the metric field $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

2. STRUCTURE OF THE DYNAMICAL VARIABLES IN THE TETRAD FORMULATION

From here on we adopt the point of view that the tetrad vectors are the fundamental variables for the gravitational field, so that the metric field $g_{\mu\nu}$ is defined by the Eq. (1) in terms of these tetrads.

In the Hamiltonian formulation for the interacting field we must specify the state of the whole system on the points x^1, x^2, x^3 of a given three-dimensional space-like hypersurface which is embeded in the four-dimensional Riemann space. In other words, we need to give the set of canonical variables $q(\vec{x})$, $p(\vec{x})$ at that "instant of time".

The three-dimensional hypersurface is characterized by the unit time-like normal

$$\ell^{\rho} = \frac{g^{\rho o}}{\sqrt{-g^{oo}}}\,, \qquad \ell_{\rho} = g_{\rho\lambda}\,\ell^{\lambda} \tag{4}$$

Given a four-vector V^{α}, we can always associate to it another four-vector W^{α} which has a vanishing time-component,

$$W^{\alpha} = \pi^{\alpha}_{\lambda}\, V^{\lambda} \tag{5.1}$$

$$\pi^{\alpha}_{\lambda} = \delta^{\alpha}_{\lambda} + \ell^{\alpha}\ell_{\lambda} \tag{5.2}$$

The quantities π^{α}_{λ} are the projection operators for the three-dimensional geometry,

$$\pi^{\alpha}_{\lambda}\, \pi^{\lambda}_{\gamma} = \pi^{\alpha}_{\gamma} \tag{5.3}$$

The vector $W^{\alpha}(x)$ is tangent to some curve lying in the hyper-surface x^{o} = constant.

Given the four covariant tetrad vectors $h_{\mu(o)}$; $h_{\mu(1)}$; $h_{\mu(2)}$ and $h_{\mu(3)}$ at the point x^{μ}, we can make the decomposition of these sixteen quantities in the form $h_{r(s)}$, $h_{r(o)}$, $h_{o(o)}$ and $h_{o(i)}$. The four components $h_{\mu(o)} = \left(h_{r(o)},\, h_{o(o)}\right)$ form a four-vector field in the Riemann space; we take this four-vector so as to define locally the space-like hypersurface [2]

$$h^{(o)}_{\mu} = \ell_{\mu} \tag{6}$$

with this condition we have that

$$h^{(o)}_{i} = -\, h_{i(o)} = 0 \tag{7.1}$$

$$h_{o(o)} = -\, h^{(o)}_{o} = -\frac{1}{\sqrt{-g^{oo}}} \tag{7.2}$$

The quantities $h_{r(s)}(x)$ depend only on variables characterizing the three-dimensional hypersurface;* in more specific terms,

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* **Another example of such quantities is given by the vector W^{α} of Eq. (5-1).**

they are invariant under any coordinate transformation which keep unchanged the hypersurface x^{o} = constant but which changes arbitrariely the adjacent hypersurface $x^{o} + \varepsilon$ = constant, where ε is an infinitesimal. Quantities with such behavior will be called D-invariants [4].

Of the above sixteen $h_{\mu(\nu)}$ it remains to be discussed only the four quantities $h_{o(\mu)}$, these variables form a local four-vector (respect to local Lorentz transformations), they are not D-invariants.

From the relation,

$$h^{o(\mu)} h_{i(\mu)} = 0$$

and using (7.1) we conclude that

$$h^{o(i)} = 0 \tag{8}$$

Now, we analyze the structure of the contravariant tetrad vectors. From Eq. (8) it follows that the three four-vectors $h^{\rho(i)}$ have vanishing time component; therefore, with the choice given by (6) the $h^{\rho(i)}$ are automatically D-invariant, *

$$\tilde{h}^{\rho(i)} = \pi^{\rho}{}_{\mu}\, h^{\mu(i)} = h^{\rho(i)}$$

With the conditions (7.1) and (8) we have dropped eight of the original set of $h_{\mu(\nu)}$ and $h^{\mu(\nu)}$, it remains as independent variables

$$h_{\mu(\alpha)} = \left(h_{i(s)},\ V_{(\mu)} = h_{o(\mu)} \right)$$

all the remaining can be written in terms of these,

* **We can also see this from the fact that,**

$$h^{(o)}_{\mu}\, h^{\mu(i)} = 0$$

since $h^{(o)}_{\mu}$ is normal to the hypersurface, it follows that $h^{\mu(i)}$ are a set of three vectors on the hypersurface.

