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INVERSION OPERATIONS IN QUANTUM FIELD THEORY*

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ABSTRACT

A systematic and rigorous presentation of the main properties of inversion operations in quantum field theory is attempted in this paper. The 1st chapter reviews the foundations of the theory; the transformations of the infinite-dimensional representation of the inhomogeneous, proper and orthochronous Lorentz group are given as well as those of the gauge group of the first kind. These are used in the 2nd chapter for a proof of the superselection rules which are essential for a clear understanding of the concept of intrinsic and relative parity of elementary particles. This proof, which does not invoke time reversal, rests on the postulate that all observables are tensor representations of the inhomogeneous, proper, orthochronous Lorentz group. The question of real and imaginary parities of fermions is discussed. After presentation of the particle-antiparticle conjugation in chapter IV, Majórana neutral fields are studied in chapter V and in particular, the question of possible interactions of hypothetical Majorana fermions. Time reversal and strong reflection are studied in chapters VI and VII.

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CHAPTER I

REVIEW OF THE FOUNDATIONS OF QUANTUM FIELD THEORY

I, 1. Lorentz transformation of field operators

In relativistic quantum theory, the wave fields are postulated to be operators which act on a Hilbert space. These operators are determined by the field equations and the commutation rules (to within a unitary transformation). These operators also depend on discrete variables - the spinor and tensor indices - which are determined by the requirement that they be elements of a finite-dimensional representation space of the inhomogeneous proper Lorentz group (from which we shall also exclude the time inversion in this chapter).

The state vectors ψ (or kets) are elements of the Hilbert space. The relativistic invariance of the theory requires that the transformed

$$\psi' = U(L)\,\psi \tag{1}$$

corresponding to a inhomogeneous proper Lorentz transformation L:

$$x^{\mu\prime} = \ell^{\mu}_{\nu}\, x^{\nu} + a^{\mu} \;; \quad g_{\mu\nu}\, \ell^{\mu}_{\lambda}\, \ell^{\nu}_{\tau} = g_{\lambda\tau} \tag{2}$$

$$\ell^{0}_{0} \geqslant 1$$

be also a possible state of the system. The operators U form an infinite dimensional representation of this Lorentz group and are unitary:

$$U(L)^{\dagger}\, U(L) = I$$

so that

$$(\psi', \Phi') = (\psi, \Phi) .$$

Let Ψ be the state vector and O(x) be the operators associated to a physical system by an observer in a frame of reference. The physicist of another Lorentz frame may either: a) ascribe to the system the same state vector Ψ and new operators $O'(x)$, or b) describe the system by means of a new state vector Ψ' and unchanged operators O(x). The first method is the Heisenberg-type of Lorentz transformation in the Hilbert space, the second is the Schrödinger type.

Both methods must be equivalent in the sense that both must give the same expectation values for the physical quantities.

Thus: $(\Psi', O(x)\Psi')_H = (\Psi', O'(x)\Psi')_S$ or:

$$(\Psi, O'(x)\Psi) = (\Psi', O(x)\Psi') \tag{3}$$

hence:

$$O'(x) = U^{\dagger}(L)O(x)U(L) \tag{4}$$

The form of $O'(x)$, on the other hand, is obtained by the requirement that the field equations be invariant under the proper Lorentz group. The study of the finite-dimensional representations (irreducible) of the proper Lorentz group yields the result that the field variables can only be scalars, spinors, four-vectors, and tensors and spinors of higher rank. Thus, a scalar field $\varphi(x)$ is such that:

$$\varphi'(x') \equiv U^{\dagger}(L)\varphi(x')U(L) = \varphi(x) \; ; \tag{5}$$

A vector field operator $A^{\mu}(x)$ satisfies:

$$A^{\mu'}(x') \equiv U^{\dagger}(L)A^{\mu}(x')U(L) = \ell^{\mu}_{\nu} A^{\nu}(x) \tag{6}$$

and for a Dirac spinor one has:

$$\psi'(x') \equiv U^{\dagger}(L)\psi(x')U(L) = D\psi(x), \tag{7}$$

where D is such that

$$D\gamma^{\mu}D^{-1} = \ell_k^{\ \mu}\gamma^k \ ; \quad \ell_k^{\ \mu} = g_{k\lambda}\,\ell^{\lambda}_{\ \nu}\,g^{\nu\mu}, \tag{8}$$

and

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} ,$$

$$g^{00} = -g^{11} = -g^{22} = -g^{33} = 1 \quad ; \quad g^{\mu\nu} = 0, \ \mu \neq \nu \tag{9}$$

and for a free spinor field:

$$(i\gamma^{\mu}\frac{\partial}{\gamma x^{\mu}} - m)\psi(x) = 0 \tag{10}$$

We remark that while the transformations U(L) which constitute an infinite-dimensional representation of the Lorentz group can be taken as unitary, the transformations such as L, D, which transform the (finite-dimensional) 4 - vector and spinor space into themselves respectively, can <u>not</u> be unitary.

For an infinitesimal proper Lorentz transformation:

$$\ell^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}_{\ \nu} \quad , \quad \epsilon^2 \ll \epsilon , \tag{11}$$

one gets:

$$D(\epsilon) = I - \frac{i}{2}\epsilon_{\mu\nu}\,\mathcal{D}^{\mu\nu}, \tag{12}$$

$$\mathcal{D}^{\mu\nu} = \frac{i}{4}\,(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}). \tag{13}$$

Thus, because:

$$(\gamma^{\mu})^{\dagger} = \gamma^{0}\,\gamma^{\mu}\gamma^{0} \tag{14}$$

one has:

$$(\mathcal{D}^{\mu\nu})^{+} = \gamma^{0}\,\mathcal{D}^{\mu\nu}\,\gamma^{0} \tag{15}$$

and

$$D(\epsilon)^{+} = \gamma^{0}\,D^{-1}\,\gamma^{0}.$$

A finite proper transformation gives

$$D = e^{\frac{1}{8}\epsilon_{\mu\nu}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})} . \tag{16}$$

For space rotations one has:

$$D_R^{+} = D_R^{-1}$$

because:

$$\gamma^{o}\,\gamma^{i}\,\gamma^{k}\,\gamma^{o} = \gamma^{i}\,\gamma^{k}\ , \qquad i,k = 1,\ 2,\ 3\ .$$

What is the form of U(L) correspomding to an infinitesimal Lorentz transformation? Let $\mathcal{L}$ be the lagrangean density constructed with the field operators O (x) and their first space-time derivatives in an invariant fashion under the proper Lorentz group and such that the variation of the action :

$$I = \int \mathcal{L}\left[O(x),\ \frac{\partial O(x)}{\partial x^{\mu}}\right]\ d^4x$$

gives rise to the field equations.

$\mathcal{L}$ is known from classical field theory. However, when the fields are operators, one must define the order in which they are taken in products, since they in general do not commute.

$\mathcal{L}$ will be taken as the expression in classical theory, the operators being ordered as <u>normal products</u>. This will be defined later and adopted for all observables such as current density $j^{\mu}(x)$, angular momentum tensor $\mathcal{M}^{\mu\nu}$, energy-momentum tensor $T^{\mu\nu}$, energy-momentum vector P^{μ} :

$$P^{\mu} = \int d^3x\ T^{o\mu}\ \left(\equiv \int d\sigma_{\alpha}\ T^{\alpha\mu}\right)\ , \tag{17}$$

$$T^{\mu\nu} = g^{\alpha\nu}\,\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial O_i(x)}{\partial x^{\mu}}\right)}\ \frac{\partial O_i(x)}{\partial x^{\alpha}} - \mathcal{L}\ g^{\mu\nu}, \tag{18}$$

$$\mathcal{M}^{\mu\nu} = \int d^3x\ \mathcal{M}^{o;\mu\nu}\ \left(\equiv \int d\sigma_{\alpha}\mathcal{M}^{\alpha;\mu\nu}\right) \tag{19}$$

(integration over space-like surface $\int d\sigma_{\alpha}$ shows the covariance of

P^{μ} and $\mathcal{M}^{\mu\nu}$),

$$\mathcal{M}^{\lambda;\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \mathcal{M}^{\lambda}_{\alpha\beta}, \tag{20}$$

$$\mathcal{M}^{\lambda}_{\alpha\beta} = T^{\lambda}_{\alpha} x_{\beta} - T^{\lambda}_{\beta} x_{\alpha} - \frac{\partial \mathcal{L}}{\left(\frac{\partial 0_i(x)}{\partial x^{\lambda}}\right)} \mathcal{D}_{ij;\alpha\beta} 0_j(x) \tag{21}$$

Here, the $\mathcal{D}_{ij;\alpha\beta}$ are the infinitesimal operators of the transformation of the field variables in spinor or tensor space, (12). Thus, for a scalar field

$$\mathcal{D}_{ij;\alpha\beta} = 0 ; \tag{22}$$

for a vector field:

$$\mathcal{D}_{ij;\alpha\beta} = \delta^{j}_{\beta} g_{i\alpha} - \delta^{j}_{\alpha} g_{i\beta}, \tag{23}$$

$$i, j, \alpha, \beta = 0, 1, 2, 3,$$

for a spinor field:

$$\mathcal{D}_{ij;\alpha\beta} = \frac{1}{8} (\gamma_{\alpha} \gamma_{\beta} - \gamma_{\beta} \gamma_{\alpha})_{ij} . \tag{24}$$

Now, then, an infinitesimal unitary transformation $U(a, \epsilon)$ in Hilbert space corresponding to an infinitesimal inhomogenous Lorentz transformation:

$$x^{\mu'} = (\delta^{\mu}_{\nu} + \epsilon^{\mu}_{\nu}) x^{\nu} + a^{\mu}, \quad \epsilon^2 \ll \epsilon, \; a^2 \ll a,$$

is of the form:

$$U(a, \epsilon) = I - i a_{\mu} P^{\mu} - \frac{i}{2} \epsilon_{\mu\nu} \mathcal{M}^{\mu\nu}, \tag{25}$$

U is unitary, the energy-momentum vector and the angular momentum tensor are hermitian. The latter are seen to be the infinitesimal operators which determine the infinite-dimensional representations (unitary) of the inhomogeneous Lorentz group. For finite transformations:

$$U(L) = \exp\left(-i a_{\mu} P^{\mu} - \frac{i}{2} \epsilon_{\mu\nu} \mathcal{M}^{\mu\nu}\right). \tag{26}$$

I,2. <u>Conditions imposed by U on the field operators.</u>

The existence of a unitary transformation U(L) which transforms O into O':

$$O'(x) = U^{+}(L)O(x)U(L) \tag{4}$$

imposes certain conditions on the field variables O(x). They must satisfy given commutation rules with P^{μ} and $\mathcal{M}^{\mu\nu}$.

Indeed, let:

$$x^{\mu'} = x^{\mu} + f^{\mu}_{\nu}\,\omega^{\nu}\,,\omega^{2} << \omega, \tag{27}$$

be an infinitesimal linear transformation on x, with infinitesimal parameters ω^{ν}. In the case of a proper Lorentz infinitesimal transformation, the index ν is a pair of indices $\alpha\beta$ and :

$$\omega^{\nu} \longrightarrow \epsilon^{\alpha\beta}$$

$$f^{\mu}_{\nu} \longrightarrow f^{\mu}_{\alpha\beta} = \delta^{\mu}_{\alpha}\, g_{\beta\nu}\, x^{\nu} - \delta^{\mu}_{\beta}\, g_{\alpha\nu}\, x^{\nu}, \quad \alpha \leq \beta\,. \tag{28}$$

In the case of a translation:

$$f^{\mu}_{\nu} = \delta^{\mu}_{\nu}, \quad \omega^{\nu} = a^{\nu}\,. \tag{29}$$

Transformation (27) induces a transformation of the field operators, in their arguments x and in their form, $O(x) \longrightarrow O'(x')$ and we shall write:

$$O'_{i}(x') = O_{i}(x) + \Omega_{ij}(x)\,\omega^{j} \tag{30}$$

The variation of the operator is:

$$\delta O_{i}(x) \equiv O'_{i}(x') - O_{i}(x) = \Omega_{ij}(x)\,\omega^{j} \tag{31}$$

while the variation in form only is

$$\bar{\delta} O_{i}(x) \equiv O'_{i}(x) - O_{i}(x) = O'_{i}(x') - O_{i}(x) - \left(O'_{i}(x') - O'_{i}(x)\right)$$

$$= \Omega_{ij}(x)\,\omega^{j} - \frac{\partial O'_{i}}{\partial x^{k}}\,\delta x^{k} = \tag{32}$$

$$= \left(\Omega_{ij}(x) - \frac{\partial O_{i}}{\partial x^{k}}\, f^{k}_{j}\right)\omega^{j}$$

On the other hand, (27) induces an infinitesimal unitary transformation $U(\omega)$ in Hilbert space which transforms the operators according to (4).

If we call $\delta U(\omega)$ the infinitesimal part of $U(\omega)$:

$$U(\omega) = I + \delta U(\omega), \tag{33}$$

then the unitarity of $U(\omega)$ gives

$$(\delta U(\omega))^{\dagger} = - \delta U(\omega)$$

$$O'_i(x) = U^{\dagger}(\omega) O_i(x) U(\omega) = \big(I - \delta U(\omega)\big) O_i(x) \big(I + \delta U(\omega)\big) = $$
$$= O_i(x) + \left[O_i(X), \delta U(\omega)\right] \tag{34}$$

where:

$$\left[A, B \right] = AB - BA .$$

Thus:

$$O_i'(x) - O_i(x) - O_i \equiv \bar{\delta}\, O_i(x) = \left[O_i(x), \delta U(\omega)\right].$$

$\delta U(\omega)$ is linear in the parameters ω, so we write:

$$\delta U(\omega) = i\, U_\nu\, \omega^\nu$$

to get:

$$\bar{\delta} O_i(x) = i\left[O_i(x), U_\nu\right]\omega^\nu . \tag{35}$$

Compare (35) with (32), you obtain:

$$i\left(\Omega_{jk}(x) - \frac{\partial O_j(x)}{\partial x^\ell} f_k^\ell\right) = \left[U_k, O_j(x)\right] . \tag{36}$$

These are the conditions imposed on the field variables $O_j(x)$.

For a translation in space-time, (29) and (36), plus the fact that then $\Omega_{jk} = 0$, will give ($U_\nu \equiv P_\nu$):

$$-i\, \frac{\partial O_i(x)}{\partial x^\nu} = \left[P_\nu, O_i(x)\right], \tag{37}$$

where P_ν is the energy - momentum vector (17).

For a homogenous proper Lorentz transformation, (28) and:

$$\Omega_{ij} \longrightarrow \Omega_{ij;\lambda\nu}(x) = D_{ij;\lambda\nu}\, O_j(x),$$

$$U_j \longrightarrow U_{\lambda\nu} = - \mathcal{M}^{\lambda}_{\nu} ,$$

will yield:

$$i\left[D_{ij;\lambda\nu}\, O_j(x) - \frac{\partial O_i(x)}{\partial x^k}\,(\delta^k_\lambda\, g_{\nu\alpha} - \delta^k_\nu\, g_{\lambda\alpha})x^\alpha\right] =$$

$$= \left[O_i(x), \mathcal{M}_{\lambda\nu}\right] \qquad (38)$$

where $\mathcal{M}_{\lambda\nu}$ is the angular momentum tensor (19).

The physical meaning of (37) and (38) will be transparent in momentum space.

I,3. Emission and absorption operators of free fields.

The free field variables must satisfy the equation:

$$(\Box + m^2)\, O(x) = 0 \qquad (39)$$

where:

$$\Box = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} .$$

We therefore define $O(P)\,\delta(P^2 - m^2)$ as the covariant Fourier transform of $O(x)$:

$$O(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\; e^{-ipx}\, O(p)\,\delta(p^2 - m^2); \quad px = p^\lambda x_\lambda \qquad (40)$$

$O(x)$ satisfies (39) because $(p^2 - m^2)\,\delta(p^2 - m^2) = 0$

We separate $O(x)$ and $O(p)$ into a positive frequency and a negative frequency part:

$$O(x) = O^{(+)}(x) + O^{(-)}(x),$$
$$O(p) = O^{(+)}(p) + O^{(-)}(p) \qquad (41)$$

where

$$O^{(+)}(p) = O(p)\, \tfrac{1}{2}(1 + \operatorname{sgn} p^o),$$
$$O^{(-)}(p) = O(p)\, \tfrac{1}{2}(1 - \operatorname{sgn} p^o). \qquad (42)$$

As p is a time-like four vector, sgn p^o, defined by:

$$\text{sgn } p^o = \begin{cases} 1 & \text{for } p^o > 0, \\ -1 & \text{for } p^o < 0, \end{cases} \tag{43}$$

is invariant under proper orthochronous Lorentz transformations.

