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FERMION-LIKE TREATMENTS OF FERMION PAIRS*

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Summary. An equation is presented by which the energy of many-fermion systems can be calculated in such an approximation as the effect of hole-hole interactions is also included besides the particle-particle interactions considered in the Bethe-Goldstone equation. Though we can correctly take account of the effect of hole motions by the Iwamoto equation when the interactions between fermions are repulsive or weakly attractive, we meet a difficulty that some of the eigen-values become complex when we apply it to a system with strongly attractive interactions. We can avoid this difficulty completely by using our equation.

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1. INTRODUCTION.

Here a new equation is presented by which the energy of many-fermion systems with strong interactions can be calculated in better approximations than by the Bethe-Goldstone equation[1].

Since Brueckner, Levinson and Mahmoud[2] proposed a powerful method for the calculations of the energy of many-fermion systems with strong interactions to which the perturbation method cannot be applied, various mathematical refinements and many applications to practical problems have been done. Their method is essentially the following: A pair of fermions is first picked up arbitrarily from the system and their motion is calculated exactly with consideration of the Pauli principle. Then the shift of the energy of these two fermions from that in case of vanishing interaction is summed up over all possible pairs. The ground state energy of the system is considered to be given by the sum of this total energy shift and the kinetic energies of fermions. The energy shifts due to correlations of more than two fermions are regarded as higher order corrections there. Mathematically, this is accomplished by solving the following Bethe-Goldstone equation:

$$(E - T)\psi = QV\psi , \tag{1}$$

where T is the kinetic energy operator of a pair of fermions, V is the potential energy between them and Q is a projection operator to states in which both fermions are above the fermi sea.

In spite of its usefulness, however, this method bears a short-point, that is, the effects of movements of holes are neglected

there which are not necessarily small. As an example, let us consider the third-order effect. In the Brueckner method, a further transition of a pair of fermions which has been excited above the fermi sea to other states above the fermi sea is taken into account (cf. Fig. 1-a), but the transition of another pair of fermions under the fermi surface to the holes which arose from the jump of the first pair is not taken into account (cf. Fig. 1.b). Consequently we can not expect qualitatively good results.

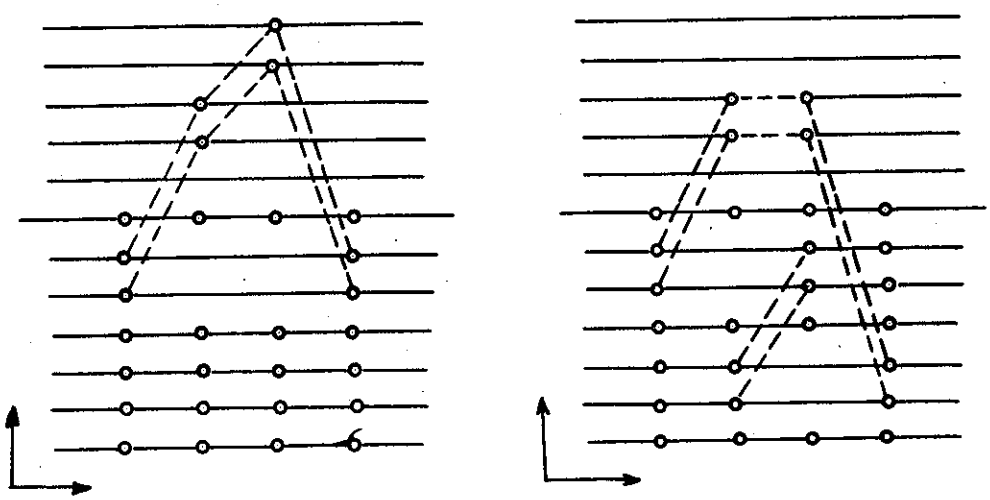

Fig. 1 - Third-order energies. The particle motion (a) is taken into account in the Brueckner theory, but the hole motion (b) is not.

This point was emphasized and improved by Iwamoto[3]. Starting from the usual secondly quantized Hamiltonian, he proceeded with his calculations in an analougous way to those which were developed by Sawada[4] and Bardeen, Cooper and Schrieffer[5] in their theories of the correlation energy of a high density electron gas and the superconductivity, respectively. That is, a product of an appropri

ate pair of creation-, or annihilation-operators for fermions were approximately substituted by a creation-, or annihilation-operator for a single boson. As conclusion, he showed that the ground state energy of the system is calculated with inclusions of the effects of hole motions by solving the following equation instead of the Bethe-Goldstone equation:

$$(E - T)\psi = (Q - P)V\psi, \tag{2}$$

where P is a projection operator to the states in which both particles are under the fermi surface.

However, we meet with a difficulty in solving this equation, when the interactions are strong and attractive. In this case, the above equation has complex eigenvalues and the total ground state energy becomes negatively infinite. This is discussed in detail in Sec. 2. Physically this corresponds to the fact that, when the interaction is attractive and strong, a pair of fermions can form a bound state with negative energy and consequently the lowest energy state is that in which an infinite numbers of pairs which are considered to be bosons enter to this negative energy level.

This difficulty has its origin in that a pair of fermion is regarded as a true boson. Pairs of fermions have a boson-like property in a point that the wave function must be symmetrical under exchanges of any pairs of fermions when the pairs are sufficiently separated. But, they are rather fermion-like when they approach enough, because more than one pair cannot enter into one state. Therefore, the validity that, stressing the symmetry property under the exchange of two pairs in different states, we regard a pair of

fermions as a boson is very doubtful. It may be rather a better approximation when a pair of fermions may form a bound state that, stressing the point that more than one pair of fermions cannot enter to a single state, we regard it as a fermion. The main difficulty with which we meet when we proceed in this manner is that the sign of some non-diagonal matrix elements of potential become incorrect. But, we may overcome this by modifying the potential itself in accordance with a guiding principle that the result must agree with Iwamoto's in the weak coupling limit. In Sec. 3, this consideration is developed. As conclusion, it is shown that the ground state energy of the many-fermion system in an approximation in which the effects of hole motions are also included is calculated by solving the following equation instead of the Bethe-Goldstone equation:

$$(E - T)\psi = (V - 2PVP)\psi . \tag{3}$$

This equation has only real eigen-values and we do not meet with such a trouble as in the case of the Iwamoto equation. In Sec. 4, the approximation used here is discussed.