$$h^{i(o)} = - h^{i(j)} V_{(j)} V^{-1}_{(o)} \tag{9.1}$$
$$h^{o(o)} = V^{-1}_{(o)} \tag{9.2}$$
$$h^{i(1)} = D^{-1} \varepsilon^{ijk} h_{j(2)} h_{k(3)} \tag{9.3}$$
$$h^{i(2)} = D^{-1} \varepsilon^{ijk} h_{j(3)} h_{k(1)} \tag{9.4}$$
$$h^{i(3)} = D^{-1} \varepsilon^{ijk} h_{j(1)} h_{k(2)} \tag{9.5}$$

where

$$D = \varepsilon^{ijk} h_{i(1)} h_{j(2)} h_{k(3)}$$

The Eqs. (9-3) through (9-5) are a consequence of the relation

$$h_{\mu(\alpha)} h^{\mu(\beta)} = \delta^{(\beta)}_{(\alpha)}$$

or equivalently of,

$$h_{i(j)} h^{i(k)} = \delta^{(k)}_{(j)}$$

These equation emphasize the fact that the $h^{i(s)}$ being D-invariant can be written as function of the D-invariant variable $h_{i(s)}$.

The spatial covariant component of the metric, the g_{rs} (which are D-invariant) are written in term of the $h_{r(i)}$ by

$$g_{rs} = h_{r(i)} h_{s(i)}$$

The inverse matrix, usually denoted by e^{rs}

$$e^{rs} = g^{rs} - \frac{g^{or} g^{os}}{g^{oo}}$$

$$e^{rs} g_{sk} = \delta^{r}_{k}$$

is presently,

$$e^{rs} = h^{r}{}_{(k)} h^{s}{}_{(k)}$$

Now, we proceed to introduce the canonical momenta. Calling by $\mathcal{L}$ the total Lagrangian density, we define the momenta as usually by means of

$$\pi^r_{(i)} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{r(i)}} \tag{12}$$

$$\chi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = K\varphi \tag{13}$$

$$\theta = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0 \tag{14}$$

where,

$$K = \frac{1}{3!} D = |h_{r(i)}| \tag{15}$$

The expression for $\pi^r{}_{(i)}$ being

$$\pi^r{}_{(i)} = 2J\left(e^{ru}\, h^v_{(i)} - e^{uv}\, h^r{}_{(i)}\right)\left\{ {o \atop uv} \right\} + \frac{1}{2} K\, h^r_{(i)}\, \varphi\psi +$$

$$+ \frac{1}{8} K\, h^r{}_{(\ell)}\, \varphi\left(\alpha_{(\ell)}\, \alpha_{(i)} - \alpha_{(i)}\, \alpha_{(\ell)}\right)\psi \tag{16}$$

From Eqs. (13-14) we can eliminate the pair of variables φ, θ from the theory,

$$\varphi = \frac{1}{K}\chi$$

it remains the pairs $h_{r(i)}$, $\pi^r_{(i)}$ and ψ , χ. With these we can define the canonical Poisson brackets,

$$\left[h_{r(i)}(\vec{x}, x^o), \pi^s_{(k)}\ (\vec{x}', x^o)\right] = \delta_{(i)(k)}\, \delta^s_r\, \delta(\vec{x}, \vec{x}') \tag{17}$$

$$\left[\psi_A(\vec{x}, x^o), \chi_B(\vec{x}', x^o)\right] = \delta_{AB}\, \delta(\vec{x}, \vec{x}')$$

3. THE CONSTRAINTS AND THE HAMILTONIAN

As consequence of the relation (16) it is possible to prove that the canonical variable are subjected to three constraints, [2]

$$M_{(i)(k)} = \pi^r_{(i)}\, h_{r(k)} - \pi^r_{(k)}\, h_{r(i)} + \frac{1}{4}\chi\left(\alpha_{(i)}\, \alpha_{(k)} - \alpha_{(k)}\, \alpha_{(i)}\right)\psi = 0, \tag{19}$$

which are primary constraints (they come from the property of invariance of the total Lagrangian density under rotations of the three-legs $h_{r(k)}$ in the hypersurface).