By performing the integration over p^o, it is clear that O (x) can be written as an integral over the 3-dimensional momentum space:

$$O(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \left\{ O^{(+)}(\vec{p}) e^{-ipx} + O^{(-)}(\vec{p})\, e^{ipx} \right\}$$

where now:

$$O^{(+)}(\vec{p}) = \frac{1}{2p^o}\, O(\vec{p}),$$

$$O^{(-)}(\vec{p}) = \frac{1}{2p^o}\, O(-\vec{p}) \;\; ; \; p^o = +(\vec{p}^2 + m^2)^{1/2} \, .$$

Note that $O^{(+)}(\vec{p})$ and $O^{(-)}(\vec{p})$ depend on the 3-dimensional vector $\vec{p}$.

One may define

$$A(\vec{p}) = \sqrt{2p^o}\; O^{(+)}(\vec{p}) \; ,$$

$$B^{\dagger}(\vec{p}) = \sqrt{2p^o}\; O^{(-)}(\vec{p})$$

where now $B^{\dagger}$ is the <u>hermitian conjugate</u> of B, so that:

$$O(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^o}} \left\{ A(\vec{p}) e^{-ipx} + B^{\dagger}(\vec{p}) e^{ipx} \right\} . \tag{44}$$

O(x), besides being an operator in Hilbert space, depends on spinor or tensor indices, α.

We separate out the two aspects in A and B by defining:

$$A_\alpha(\vec{p}) = \sum_r a(r,\vec{p}) u_\alpha(r,\vec{p}),$$

$$B^{+}_\alpha(\vec{p}) = \sum_r b^{+}(r,\vec{p})\, v_\alpha(r,\vec{p}),$$

where the sum over r refers to polarization states: a and b are operators in Hilbert space: u_α and v_α are spinor or tensor functions

Thus:

$$O_\alpha(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2p^0}} \left\{ a(r,\vec{p})\, u_\alpha(r,\vec{p})e^{-ipx} + b^\dagger(r,\vec{p})v_\alpha(r,\vec{p})e^{ipx}\right\},$$

$$O^\dagger_\alpha(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2p^0}} \left\{ b(r,\vec{p})v^\dagger_\alpha(r,\vec{p})e^{-ipx} + a^\dagger(r,\vec{p})u^\dagger_\alpha(r,\vec{p})e^{ipx}\right\}, \tag{45}$$

where summation over r is understood.

Thus for a <u>scalar field</u> $\varphi(x)$:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2p^0}} \left\{ a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})e^{ipx}\right\},$$

$$\varphi^\dagger(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2p^0}} \left\{ b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}\right\}, \tag{46}$$

which will be hermitian for a = b.

For a <u>spinor field</u>:

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{dp^0}} \left\{ a(r,\vec{p})\,u(r,\vec{p})e^{-ipx} + b^\dagger(r,\vec{p})v(r,\vec{p})e^{ipx}\right\}, \tag{47}$$

$$\overline{\psi}(x) = \psi^\dagger(x)\,\gamma^0 = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{\sqrt{2p^0}} \left\{ b(r,\vec{p})\,\bar{v}(r,\vec{p})e^{-ipx} + a^\dagger(r,\vec{p})\,\bar{u}(r,\vec{p})e^{ipx}\right\},$$

the spinor index α being omitted and where $\overline{\psi} = \psi^\dagger\gamma^0$ and:

$$(\gamma^k p_k - m)\, u(r,\vec{p}) = 0, \quad (\gamma^k p_k + m)\, v(r,\vec{p}) = 0,$$

$$\bar{u}(r,\vec{p})(\gamma^k p_k - m) = 0, \quad \bar{v}(r,\vec{p})(\gamma^k p_k + m) = 0\,. \tag{48}$$

Normalization of the u's and v's is taken as follows. First

$$u^\dagger(r',\vec{p})\, u(r,\vec{p}) = 2p^0\,\delta_{r'r}$$

$$v^\dagger(r',\vec{p})\, v(r,\vec{p}) = 2p^0\,\delta_{r'r},\ p^0>0, \tag{49}$$

then it follows from the equations (48) that:

$$\bar{u}\,(r',\vec{p})\,u\,(r,\,\vec{p}) = 2m\,\delta_{r'r}\quad ,$$
$$\bar{v}\,(r',\vec{p})\,v\,(r,\,\vec{p}) = -2m\,\delta_{r'r}\,, \qquad (49)'$$
$$v^{\dagger}(r,'-\vec{p})\,u\,(r,\,\vec{p}) = 0\ .$$

For the electromagnetic field:

$$A_{\mu}\,(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2k^0}}\sum_{\lambda}\ e_{\mu}\,(\lambda,\vec{k})\left\{a(\lambda,\vec{k})e^{-ikx} + a^{\dagger}(\lambda,\vec{k})e^{ikx}\right\}. \qquad (50)$$

$e_{\mu}\,(\lambda,\vec{k})$ is the polarization vector, $k^{0} = |\vec{k}|$.

We now wish to show that the commutation rules (37) and (38) permit to interpret the operators a,b as absorption operators of particles with momentum $\vec{p}$ and spin r, $a^{\dagger}$, $b^{\dagger}$ as emission operators.

That the b's refer to antiparticles while the a's refer to particles follows from the commutation rules of the operators with the charge (next paragraph).

Consider (37) in the case in which 0 (x) is a spinor field $\Psi(x)$. We have

$$[P_{\nu}\,,\,a(r,\vec{p})]\,u\,(r,\vec{p}) = -\,p_{\nu}a(r,\vec{p})\,u(r,\vec{p})\ ,$$
$$[P_{\nu}\,,\,b^{\dagger}(r,\vec{p})]\,v(r,\vec{p}) = p_{\nu}b^{\dagger}(r,\vec{p})\,v(r,\vec{p})\quad . \qquad (51)$$

Let Ψ (K) be a state vector with momentum K:

$$P_{\nu}\Psi\,(K) = K_{\nu}\Psi\,(K)\ ,$$

the K_{ν} being numbers. We then have:

$$[P_{\nu}\,,a(r,\vec{p})]\,\Psi\,(K) = P_{\nu}\,a(r,\vec{p})\Psi\,(K) - a(r,\vec{p})\,P_{\nu}\Psi(K) =$$
$$= (P_{\nu} - K_{\nu})\,a(r,\vec{p})\Psi(K)$$

which should be equal, according to (51), to:

$$-\,p_{\nu}\,a(r,\vec{p})\Psi(K)$$

(we got rid of $u(r,\vec{p})$ in (51) by multiplying on the left by $u^{+}(r,\vec{p})$ and using (49)).

So

$$P_{\nu}\left(a(r,\vec{p})\Psi(K)\right) = (K_{\nu} - p_{\nu})\left(a(r,\vec{p})\Psi(K)\right). \qquad (52)$$

Thus $a(r,\vec{p})\Psi(K)$ is an eigen vector of P_{ν} with momentum K - p, hence the operator $a(r,\vec{p})$ destroys a particle of momentum p.

For $a^{+}(r,\vec{p})$, the operation is creation of particle. Analogously for $b(r,\vec{p})$, $b^{+}(r,\vec{p})$.

With (38), one obtains for spinors:

$$\left[a(r,\vec{p}), \mathcal{M}_{\lambda\nu}\right] = (S_{\lambda\nu} + L_{\lambda\nu})\, a(r,\vec{p}),$$

where:

$$S_{\lambda\nu} = \frac{1}{2p^{0}} u^{+} \frac{i}{4} (\gamma_{\lambda}\gamma_{\nu} - \gamma_{\nu}\gamma_{\lambda}) u(r,\vec{p}), \qquad (53)$$

$$L_{\lambda\nu} = x_{\lambda}p_{\nu} - x_{\nu}p_{\lambda},$$

and thus $a(r,\vec{p})$ is also absorption of a particle with spin $S_{\lambda\nu}$ and orbital angular momentum $L_{\lambda\nu}$.

Now we want to show that a and a^{+} refer to particles, b, b^{+}, to antiparticles.

I,4. Particle and antiparticle operators.

Assume that the wave field operators O(x) are non-hermitian. As the lagrangean must be hermitian, it can only depend on combinations of the fields like $O^{+}(x)\,O(x)$. Therefore, the lagrangean will be unchanged if the fields are multiplied by an arbitrary phase factor $e^{i\alpha}$:

$$O'(x) = e^{i\alpha}O(x),$$
$$O'^{+}(x) = e^{-i\alpha}O^{+}(x). \qquad (54)$$

The induced transformation in Hilbert space, $U(\alpha)$, will be written:

$$U(\alpha) = e^{iQ\alpha} , \qquad (55)$$

where Q is an hermitian operator, the charge of the field. We have

$$O'(x) = U^{+}(\alpha)\, O(x)\, U(\alpha) ,$$

so that for infinitesimal α :

$$(1 + i\alpha)O(x) = (I - iQ\alpha)\, O(x)(I + iQ\alpha) ,$$

hence:

$$[O(x), Q] = O(x),$$

$$[O^{+}(x), Q] = -O^{+}(x) . \qquad (56)$$

That Q is the charge operator follows from classical theory where it is shown that invariance of $\mathcal{L}$ under gauge transformation leads to a conserved vector $j^{\nu}(x)$, the current:

$$j^{\nu}(x) = i \left\{ O_i^{*}(x) \frac{\partial \mathcal{L}}{\partial(\frac{\partial O_i^{*}}{\partial x^{\nu}})} - \frac{\partial \mathcal{L}}{(\frac{\partial O_i}{\partial x^{\nu}})} O_i(x) \right\} . \qquad (57)$$

This will be taken over into quantum theory, with the nomal product as the ordering criterion. Now the only, first order in j, invariants formed with $j^{\nu}(x)$ are $\frac{\partial j^{\nu}}{\partial x^{\nu}}$, which vanishes, and Q = $= \int d\sigma_{\mu}\, j^{\mu}(x) = \int d^{3}x\, j^{0}(x)$, which is the charge. This is the only one available to be put in (55).

Now suppose that O(x) is $\psi(x)$ as given by (47). We get:

$$\begin{aligned} [a(r,\vec{p}), Q] &= a(r,\vec{p}), \\ [b^{+}(r,\vec{p}), Q] &= b^{+}(r,\vec{p}), \\ [a^{+}(r,\vec{p}), Q] &= -a^{+}(r,\vec{p}), \\ [b(r,\vec{p}), Q] &= -b(r,\vec{p}) . \end{aligned} \qquad (58)$$

The difference in sign in the commutation rules (58) for a^{+} and b^{+}

gives us the clue to the interpretation. Let q be the charge of a state $\psi(q)$ (in units of e): $Q\psi(q) = q\psi(q)$,

then it follows from (58) that:

$$a^{+}(r,\vec{p})\psi(q) \text{ is a state with charge } q+1;$$

$$b^{+}(r,\vec{p})\psi(q) \text{ is a state with charge } q-1.$$

So a, a^{+} are operators which refer to particles, b, b^{+}, to antiparticles (with charge opposite to that of particles).

I,5. Lorentz transformation of emission and absorption operators.

From (4) and (45), plus the fact that

$$O'(x') = S\,O(x),$$

where S is a matrix which acts on the spinor or tensor indices of O(x), as illustrated by (5), (6) and (7), we obtain:

$$\int \frac{d^3p}{\sqrt{2p^0}} \sum_r \left\{ U^{+} a(r,\vec{p}) U v(r,\vec{p}) e^{-ipx'} + U^{+} b^{+}(r,\vec{p})\, U v(r,\vec{p}) e^{ipx'} \right\} =$$

$$\int \frac{d^3p}{\sqrt{2p^{0'}}} \sum_r \left\{ a(r,\vec{p})\, S u(r,\vec{p}) e^{-ipx} + b^{+}(r,\vec{p})\, S v(r,\vec{p}) e^{ipx} \right\} .$$

Change the integration variable of the left-hand side into p', such that $p'x' = px$, note that $\frac{d^3p'}{p^{0'}} = \frac{d^3p}{p^0}$

and $u(r,\vec{p}\,') = S\,u(r,\vec{p})$, you get:

$$\sqrt{p^{0'}}\; U^{+} a(r,\vec{p}\,')\, U = \sqrt{p^0}\; a(r,\vec{p}), \tag{59}$$

which holds for all emission and absorption operators.

For a pure spatial rotation:

$$R^{-1} a(r,\vec{p}\,') R = a(r,\vec{p}) . \tag{60}$$

As a check, verify that the energy-momentum vector being of the form:

$$P^{\mu} = \int d^3p\; p^{\mu} \left\{ a^{+}(r,\vec{p})\, a(r,\vec{p}) + b^{+}(r,\vec{p})\, b(r,\vec{p}) \right\}$$

(59) gives:

$$U^\dagger P^\mu U = \ell^\mu_\nu \; P^\nu .$$

I,6. Lagrangean, energy-momentum tensor and current of classical fields.

Complex scalar field: $\varphi(x)$:

$$\mathcal{L} = g_{\mu\nu} \frac{\partial \varphi^*(x)}{\partial x_\mu} \frac{\partial \varphi(x)}{\partial x_\nu} - m^2 \varphi^*(x)\varphi(x),$$

$$T^{\mu\lambda} = \frac{\partial \varphi^*(x)}{\partial x_\mu} \frac{\partial \varphi(x)}{\partial x_\lambda} + \frac{\partial \varphi^*(x)}{\partial x_\lambda} \frac{\partial \varphi(x)}{\partial x_\mu} - \mathcal{L}\, g^{\mu\lambda}, \tag{61}$$

$$j^\nu = i g^{\nu\alpha}\left(\varphi^*(x) \frac{\partial \varphi(x)}{\partial x^\alpha} - \frac{\partial \varphi^*(x)}{\partial x^\alpha}\varphi(x)\right) .$$

Substitution of (46) in these formulae (where one replaces hermitian conjugation † by complex conjugation *) gives:

$$\begin{aligned} P^\mu &= \int d^3k \left\{ a^*(\vec{k})\, a(\vec{k}) + b(\vec{k})\; b^*(\vec{k}) \right\} k^\mu , \\ Q &= \int d^3k \left\{ a^*(\vec{k})\, a(\vec{k}) - b(\vec{k})\; b^*(\vec{k}) \right\} . \end{aligned} \tag{62}$$

Spinor field $\psi(x)$:

$$\mathcal{L} = i\, \overline{\psi}(x)\gamma^\mu \frac{\partial \psi(x)}{\partial x^\mu} - m\overline{\psi}(x)\psi(x),$$

$$T^{\mu\lambda} = \frac{1}{2}\, g^{\alpha\mu} \left(\overline{\psi}(x)\gamma^\lambda \frac{\partial \psi(x)}{\partial x^\alpha} - \frac{\partial \overline{\psi}(x)}{\partial x^\alpha}\gamma^\lambda \psi(x) \right), \tag{63}$$

$$j^\nu = \overline{\psi}(x)\gamma^\nu \psi(x),$$

which, with (47) gives:

$$\begin{aligned} P^\mu &= \int d^3p \sum_r \left\{ a^*(r,\vec{p})\, a(r,\vec{p}) - b(r,\vec{p}) b^*(r,\vec{p}) \right\} p^\mu , \\ Q &= \int d^3p \sum_r \left\{ a^*(r,\vec{p})\, a(r,\vec{p}) + b(r,\vec{p}) b^*(r,\vec{p}) \right\} . \end{aligned} \tag{64}$$

I,7. Fundamental commutation rules of the field operators. Normal products. Spin and statistics.

In (61) and (63) we have written the complex conjugate of the classical field variables at the left of the field variables, in x - space. If we keep this order, it is seen from (46) that in momentum space, a^* will be at the left of a, b at the left of b^*. This order is irrelevant in classical theory, where the fields are ordinary functions. In quantum theory, it is important because the fields are operators.

On the other hand, (62) shows that the energy is positive definite for scalar fields and the charge is not (this is true for boson fields, i.e., tensor fields). But by (64) the energy would be non-positive definite for spinor fields while the charge would be positive definite. We do not want the energy to have negative values. How can we prevent this for spinor fields?