2. COMPLEX EIGEN-VALUES OF THE IWAMOTO EQUATION.

For completeness and later convenience, we first repeat briefly the Iwamoto theory. The Hamiltonian of a many-fermion system in the secondly quantized form is written as follows:

$$H = \sum \frac{\mathbb{p}_s^2}{2M} c_s^\dagger c_s + \frac{1}{2} \sum V(s,t;u,v)\, c_s^\dagger c_t^\dagger c_v c_u \,, \qquad (4)$$

where $c_s^\dagger$ and c_s are creation-, and annihilation-operators for single fermions respectively, $V(s,t;u,v)$ is the Fourier transform of the interaction potential, M is the mass of the fermions, the suffix on $c^\dagger$ and c, for instance s, denotes the momentum $\mathbb{p}_s$ and the spin state σ_s and the isotopic spin state τ_s of the fermion, and the whole system is enclosed in a large normalization box with volume Ω for simplicity. It is more convenient for us to transform this Hamiltonian into the form analogous to that of Dirac's positron theory. To do this, we introduce new operators a_s, b_s and their complex conjugate which have the following properties:

$$\begin{aligned} &a_s = c_s, \text{ for } |\mathbb{p}_s| > p_F;\ a_s = 0, \text{ for } |\mathbb{p}_s| < p_F, \\ &b_s = c_s^\dagger, \text{ for } |\mathbb{p}_s| < p_F;\ b_s = 0, \text{ for } |\mathbb{p}_s| > p_F. \end{aligned} \qquad (5)$$

Here p_F is the Fermi momentum and in the case of the nuclear matter $p_F = (9\pi/8)^{1/3} r_o^{-1}$ where r_o is the mean interparticle distance. Hereafter we call a_s and b_s the annihilation operators of a 'particle' and 'hole' respectively. Rewriting eq. (4) in term of a_s and b_s, we have

$$H = \sum_{|\mathbb{p}_s| < p_F} \frac{\mathbb{p}_s^2}{2M} + T + V_{SI} + V_I + V_S + V_R + V_{odd}, \qquad (6)$$

where

$$T = \sum \frac{\mathbb{p}_s^2}{2M} \left(a_s^+ a_s - b_s^+ b_s \right), \qquad (7)$$

$$V_{SI} = (\tfrac{1}{2}) \sum V(s,t;u,v) \left[a_s^+ a_t^+ b_v^+ b_u^+ + b_s b_t a_v a_u \right], \qquad (8)$$

$$V_I = (\tfrac{1}{2}) \sum V(s,t;u,v)\left[a_s^+ a_t^+ a_v a_u + b_s b_t b_v^+ b_u^+\right] , \quad (9)$$

$$V_S = (\tfrac{1}{2}) \sum V(s,t;u,v)\left[a_s^+ b_t a_v b_u^+ + b_s a_t^+ b_v^+ a_u\right] , \quad (10)$$

$$V_R = (\tfrac{1}{2}) \sum V(s,t;u,v)\left[a_s^+ b_t b_v^+ a_u + b_s a_t^+ a_v b_u^+\right] , \quad (11)$$

V_{odd} = terms of odd number of creation operators.

According to Brueckner, Iwamoto neglected completely the interaction between holes and particles, that is, he neglected V_S, V_R and V_{odd}, but he kept all other terms. Putting $\mathbb{p}_s = \mathbb{K}+\mathbb{k}$, $\mathbb{p}_t = \mathbb{K}-\mathbb{k}$, $\mathbb{p}_u = \mathbb{K}+\mathbb{k}'$ and $\mathbb{p}_v = \mathbb{K}-\mathbb{k}'$ and limiting the summation over $\mathbb{k}$ and $\mathbb{k}'$ to the half-spaces k_z, $k_z' > 0$, we can write the interaction Hamiltonian in this approximation as follows:

$$H_{int} = \sum_{\mathbb{K}} {\sum_{s,t}}' V(s,t)\left[a_{s+}^+ a_{s-}^+ b_{t-}^+ b_{t+}^+ + b_{s+} b_{s-} a_{t-} a_{t+} + \right.$$

$$\left. + a_{s+}^+ a_{s-}^+ a_{t-} a_{t+} + b_{s+} b_{s-} b_{t-}^+ b_{t+}^+\right] , \quad (13)$$

where s+ denotes a state with momentum $\mathbb{K} + \mathbb{k}_s$, spin state σ_{s+} and isotopic spin state τ_{s+}, s- denotes a state with momentum $\mathbb{K} - \mathbb{k}_s$, spin state σ_{s-} and isotopic spin state τ_{s-}, $\sum_s'$ means the summation over s+ and s- under the restriction that $\mathbb{K}$ is fixed and $k_{sz} > 0$, and

$$V(s,t) = V(s+, s-; t+,t-) - V(s+,s-; t-,t+). \quad (14)$$

The next important point in the Brueckner method is to consider that, once a pair of fermions is excited above the fermi surface, we need to take account of all effects of interactions

between themselves, but the effects of splitting of this pair or interactions between different pairs is negligible. Mathematically, this corresponds to regard a product of a pair of particle operators as a single operator. Extending this idea to the case of a pair of holes, Iwamoto introduced the following operators:

$$A_s(\mathbb{K}) = \begin{cases} a_{s-}\, a_{s+} & \text{for } s > p_F \\ 0 & \text{for other cases,} \end{cases}$$

$$B_s(\mathbb{K}) = \begin{cases} b_{s+}\, b_{s-} & \text{for } s < p_F \\ 0 & \text{for other cases,} \end{cases} \tag{15}$$

where $s > p_F$ and $s < p_F$ mean respectively $|\mathbb{K} \pm \mathbb{k}_s| > p_F$ and $|\mathbb{K} \pm \mathbb{k}_s| < p_F$, and then, on the analogy of the Sawada theory and the Bardeen-Cooper-Schrieffer theory, he assumed that these operators satisfies the usual boson commutation relations:

$$\left[A_s(\mathbb{K}),\ A_t^{\dagger}(\mathbb{K}')\right]_- = \delta_{\mathbb{K},\mathbb{K}'}\,\delta_{s,t},$$

$$\left[B_s(\mathbb{K}),\ B_t^{\dagger}(\mathbb{K}')\right]_- = \delta_{\mathbb{K},\mathbb{K}'}\,\delta_{s,t}, \tag{16}$$

all others are zero.

In this approximation, it is shown that the kinetic energy of the system T is equivalently written as

$$T = \sum_{\mathbb{K}} \sum_{s}{}' \varepsilon_s(\mathbb{K})\, \omega_s(\mathbb{K}) \left[A_s^{\dagger}(\mathbb{K})\, A_s(\mathbb{K}) + B_s^{\dagger}(\mathbb{K})\, B_s(\mathbb{K})\right], \tag{17}$$

where

$$\omega_s(K) = \left(\mathbb{K}^2 + \mathbb{k}_s^2 - p_F^2\right)/M \tag{18}$$

and $\epsilon_s(K)$ is an indefinite metric function defined by

$$\epsilon_s(K) = \begin{cases} +1 & \text{for } s > p_F \\ -1 & \text{for } s < p_F \\ 0 & \text{for other cases.} \end{cases} \tag{19}$$

As the result, the following reduced Hamiltonian is obtained:

$$H = \sum_{|\mathbb{p}_s| < p_F}{}' \frac{\mathbb{p}_s^2}{2M} + \sum_{\mathbb{K}} H(\mathbb{K}) , \tag{20}$$

$$H(\mathbb{K}) = \sum_s{}' \epsilon_s(K)\ \omega_s(K) \left[A_s^+(\mathbb{K})\, A_s(\mathbb{K}) + B_s^+(\mathbb{K})\, B_s(\mathbb{K})\right]$$

$$+ \sum_{s,t}{}' \left[A_s^+(\mathbb{K}) + B_s(\mathbb{K})\right] V(s,t) \left[A_t(\mathbb{K}) + B_t^+(\mathbb{K})\right] .$$

This $H(\mathbb{K})$ can be easily diagonilized by introducing new boson operators:

$$\alpha_n(\mathbb{K}) = \begin{cases} \sum_s{}' \psi_n^+(s, \mathbb{K}) \left[A_s(\mathbb{K}) - B_s^+(\mathbb{K})\right], & \text{for } n \in M(A) \\ 0, & \text{for } n \in M(B) \end{cases}$$

$$\beta_n(\mathbb{K}) = \begin{cases} -\sum_s{}' \psi_n^+(s, \mathbb{K}) \left[A_s(\mathbb{K}) - B_s^+(\mathbb{K})\right], & \text{for } n \in M(B) \\ 0, & \text{for } n \in M(A) \end{cases}$$

where $\psi_n(s, \mathbb{K})$'s are a complete set of orthonormal functions in the following sense:

$$\sum_s{}' \psi_n^+(s)\, \varepsilon_s\, \psi_m(s) = \varepsilon_n\, \delta_{nm}\,, \tag{22}$$

$$\sum_n \psi_n(s)\, \varepsilon_n \psi_n^+(t) = \varepsilon_s\, \delta_{st}\,, \tag{23}$$

and are the solutions of an integral equation:

$$(E_n - \omega_s)\, \psi_n(s) = \sum_t{}' \varepsilon_s\, V(s,t)\, \psi_n(t), \tag{24}$$

and M(A) is a set of states which are considered as the shifted ones of levels with $s > p_F$, M(B) is another set of states which are considered as the shifted ones of levels with $s < p_F$ and $\varepsilon_n(\mathbb{K})$ is a new indefinite metric function which is defined by

$$\varepsilon_n(K) = \begin{cases} +1, & \text{for } n \in M(A) \\ -1, & \text{for } n \in M(B). \end{cases} \tag{25}$$

In terms of α_n and β_n, H becomes diagonal:

$$H = E_G + \sum_{\mathbb{K},n} \varepsilon_n(\mathbb{K})\, E_n(\mathbb{K}) \left[\alpha_n^+(\mathbb{K})\, \alpha_n(\mathbb{K}) + \beta_n^+(\mathbb{K})\, \beta_n(\mathbb{K})\right], \tag{26}$$

$$E_G = \sum_{|p_s| < p_F} \frac{p_s^2}{2M} + \sum_{\mathbb{K}} \left[\sum_{n \in M(B)} \varepsilon_n(\mathbb{K})\, E_n(\mathbb{K}) - \sum_{s<p_F}{}' \varepsilon_s(\mathbb{K})\, \omega_s(\mathbb{K})\right], \tag{27}$$

and the ground state energy is given by E_G itself. Eq. (24) is

the momentum representation of the Iwamoto equation.

Now, when the interaction potencial is repulsive or it is attractive but very weak, this equation is very powerful. However, when the interaction potential is attractive and strong, this has solutions with complex eigen-values. In that case, eqs. (22) and (23) do not hold, namely, the eigen-functions with complex eigen-values are neither normalizable as eq. (22) because their 'length' becomes zero owing to the indefinite metric nor orthogonal with each other. We here show this difficult situation by a simple example of the separable potential.