As in the problem for free gravitational fields,[3] there exists four constraints coming as result of four of the field equations, namely the equations,

$$G_{\mu o} - T_{\mu o} = 0 \;*$$

These constraints are called secondary constraints. It can be shown that the existence of these constraints is associated to the invariance of the total Lagrangian density under coordinate transformations.[7]

The explicit form for these constraints is presently

$$\mathcal{H}_s = \mathcal{H}_{sG} + \mathcal{H}_{sS} = 0 \qquad (20)$$

$$\mathcal{H}_L = \mathcal{H}_{LG} + \mathcal{H}_{LS} = 0 \qquad (21)$$

where,

$$\mathcal{H}_{sG} = p^{uv}\, g_{uv,s} - 2(p^{uv}\, g_{us}),\, v \qquad (22)$$

$$\mathcal{H}_{LG} = K^{-1}\left(p^{rs}\, p_{rs} - \frac{1}{2}\, p^r_r\, p^s_s\right) + K e^{rs}\, S_{rs} \qquad (23)$$

with

$$p^{rs} = \frac{1}{4}\left(\pi^r_{(i)}\, h^s_{(i)} + \pi^s_{(i)}\, h^r_{(i)}\right) - \frac{1}{4}\, e^{rs} \chi\, \psi \qquad (24)$$

$$p_{rs} = g_{ri}\, g_{sj}\, p^{ij}$$

In eq. (23) the symbol S_{rs} denotes the Ricci tensor build up with the three-dimensional metric g_{rs} and its inverse e^{rs}. Both of these are in turn given in function of the present set of variables by the use of Eqs. (10), (11) and (9-3) through (9-5).

$$\mathcal{H}_{sS} = \frac{\partial \psi}{\partial x^s} - \frac{1}{2}\, K \frac{\partial}{\partial x^s}\, (K^{-1} \chi\, \psi) +$$

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* $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R$ is the Einstein's tensor.

$$+\frac{1}{8}\, h^{u}_{(i)} \frac{\partial h_{u(j)}}{\partial x^{s}} \chi \left(\alpha_{(i)}\, \alpha_{(j)} - \alpha_{(j)}\, \alpha_{(i)}\right)\psi -$$

$$-\frac{1}{8}\frac{\partial}{\partial x^{u}} \left\{ h^{u}_{(i)}\, h_{s(j)} \chi \left(\alpha_{(i)}\, \alpha_{(j)} - \alpha_{(j)}\, \alpha_{(i)}\right)\psi \right\}, \qquad (26)$$

$$\mathcal{H}_{LS} = - h^{i}_{(j)} \chi\, \alpha_{(j)} \frac{\partial \psi}{\partial x^{i}} + \frac{1}{2} K\, h^{i}_{(j)} \frac{\partial}{\partial x^{i}} \left(K^{-1} \chi \alpha_{(j)} \psi \right) +$$

$$+ i m \beta \psi - \frac{1}{24}\, \varepsilon_{(i)(j)(k)}\, h^{r}_{(i)}\, h^{u}_{(j)}\, h_{u(k),\, r} \chi \alpha(1)\, \alpha(2)\, \alpha(3) \qquad (27)$$

The Hamiltonian is a linear combination of these constraints,

$$H = \int d_3 \left[(-g^{oo})^{-\frac{1}{2}} \mathcal{H}_L + g_{so}\, e^{rs} \mathcal{H}_r \right]. \qquad (28)$$

4. THE LINEAR STRUCTURE FOR THE COUPLED SYSTEM

The variation of the components $\psi_A(x)$ of the spinor field at the point x^u induced by the generator

$$G = - \int \Lambda^{r} \mathcal{H}_r \, d_3 x \, * \qquad (29)$$

is given by the Poisson bracket,

$$\delta \psi_A(x) = \left[\psi_A(x),\, G \right] . \qquad (30)$$

As it is known, the functional G of the canonical variables, is the generator of infinitesimal coordinate transformations for the three-dimensional geometry.