First, we remark that in quantum theory, we want that the energy-momentum and the charge of free scalar fields be of the form

$$\begin{aligned} P^\mu &= \int d^3k \, [n_+ (\vec{k}) + n_- (\vec{k})] \, k^\mu \, , \\ Q &= \int d^3k \, [n_+ (\vec{k}) - n_- (\vec{k})] \, , \end{aligned} \tag{65}$$

where $\int n_+(\vec{k}) d^3k$ and $\int n_-(\vec{k}) d^3k$ are operators of which the eigenvalues are non-negative integral numbers, which are, respectively, the number of positive and negative particles. This is obtained if we impose the following commutation rules:

$$[a(\vec{k}), a^+ (\vec{k}')] = \delta(\vec{k} - \vec{k}'),$$

$$[b(\vec{k}), b^{+}(\vec{k'})] = \delta(\vec{k} - \vec{k'})$$

$$[a(\vec{k}), a(\vec{k'})] = [b(\vec{k}), b(\vec{k'})] = [a(\vec{k}), b(\vec{k'})] = [a(\vec{k}), b^{+}(\vec{k})] = 0. \tag{66}$$

Then

$$n_{+}(\vec{k}) = a^{+}(\vec{k})\, a\, (\vec{k}),$$
$$n_{-}(\vec{k}) = b^{+}(\vec{k})\, b\, (\vec{k}). \tag{67}$$

Indeed, it follows from (52) that if $\Psi(0)$ is the vacuum state ($P_\nu \Psi(0) = 0$):

$$P_\nu(a(\vec{p})\Psi(0)) = -p_\nu(a(\vec{p})\Psi(0))$$

and as the energy P_o must be positive definite, then

$$a(\vec{p})\,\Psi_o = 0$$

for $p_o > 0$.

From (51), one deduces that:

$$\psi(n_1(\vec{k}_1), \dots n_s(\vec{k}_s)) \equiv [a^{+}(\vec{k}_1)]^{n_1} \dots [a^{+}(\vec{k}_s)]^{n_s}\psi(0)$$

is a state with n_1 particles with momentum $\vec{k}_1, \dots, n_s$ particles with momentum $\vec{k}_s$.

Application of the commutation rules (66) and of the last two equations will show in a straightforward way that:

$$\int a^{+}(\vec{k})\, a\, (\vec{k})\, d^3k\, \psi(n_1(\vec{k}_1), \dots, n_s(\vec{k}_s)) =$$
$$= [n_1(\vec{k}_1) + \dots + n_s(\vec{k}_s)]\, \psi(n_1(\vec{k}_1), \dots, n_s(\vec{k}_s)).$$

However, it is also possible to impose, alternatively, the following <u>anticommutation</u> rules:

$$\{a(\vec{k}), a^{+}(\vec{k'})\}_+ = a(\vec{k})\, a^{+}(\vec{k'}) + a^{+}(\vec{k'})\, a\, (\vec{k}) = \delta(\vec{k} - k'),$$

$$\{b(\vec{k}), b^{+}(\vec{k'})\}_+ = \delta(\vec{k} - \vec{k'}),$$

$$\left\{a(\vec{k}),a(\vec{k}')\right\}_+ = \left\{b(\vec{k}),b(\vec{k}')\right\}_+ = \left\{a(\vec{k}),b(\vec{k}')\right\}_+ = \left\{a(\vec{k}),b^\dagger(\vec{k}')\right\}_+ = 0. \qquad (68)$$

In this case, the number operators will have as eigenvalues only 0 and 1. Replace $a(\vec{k})$ by $a(r,\vec{k})$, $b(\vec{k})$ by $b(r,\vec{k})$ and $\delta(\vec{k}-\vec{k}')$ by $\delta_{rr'}(\vec{k}-\vec{k}')$, when there is a polarization index r.

Which quantization procedure must we adopt, (66) or (68)?

If we chose (66) for the scalar field, you see that the expression (62) will not give (65) but rather:

$$P^o = \int d^3k\left[n_+(\vec{k}) + n_-(\vec{k})\right]k^o \quad + \text{ infinite energy due to}$$

the δ - term in (66).

The free-field vacuum state must be the one with least energy, i.e., zero. To get rid of this infinite energy for the vacuum, we may redefine all the operators like the lagrangean, energy momentum tensor, current, etc., such as (61), (62), (63), (64), by imposing the condition that the products of field operators be <u>normal products</u>, i.e., that they be ordered by displacing the emission operators always to the left of absorption operators, the sign of the commutation or anticommutation of the operators being taken in this displacement, according to (66) or (68), but the δ-term being discarded. Represent the normal product of two operators $O_1\ O_2$ by $:O_1\ O_2:$

Thus, if b, $b^\dagger$, obey (66) :

$$:b(\vec{k})\ b^\dagger(\vec{k}): \ = \ b^\dagger(\vec{k})\ b(\vec{k});$$

if, however, they satisfy (68):

$$:b(\vec{k})\ b^\dagger(\vec{k}): \ = -b^\dagger(\vec{k})\ b(\vec{k}).$$

Let us then assume that the quantities (61), (63) are, in quantum theory, normal products. The vacuum expectation value of P^μ, Q, etc., will vanish, as required.

But now we see which quantization rule to adopt for which type of field: The scalar (and, in general, the tensor) field must obey the commutation rule (66). The spinor fields must obey the anticommutation rules (68).

Thus, (64) is:

$$\begin{aligned} P^\mu &= :\int d^3p \sum_r \left\{ a^\dagger(r,\vec{p})\, a(r,\vec{p}) - b(r,\vec{p})\, b^\dagger(r,\vec{p}) \right\} p^\mu : = \\ &= \int d^3p \sum_r \left\{ a^\dagger(r,\vec{p})\, a(r,\vec{p}) + b^\dagger(r,\vec{p})\, b(r,\vec{p}) \right\} p^\mu , \\ Q &= :\int d^3p \sum_r \left\{ a^\dagger(r,\vec{p})\, a(r,\vec{p}) + b(r,\vec{p})\, b^\dagger(r,\vec{p}) \right\} : = \\ &= \int d^3p \sum_r \left\{ a^\dagger(r,\vec{p})\, a(r,\vec{p}) - b^\dagger(r,\vec{p})\, b(r,\vec{p}) \right\} , \end{aligned} \tag{69}$$

for spinor fields, thanks to (68).

And (62) is:

$$\begin{aligned} P^\mu &= :\int d^3k \left\{ a^\dagger(\vec{k})\, a(\vec{k}) + b(\vec{k})\, b^\dagger(\vec{k}) \right\} k^\mu : = \\ &= \int d^3k \left\{ a^\dagger(\vec{k})\, a(\vec{k}) + b^\dagger(\vec{k})\, b(\vec{k}) \right\} k^\mu , \\ Q &= :\int d^3k \left\{ a^\dagger(\vec{k})\, a(\vec{k}) - b(\vec{k})\, b^\dagger(\vec{k}) \right\} : = \\ &= \int d^3k \left\{ a^\dagger(\vec{k})\, a(\vec{k}) - b^\dagger(\vec{k})\, b(\vec{k}) \right\} , \end{aligned} \tag{70}$$

for scalar fields (and in general for tensor fields, if you include a polarization variable λ), thanks to (66).

You see that the requirement that the energy be always positive - definite imposes that tensor fields be quantized according to (66), spinor fields, according to (68).

As the occupation numbers, eigenvalues of the integrals of (67), can be any non-negative integer, in the case (66), this corresponds to the Bose-Einstein statistics, and the tensor fields and particles are called bosons.

The occupation numbers, in the case (68), can only be 0 and 1; this corresponds to statistics which incorporate the Pauli exclusion principle, the Fermi-Dirac statistics, and the spinor fields are accordingly fermions.

This connection between spin and statistics was discovered in 1940 by W. Pauli.

The exclusion principle, in case (68), results from the fact that:

$$a(r,\vec{p})\,a(r,\vec{p}) = a^{+}(r,\vec{p})\,a^{+}(r,\vec{p}) = b(r,\vec{p})\,b(r,\vec{p}) = b^{+}(r,\vec{p})\,b^{+}(r,\vec{p}) = 0.$$

If you now work back from (66) and (68), through (45), you will find the commutation rules in coordinate space:

$$[\varphi^{+}(x), \varphi(y)] = i\Delta(x-y); \quad [\varphi(x), \varphi(y)] = 0$$

for a scalar field;

$$[A^{\mu}(x), A^{\nu}(y)] = -ig^{\mu\nu} D(x-y) \tag{71}$$

for the electromagnetic field;

$$\{\psi_{\alpha}(x), \overline{\psi}_{\beta}(y)\}_{+} = -iS_{\alpha\beta}(x-y); \; \{\psi_{\alpha}(x), \psi_{\beta}(y)\}_{+} = 0$$

for a spinor field. Here:

$$\Delta(x) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{k^0}\, e^{+i\vec{k}\cdot\vec{x}} \sin k^0 x^0 \,,$$

$$k^0 = (\vec{k}^2 + m^2)^{\frac{1}{2}} \; ; \; D(x) = \Delta(x) \quad \text{for} \quad m = 0 \; ;$$

$$S(x) = -(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + m)\Delta(x). \tag{72}$$

These functions satisfy:

$$(\Box + m^2)\,\Delta\,(x) = 0,$$

$$\Delta(x) = 0 \quad \text{for a space-like vector } x,$$

$$\left(\frac{\partial \Delta(x)}{\partial x^0}\right)_{x^0=0} = -\delta(\vec{x}), \tag{73}$$

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\, S\,(x) = 0$$

and are the ones which solve the Cauchy problem for free fields.

Thus the <u>commutation rules (71) hold only for field operators</u> which satisfy the free-field equations.

On the other hand, <u>the more restricted rules</u>:

$$\left[\varphi^+(x),\varphi(y)\right] = 0 \text{ for } (x - y) \text{ a space-like vector},$$

$$\left[\frac{\partial}{\partial x^0}\varphi^+(x),\varphi(y)\right]_{y^0=x^0} = -i\,\delta(\vec{x}-\vec{y}), \tag{74}$$

$$\left[A^\mu(x),\ A^\nu(y)\right] = 0 \text{ for } (x-y) \text{ space-like},$$

$$\left[\frac{\partial A^\mu(x)}{\partial x^0},\ A^\nu(y)\right]_{y^0=x^0} = ig^{\mu\nu}\delta(\vec{x}-\vec{y}),$$

$$\left\{\psi(x),\ \psi^+(y)\right\}_+ = \delta(\vec{x}-\vec{y}) \quad \text{for } x^0 = y^0\ ,$$

still <u>hold when there is an interaction</u> among the fields, which does not involve field derivatives.

I,8. <u>Interactions</u>

The physical processes are due to the interaction among fundamental fields. An interaction between two wave fields $0_1(x)$ and $0_2(x)$ is taken into account by adding to the two free-field lagrangeans of $0_1(x)$ and $0_2(x)$, a third term $\mathcal{L}'$, the interaction lagrangean, which is constructed as a normal product of expressions formed with $O_1(x)$ and $O_2(x)$, in such a way that $\mathcal{L}'$ be hermitian and invariant under the proper Lorentz group. Further requirements may be imposed on $\mathcal{L}'$ for specific cases: the coupling of charged fields with the electromagnetic field must be gauge - invariant, the interaction of nucleons with pions, in the absence of electromagnetic effects, must be charge independent, the strong couplings of baryons with mesons must be, as far as we now know, invariant under space reflection, charge conjugation and time reversal, separately. The last requirement must not be imposed for weak interactions, such as the Fermi coupling among spinor fields, except for invariance under the reversal. Experiment is our guide in the choice of these impositions, as it has been, so far, for the Lorentz invariance of the theory.

The interaction Lagrangean gives rise to terms in each field equation which make it depend on the other fields. The wave equations are coupled together.

We shall consider only local interactions, for which $O_1(x)$ and $O_2(x)$ are taken at the same point x, in $\mathcal{L}'$.

It follows from (7) and (15) that the adjoint of a Dirac spinor field, $\bar{\psi}(x)$, transforms, under a proper Lorentz transformation, in the following way:

$$\bar{\psi}'(x') = \bar{\psi}(x)\, D^{-1} \tag{75}$$

Then, with the help of (7), (8) and (75), you will show that, if $\psi_1(x)$ and $\psi_2(x)$ are two spinor fields (representing, for example, protons and neutrons, respectively), $\bar{\psi}_1(x)\,\psi_2(x)$ is invariant with respect to the Lorentz group, a scalar S; $\bar{\psi}_1(x)\,\gamma^\mu\,\psi_2(x)$ is a four-

V; $\frac{i}{2}\bar{\psi}_1(x)[\gamma^\mu,\gamma^\nu]\psi_2(x)$ is an antisymmetric tensor T. It will be shown in the next chapter that, in addition, the following forms, $\bar{\psi}_1(x)\gamma^\mu\gamma^5\psi_2(x)$ and $i\bar{\psi}_1(x)\gamma^5\psi_2(x)$ are a pseudovector (or axial vector) A and pseudoscalar P, respectively, because under space reflections they behave as such (here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$).

Thus we have the table

Tensor character	Bilinear form	Hermitian conjugate
S	$\bar{\psi}_1(x)\psi_2(x)$	$\bar{\psi}_2(x)\psi_1(x)$
V	$\bar{\psi}_1(x)\gamma^\mu\psi_2(x)$	$\bar{\psi}_2(x)\gamma^\mu\psi_1(x)$
T	$\frac{i}{2}\bar{\psi}_1(x)[\gamma^\mu,\gamma^\nu]\psi_2(x)$	$\frac{i}{2}\bar{\psi}_2(x)[\gamma^\mu,\gamma^\nu]\psi_1(x)$
A	$\bar{\psi}_1(x)\gamma^\mu\gamma^5\psi_2(x)$	$\bar{\psi}_2(x)\gamma^\mu\gamma^5\psi_1(x)$
P	$i\bar{\psi}_1(x)\gamma^5\psi_2(x)$	$i\bar{\psi}_2(x)\gamma^5\psi_1(x)$

Remember that we have chosen:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \;, \; (\gamma^5)^\dagger = \gamma^5,$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 .$$

The following are some examples of interacting fields:

1) <u>Charged fermions and electromagnetic field</u>:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}' ,$$

$$\mathcal{L}_1 = -\frac{1}{2} : \frac{\partial A^\mu(x)}{\partial x^\nu} \frac{\partial A_\mu(x)}{\partial x_\nu} : ,$$

$$\mathcal{L}_2 = : i\bar{\psi}(x)\gamma^\mu \frac{\partial \psi(x)}{\partial x^\mu} - m\bar{\psi}(x)\psi(x) : , \tag{77}$$

$$\mathcal{L}' = -e : \bar{\psi}(x)\gamma^\mu\psi(x) A_\mu(x) : .$$

The field equations of motion are:

$$\left\{ \gamma^{\mu}\left(i \frac{\partial}{\partial x^{\mu}} - e A_{\mu}(x)\right) - m \right\} \psi(x) = 0 ,$$

$$\Box A^{\mu}(x) = e \bar{\psi}(x) \gamma^{\mu} \psi(x) , \tag{78}$$

and the Lorentz supplementary condition on the state vectors:

$$\frac{\partial A_{\mu}^{(+)}(x)}{\partial x_{\mu}} \Psi = 0 . \tag{79}$$

The equations are obtained in the following way. With $\mathcal{L}$, one obtains $T^{\mu\nu}$ and P^{ν}. With P^{ν}, one uses equation (37), where $O_i(x)$ is $\psi(x)$ and $A^{\mu}(x)$. To calculate the commutator in (37), one needs commutation rules for the fields. These cannot be (71) because they would lead to free field equations. The commutation rules (74), which refer to fields taken at the same time, and the additional condition that $\psi(x)$ and $A_{\mu}(y)$ commute for $y^{0} = x^{0}$, will give the equations (78).

Commutation rules (74) can be written in a more general fashion, if σ designates a space-like surface and $F(x)$ is an arbitrary function:

$$\int d\sigma_{\mu} \left[\varphi^{\dagger}(x), \varphi(y)\right] = 0 \quad \text{for } y \text{ in } \sigma ;$$

$$\int d\sigma_{\mu} \left[\frac{\partial \varphi^{\dagger}(x)}{\partial x_{\mu}}, \varphi(y)\right] F(x) = -iF(y), \quad y \text{ in } \sigma ;$$

$$\int d\sigma_{\mu} \left[A^{\lambda}(x), A^{\nu}(y)\right] = 0, \quad y \text{ in } \sigma ; \tag{86}$$

$$\int d\sigma_{\mu} \left[\frac{\partial A^{\lambda}(x)}{\partial x_{\mu}}, A^{\nu}(y)\right] F(x) = i g^{\lambda\nu} F(y), \quad y \text{ in } \sigma ;$$

$$\int d\sigma_{\mu} \left\{\psi(x), \bar{\psi}(y) \gamma^{\mu}\right\}_{+} F(x) = F(y), \quad y \text{ in } \sigma ;$$

$$\int d\sigma_{\mu} \left\{\psi(x), \psi(y)\right\} = 0, \quad y \text{ in } \sigma ;$$

$$\int d\sigma_{\mu} \left[A_{\mu}(x), \psi(y)\right] = 0, \quad y \text{ in } \sigma .$$

Equations (78) give the time development of the field operators and this constitutes the so-called Heisenberg representation or picture, in which the state vectors ψ are regarded as fixed in time.