Putting

$$V(s,t) = G\, v(s)\, v(t) \tag{28}$$

in the Iwamoto equation (24), we can easily solve it:

$$\psi_n(s) = G \frac{\varepsilon_s\, v(s)}{E_n - \omega_s}\, C_n, \tag{29}$$

where

$$C_n = \sum_s{}' v(s)\, \psi_n(s) . \tag{30}$$

Substituting eq. (29) itself into eq. (30) and dividing by C_n, we have a secular equation by which the eigen-values should be decided:

$$1 = G \sum_s{}' \frac{\varepsilon_s\, v(s)^2}{E_n - \omega_s} . \tag{31}$$

We plotted the right-hand side of this equation as a function of E_n in Fig. 2 of which (a) is for a repulsion ($G > 0$), (b) is for a weak attraction ($G < 0$) and (c) is for a strong attraction ($G \ll 0$).

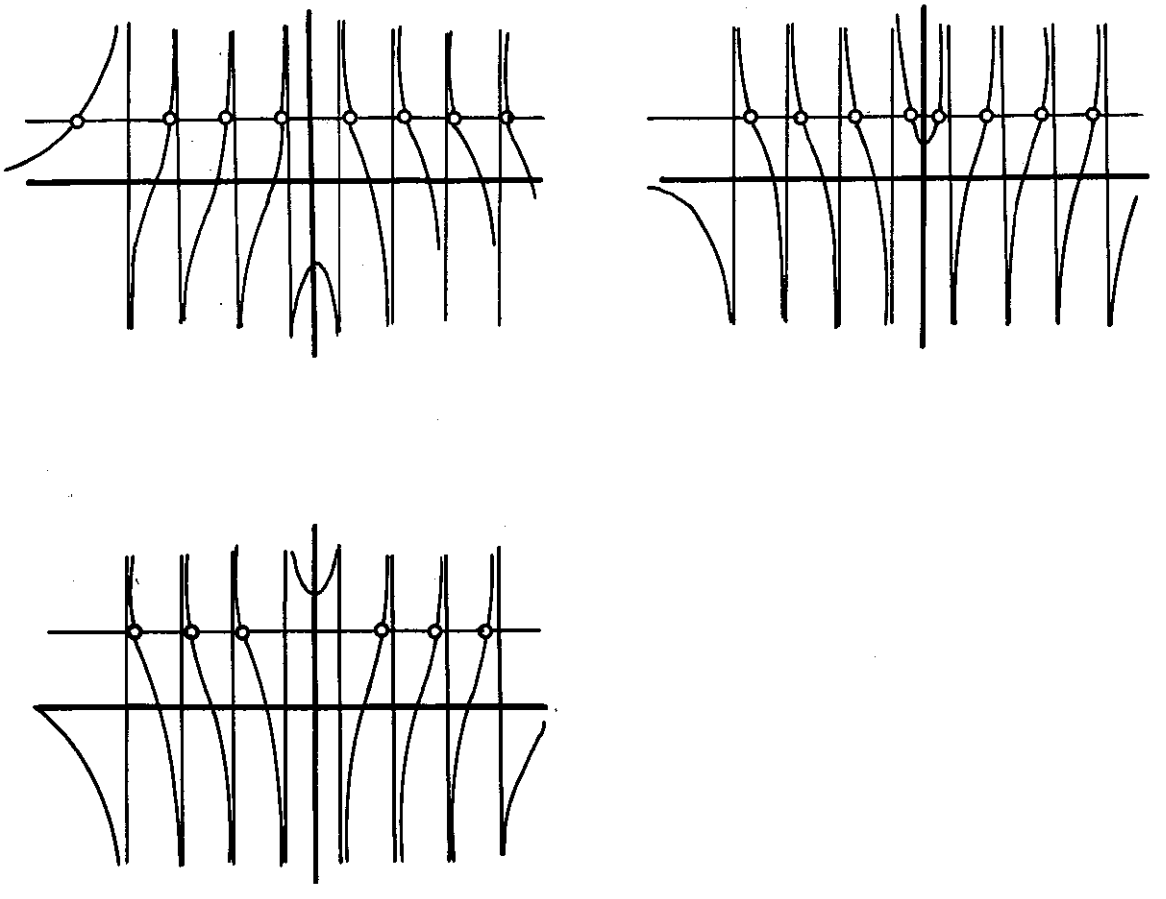

Fig. 2 - Plot of secular equation (31). Small circles give the eigen-values E_n's. (a) Case of repulsion, $G > 0$, (b) Case of weak attraction, $G < 0$, (c) Case of strong attraction, $G \ll 0$.

In the figures, fine virtical lines show positions of ω_s's. The abscissas of the points at which the curves cross a line of ordinate 1 and which are shown by small circles in the figures give the eigen-values E_n.