In the weak field approximation, up to first order terms, we get the following expression for the Poisson bracket of g_{rs} with G,

$$\delta g_{rs}(x) = \left[g_{rs}(x),\, G \right] = -\Lambda_{r;s}(x) - \Lambda_{s,r}(x) . \qquad (31)$$

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* Λ^r are first order infinitesimals.

Thus, we can think of G as the generator of gauge transformations * (this holds only in the weak field approximation) on the gravitational potentials, and to consider $\delta\psi$ of Eq. (30) as a gauge transformation on the components of ψ.

The explicit expression for this variation is (in the complete theory)

$$-\delta\psi = \Lambda^r\ \psi_{,r} + \frac{1}{8}\Lambda^r\ h^u_{(i)}\ h_{u(j),r}\ \alpha_{[(i)}\ \alpha_{(j)]}\ \psi + \frac{1}{8}\Lambda^r{}_{,u}\ h^u_{(i)}\ h_{r(j)}\ \alpha_{[(i)}\ \alpha_{(j)]}\ \psi \tag{32}$$

In the weak field approximation,

$$h_{u(j)} = \overset{\circ}{g}_{uj} + k\ \gamma_{uj}$$
$$h^u_{(j)} = \delta^\mu_j - k\ \gamma_j{}^u$$
$$\gamma_j{}^u = \overset{\circ}{g}_{jk}\ \gamma^{ku}$$

(k a first order infinitesimal).

Retaining only the first order terms, we obtain for $\delta\psi$,

$$-\delta\psi \simeq \Lambda^r\ \psi_{,r} + \frac{1}{8}\Lambda_{i,j}\ \alpha_{[j}\ \alpha_{i]}\ \psi \tag{33}$$

As in the corresponding problem for electromagnetic fields,[8] we try to find out a gauge invariant quantity build up with the components of the spinor field.

We define the quantity,

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* **These gauge transformations are defined on points of the three-dimensional hypersurface.**

$$\psi^* = \int \psi(x)\, e^{iC(x)}\, d_3x \qquad (34)$$

where

$$C(x) = \int d_3 x' \left[C^s{}_{(j)} (x,x')\, h_{s(j)}(x') + \frac{1}{m}\, b^{[sr]}_{(j)}(x,x') \cdot \left(\frac{\partial h_{s(j)}}{\partial x^r} - \frac{\partial h_{r(j)}}{\partial x^s} \right) \right] \qquad (35)$$

(m is the mass of the spinor particle).

The coefficients $C^s{}_{(j)}$ and $b^{[sr]}_{(j)}$ are to be determined so that ψ^* has a null Poisson bracket * (up to first order terms) with the generator G.

$$\delta\psi^* = 0$$

With this requirement, it is possible to show that, **

$$C^{sj}(\vec{x},\vec{x'}) = i g^{osj}\, \delta(\vec{x},\vec{x'}) \qquad (36)$$

$$b^{[sr]}_{j,j}(\vec{x},\vec{x'}) = \frac{im}{8}\, \alpha^{[s}\, \alpha^{r]}\, \delta(\vec{x},\vec{x'}) \qquad (37)$$

The Eq. (37) giving the divergence of $b^{[sr]}_j$ is similar to the relation found by Dirac [8] in the correspondent problem for electromagnetic fields.

Thus, a quantity ψ^* of the form given by Eqs. (34), (35), (36) and (37) represents a gauge invariant quantity build up with the spinor field ψ and three-legs $h_{s(j)}$.

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* We also impose that these coefficients are C-number functions (not dependent on the canonical variables).

** Provided that $\wedge^r$ is equal to zero on the boundary of the three-dimensional space.

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