(77) is gauge-invariant, i.e., invariant under the transformations:

$$A'_\mu(x) = A_\mu(x) \frac{\partial\Lambda(x)}{\partial x^\mu} ,$$
$$\psi'(x) = e^{ie\Lambda(x)}\psi(x), \tag{81}$$
$$\bar{\psi}'(x) = e^{-ie\Lambda(x)}\bar{\psi}(x),$$

where $\Lambda(x)$ is an ordinary function which obeys the equation $\Box\Lambda(x) = 0$.

You will notice that $\mathcal{L}'$ was made out of the contraction of $A_\mu(x)$ with $\bar{\psi}(x)\gamma^\mu\psi(x)$, which guarantees its relativistic invariance. A contraction of $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ with $A_\mu(x)$ would destroy gauge-invariance (under (81)). Thus, it seems that the gauge-invariance requirement leads to a space-reflection invariant coupling.

The fact that the electromagnetic effects are described by the interaction (77), without the need of an additional term in the lagrangean, of the form

$$\frac{1}{2}\bar{\psi}(x)[\gamma^\mu,\gamma^\nu]\psi(x)F_{\mu\nu}(x) \quad \text{, where} \quad F_{\mu\nu}(x) = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} ,$$

(such a term results in the second order equation), has led Gell-Mann to propose the principle of minimal electromagnetic interaction, which restricts all direct electromagnetic couplings to those with $A_\mu(x)$ in the lagrangean.

2) Nucleons and pions. Charge independence and charge symmetry.

Positive and negative pions are described by a non-hermitian pseudo-scalar field operator $\varphi(x)$, neutral pions by a hermitian pseudo-scalar field $\varphi_3(x)$. The proton field is a Dirac spinor $\psi_p(x)$, the neutron field, another Dirac spinor $\psi_n(x)$.

The lagrangean is:

$$\mathcal{L} = \mathcal{L}_n + \mathcal{L}_\pi + \mathcal{L}' ,$$

$$\begin{aligned} \mathcal{L}_n &= :i\,\bar{\psi}_p(x)\gamma^\mu \frac{\partial \psi_p(x)}{\partial x^\mu} - M\, \bar{\psi}_p(x)\psi_p(x): + \\ &+ :i\bar{\psi}_n(x)\gamma^\mu \frac{\partial \psi_n(x)}{\partial x^\mu} - M\,\bar{\psi}_n(x)\psi_n(x): , \\ \mathcal{L}_\pi &= -:(\mu^2 \varphi^+(x)\varphi(x) - \frac{\partial\varphi^+(x)}{\partial x^\mu}\frac{\partial\varphi(x)}{\partial x_\mu}: - \\ &- \frac{1}{2}:(\mu^2\varphi_3^2(x) - \frac{\partial\varphi_3(x)}{\partial x^\mu}\frac{\partial\varphi_3(x)}{\partial x_\mu}: , \\ \mathcal{L}' &= ig_c :\left[(\bar{\psi}_p(x)\gamma^5\psi_n(x))\varphi(x) + \right. \\ &\left. + (\bar{\psi}_n(x)\gamma^5\psi_p(x)\,\varphi^+(x)\right]: + \\ &+ ig_p:\bar{\psi}_p(x)\gamma^5\psi_p(x)\varphi_3(x): + ig_n:\bar{\psi}_n(x)\gamma^5\psi_n(x)\varphi_3(x): , \end{aligned} \tag{82}$$

with an obvious meaning for the different terms and the coupling constants g_c, g_p, g_n; and the g_c terms couple charged pions with transitions between neutron and proton states; the g_p and g_n terms couple neutral pions with proton and neutron states respectively; μ is the pion mass, M the mass of neutron and proton, assumed equal.

The charge-independent theory, first proposed by Kemmer, makes

$$\frac{1}{\sqrt{2}} g_c = g_p = -g_n \equiv g \ . \qquad (83)$$

Introduce an 8-component spinor:

$$\psi(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} \qquad (84)$$

and the isobaric spin matrices τ_1, τ_2, τ_3 such that:

$$\tau_1 \psi(x) = \begin{pmatrix} \psi_n(x) \\ \psi_p(x) \end{pmatrix} , \qquad \tau_2 \psi(x) = \begin{pmatrix} -i\psi_n(x) \\ i\psi_p(x) \end{pmatrix} ,$$

$$\tau_3 \psi(x) = \begin{pmatrix} \psi_p(x) \\ -\psi_n(x) \end{pmatrix} . \qquad (85)$$

Then:

$$\tau_+ = \frac{1}{\sqrt{2}} (\tau_1 + i\tau_2), \quad \tau_- = \frac{1}{\sqrt{2}}(\tau_1 - i\tau_2) , \qquad (86)$$

give:

$$\bar{\psi}(x) \tau_+ \Gamma \psi(x) = \sqrt{2}\ \bar{\psi}_p(x) \Gamma \psi_n(x) ,$$

$$\bar{\psi}(x) \tau_- \Gamma \psi(x) = \sqrt{2}\ \bar{\psi}_n(x) \Gamma \psi_p(x) ,$$

where Γ is any of the 16 γ - matrices which act on $\psi(x)$ as follows:

$$\Gamma\psi(x) = \begin{pmatrix} \Gamma\psi_p(x) \\ \Gamma\psi_n(x) \end{pmatrix}, \quad [\Gamma, \tau_i] = 0 \ . \qquad (87)$$

One then finds, with (83):

$$\mathcal{L}' = ig: \sum_{i=1}^{3} \bar{\psi}(x) \tau_i \gamma^5 \psi(x) \varphi_i(x): , \qquad (88)$$

where:

$$\varphi(x) = \frac{1}{\sqrt{2}} (\varphi_1(x) - i\varphi_2(x)) \qquad (89)$$

and $\varphi_1(x)$ and $\varphi_2(x)$ are hermitian.

We see that now $\psi(x)$, as defined by (84), has two kinds of spinor indices, one $\alpha = 1, 2, 3, 4$, defines it as a 4-component Dirac spinor, the other $i = 1, 2$, defines it as a 2-component spinor in the isobaric spin space. The matrices τ_1, τ_2, τ_3 are the infinitesimal operators which generate a rotation in this space. The pseudoscalar field has an isobaric spin index $i = 1, 2, 3$, which defines it as a 3-vector in the isobaric spin space. The coupling (88) is Lorentz-invariant and, in addition, invariant under rotations in the isobaric spin space. This is the requirement of charge independence, which holds for (83).

Thus we can write (82) in a more compact form:

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi + \mathcal{L}' ,$$

$$\mathcal{L}_N = :\bar{\psi}(x)(i\gamma^\alpha \frac{\partial}{\partial x^\alpha} - M)\psi(x): ,$$

$$\mathcal{L}_\pi = -\frac{1}{2}\sum_{i=1}^{3} :(\mu^2\varphi_i^2(x) - \frac{\partial\varphi_i(x)}{\partial x^\alpha}\frac{\partial\varphi_i(x)}{\partial x_\alpha}): ,$$

$$\mathcal{L}' = ig : \sum_{i=1}^{3} \bar{\psi}(x)\,\tau_i\,\gamma^5\psi(x)\,\varphi_i(x): .$$

A special rotation in the isobaric spin space is the one around the first axis in this space by an angle π. The transformed pion field $\varphi_i'(x)$ will be

$$\varphi_1'(x) = \varphi_1(x) \;,\; \varphi_2'(x) = -\varphi_2(x) \;,\; \varphi_3'(x) = -\varphi_3(x) \;. \tag{90}$$

One then sees that the nucleon field $\psi(x)$ transforms in the following way:

$$\psi'(x) = \tau_1\psi(x) \;. \tag{91}$$

In terms of $\varphi(x)$, (89), $\psi_p(x)$ and $\psi_n(x)$, (84) one gets:

$$\begin{aligned} \varphi'(x) &= \varphi^{+}(x) \;, \\ \varphi^{+\prime}(x) &= \varphi(x), \\ \psi_p'(x) &= \psi_n(x) \;, \\ \psi_n'(x) &= \psi_p(x) \;. \end{aligned} \qquad (92)$$

This is the so-called charge symmetry operation, under which protons are replaced by neutrons, neutrons by protons, and positive and negative pions are interchanged. Nuclei have this symmetry, as known from the fact that the energies of the ground states of two mirror nuclei are equal, except for a small amount due to the coulomb energy of protons.

While a charge-independent theory is automatically charge-symmetric, the reverse is not true, as is obvious by the above considerations.

3) Fermi interactions

This is the coupling which we want to describe processes like the neutron beta-decay:

$$n \longrightarrow p + e \quad \bar{\nu}$$

the neutron decays into a proton, an electron and an antineutrino.

With the help of the table (76), we can construct an interaction lagrangean suitable for this purpose:

$$\mathcal{L}' = :(\bar{\psi}_p \psi_n)(\bar{\psi}_e [C_s - C_s' \gamma^5] \psi_\nu) +$$

$$+ (\bar{\psi}_p \gamma^\mu \psi_n)(\bar{\psi}_e \gamma_\mu [C_V - C_V' \gamma^5] \psi_\nu) +$$

$$+ (\bar{\psi}_p \tfrac{1}{2} [\gamma^\mu, \gamma^\nu] \psi_n)(\bar{\psi}_e \tfrac{1}{2} [\gamma_\mu, \gamma_\nu] (C_T - C_T' \gamma^5) \psi_\nu) +$$

$$+ (\bar{\psi}_p \gamma^\mu \gamma^5 \psi_n)(\bar{\psi}_e \gamma_\mu \gamma^5 [C_A - C'_A \gamma^5] \psi_\nu) +$$

$$+ (\bar{\psi}_p i \gamma^5 \psi_n)(\bar{\psi}_e i \gamma^5 [C_p - C'_p \gamma^5] \psi_\nu) +$$

$$+ (\bar{\psi}_n \psi_p)(\bar{\psi}_\nu [C_s^* + C_s'^* \gamma^5] \psi_e) +$$

$$+ (\bar{\psi}_n \gamma^\mu \psi_p)(\bar{\psi}_\nu \gamma_\mu [C_V^* - C_V'^* \gamma^5] \psi_e) +$$

$$+ (\bar{\psi}_n \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \psi_p)(\bar{\psi}_\nu \tfrac{1}{2} [\gamma_\mu, \gamma_\nu](C_T^* + C_T'^* \gamma^5) \psi_e) +$$

$$+ (\bar{\psi}_n \gamma^\mu \gamma^5 \psi_p)(\bar{\psi}_\nu \gamma_\mu \gamma^5 [C_A^* - C_A'^* \gamma^5] \psi_e) +$$

$$+ (\bar{\psi}_n i \gamma^5 \psi_p)(\bar{\psi}_\nu i \gamma^5 [C_p^* + C_p'^* \gamma^5] \psi_e):$$

$\mathcal{L}'$ is hermitian and Lorentz invariant. Invariance under space reflection, charge conjugation and time reversal depends on the choice of the coupling constants C and C', as will be seen in the next chapter. The last five terms are the hermitian conjugate of the first five and describe the inverse process:

$$p \rightarrow n + \bar{e} + \nu \, .$$

The fields are taken at the same point x.

Generalizations to process like muon $e + \nu + \bar{\nu}$ can be made.

4) <u>Pions and the electromagnetic field.</u>

In this case, the interaction lagrangean is:

$$\mathcal{L}' = ie: A^\mu (x)(\varphi^+ (x) \frac{\partial \varphi}{\partial x^\mu} - \frac{\partial \varphi^+}{\partial x^\mu} \varphi(x)) -$$

$$- e^2 A^\mu (x) A_\mu(x) \varphi^+(x) \varphi(x). \quad (94)$$

CHAPTER II

Space Reflections and Parity: Bose Fields

II, 1. Parity in non-relativistic quantum mechanics

Let
$$-\frac{1}{2m}\nabla^2\psi(\vec{x},t) + V(r)\psi(\vec{x},t) = i\,\frac{\partial\psi(\vec{x},t)}{\partial t} \tag{95}$$
be the Schrödinger equation of a particle which moves in a central field in which it has a time-independent potential energy $V(r)$, $r = |\vec{x}|$.

It is well known that the stationary states of this system are specified by three quantum numbers n, ℓ, m and the corresponding solutions of the wave equation are
$$\psi_{n\ell m}(\vec{x},t) = R_{n\ell}(r)\,Y_{\ell m}(\theta,\varphi)e^{-iE_n t} \tag{96}$$
where r, θ, φ are the polar coordinates, $R_{n\ell}(r)$ is the radial wave function, $Y_{\ell m}(\theta,\varphi)$ are the spherical harmonics, E_n, the energy of the state. Normalization coefficients are included in the functions.

Let us now introduce new, space-reflected coordinates:
$$\vec{x}' = -\vec{x}, \quad t' = t \tag{97}$$
and look for the transformed wave functions
$$\psi'(\vec{x}', t') = P\psi(\vec{x}', t) \tag{98}$$
which satisfy the same equation in the new system, as $\psi(\vec{x},t)$ does in the old one:
$$-\frac{1}{2m}\nabla'^2\psi'(\vec{x}',t') + V'(r')\psi'(\vec{x}', t') = i\,\frac{\partial\psi'(\vec{x}',t')}{\partial t'}. \tag{99}$$

Because:

$$V'(r') = V(r)$$
$$\nabla'^2 = \nabla^2 \, , \quad t' = t$$

we see that the coefficients of ψ' in (99) are the same as those of ψ in (95).

We can therefore write:

$$\psi'(\vec{x}',t) = P\psi(\vec{x}',t) = \epsilon\,\psi(\vec{x},t) \, ,$$

where ϵ is an indeterminate phase factor:

$$\epsilon\epsilon^* = 1 \, .$$

Thus:

$$P\psi(\vec{x},t) = \epsilon\,\psi(-\vec{x},t).$$

But by a well known property of $Y_{\ell m}(\theta,\varphi)$:

$$\psi_{n\ell m}(-\vec{x},t) = (-1)^{\ell}\,\psi_{n\ell m}(\vec{x},t).$$

So:

$$P\,\psi_{n\ell m}(\vec{x},t) = (-1)^{\ell}\epsilon\psi_{n\ell m}(\vec{x},t) \, . \tag{100}$$

We see that, under the parity operation P (in the space of wave functions), the wave function of a state with angular momentum quantum number ℓ acquires a phase $(-1)^{\ell}$ relative to that of the s-state wave function. This phase is called the parity of the state.

We also see that the complex number of modulus one, ϵ, is indeterminate, as a phase factor of the (complex) wave function is. The relative parity of a state with respect to the S-state is well determined and independent of ϵ, namely $(-1)^{\ell}$.

One may, however, arbitrarily wish to attribute an even parity to the S-state and thus choose $\epsilon = 1$. As V(r) is even, it is conventional to say that the non-relativistic spinless particle described by (95) has an <u>even intrinsic parity</u>, it is a scalar particle.

The choice $\epsilon = -1$ gives an odd parity to the S-state and the particle is accordingly called a pseudoscalar particle, i.e., it has an <u>odd intrinsic parity</u>.

II, 2. <u>Parity in quantum field theory</u>.

The proper and orthochronous Lorentz transformations of fields and state vectors are determined by the requirement that the coupled field equations, or the complete lagrangean, be invariant under such transformations. It follows from (3), (4) and (5), (6),(7) that one has in the case:

$$(\Psi', O'_i(x')\Psi') = A_{ij}(\Psi, O_j(x)\Psi), \tag{101}$$

where A is a matrix which acts on the spinor or tensor index i of the operator O(x).