As seen from Fig. 2-a and 2-b, we have a complete number of eigen-values which correspond to each ω_s^0s one by one,

when the interaction is repulsive, or attractive but weak. However, as seen from Fig. 2-c, we cannot find any eigen-values corresponding to two ω_s's which are nearest to zero. Namely, the eigen-values which should correspond to these ω_s's become complex in this case. Since the secular equation is real, these two complex eigen-values must be complex-conjugate with each other. Giving marks a and b to these eigen-states, we may write the eigen-values as $E_a = \epsilon + i\sigma$ and $E_b = \epsilon - i\sigma$. The eigen-functions of these levels ψ_a and ψ_b are orthogonal to those of other levels with real eigen-values but they themselves do not satisfy eq. (22), that is, we may have rather the following relations:

$$\sum_s{}' \psi_a^+(s)\, \epsilon_s\, \psi_a(s) = 0, \quad \sum_s{}' \psi_b^+(s)\, \epsilon_s\, \psi_b(s) = 0,$$

$$\sum{}' \psi_a^+(s)\, \epsilon_s\, \psi_b(s) = 1.$$

Therefore we can not transform A_s and B_s as in eq. (21) by using ψ's. For this purpose, let us construct a new set of functions φ_n's:

$$\begin{aligned} \varphi_n(s) &= \psi_n(s) \qquad \text{when } n \neq a \text{ and } b \\ \varphi_\alpha(s) &= \left[\psi_a(s) + \psi_b(s)\right]/\sqrt{2}\,, \\ \varphi_\beta(s) &= \left[\psi_a(s) - \psi_b(s)\right]/\sqrt{2}\,, \end{aligned} \tag{33}$$

If we consider that α belongs to M(A) and β to M(B), we can see that φ_n's satisfy eqs. (22) and (23) though they do not

satisfy eq. (24). Expanding A's and B's by these functions, we can transform the Hamiltonian eq. (20) as follows:

$$H = \sum_{|p_s|<p_F} \frac{p_s^2}{2M} + \sum_{\mathbb{K}} \Bigg[\sum_{\substack{n \in M(B) \\ n \neq \beta}} \varepsilon_n(\mathbb{K})\, E_n(\mathbb{K}) - \sum_{s<p_F}{}' \varepsilon_s(\mathbb{K})\omega_s(\mathbb{K}) \tag{34}$$

$$+ \sum_{n=\alpha,\beta} \varepsilon_n E_n \left[\alpha_n^+(\mathbb{K})\,\alpha_n(\mathbb{K}) + \beta_n^+(\mathbb{K})\,\beta_n(\mathbb{K})\right]\Bigg] + h,$$

$$h = -\epsilon + \epsilon(\alpha^+\alpha - \beta^+\beta) + i\sigma(\alpha^+\beta^+ - \alpha\beta), \tag{35}$$

where we have written simply α and β instead of α_α and β_β. The second and third terms in eq. (35) are commutative with each other, and we can obtain state vectors which are the eigen-states of each terms. If Ψ is a super-position of the states in which the difference of the numbers of α-quanta and β-quanta is constant, Ψ is an eigen-state of the second term. Therefore, if $\epsilon > 0$ for instance, a state in which α-quanta are infinitely more than β-quanta is considered to be the ground state, which has infinite negative energy. In case of $\epsilon = 0$, we have not this type of difficulty, but we can obtain no normalizable state vector, anyway, because Ψ is a divergent superposition of infinite number of states as seen from the structure of the third term of eq. (35).

Physically, this may correspond to the following fact: when the interaction is strongly attractive, a pair of fermions can form a bound state with negative energy. Since we considered a pair of fermions as a boson, infinite numbers of these bosons

should appear in the lowest energy state. This contradicts the true situation of nuclei.

3. FERMION-LIKE TREATMENTS OF PAIRS.

As shown in the preceding section, we usually treat a pair of fermions as a boson. But, we hardly consider this a good approximation, when a pair of fermions can form a bound state with negative energy. To examine this point more in detail, let us calculate commutators and anticommutators of A_s, B_s, etc. From eq. (15), the commutation relations become

$$\left[A_s(\mathbb{K}),\ A_t^+(\mathbb{K}')\right]_- = \delta_{\mathbb{K}\mathbb{K}'}\,\delta_{st} - a_{t+}^+ a_{s+}\,\delta_{t-,s-} - a_{t-}^+ a_{s-}\,\delta_{t+,s+},$$

$$\left[B_s(\mathbb{K}),\ B_t^+(\mathbb{K}')\right]_- = \delta_{\mathbb{K}\mathbb{K}'}\,\delta_{st} - b_{t+}^+ b_{s+}\,\delta_{t-,s-} - b_{t-}^+ b_{s-}\,\delta_{t+,s+}, \tag{36}$$

all other commutators are zero.

On the other hand, their anticommutators are

$$\left[A_s(\mathbb{K}),\ A_t^+(\mathbb{K}')\right]_+ = \delta_{\mathbb{K}\mathbb{K}'}\,\delta_{st} - a_{t+}^+ a_{s+}\delta_{t-,s-} - a_{t-}^+ a_{s-}\delta_{s+,t+} + 2a_{t+}^+ a_{t-}^+ a_{s-} a_{s+},$$

$$\left[B_s(\mathbb{K}),\ B_t^+(\mathbb{K}')\right]_+ = \delta_{\mathbb{K}\mathbb{K}'}\,\delta_{st} - b_{t+}^+ b_{s+}\delta_{s-,t-} - b_{t-}^+ b_{s-}\delta_{s+,t+} + 2\,b_{t+}^+ b_{t-}^+ b_{s-} b_{s+}, \tag{37}$$

$$\left[A_s(\mathbb{K}),\ A_t(\mathbb{K})\right]_+ = 2\, a_{s-}\, a_{s+}\, a_{t-}\, a_{t+},\ \text{etc.}$$

From these we can see that the relation between two operators belonging to different states is completely boson-like, for example as $\left[A_s(\mathbb{K}),\ B_t^+(\mathbb{K}')\right]_- = 0$, but the relation between two operators belonging to the same state is very different from that of true bosons. When the state vectors which are eigen-solutions of the reduced Hamiltonian are very similar to the free ground state, that is, the vacuum in our representation, the expectation values of the second and third terms of eq. (36) are considered to be very small. In this case, we may regard that A_s, B_s, etc. satisfy the usual boson commutation relations, for example as $\left[A_s(\mathbb{K}),\ A_s^+(\mathbb{K})\right]_- = 1$. However, when the eigen-state is very different from the vacuum, the probability with which the single particle-, or hole-level is excited, that is, the expectation values of the second and third terms of eq. (36), is not considered to be small, and then we cannot regard them as boson operators. In this case, it is rather better to consider that they satisfy the fermion-like commutation relation, for example as $\left[A_s(\mathbb{K}),\ A_s^+(\mathbb{K})\right]_+ = 1$. As obvious from eq. (37), when the eigen-states are very similar to the free ground state, the last three terms are very small, and when the eigen-states are very different from the free ground state and the probability of the excitation of single particle states is nearly equal to one, the sum of the second and third terms almost cancels out with the fourth term. That is, the anti-

commutation relations hold in much wider regions. Thus we should rather take the anticommutation relations for two operators belonging to the same level instead of the usual commutation relations.