Thus, for a relativistically invariant operator

$$F\left(O_i(x), \frac{\partial O_i(x)}{\partial x^\mu}\right)$$

formed with the operator and its derivatives, such as the lagrangean, one has:

$$\left(\Psi', F'\left(O'_i(x'), \frac{\partial O'_i(x')}{\partial x^{\mu'}}\right)\Psi'\right) = \left(\Psi, F\left(O_i(x), \frac{\partial O_i(x)}{\partial x^\mu}\right)\Psi\right). \tag{102}$$

From (101) and: either $\psi' = \psi$, $O'(x) = U^+O(x)U$ in the Heisenberg-Lorentz transformation, or $\psi' = U\psi$ and $O'(x) = O(x)$ in the Schrödinger-Lorentz transformation, we obtain:

$$U\,O(x)\,U^{-1} = A^{-1}\,O\,(L^{-1}x)\ . \tag{103}$$

We shall assume the same relations for the space reflections and call P the parity operation in Hilbert space and S the matrix which acts on the spinor or tensor indices, R the special case of the 4-vector indices:

$$PO(x)P^{-1} = s\ S^{-1}O\ (R^{-1}x), \tag{104}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{105}$$

and s is a phase factor, $s^* s = 1$, also allowed in (103).

II,3. <u>Parity transformed of spinless fields</u>

Consider a <u>hermitian</u> spinless field $\varphi_o(x)$. Then S must be the identity. (104) and (105) give:

$$P\ \varphi_o(x)\ P^{-1} = s\ \varphi_o(-\vec{x}, x^o)\ . \tag{106}$$

If the free-field lagrangean is invariant under P:

$$s^2 = 1\ . \tag{107}$$

We call the field <u>scalar</u> for s = +1, <u>pseudoscalar</u> for s = = -1, if the vacuum state Ψ_o is even under P.

Let $\varphi(x)$ be a <u>non-hermitian</u> spinless field. Then:

$$P\,\varphi(x)\ P^{-1} = s\,\varphi(-\vec{x},x^o)\ ,$$

$$P\,\varphi^{\dagger}(x)\ P^{-1} = s^*\ \varphi^+(-x,x^o)\ . \tag{108}$$

Invariance of the free-field lagrangean imposes that s is a complex number of modulus 1:

$$s^*\ s = 1. \tag{109}$$

P is a unitary operator: it conserves the commutation rules (71) for $\varphi(x)$.

(46) and (108) will give you the transformed of the emission and absorption operators:

$$\begin{aligned} P\ a\ (\vec{k})P^{-1} &= s\ a\ (-\vec{k})\ , \\ P\ b\ (\vec{k})P^{-1} &= s^{*} b\ (-\vec{k}), \\ P\ a^{+}(\vec{k})P^{-1} &= s^{*} a^{+}(-\vec{k}), \\ P\ b^{+}(\vec{k})P^{-1} &= s\ b^{+}(-\vec{k})\ . \end{aligned} \tag{110}$$

The amplitude of a state with n particles is:

$$\Psi_n = \int F_n\ (\vec{k}_1,\dots\vec{k}_n)\ a^{+}\ (\vec{k}_1)\dots a^{+}\ (\vec{k}_n)\,\Psi_o\ d^3k_1\ \dots\ d^3k_n \tag{111}$$

where Ψ_o is the vacuum state.

The convention that the latter is even:

$$P\,\Psi_o = \Psi_o$$

leads to, because of (110):

$$P\,\Psi_n = (s^{*})^n \int F_n\ (-\vec{k}_1,\dots-\vec{k}_n)\ a^{+}(\vec{k}_1)\dots a^{+}(\vec{k}_n)\Psi_o d^3k_1\dots d^3k_n\ ,$$

so that, defining PF_n by :

$$P\Psi_n = \int \big(P\ F_n(\vec{k}_1,\dots\vec{k}_n)\big)\ a^{+}\ (\vec{k}_1)\dots\ a^{+}(\vec{k}_n)\,\Psi_o\ d^3k_1\dots\ d^3k_n\ ,$$

we have:

$$P\ F_n\ (\vec{k}_1,\dots\vec{k}_n) = (s^{*})^n\ F_n\ (-\vec{k}_1,\dots,-\vec{k}_n)\ .$$

The wave function of the n particles in coordinate space $\Phi\ (\vec{x}_1,\dots\vec{x}_n)$, Fourier-transformed of F_n:

$$F_n(\vec{k},\dots\vec{k}_n) = \frac{1}{(2\pi)^{3n/2}} \int e^{i(\vec{k}_1\cdot\vec{x}_1+\dots+\vec{k}_n\cdot\vec{x}_n)}\,\Phi_n(\vec{x}_1,\dots\vec{x}_n)d^3x_1\dots d^3x_n$$

will be transformed in the following way:

$$P\,\Phi_n\ (\vec{x}_1,\dots,\ \vec{x}_n) = (s^{*})^n\ \Phi_n(-\vec{x}_1,\dots,-\vec{x}_n), \tag{112}$$

The same result holds when the particles interact with an even static potential. If there are n_1 particles with angular momentum $\ell_1, \ldots n_j$, with angular momentum ℓ_j, then:

$$\Phi(-\vec{x}_1, \ldots - \vec{x}_n) = (-1)^{n_1\ell_1 + \ldots + n_j\ell_j}\, \Phi(\vec{x}_1, \ldots \vec{x}_n).$$

Therefore:

$$P\,\Phi_n(\vec{x}_1, \ldots, \vec{x}_n) = (s^*)^n\,(-1)^{n_1\ell_1 + \ldots + n_j\ell_j}\, \Phi(\vec{x}_1, \ldots, \vec{x}_n) .$$

As in II.1, we make the convention to call the particles scalar if s is arbitrarily chosen + 1, pseudoscalar if s = -1. The parity of a n-scalar-meson-state is $(-1)^{n_1\ell_1 + \ldots + n_j\ell_j}$, that of a n-pseudoscalar meson state is

$$-(-1)^{n_1\ell_1 + \ldots + n_j\ell_j}, \quad n_1 + n_2 + \ldots + n_j = n .$$

II,4. Parity of photons

In classical electrodynamics, one assumes that the charge density is invariant under space reflection:

$$\rho'(x) = \rho(-\vec{x}, x^0) .$$

It follows that charge-conservation invariance:

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

imposes:

$$j'(x) = -j(-\vec{x}, x^0) .$$

Invariance of Maxwell's equations will then lead to a polar electric field $\vec{E}$ and an axial magnetic field $\vec{H}$:

$$\vec{E}'(x) = -\vec{E}(-\vec{x}, x^0),$$

$$\vec{H}'(x) = \vec{H}(-\vec{x}, x^0) ,$$

hence:

$$\vec{A}'(x) = -\vec{A}(-\vec{x}, x^0),$$

$$A_0'(x) = A_0(-\vec{x}, x^0) .$$

We want these relations to hold in quantum theory, consistently with $S \equiv R$, $s = 1$, in (104), (105):

$$P \vec{A}(x) P^{-1} = -\vec{A}(-\vec{x}, x^o) ,$$
$$P A_o(x) P^{-1} = A_o(-\vec{x}, x^o) . \qquad (113)$$

The Fourier development (50) and the supplementary condition (79) give:

$$k^\mu \sum_\lambda e_\mu(\lambda,\vec{k})\, a(\lambda,\vec{k})\Psi = 0$$

and this, together with the choice:

$$\begin{aligned} k^\mu / |\vec{k}| &= (1, 0, 0, 1), \\ e^\mu(0,\vec{k}) &= (1, 0, 0, 0), \\ e^\mu(1,\vec{k}) &= (0, 1, 0, 0), \\ e^\mu(2,\vec{k}) &= (0, 0, 1, 0), \\ e^\mu(3,\vec{k}) &= (0, 0, 0, 1), \end{aligned} \qquad (113)'$$

imposes that:

$$a(0, \vec{k})\Psi = a(3, \vec{k})\Psi . \qquad (114)$$

The consequence of (114) is that the time-like and the longitudinal components of $a(\lambda, \vec{k})$ do not contribute to observables. Thus, the energy-momentum is:

$$P^\nu \Psi = \int d^3k\, k^\nu \left[a^+(1,\vec{k})\, a(1,\vec{k}) + a^+(2,\vec{k})\, a(2,\vec{k})\right]\Psi$$

and $P_o \Psi$ is positive definite.

We therefore need to consider only the development:

$$\vec{A}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^o}} \sum_{\lambda=1,2} \vec{e}(\lambda,\vec{k}) \left\{ a(\lambda,\vec{k})e^{-ikx} + a^+(\lambda,\vec{k})e^{ikx} \right\}$$

(115)

and the rules:

$$[a(1, \vec{k}), a^{\dagger} (1, \vec{k})] = [a (2, \vec{k}), a^{\dagger} (2, \vec{k})] = \delta(k - \vec{k}'),$$

$$[a (2, \vec{k}), a(2, \vec{k})] =[a (1, \vec{k}),a(1, \vec{k})] = [a(1,\vec{k}), a(2,\vec{k})] = \tag{116}$$

$$= [a(1, \vec{k}), a^{\dagger} (2, \vec{k})] = 0 .$$

The number of <u>linearly polarized photons</u> is $\int a^{\dagger}(\lambda, \vec{k})\, a(\lambda, \vec{k})\, d^3k$ with polarization $\lambda = 1$ or 2.

The spin angular momentum of the field (115) is, from (19), (21) and $\mathcal{L}_1$ in (77):

$$S^{jk} = :\int d^3x \left(\frac{\partial A^j}{\partial x^0} A^k - A^j \frac{\partial A^k}{\partial x^0}\right):$$

or:

$$S^3 = i\int d^3 k \left[a^{\dagger}(1,\vec{k})\, a(2,\vec{k}) - a^{\dagger}(2,\vec{k})\, a(1,\vec{k})\right] . \tag{117}$$

The one-linearly-polarized photon state is not an eigen-state of S^3. You will show, with the help of (117) and (116), that

$$S^3 \left[a^{\dagger}(1,\vec{k}) \Psi_0\right] = -i\, a^{\dagger} (2,\vec{k})\Psi_0 ,$$

$$S^3 \left[a^{\dagger}(2, \vec{k}) \Psi_0\right] = i\, a^{\dagger}(1, \vec{k})\Psi_0 . \tag{118}$$

If you now form the linear combinations:

$$a(R, \vec{k}) = \frac{1}{\sqrt{2}} \left[a(1,\vec{k}) + ia (2, \vec{k})\right] ,$$

$$a(L,\vec{k}) = \frac{1}{\sqrt{2}}\left[a(1, \vec{k}) - i\, a(2, \vec{k})\right], \tag{119}$$

you will find that:

$$S^3\left[a^{\dagger}(R,\vec{k})\Psi_0\right] = a^{\dagger}(R,\vec{k}) \Psi_0,$$

$$S^3\left[a^{\dagger}(L,k) \Psi_0\right] = -a^{\dagger}(L,\vec{k}) \Psi_0 . \tag{120}$$

(120) suggests to call $a^{\dagger}(R,\vec{k})\psi_o$ a one-right-circularly-polarized-photon-state, $a^{\dagger}(L, k)\psi_o$, a one-left-circularly-polarized-photon state.

$\int a^{\dagger}(R,\vec{k})\, a(R,\vec{k})\, d^3k$ and $\int a^{+}(L, \vec{k})\, a(L, \vec{k})\, d^3k$ are the number operators of right-and-left-circular photons, since (116) and (119) give the necessary commutation rules for the emission and absorption operators of right and left photons:

(115) can also be written:

$$\vec{A}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^o}} \Big\{ \left[\vec{e}(R,\vec{k})\, a(R,\vec{k}) + \vec{e}(L,\vec{k})\, a(L,\vec{k}) \right] e^{-ikx} + \left[\vec{e}^{\,*}(R,\vec{k})\, a^{+}(R,\vec{k}) + \vec{e}^{\,*}(L,\vec{k})\, a^{+}(L,\vec{k}) \right] e^{ikx} \Big\}, \tag{121}$$

with

$$\vec{e}(R,\vec{k}) = \frac{1}{\sqrt{2}} \left(\vec{e}(1,\vec{k}) - i\, \vec{e}(2, \vec{k}) \right),$$
$$\vec{e}(L, \vec{k}) = \frac{1}{\sqrt{2}} \left(\vec{e}(1, \vec{k}) + i\, \vec{e}(2,\vec{k}) \right). \tag{122}$$

Now, back to the parity transformation (113). First, observe that the three vectors $\vec{e}(1, \vec{k})$, $e(2, \vec{k})$ and $\vec{k}$ are related by

$$\frac{\vec{k}}{|\vec{k}|} = \vec{e}(1, \vec{k}) \wedge \vec{e}(2, \vec{k}) \tag{123}$$

From this:

$$\frac{-\vec{k}}{|\vec{k}|} = \vec{e}(1, -\vec{k}) \wedge \vec{e}(2,-\vec{k}) = -\vec{e}(1, \vec{k}) \wedge \vec{e}(2, \vec{k}).$$

We choose:

$$\vec{e}(1, -\vec{k}) = \vec{e}(1, \vec{k}),\ \vec{e}(2, -\vec{k}) = -\vec{e}(2, \vec{k}), \tag{124}$$

which means that a right-handed system goes over into a left-handed system, as it must be:

$$\vec{e}(R, -\vec{k}) = \vec{e}(L, \vec{k}). \tag{125}$$

Now then, (113), (115) and (124) will give us:

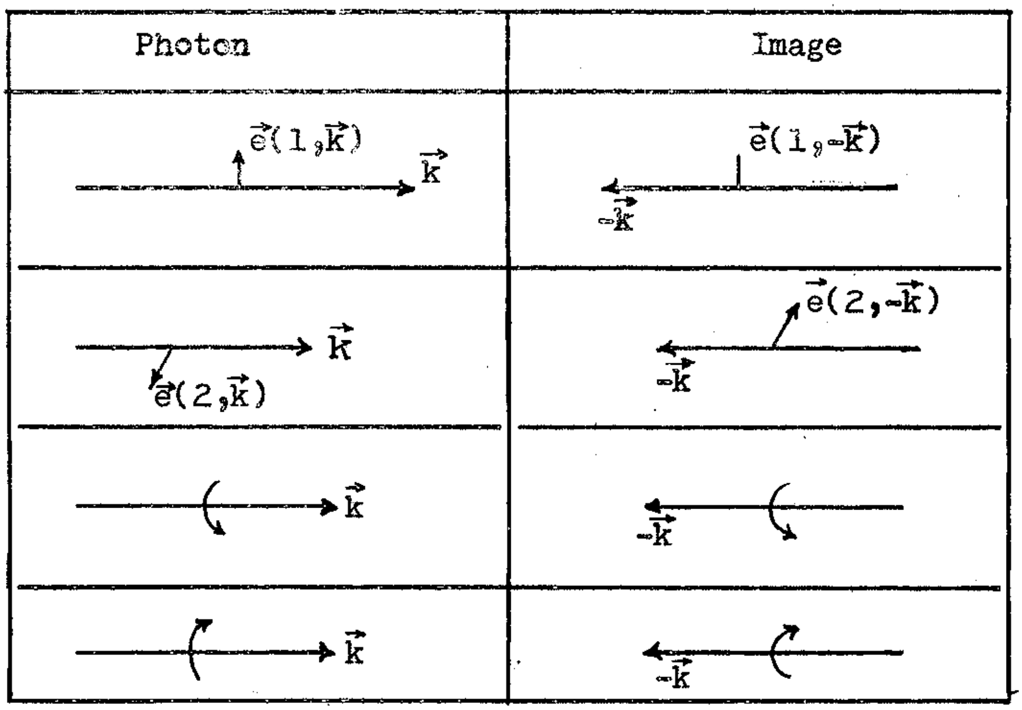
$$
\begin{aligned}
P\, a(1, \vec{k})\, P^{-1} &= -a(1, -\vec{k}),\\
P\, a(2, \vec{k})\, P^{-1} &= +a(2, -\vec{k}),\\
P\, a^{\dagger}(1, \vec{k})\, P^{-1} &= -a^{\dagger}(1, -\vec{k}),\\
P\, a^{\dagger}(2, \vec{k})\, P^{-1} &= +a^{\dagger}(2, -\vec{k}).
\end{aligned} \tag{126}
$$

From (119), then:

$$
\begin{aligned}
P\, a(R, \vec{k})\, P^{-1} &= -a(L, -\vec{k}),\\
P\, a(L, \vec{k})\, P^{-1} &= -a(R, -\vec{k}),\\
P\, a^{\dagger}(R, \vec{k})\, P^{-1} &= -a^{\dagger}(L, -\vec{k}),\\
P\, a^{\dagger}(L, \vec{k})\, P^{-1} &= -a^{\dagger}(R, -\vec{k}).
\end{aligned} \tag{127}
$$

Thus a one-linearly-polarized-photon state is an eigenstate of P but not of S^3; a one-circularly-polarized-photon state is an eigenstate of S^3 but not of P.

P transforms a right-photon moving in the $\vec{k}$-direction into left-photon moving into $-\vec{k}$-direction. See the table below (always under the assumption $P\Psi_o = \Psi_o$):



(128)

II, 5. Parity and spin of two-photon states. Selection rules for the decay of a neutral boson into two photons.

The following are the possible state vectors of two photons with momenta $\vec{k}$ and $-\vec{k}$:

$$
\begin{aligned}
\psi(+R, -R) &\equiv a^{+}(R, \vec{k})\, a^{+}(R, -\vec{k})\psi_o \,, \\
\psi(+L, -L) &\equiv a^{+}(L, \vec{k})\, a^{+}(L, -\vec{k})\psi_o , \\
\psi(+R, -L) &\equiv a^{+}(R, \vec{k})\, a^{+}(L, -\vec{k})\psi_o , \\
\psi(+L, -R) &\equiv a^{+}(L, \vec{k})\, a^{+}(R, -\vec{k})\psi_o ,
\end{aligned}
\tag{129}
$$

of which the following ones: $\psi(+R, -R) + \psi(+L, -L)$, $\psi(+R, -R) - \psi(+L, -L)$, $\psi(+R, -L)$, $\psi(+L, -R)$, are the eigenstates of both P and S^3, as you will easily show. The table (129) gives the corresponding eigenvalues:

Eigenvalues and Eigenstates of P and S^3

	$\psi(+R, -L)$	$\psi(+L, -R)$	$\psi(+R, -R)+\psi(+L, -L)$	$\psi(+R, -R)-\psi(+L,-L)$
P	even	even	even	odd
S	2	-2	0	0

(129)

If we make use of (119) we shall be able to write:

$$
\psi(+R,-R) + \psi(+L,-L) = (a^{+}(1,\vec{k})\, a^{+}(1,-\vec{k}) - a^{+}(2,\vec{k})\, a^{+}(2,-\vec{k})\psi_o
\tag{130}
$$

which shows that in this state the two photons have parallel polarization planes, with equal probability for the polarization being

$\lambda = 1$ and $\lambda = 2$. Also:

$$\Psi(+R,-R) - \Psi(+L,-L) = -i \left(a^{\dagger}(1,\vec{k})\, a^{\dagger}(2,-\vec{k}) + a^{\dagger}(2,\vec{k})\, a^{\dagger}(1,-\vec{k}) \right) \Psi_0 \tag{131}$$

in this state, the two photons have perpendicular polarization planes. Finally:

$$\Psi(+R,-L) = \frac{1}{2} \Big\{ a^{\dagger}(1,\vec{k})\, a^{\dagger}(1,-\vec{k}) + a^{\dagger}(2,+\vec{k})\, a^{\dagger}(2,-\vec{k}) + + i\, a^{\dagger}(1,\vec{k})\, a^{\dagger}(2,-\vec{k}) - i\, a^{\dagger}(2,\vec{k})\, a^{\dagger}(1,-\vec{k}) \Big\} \Psi_0 \tag{132}$$

$$\Psi(+L,-R) = \frac{1}{2} \Big\{ a^{\dagger}(1,\vec{k})\, a^{\dagger}(1,-\vec{k}) + a^{\dagger}(2,\vec{k})\, a^{\dagger}(2,-\vec{k}) - - i\, a^{\dagger}(1,\vec{k})\, a^{\dagger}(2,-\vec{k}) + i\, a^{\dagger}(2,\vec{k})\, a^{\dagger}(1,-\vec{k}) \Big\} \Psi_0 ,$$

the two photons, in each of these states have equal chances that their polarization planes be parallel or perpendicular.

From (129) and the interpretation given in (130), (131) and (132) we are now capable of giving selection rules for the decay of a neutral boson of spin 0 or 1 into two photons, which are indicated in the table (133), where $||$ means that the polarization planes are parallel, $\perp$ means that they are perpendicular to each other. The interaction responsible for the decay must be invariant under proper Lorentz transformations and under space reflections, conserving angular momentum J and parity P:

P \ J	0	1
even	$\|\|$	forbidden
odd	$\perp$	forbidden

(133)

The result is important:

1) a spin 1 neutral meson (vector or pseudovector) cannot decay into two photons:

2) positronium (bound state of an electron and a positron) in the triplet S-state, ^3S, cannot decay into two photons:

3) a scalar neutral meson can decay only into two $\parallel$ photons:

4) a pseudoscalar neutral meson decays into two $\perp$ photons;

5) positronium in the singlet S state ^1S, is odd, as will be shown later, and thus decays into two $\perp$ photons.

II, 6. Determination of the parity of bosons.

It was emphasized, from the beginning of this chapter, that the definition of the intrinsic parity of a particle depends on the arbitrary choice of the value + 1 or − 1 for a number s which is indeterminate to the extent that it must satisfy the relation $s^2 = 1$ for neutral bosons and $s^* s = 1$ for charged bosons, and on the arbitrary choice $\omega_o = 1$ in $P\Psi_o = \omega_o \Psi_o$.

How are we to make this choice for the particles which exist in Nature?

A well-defined procedure for bosons is the following: first step, choose the electric field as a polar vector, which fixes s, as was done in (113); and 2 define the vacuum state as even:

$$P\Psi_o = \Psi_o. \tag{134}$$

This determines the parity of any state which contains only photons. Next, to determine the parity of pions, look for the decay of neutral pions into photons. As they do decay into two photons, the π_o cannot have spin 1. Disregarding the possibility of higher spins, as supported by other considerations, π_o must be spinless. Then, in principle, <u>the second step</u> is to determine whether the two photons have parallel or perpendicular polarization planes. A pseudoscalar π_o leads to the latter. <u>The third step</u>, to determine the parity of the charged pions is to be guided by the charge-independence of the interaction of pions with nucleons, as described in I, 8; item 2), and to state, <u>as a new</u>, very reasonable, <u>assumption</u>, that the charged and neutral pion fields, components of a vector in the isobaric spin space, have the same parity transformation properites. This last assumption is equivalent to stating that $\bar{\psi}_p \gamma^5 \psi_n$ transforms in the same way as $\bar{\psi}_p \gamma^5 \psi_p$, $\bar{\psi}_n \gamma^5 \psi_n$, i.e., as a pseudoscalar, as follows from the assumed parity-invariance of the interaction lagrangean in (82).

The remaining bosons, so far known experimentally, are the K-mesons, which decay into pions, but the determination of their parity is more involved because the corresponding coupling does not conserve parity.

Summing up, you will see that (104) combined with:

$$P\Psi_o = \omega_o \Psi_o \, , \tag{135}$$

where $\omega_o^* \omega_o = 1$, gives:

$$P^2 O(x)\Psi_o = \omega^2 O(x)\Psi_o, \tag{136}$$

where $\omega = s\,\omega_o$.

Our assumption (134) fixes $\omega_o = 1$. The procedure described above fixes s = 1 for photons, s = -1, for pions, giving for any state vector Ψ constructed by application of Bose field operators O(x) to the vacuum state:

$$P^2\Psi = \Psi. \tag{137}$$

It may also be appropriate to emphasize that the third step described above, namely, the identification of the parity of charged pions with that of neutral pions is a new arbitrary assumption, independent of that made for photons. This is because charged fields have an arbitrary phase factor $e^{i\alpha}$, as described in 1,4. The fact that all observables must be invariant with respect to this phase transformation leads to the conclusion that the phases (or parities) of state vectors belonging to different charges cannot be compared.

In fact, let Ω be such an observable. Let Ψ be an eigenstate of the charge Q with eigenvalue q:

$$Q\Psi = q\Psi \tag{138}$$

and Ψ' an eigenstate of Q with eigenvalue q':

$$Q\Psi' = q'\Psi' . \tag{139}$$

Since:

$$\Omega = e^{-iQ\alpha}\,\Omega\, e^{iQ\alpha} , \tag{140}$$

we have:

$$\begin{aligned}(\Psi, \Omega\Psi') &\equiv (\Psi, e^{-iQ\alpha}\Omega\, e^{iQ\alpha}\Psi') = \\ &= (e^{iQ\alpha}\Psi, \Omega\, e^{iQ\alpha}\Psi') = e^{i(q-q')}(\Psi, \Omega\,\Psi')\end{aligned} \tag{141}$$

therefore

$$(\Psi, \Omega\,\Psi') = 0 \tag{142}$$

unless $q = q'$.

Now, given Ψ and Ψ', form the linear combination $\Phi = \Psi + \Psi'$ then:

then:

$$\Phi' = P^2 \Phi = \omega^2 \Psi + \omega'^2 \Psi' = P^2 (\Psi + \Psi'), \tag{143}$$

where ω and ω' are phase factors. If Ψ describes neutral boson states, the assumption made in step one fixed ω. To determine ω', as a consequence of this, I could measure the expectation values of an observable Ω in the Φ' - state, in the Ψ-state and in the Ψ'-state, and the transition probability between Ψ and Ψ':

$$(\Phi', \Omega \Phi') = (\Psi, \Omega \Psi) + (\Psi', \Omega \Psi') + (\omega^* \omega')^2 (\Psi, \Omega \Psi') + (\omega'^* \omega)^2 (\Psi', \Omega \Psi). \tag{144}$$

This is, however, impossible if $q \neq q'$, because of (142). Therefore, the choice of ω' for charged systems is independent of that of ω for neutral systems.

We shall see in the next chapter that a similar selection rule operates between states with integral total angular momenta and states with half-integral total angular momenta, so that the parity of a fermion cannot be deduced from that of bosons.

CHAPTER III

SPACE REFLECTION AND PARITY: FERMI FIELDS

III, 1. Parity transformed of spinor fields.

The determination of the matrix S of formula (104) when O(x) is a Dirac spinor field $\psi(x)$ is obtained by requiring that the free-field equation (10), or the free-field lagrangean $\mathcal{L}_2$ in (77), be invariant under space reflection. One finds $S = \gamma^0$. Call η the phase factor s in (104):

$$P\,\psi(x)\,P^{-1} = \eta\,\gamma^0\,\psi(-\vec{x}, x^0) \tag{145}$$

where $\eta^*\eta = 1$. You will see that P preserves the anticommutation rules for $\psi(x)$.

One also has:

$$\begin{aligned} P\,\psi^\dagger(x)\,P^{-1} &= \eta^*\,\psi^\dagger(-\vec{x}, x^0)\,\gamma^0 \\ P\,\bar{\psi}(x)\,P^{-1} &= \eta^*\,\bar{\psi}(-\vec{x}, x^0)\,\gamma^0. \end{aligned} \tag{146}$$

The Fourier integral (47), and (145), lead to:

$$\int \frac{d^3p}{\sqrt{2p^0}} \sum_r \left\{ P\,a(r,\vec{p})\,P^{-1}u(r,\vec{p})e^{-ipx} + Pb^\dagger(r,\vec{p})P^{-1}v(r,\vec{p})e^{ipx} \right\} =$$

$$= \eta \int \frac{d^3p}{\sqrt{2p^0}} \sum_{r'} \left\{ a(r',-\vec{p})\,\gamma^0 u(r',-\vec{p})e^{-ipx} + b^\dagger(r',-\vec{p})\,\gamma^0 v(r',-\vec{p})e^{ipx} \right\}. \tag{147}$$

From (48), you will show that:

$$(\gamma^k p_k - m)\,\gamma^0 u(r',-\vec{p}) = 0 \qquad (\gamma^k p_k + m)\,v(r',-\vec{p}) = 0 \tag{148}$$

so that we can take

$$\begin{aligned} u(r,\vec{p}) &= \epsilon_1\,\gamma^0 u(r',-\vec{p}) \\ v(r,\vec{p}) &= \epsilon_2\,\gamma^0 v(r',-\vec{p}) . \end{aligned} \tag{149}$$

We fix the factors ϵ_1 and ϵ_2 in the special solutions u and v for a fermion moving along p_z, with the spin parallel or antiparallel to p_z, which are, except for a normalization coefficient:

$E > 0$	$r = \frac{\sigma_z p_z}{\lvert p_z \rvert} = 1$	$r = -1$
$u_1(r,\vec{p})$	1	0
$u_2(r,\vec{p})$	0	1
$u_3(r,\vec{p})$	$p_z/(E+m)$	0
$u_4(r,\vec{p})$	0	$-p_z/(E+m)$

(150)

$E < 0$	$r = 1$	$r = -1$
$v_1(r, \vec{p})$	$p_z/(\lvert E\rvert + m)$	0
$v_2(r, \vec{p})$	0	$-p_z/(\lvert E\rvert + m)$
$v_3(r, \vec{p})$	1	0
$v_4(r, \vec{p})$	0	1

Then, with $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$:

$$\epsilon_1 = 1, \quad \epsilon_2 = -1 . \tag{151}$$

Thus, (147), (149) and (151) will give:

$$\begin{aligned} P\, a\,(r, \vec{p})\, P^{-1} &= \eta\, a\,(r', -\vec{p}) \\ P\, b^{\dagger}(r, \vec{p})\, P^{-1} &= -\eta\, b^{\dagger}(r', -\vec{p}) \\ P\, a^{\dagger}(r, \vec{p})\, P^{-1} &= \eta^{*} a^{\dagger}(r', -\vec{p}) \\ P\, b\,(r, \vec{p})\, P^{-1} &= -\eta^{*} b(r', -\vec{p}) . \end{aligned} \tag{152}$$

It is clear why we used r' in the second-hand side. When you reflect $\vec{p}$, the helicity r changes sign, since σ does not. So when we change $\vec{p}$ into $-\vec{p}$ in the columns in (150), we have to interchange r = 1 and r = -1.

III, 2. Parity of fermion states

How are we to define the parity of a fermion?

If we adopt the convention (134), (137), namely that the repeated application of P to any state vector gives back this state vector:

$$P^2 \Psi = \Psi \tag{137}$$

then for a one-particle state Ψ_1 :

$$\Psi_1 = \bar{\psi}(x) \Psi_o \ .$$

I obtain

$$P^2 \Psi_1 = (\eta^*)^2 \Psi_1 \tag{153}$$

since

$$P^2 \Psi_o = \Psi_o \ .$$

(137) and (153) give then:

$$\eta^2 = 1$$

and we are free to choose $\eta = +1$ or $\eta = -1$. For any of these two choices we see that a one-particle and a one-antiparticle states have opposite relative parity.

Yang and Tiomno have proposed some years ago, that one might also make the choice $\eta = +i$ or $\eta = -i$. Their argument is essentially the following: Apply again the parity operation to (145), to obtain:

$$P^2 \psi(x) P^{-2} = \eta^2 \psi(x) . \qquad (154)$$

This corresponds to returning to the original point x in space-time and the spinor ψ, due to its double-valuedness, may change its sign; hance:

$$\eta^2 = \pm 1 . \qquad (155)$$

Of course, η is an arbitrary phase factor, in (145), and the choice of its value is to a certain extent arbitrary. If we adopt the convention (137), or in general, that the phase factor of the phase in:

$$P^2 \Psi_i = \omega \Psi_i \qquad (137)'$$

is <u>independent</u> of the state vector Ψ_i, then:

$$\eta^2 = 1 . \qquad (156)$$

Suppose that you choose to say, with Yang and Tiomno, that there may exist two families of Dirac spinor fields in Nature, one, the "real" family, for which $\eta = +1$ (or $\eta = -1$), the other, the "imaginary" family, for which $\eta = +i$ (or $\eta = -i$). Then, the convention (137)' will not hold any more, the phase ω_j in:

$$P^2 \Psi_j = \omega_j \Psi_j \qquad (157)$$

will depend on the state vector Ψ_j. Thus, if you assume that for the vacuum $\omega_o = 1$, then

$$P^2 \Psi[n_r, n_i] = (-1)^{n_i} \Psi[n_r, n_i] \qquad (158)$$

where n_r is the number of particles of the "real"-family, n_i, the number of particles of the "imaginary"-family present in the state Ψ.

You will note, from (152), that the intrinsic parity of a particle is opposite to that of an antiparticle of the "real" -

family, while the intrinsic parity of a particle is the same as that of an antiparticle of the "imaginary" -family. The physically significant quantity is, however, the relative parity of a particle-antiparticle S-state and this is odd, as is clear from (152).