Namely, particles and holes described by A_s and B_s behave like bosons in the point that the state vectors must be symmetrical under the exchange of two of them in different states, but they behave like fermions in the point that only one of them can enter to a single level. When the potential is repulsive of weakly attractive, we may consider that the eigen-states are similar to the free ground state and the probability with which each single particle or hole level is occupied is so small that we may neglect the probability that more than one particle or hole enter to a single level at the same time. This means that it is rather a good approximation in this case to substitute the anticommutators by the usual commutators. But, when the potential is strongly attractive, we cannot use this approximation but should use rather another approximation that two operators belonging to different states statisfy the boson commutation relations but two operators belonging to the same state satisfy the fermion commutation relations:

$$\left[A_s(\mathbb{K}),\ A_s^+(\mathbb{K})\right]_+ = 1, \quad \left[A_s(\mathbb{K}),\ A_s(\mathbb{K})\right]_+ = 0,$$

$$\left[B_s(\mathbb{K}),\ B_s^+(\mathbb{K})\right]_+ = 1, \quad \left[B_s(\mathbb{K}),\ B_s(\mathbb{K})\right]_+ = 0, \qquad (38)$$

$$\left[A_s(\mathbb{K}),\ A_t^+(\mathbb{K}')\right]_- = 0 \quad \text{for } s \neq t \quad \mathbb{K} \neq \mathbb{K}', \text{ etc.}$$

because we cannot consider that the probability with which each single particle states are occupied is small so that we cannot believe that A_s, B_s, etc. satisfy the usual commutation relations eq. (16).

This assumption may be a good approximation but it is so difficult to treat them mathematically that we cannot solve the Hamiltonian in any methods other than the perturbation.

Then let us try last possibility that, stressing the fact that only one particle or hole can enter to a single level, we regard that they behave approximately as fermions. The main difficulty with which we meet in this approximation is that some matrix elements of the interaction become incorrect in their signs due to the change of simmetry properties. That is, the sign of matrix elements of hole-hole interactions becomes incorrect. But, we may overcome this difficulty by modifying the interaction itself so that it gives correct signs of matrix elements. In the other words, according to the guiding principle that the energy calculated under this assumption must agree with that calculated in the usual way when the perturbation method is applicable, we may construct a new interaction Hamiltonian. For this purpose, we may use the following interaction Hamiltonian in our case:

$$H_{int} = \sum_{\mathbb{K}} \sum_{s<p_F} 2\, V(s,s) + \sum_{\mathbb{K}} \sum_{s}{}' \left[A_s^+(\mathbb{K}) + B_s(\mathbb{K})\right]$$

$$\times V_M(s,t) \left[A_t(K) + B_t^+(K)\right], \qquad (39)$$

where A_s, B_s, etc. satisfy the anticommutation relations:

$$\left[A_s(\mathbb{K}),\ A_t^+(\mathbb{K}')\right]_+ = \delta_{\mathbb{K}\mathbb{K}'}\ \delta_{st}\ ,$$
$$\left[B_s(\mathbb{K}),\ B_t^+(\mathbb{K}')\right]_+ = \delta_{\mathbb{K}\mathbb{K}'}\ \delta_{st}\ , \tag{40}$$

all other anticommutators are zero, and $V_M(s,t)$ is defined by

$$V_M(s,t) = \begin{cases} -\ V(s,t) & \text{for } s < p_F \quad t < p_F \\ +\ V(s,t) & \text{for other cases.} \end{cases} \tag{41}$$

It is easily shown that the matrix elements of this interaction Hamiltonian are completely equal to those of eq. (13) with eqs. (15) and (16).

Consequently, in orther to calculate the energy of many-fermion system approximately, we may alternatively use the following reduced Hamiltonian:

$$H = \sum_{|p_s|<p_F} \frac{p_s^2}{2M} + \sum_{\mathbb{K}} \sum_{s<p_F}{}' 2\ V(s,t) + \sum_{\mathbb{K}} H(\mathbb{K})\ , \tag{42}$$

$$H(\mathbb{K}) = \sum_s{}'\ \varepsilon_s\ \omega_s\left[A_s^+(\mathbb{K})\ A_s(\mathbb{K}) + B_s^+(\mathbb{K})\ B_s(\mathbb{K})\right] +$$

$$+ \sum_{s,t}{}'\left[A_s^+(\mathbb{K}) + B_s(\mathbb{K})\right]\ V_M(s,t)\left[A_t(\mathbb{K}) + B_t^+(\mathbb{K})\right]\ ,$$

where A_s, B_s, etc. are this time the fermion operators which satisfy eq. (40). We can easily diagonalize $H(\mathbb{K})$ by intro-

ducing new operators α_n, β_n, etc. defined by the following equations:

$$\alpha_n(K) = \begin{cases} \sum_s \psi_n^+(s,\mathbb{K})\left[A_s(\mathbb{K}) + B_s^+(\mathbb{K})\right], & \text{for } n \in M(A) \\ 0\ , & \text{for } n \in M(B) \end{cases}$$