III, 3. Superselection rule of Wick-Wightman-Wigner

After the paper of Yang and Tiomno (See bibliography), the question of the intrinsic and relative parity of bosons and fermions was greatly clarified by an article of Wick, Wightman and Wigner. They essentially showed:
1) the parity of a neutral boson system of particles which can decay into pure photon states is well-defined if one makes a convention on the parity of a photon state; 2) the parity of charged boson states cannot be compared with that of neutral boson states, as shown in the preceding chapter, and thus needs a new, independent convention; 3) the parity of half-integral total angular momentum states cannot be deduced from that of integral total angular momentum states and thus, also needs another, independent, assumption. The last statement constitutes what is called the Wick - Wightman - Wigner superselection rule. In their paper, these authors gave a proof of it by using the operation of time reversal. This has the inconvenience that you may have, in principle, interactions which are not invariant under time reversal. The proof we shall give now is simple and general for all theories invariant under the proper, orthochronous Lorentz group. In such theories, all observables Ω must be relativistically covariant, i.e., tensors:

$$\ell_\alpha^{\alpha'}\ell_\beta^{\beta'} \ldots \Omega_{\alpha'\beta'\ldots}(L^{-1}x) = U^{-1}\Omega_{\alpha\beta\ldots}(x) \quad U \tag{159}$$

where U is given by (26). In particular, for the element R of the group U, corresponding to the rotation by the angle 2π around the z -axis, we have:

$$\Omega_{\alpha\beta\ldots} = R^{-1}\,\Omega_{\alpha\beta\ldots}\,R \tag{160}$$

where

$$R = e^{+iJ_3\varphi}\,, \quad J_3 = M^{12},\ \varphi = 2\pi\ .$$

Let Ψ be an eigenstate of J_3 with eigenvalue m:

$$J_3\Psi = m\,\Psi \tag{161}$$

and Ψ' another one with eigenvalue m':

$$J_3\Psi' = m'\Psi'\ .$$

We then have, for the transition amplitude of Ω between these two states:

$$\begin{aligned}(\Psi,\Omega\Psi') &\equiv (\Psi,\ R^{-1}\Omega R\Psi') = (R\Psi,\Omega\ R\Psi') = \\ &= e^{i(m'-m)2\pi}(\Psi,\Omega\Psi').\end{aligned} \tag{162}$$

Hence:

$$(\Psi,\Omega\Psi') = 0 \quad \text{unless } m - m' = \text{integral number}. \tag{163}$$

Clearly, if m is an integer and m' a half-odd integer, $(\Psi,\Omega\Psi') = 0$.

Let us call Ψ_A the Ψ's with m integral, Ψ_B the Ψ's with m half-integral and make the combination:

$$\Psi = \Psi_A + \Psi_B\ .$$

Then

$$P^2\,\Psi = \omega_A\Psi_A + \omega_B\Psi_B \tag{164}$$

according to (157).

Now, by (164) and (163):

$$(P^2\psi,\Omega P^2\psi) = (\psi_A,\Omega\psi_A) + (\psi_B,\Omega\psi_B) = (\psi,\Omega\psi). \qquad (165)$$

The term in $\omega_A^* \omega_B$, which would allow us to deduce ω_B from an assignment given to ω_A is missing because of (163) and, thus, its determination is arbitrarily independent of ω_A. Both states $P^2\psi$ and ψ are indistinguishable for the computation of expectation values of observables, or in other words P^2 commutes with Ω.

III, 4. Further digression on the "real" - and "imaginary" - fermions.

There is a subtle point at the end of the above proof. We did not make any assumption about whether P^2 commutes with all observables Ω. This was proved by (165):

$$P^{-2}\Omega P^2 = \Omega \qquad (166)$$

P being unitary. It is also a consequence of (142), (143) and (144). Therefore, as Ω is hermitian and P is unitary we have: $P\Omega P^{-1} =$ $= \pm\Omega$.

This makes it impossible for us to decide physically between the assumption (137)', for which $\eta^2 = 1$, and the assumption (157), for which η^2 is arbitrary and can be taken, in particular, equal to + 1 for some spinors and -1 for others, as indicated in (158).

As a result of (166) you cannot have a transition between a state with an even number of "imaginary"-fermions and a state with an odd number of such particles. If both states have the same total angular momentum and charge, as for example:

1 "real"-neutron + 1 "imaginary"-neutron $\longrightarrow$

$\longrightarrow$ 2 "real"-neutrons (167)

then (166) forbids the transition:

$$(\Psi_i + \Psi_f, \Omega(\Psi_i + \Psi_f)) = (\Psi_i + \Psi_f, P^{-2}\Omega P^2(\Psi_i + \Psi_f))$$
$$= (-\Psi_i + \Psi_f, \Omega(-\bar{\Psi}_i + \Psi_f))$$

hence:

$$(\Psi_f, \Omega\Psi_i) = 0 \qquad (168)$$

where Ψ_i and Ψ_f are the initial and final states in (167).

The unsatisfactory circumstance in (168) is that we would like to have a quantity, like angular momentum and charge, that would not be conserved in (168). The nucleon number cannot be invoked, since real-fermions and imaginary-fermions, if both have particles and antiparticles, have a nucleon number which can be chosen to be conserved in (167).

A bose field, like the pion field, which interacts with both real and imaginary neutrons, and which coupling is conserved under parity, cannot make the transition (167). But another field like a vector field, whose coupling is not invariant under the parity operation - a weak coupling - can carry the transition (167).

We shall see, in chapter V, that for neutral Majorama fermions, which belong to the imaginary family, one may have a physical justification for (168).

III, 5. Parity-transformed of spinor bilinear forms. Parity of neutron and proton.

(145) and (146) justify now the names S, V, T, A, P,

given to the operators in the table (76). You will easily find

	Bilinear form	Parity transformed
S	$\bar{\psi}_1(x)\ \psi_2(x)$	$\eta_1^* \eta_2\ \bar{\psi}_1(-\vec{x}, x^0)\ \psi_2(-\vec{x}, x^0)$
V	$\bar{\psi}_1(x)\gamma^0\psi_2(x)$ $\bar{\psi}_1(x)\vec{\gamma}\ \psi_2(x)$	$\eta_1^* \eta_2\ \bar{\psi}_1(-\vec{x}, x^0)\gamma^0\ \psi_2(-\vec{x}, x^0)$ $-\eta_1^* \eta_2\ \bar{\psi}_1(-\vec{x}, x^0)\vec{\gamma}\ \psi_2(-\vec{x}, x^0)$
T	$\frac{i}{2}\ \bar{\psi}_1(x)[\gamma^i, \gamma^k]\ \psi_2(x)$ $\frac{i}{2}\ \bar{\psi}_1(x)[\gamma^0, \gamma^k]\ \psi_2(x)$	$\eta_1^* \eta_2\ \frac{i}{2}\ \bar{\psi}_1(-\vec{x}, x^0)[\gamma^i, \gamma^k]\psi_2(-\vec{x}, x^0)$ $-\eta_1^* \eta_2\ \frac{i}{2}\ \bar{\psi}_1(-\vec{x}, x^0)[\gamma^0, \gamma^k]\psi_2(-\vec{x}, x^0)$
A	$\bar{\psi}_1(x)\gamma^0\gamma^5\psi_2(x)$ $\bar{\psi}_1(x)\vec{\gamma}\gamma^5\psi_2(x)$	$-\eta_1^* \eta_2\ \bar{\psi}_1(-\vec{x}, x^0)\gamma^0\gamma^5\psi_2(-\vec{x}, x^0)$ $\eta_1^* \eta_2\bar{\psi}_1(-\vec{x}, x^0)\vec{\gamma}\gamma^5\ \psi_2(-\vec{x}, x^0)$
P	$i\,\bar{\psi}_1(x)\gamma^5\psi_2(x)$	$-\eta_1^* \eta_2\ i\ \bar{\psi}_1(-\vec{x}, x^0)\gamma^5\ \psi_2(-\vec{x}, x^0)$

(169)

The geometric character shows up for $\psi_2 = \psi_1$, when the phase factors disappear.

We now see that if the constants C' in (93) are different from zero, the Fermi coupling is not space-reflection invariant.

Consider now the coupling $\mathcal{L}'$ in (82). Our choice of the parity of charged bosons, made in (II,6), imposes now, by the requirement that (82) be invariant under the parity operation, that the phases, or parities, of the neutron and proton be equal. The alternative choice would be possible as well.

CHAPTER IV

PARTICLE - ANTIPARTICLE CONJUGATION

IV, 1. Definition of particle-antiparticle conjugation.

This operation is defined as the transformation of particle states into antiparticle states.

It is frequently called charge conjugation for obvious reasons, but it also applies to neutral particles, for which particles are never identical to antiparticle states.

We shall abbreviate it into $\mathcal{C}$ -conjugation , where the letter $\mathcal{C}$ designates the operator in Hilbert space.

Consider (45). Then by definition:

$$\mathcal{C}\, a^{+}(r,\vec{p})\,\Psi_o = \epsilon^{*}\, b^{+}(r,\vec{p})\,\Psi_o \qquad (170)$$

$$\mathcal{C}\, b^{+}(r,\vec{p})\,\Psi_o = \epsilon\, a^{+}(r,\vec{p})\,\Psi_o$$

where ϵ is an arbitrary phase factor.

If we assume, as reasonable, that:

$$\mathcal{C}\ \Psi_o = \Psi_o \qquad (171)$$

then

$$\mathcal{C}\ a^{+}(r,\vec{p})\,\mathcal{C}^{-1} = \epsilon^{*}\ b^{+}(r,\vec{p}) \qquad (172)$$

and

$$\mathcal{C}^{2}\ a^{+}(r,\vec{p})\ \mathcal{C}^{-2} = a^{+}(r,\vec{p}) \qquad (173)$$

and analogously for b^{+}. $\mathcal{C}$ conserves the commutation rules, as U and P do, and will be taken as unitary.

From (45), we see that if there exists a matrix C' which acts on the spinor or tensor indices of O(x), such that

$$u_\alpha(r,\vec{p}) = C'_{\alpha\beta} v_\beta^{+}(r,\vec{p})$$
$$v_\alpha(r,\vec{p}) = C'_{\alpha\beta} u^{\dagger}(r,\vec{p}) \qquad (174)$$

then:

$$\mathcal{C}\, O_\alpha(x)\, \mathcal{C}^{-1} = \epsilon\, C'_{\alpha\beta}\, O^{\dagger}_\beta(x) \ . \qquad (175)$$

The reader will find easily that for a <u>spinless non-hermitian</u> field $\varphi(x)$, one has:

$$\mathcal{C}\varphi(x)\,\mathcal{C}^{-1} = \epsilon\, \varphi^{+}(x) \qquad (176)$$
$$\mathcal{C}\,\varphi^{+}(x)\,\mathcal{C}^{-1} = \epsilon^{*}\varphi(x)$$

and that the free-field lagrangean and energy-momentum tensor are <u>in</u>variant under $\mathcal{C}$. The current, however, changes sign, as expected:

$$j^{\nu} = i : (\varphi^{+}\frac{\partial\varphi}{\partial x_\nu} - \frac{\partial\varphi^{+}}{\partial x_\nu}\varphi): \; = -\,\mathcal{C}\, j^{\nu}\,\mathcal{C}^{-1} \ . \qquad (177)$$

The field orbital angular momentum does not change sign.

For a <u>spinless hermitian field</u>, one has:

$$\mathcal{C}\,\varphi_o(x)\,\mathcal{C}^{-1} = \epsilon_o\,\varphi_o(x) \qquad (178)$$

and here $\epsilon_o{}^2 = 1$. ϵ_o is called the $\mathcal{C}$ - conjugation parity.

IV, 2. <u>Eigenstates of $\mathcal{C}$</u>.

We shall call <u>charge</u> of a state the number of its particles minus the number of its antiparticles. This is also the meaning of the operator Q in (55). It can be electric charge, the nucleon number, the lepton number, etc.

Now we prove that <u>the only possible eigenstates of $\mathcal{C}$ are those with total charge zero</u>. This is trivial. If q is the eigenvalue of Q in the state Ψ :

$$Q \Psi = q \Psi \tag{179}$$

and if Ψ is an eigenstate of $\mathcal{C}$ with eigenvalue c,

$$\mathcal{C} \Psi = c \Psi , \tag{180}$$

then

$$\mathcal{C} Q \Psi = \mathcal{C} Q \mathcal{C}^{-1} \mathcal{C} \Psi = - Q \mathcal{C} \Psi = - Q c \Psi$$

so:

$$\mathcal{C} q \Psi = qc \Psi = - qc \Psi ,$$

hence:

$$q = 0. \tag{181}$$

Now (170), (171) and (181) show that $c = \pm 1$.

The two eigen states, Ψ_+ and Ψ_- , corresponding to $c = + 1$ and $c = - 1$ in (180), are orthogonal because $\mathcal{C}$ is unitary, and thus any state vector Φ can be written

$$\Phi = \alpha \Psi_+ + \beta \Psi_- \tag{182}$$

with the normalization: $\alpha^* \alpha + \beta^* \beta = 1$.

Examples constructed with (170):

$$\Psi_+ = \frac{1}{\sqrt{2}} (a^+(r,\vec{p}) b^+(r',\vec{p}') + b^+(r,\vec{p}) a^+(r',\vec{p}')\Psi_0$$

$$\Psi_- = \frac{1}{\sqrt{2}} (a^+(r,\vec{p}) b^+(r',\vec{p}') - b^+(r,\vec{p}) a^+(r',\vec{p}')) \Psi_0 . \tag{183}$$

It is interesting to note that the relativistic invariance of the theory, which gave rise to (166), by means of the superselection rules, also leads to

$$\mathcal{C}^2 \Omega \mathcal{C}^{-2} = \Omega \tag{184}$$

and hence to:

$$\mathcal{C} \Omega \mathcal{C}^{-1} = \pm \Omega .$$

IV,3. $\mathcal{C}$-conjugation of spinor fields.

The consideration of (47) and (48) and of (170),(174), (175), leads us straightforwardly to:

$$\mathcal{C}\,\psi(x)\,\mathcal{C}^{-1} = \epsilon\, C\bar{\psi}^{T}(x)$$

$$u(r,\vec{p}) = C\,\bar{v}^{T}(r,\vec{p}) \qquad (185)$$

$$v(r,\vec{p}) = C\,\bar{u}^{T}(r,\vec{p})$$

where T means transposition in spinor space (not in Hilbert space) and C is such that:

$$C^{-1}\gamma^{k}C = -(\gamma^{k})^{T} \qquad (186)$$

$$C^{T} = -C$$

besides being unitary. The antisymmetric nature of C is needed for the consistency of the last two relations of (185).

From (185), it follows that:

$$\mathcal{C}\,\bar{\psi}^{T}(x)\,\mathcal{C}^{-1} = \epsilon^{*}\,C^{-1}\psi(x) \qquad (187)$$

$$\mathcal{C}\,\bar{\psi}(x)\,\mathcal{C}^{-1} = -\epsilon^{*}\,\psi^{T}(x)\,C^{-1}.$$

IV,4. $\mathcal{C}$-conjugation of the spinor bilinear covariants.

From (185), (186) and (187) we obtain the $\mathcal{C}$-conjugated of the bilinear forms in (76). It is important, however, to emphasize once again that the physical quantities are not those expressions in (76) but rather these expressions taken as normal products.

Thus for $j^{\mu}(x) = :\bar{\psi}(x)\gamma^{\mu}\psi(x):$ we get:

$$\mathcal{C}\,j^{\mu}(x)\,\mathcal{C}^{-1} = -:\psi^{T}(x)\,C^{-1}\gamma^{\mu}C\,\bar{\psi}^{T}(x): = :\psi^{T}(x)\gamma^{\mu T}\bar{\psi}^{T}(x):$$

$$= -:\bar{\psi}(x)\gamma^{\mu}\psi(x):.$$

In the same way one obtains the following table:

	Normal bilinear forms	$\mathbb{C}$ - conjugated
S	$:\bar{\psi}_1(x)\,\psi_2(x):$	$\epsilon_1^*\epsilon_2 : \bar{\psi}_2(x)\,\psi_1(x):$
V	$:\bar{\psi}_1(x)\gamma^\mu\,\psi_2(x):$	$-\epsilon_1^*\epsilon_2 : \bar{\psi}_2(x)\gamma^\mu\psi_1(x):$
T	$\frac{1}{2}\,\bar{\psi}_1(x)[\gamma^\mu, \gamma^\mu]\,\psi_2(x):$	$-\epsilon_1^*\epsilon_2 : \frac{1}{2}\,\bar{\psi}_2(x)[\gamma^\mu, \gamma^\nu]\,\psi_1(x):$
A	$:\bar{\psi}_1(x)\gamma^\mu\gamma^5\,\psi_2(x):$	$\epsilon_1^*\epsilon_2 : \bar{\psi}_2(x)\gamma^\mu\gamma^5\psi_1(x):$
P	$:i\bar{\psi}_1\,(x)\gamma^5\psi(x):$	$\epsilon_1^*\,\epsilon_2 i : \bar{\psi}_2(x)\gamma^5\,\psi_1(x):$

(188)

Note that the spin term $:\bar{\psi}\gamma^0[\gamma^i, \gamma^k]\psi:$ does not change sign under C.