$$\beta_n^+(\mathbb{K}) = \begin{cases} \sum_s \psi_n^+(s,\mathbb{K})\left[A_s(\mathbb{K}) + B_s^+(\mathbb{K})\right], & \text{for } n \in M(B) \\ 0\ , & \text{for } n \in M(A) \end{cases}$$

which satisfy this time the usual anticommutation relations:

$$\begin{aligned} \left[\alpha_n(\mathbb{K}),\ \alpha_m^+(\mathbb{K})\right]_+ &= \delta_{\mathbb{K}\mathbb{K}'}\,\delta_{nm}\ , \\ \left[\beta_n(\mathbb{K}),\ \beta_m^+(\mathbb{K})\right]_+ &= \delta_{\mathbb{K}\mathbb{K}'}\,\delta_{nm}\ , \end{aligned} \tag{44}$$

all other anticommutators are zero,

and $\psi_n(s)$'s now satisfy usual orthonormality and closure property:

$$\begin{aligned} \sum_s \psi_n^+(s)\,\psi_m(s) &= \delta_{nm}\ , \\ \sum_n \psi_n(s)\,\psi_n^+(t) &= \delta_{st}\ , \end{aligned} \tag{45}$$

and they are solutions of an integral equation:

$$(E_n - \omega_s)\,\psi_n(s) = \sum_t{}' V_M(s,t)\,\psi_n(t). \tag{46}$$

In terms of these α_n, β_n, etc. we have the following diagonalized

Hamiltonian:

$$H = E_G + \sum_{\mathbb{K}} \sum_{n} \epsilon_n(\mathbb{K})\, E_n(\mathbb{K}) \left[\alpha_n^+(\mathbb{K})\, \alpha_n(\mathbb{K}) + \beta_n^+(\mathbb{K})\, \beta_n(\mathbb{K}) \right]. \tag{47}$$

$$E_G = \sum_{|\mathbb{p}_s| < p_F} \frac{\mathbb{p}_s^2}{2M} + \sum_{\mathbb{K}} \left[\sum_{s<p_F}{}' 2V(s,s) \right.$$

$$\left. + \sum_{s<p_F} \epsilon_s(\mathbb{K})\, \omega_s(\mathbb{K}) - \sum_{n \epsilon M(B)} \epsilon_n(\mathbb{K})\, E_n(\mathbb{K}) \right], \tag{48}$$

and the ground state energy is given by E_G itself.

Eq. (46) has always real eigen-values, since the potential is 'Hermitian' and the eigen-solutions are orthonormal in usual way without using indefinite metric functions. We can easily show this by the example of the separable potential eq. (28). Eq. (46) becomes in this case

$$\begin{aligned} (E_n - \omega_s)\, \psi_n(s) &= G\, v(s)\, (C_A + C_B) \quad s > p_F \\ (E_n - \omega_s)\, \psi_n(s) &= G\, v(s)\, (C_A - C_B) \quad s < p_F \end{aligned} \tag{49}$$

and

$$C_A = \sum_{}{}' v(s)\, \psi_n(s), \quad C_B = \sum_{s<p_F}{}' v(s)\, \psi_n(s). \tag{50}$$

Eliminating C_A and C_B from eqs. (49) and (50), we have a secular equation

$$1 = G \sum_{s>p_F}{}' \frac{v(s)^2}{E_n - \omega_s} - G \sum_{s<p_F}{}' \frac{v(s)^2}{E_n - \omega_s} + 2G^2 \sum_{s>p_F} \frac{v(s)^2}{E_n - \omega_s} \sum_{t<p_F} \frac{v(t)^2}{E_n - \omega_t} . \tag{51}$$

We plotted the right-hand side of this equation as a function of E_n in Fig. 3. Even when the potential is strongly attractive (c.f. Fig. 3-c), we see only the translation of minimum point to the negative direction and we have always complete numbers of real solutions.

When the perturbation is applicable, it is directly shown by explaining eqs. (31) and (51) that both eqs. (27) and (48) agree with each other.

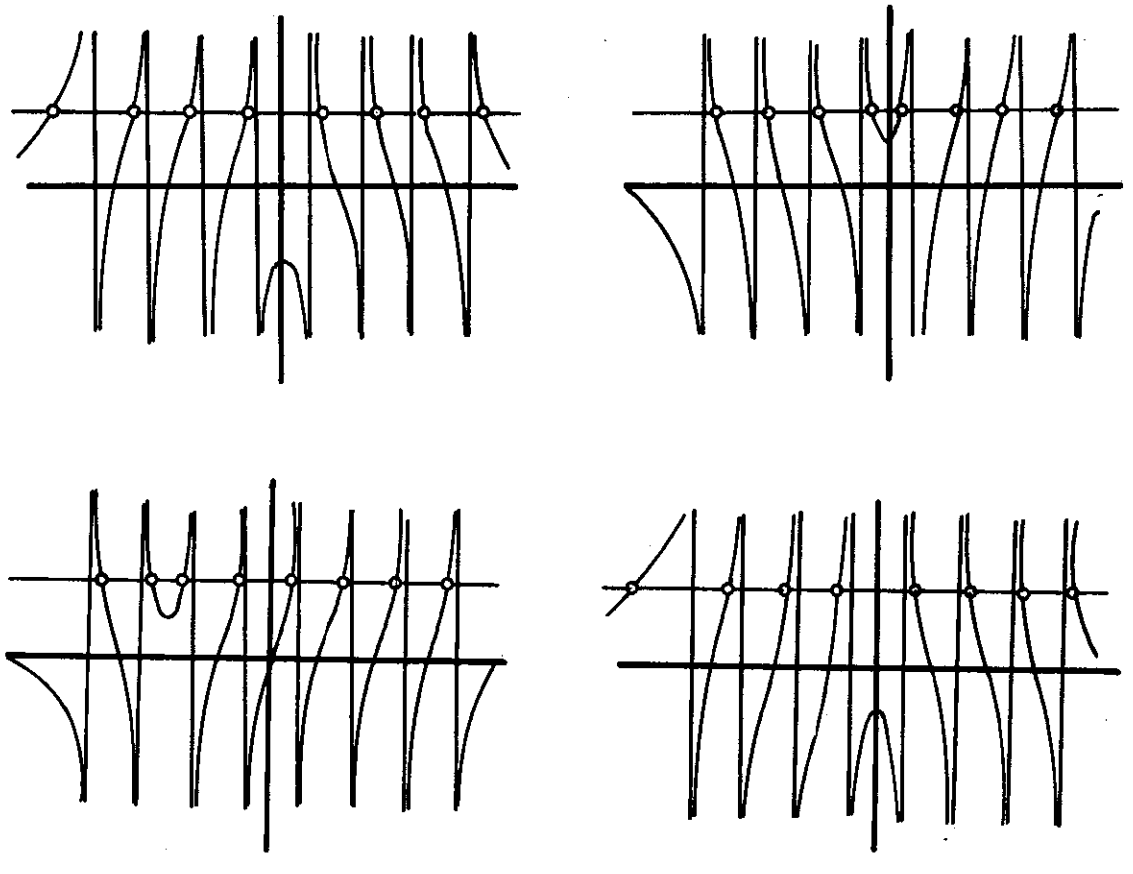

Fig. 3 - Plot of secular equation (51). (a) Repulsion, (b) Weak attraction, (c) Strong attraction, (d) Very strong attraction.

4. DISCUSSIONS.

Using the Iwamoto equation, we could take account of the effect of hole motions which was neglected in the Brueckner method. But, if we apply the Iwamoto method to the system with strongly attractive interactions, we meet a difficulty of complex eigen-values in relation with the appearences of bound states. This is originated in that we have fixed our eyes upon the fact that the wave function must be symmetrical under the exchange of two pairs of fermions in different states and have treated them as bosons, in spite of that only one of them can enter to one level as a matter of fact. Improving this point, we treated them as fermions in the preceding section. In this case, the sign of some matrix elements becomes wrong due to incorrect symmetry property of the wave functions. To avoid this difficulty, we modified the interaction itself beforehand so that the perturbation limit of the ground state energy of this method agrees with usual ones. As conclusion, it was shown that the ground state energy is calculated in the approximation that the effect of hole motions is included, by solving eq. (46), deciding the eigen-values E_n's and substituting them into eq. (48), without meeting any difficulty of complex eigen-values even for the case of strongly attractive interactions.

The ground state vector in the present approximation is easily calculated by transforming the definition

$$\beta_n \Psi_G = 0 , \qquad n \in M(B) \tag{52}$$

into the A and B representation by using eq. (43) with the result,

$$\Psi_G \propto \exp\left[- \sum_{\mathbb{K}} \sum_{s,t}{}' \sum_{n} \psi_n(s,\mathbb{K})\, \varepsilon_n(\mathbb{K})\, \psi_n^+(t,\mathbb{K}) \times A_s^+(\mathbb{K})\, B_t^+(\mathbb{K}) \right] \Phi_G , \quad (53)$$

where Φ_G is the vacuum of the A and B representation.

To clarify the physical meaning of the approximation, however, it is more convenient to introduce new operators C's as follows: (For simplicity, we discuss only the sub-Hamiltonian with $\mathbb{K} = 0$).

$$G_s = \begin{cases} A_s(0) & \text{for } s > p_F , \\ B_s^+(0) & \text{for } s < p_F , \end{cases} \quad (54)$$

In terms of this C_s, H(0) becomes

$$H(0) = \sum_{s,t}{}' C_s^+ \left[\omega_s\, \delta_{st} + V_M(s,t) \right] C_t . \quad (55)$$

This is easily diagonalized by a canonical transformation,

$$C_s = \sum_n \gamma_n\, \psi_n(s) , \qquad \left[\gamma_n, \gamma_m^+ \right]_+ = \delta_{nm} , \quad (56)$$

with the result

$$H(0) = \sum_n E_n(0)\, \gamma_n^+\, \gamma_n . \quad (57)$$

The ground state vector of this sub-Hamiltonian is given by

$$\Psi_G(0) = \prod_{n \in M(B)} \left(\sum_s \psi_n^+(s)\, C_s^+ \right) \Phi_o , \quad (58)$$

where Φ_0 is the vacuum state vector. If we consider C_s as an approximation of $c_{s-}\, c_{s+}$, we may interpret this as follows: we first pair the fermions with $\mathbb{K} = 0$ and considering the interactions only within the pair but modified due to the Pauli principle we solve the two-body problem exactly, and then we put all pairs in the two-particle levels thus obtained from the bottom one after another; this is the ground state in our approximation. In this case, the eigen-values of H(0) give the shifts of energies from that of the free systems.

If there are some eigen-functions with finite extensions among ψ_s's, or in case of finite nuclei someones with extensions smaller than the dimension of the nuclei, the pairs in those states are considered to form a kind of bound states. In fact, the energy of these pairs becomes negative. The word 'bound state' used up to now has had of course this meaning and it does not mean that the same two-particle bound state as that in the free space is formed in the many-fermion system.

Extending this, we may consider our approximation as follows: we first pair the fermions with a constant $\mathbb{K}$ and then, considering only the interactions mentioned before, we calculate the energy shifts as explained above in case of $\mathbb{K} = 0$; we may consider as the ground state energy the sum of these energy shifts over different $\mathbb{K}$ and the kinetic energy of each fermion.

The application to actual problems will be presented separately in collaborations with other members of the University of São Paulo.

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