Of course the phase factors ϵ are restricted by requirements on the coupling with other fields. Thus, $\mathbb{C}$ -conjugation invariance of the pion nucleon coupling $\mathcal{L}'$ in (82) gives:

$$\overset{*}{\epsilon}\,(p)\,\epsilon\,(n)\,\epsilon\,(\pi^{\pm}) = 1$$
$$\epsilon\,(\pi^0) = 1 \, . \qquad (189)$$

The same requirement of $\mathbb{C}$ - conjugation invariance determines uniquely the phase-factor ($\mathbb{C}$ -conjugation parity) of <u>neutral</u> (hermitian) bose fields which interact with fermi fields. Thus, if:

$$\mathbb{C} : \bar{\psi}\,(x)\,\Gamma\psi(x) : B(x)\,\mathbb{C}^{-1} = \pm : \bar{\psi}\,(x)\Gamma\,\psi(x) : B(x) \;,$$

the following table results:

Neutral bose field B(x)	ϵ(B)
Scalar	1
Vector	-1
Tensor	-1
Pseudovector	1
Pseudoscalar	1

(190)

Clearly, a superselection rule operates here: from the assignment of the phase-factor of a neutral state you cannot deduce the phase-factor of a charged state, and this follows from (184). Thus in (189), $\epsilon(\pi^o) = 1$ but we are free to choose $\epsilon(\pi^{\pm})$, and once fixed the latter you are free to choose $\epsilon(p)$ and $\epsilon(n)$ satisfying the relation (189). In the last case, the superselection rule referred to states with integral and states with half-integral angular momenta.

The reader will now easily show the following consequences of the assumption of $\mathcal{C}$ -conjugate-invariant interactions:

1) the coupling of neutral scalar mesons with fermions cannot be the sum of a scalar and a vector coupling (these terms mean the coupling of S with the scalar field and of V with the 4-gradient of this field, respectively);

2) the coupling of neutral pseudovector mesons with fermions cannot be the sum of a pseudovector and a pseudotensor coupling:

3) a reaction among neutral bosons is forbidden if the number of

vector couplings and tensor couplings with intermediate fermi fields, is odd. This is because, by assumption, the interaction, and hence the S-matrix, responsible for the reaction is invariant under $\mathbb{C}$:

$$\mathbb{C} \, S \, \mathbb{C}^{-1} = S \, . \tag{191}$$

Then if the reaction is p initial bosons $\rightarrow$ q final bosons:

$$\Lambda = (\Psi_o, a_q \ldots a_1 \, S a_1^+ \ldots a_p^+ \Psi_o) \equiv (\Psi_o, \mathbb{C}^+ \mathbb{C} a_q \ldots a_1 S a_1^+ \ldots a_p^+ \mathbb{C}^+ \mathbb{C} \Psi_o)$$

$$= (\Psi_o, \mathbb{C} a_q \mathbb{C}^{-1} \ldots \mathbb{C} a_1 \mathbb{C}^{-1} S \, \mathbb{C} a_1^+ \mathbb{C}^{-1} \ldots \mathbb{C} a_p^+ \mathbb{C}^{-1} \Psi_o) . \tag{192}$$

Now, each a and a^+ transforms in the negative one, if the couplings are vector and tensor, according to (190); taking also (191) into account you will see that (192) is equal to:

$$\Lambda = (-1)^{n(V) + n(t)} \Lambda \, .$$

<u>Corollaries:</u> a) neutral vector meson $\rightarrow$ 2 photons, is absolutely forbidden (Sakata and Tanikawa);

b) neutral pseudovector meson $\rightarrow$ 3 photons, is absolutely forbidden;

c) neutral pseudovector meson $\rightarrow$ 2 photons is forbidden by (space-reflection) parity conservation;

d) transitions involving an odd number of external photon lines are absolutely forbidden;

e) neutral spinless meson $\rightarrow$ 3 photons is absolutely forbidden.

IV,5. <u>$\mathbb{C}$-conjugation parity of photon states</u>

If electrodynamics is invariant under particle-antiparticle conjugation, it follows from table (190) that the $\mathbb{C}$-conjugation parity of the electromagnetic field is -1:

$$\mathbb{C} \, A^\mu(x) \, \mathbb{C}^{-1} = -A^\mu(x) \tag{193}$$

which leads to
$$\mathbb{C}\, a(\lambda, \vec{k})\,\mathbb{C}^{-1} = -a(\lambda, \vec{k})$$
$$\mathbb{C}\, a^{+}(\lambda, \vec{k})\,\mathbb{C}^{-1} = -a^{+}(\lambda, \vec{k}) \;. \tag{194}$$

With the choice (171), the $\mathbb{C}$-conjugation parity of a n-photon state is $(-1)^n$:

$$\mathbb{C}\, a^{+}(\lambda_1, \vec{k}_1)\ldots a^{+}(\lambda_n, \vec{k}_n)\,\Psi_o = (-1)^n a^{+}(\lambda_1, \vec{k}_1)\ldots a^{+}(\lambda_n, \vec{k}_n)\,\Psi_o \;. \tag{195}$$

IV,6. $\mathbb{C}$-conjugation parity of positronium. Selection rules.

The state-vector of positronium in a stationary state can be developed in a perturbation series. As this is a neutral system, it may be an eigenstate of the operator $\mathbb{C}$. This will then be true for each term of the series. To determine the $\mathbb{C}$-conjugation parity, we will, therefore, need to consider only the lowest order term, which is, namely, in the center-of-momentum system (see (111)):

$$\Psi = \sum_{r,r'} \int d^3p \;\; F(r,r',\vec{p}) \;\, b^{+}(r,\vec{p}) \;\, a^{+}(r',-\vec{p})\,\Psi_o \;.$$

We then have (see (170), (171), (172)):

$$\mathbb{C}\;\Psi = \sum_{r,r'} \int d^3p\, F(r,r',\vec{p}) \;\, a^{+}(r,\vec{p}) \;\, b^{+}(r',-\vec{p})\,\Psi_o$$
$$= -\sum_{r,r'} \int d^3p \; F(r',r,-\vec{p}) \;\, b^{+}(r,\vec{p}) \;\, a^{+}(r',-\vec{p})\,\Psi_o$$

by changing $\vec{p}$ into $-\vec{p}$, r with r', and taking into account the anti-commutativity of a and b.

Thus the definition:

$$\mathbb{C}\;\Psi = \sum_{r,r'} \int d^3p \; (\mathbb{C}\; F(r,r',\vec{p}) \;\; b^{+}(r,\vec{p}) \;\, a^{+}(r',-\vec{p})\,\Psi_o$$

leads to the transformation of the wave function F :

$$\mathbb{C}\; F\;(r,r',\vec{p}) = -F(r',r,-\vec{p}) \;. \tag{196}$$

vector couplings and tensor couplings with intermediate fermi fields, is odd. This is because, by assumption, the interaction, and hence the S-matrix, responsible for the reaction is invariant under $\mathbb{C}$:

$$\mathbb{C}\, S\, \mathbb{C}^{-1} = S \, . \tag{191}$$

Then if the reaction is p initial bosons $\rightarrow$ q final bosons:

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Now, each a and a^+ transforms in the negative one, if the couplings are vector and tensor, according to (190); taking also (191) into account you will see that (192) is equal to:

$$\Lambda = (-1)^{n(V) + n(t)} \Lambda \, .$$

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The state-vector of positronium in a stationary state can be developed in a perturbation series. As this is a neutral system, it may be an eigenstate of the operator $\mathbb{C}$. This will then be true for each term of the series. To determine the $\mathbb{C}$-conjugation parity, we will, therefore, need to consider only the lowest order term, which is, namely, in the center-of-momentum system (see (111)):

$$\Psi = \sum_{r,r'} \int d^3p \;\; F(r,r',\vec{p}) \; b^{+}(r,\vec{p}) \; a^{+}(r',-\vec{p})\Psi_o \; .$$

We then have (see (170), (171), (172)):

$$\mathbb{C}\;\Psi = \sum_{r,r'} \int d^3p\, F(r,r',\vec{p}) \; a^{+}\,(r,\vec{p}) \; b^{+}(r',-\vec{p})\Psi_o$$
$$= - \sum_{r,r'} \int d^3p \; F\,(r',r,-\vec{p}) \; b^{+}(r,\vec{p}) \; a^{+}(r',-\vec{p})\Psi_o$$

by changing $\vec{p}$ into $-\vec{p}$, r with r', and taking into account the anti-commutativity of a and b.

Thus the definition:

$$\mathbb{C}\;\Psi = \sum_{r,r'} \int d^3p \; (\mathbb{C}\; F(r,r',\vec{p}) \;\; b^{+}(r,\vec{p}) \; a^{+}(r',-\vec{p})\Psi_o$$

leads to the transformation of the wave function F :

$$\mathbb{C}\; F\,(r,r',\vec{p}) = -F(r',r,-\vec{p}) \; . \tag{196}$$

The effect of the $\mathbb{C}$ -conjugation is to exchange the relative momenta and the polarizations of electron and positron, with a negative sign for the resulting amplitude. As

$$r' = -\frac{\vec{\sigma}'\cdot\vec{p}}{|\vec{p}|} \quad , \quad r = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \quad ,$$

$r \longrightarrow r'$, $\vec{p} \longrightarrow -\vec{p}$ lead to $\vec{\sigma} \longrightarrow \vec{\sigma}'$.

Let ℓ be the orbital angular momentum quantum number of the state. The exchange $\vec{p} \longrightarrow -\vec{p}$ corresponds to an exchange of position (space-reflection of the relative coordinates). Let s be the total spin quantum number of positronium. We see that:

$$F(r', r, -\vec{p}) = -(-1)^{\ell+s+1} F(r, r', \vec{p})$$

and so:

$$\mathbb{C}\, F(r, r', \vec{p}) = (-1)^{\ell+s} F(r, r', \vec{p}) \tag{197}$$

$(-1)^{\ell+s}$ is the $\mathbb{C}$ -conjugation parity of positronium.

The following table results:

Positronium state	$\mathbb{C}$ - parity	$\mathbb{C}$ -parity of n photons	Selection rule
$^3S(\ell = 0, \quad s = 1)$	-1	$(-1)^n$	^3S cannot decay into even number of photons
$^1S(\ell = 0, \quad s = 0)$	1	$(-1)^n$	^1S cannot decay into odd number of photons

IV, 7. Fermi interactions: condition on the coupling constants by $\mathbb{C}$ -conjugation invariance requirement.

Straightforward application of formulas (185), (186) and (187) to the lagrangean (93) will show that if:

$$\mathbb{C}\,\mathcal{L}'\,\mathbb{C}^{-1} = \mathcal{L}' \qquad (199)$$

then one must have

$$\epsilon^*(p)\,\epsilon(n)\,\epsilon^*(e)\,\epsilon(\nu) = 1$$

and, moreover:

$$C_i = C_i^*,\ C'_i = -C'^*_i\ ,\ i = S,\ V,\ T,\ A,\ P. \qquad (200)$$

Experiments disprove (200) and, therefore, (199) is not true.

CHAPTER V

MAJORANA NEUTRAL FIELDS

PARITY OF MAJORANA NEUTRAL FERMIONS

V,1. Definition of Majorana neutral fields.

We are now in condition to define a Majorana neutral field. Denote it by M(x). We shall define it as a field which is identical, except for a phase factor η , to its particle-antiparticle conjugated:

$$M(x) = \eta \, \mathbb{C} \, M(x) \, \mathbb{C}^{-1} . \tag{201}$$

Repetition of $\mathbb{C}$ -conjugation gives:

$$\mathbb{C} \, M(x) \, \mathbb{C}^{-1} = \eta^2 M(x) = M(x)$$

hence:

$$\eta = \pm 1 . \tag{202}$$

Thus a Majorana neutral field is such that:

$$M(x) = \pm \, \mathbb{C} \, M(x) \, \mathbb{C}^{-1} . \tag{203}$$

The hermitian bose fields are examples of Majorana neutrals, such as the neutral pion and the electromagnetic field; neutral pions and photons are Majorana neutrals, they are identical to their antiparticles. For the electromagnetic field, the - sign is taken in (203).

V,2. Dirac and Majorana fermi fields.

We shall call Dirac fermi field a spinor field which does not satisfy (201). These are the fermions so far found in the world such as electrons, muons, protons, and the charged baryons. They may also describe neutral fermions, such as neutrons, neutrinos and probably the neutral strange baryons. Their antiparticles are distinct from the corresponding particles, by some physical property such as the magnetic moment, nucleon or lepton number. That their electrical charge is zero leads to state that for them $e = 0$. But for the neutral Dirac fermions, as for the charged ones,

$$\bar{\psi}(x)\gamma^{\mu}\psi(x): ,$$

as well as the other covariants of the table (76), does not vanish. This is not a satisfactory explanation and we do not know a better one.

Majorana fermi fields, i.e., spinors which satisfy (201), have not been found in nature, as yet. It may be worthwhile to investigate some of their properties, since we cannot exclude their existence a priori.

V, 3. Bilinear covariants of Majorana fermions. Commutation rules.

It follows from the definition (201) and from inspection of table (188) that the V and T bilinear forms, as normal products, vanish identically for a given M(x): $:\bar{M}(x)\gamma^{\mu}M(x): =$

$$= \bar{M}(x)[\gamma^{\mu},\gamma^{\nu}]\, M(x): \equiv 0$$

	Majorana bilinear forms
S	$:\bar{M}(x)\, M(x):$
V	0
T	0
A	$:\bar{M}(x)\gamma^{\mu}\gamma^{5} M(x):$
P	$i:\bar{M}(x)\gamma^{5} M(x):$

Of course, this statement was based on the anticommutation properties of this field, which we shall now find. First, the spin of the field, namely, $-\frac{i}{4}\,\overline{M}(x)\,\gamma^{o}[\gamma^{i},\gamma^{k}]\,M(x)$, is well defined.

Now, by (185) and (187), and (201), we have:

$$M(x) = \pm\, C\, \overline{M}^{T}(x) \qquad \overline{M}(x) = \mp\, M^{T}(x)\, C^{-1}\,. \tag{205}$$

The definition (201), (202) <u>must now be completed by the requirement that, in a particular representation</u> of the γ-matrices, M(x) <u>be hermitian in Hilbert space</u>. This is obtained by choosing, as well-known:

$$\gamma^{o} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = -\gamma^{o*} = -\gamma^{o^{T}} \tag{206}$$

and:

$$C = \pm\, \gamma^{oT} = \mp\, \gamma^{o}. \tag{207}$$

This is the Majorana representation, in which all γ^{μ},s are imaginary.

We take the + sign in (205), to avoid unnecessary writing, and C in the Majorana representation is $-\gamma^{o}$.

The Fourier development of M(x) is:

$$M(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^{3}p}{\sqrt{2p^{o}}} \sum_{r} \left\{ C(r,\vec{p})\, u(r,\vec{p})e^{-ipx} + C^{+}(r,\vec{p})C\overline{u}^{T}(r,\vec{p})e^{ipx} \right\} \tag{208}$$

where $u(r,\vec{p})$ and $u(r,\vec{p})$ are given in (48) and C, in (186).

The anticommutation rules:

$$\left\{c(r,\vec{p}),\, c^{+}(r',\vec{p}')\right\}_{+} = \delta_{rr'}\,\delta(\vec{p}-\vec{p}') \qquad \left\{c(r,\vec{p}),\, c(r',\vec{p}')\right\}_{+} = \left\{c^{+}(r,\vec{p}),\, c^{+}(r',\vec{p}')\right\}_{+} = 0 \tag{209}$$

lead to the following ones for free fields:

$$\{M_\alpha(x),\ M_\beta(y)\}_+ = i\ S_{\alpha\lambda}(x-y)\ C_{\lambda\beta} \tag{210}$$

where S is given in (72); and

$$\{M_\alpha(x),\ \overline{M}_\beta\ (y)\} = -i\ S_{\alpha\beta}\,(x-y)\ . \tag{211}$$

Thus, $M_\alpha(x)$ and $M_\beta(y)$ have an anticommutator different from zero. This shows that in the representation (206), in which M(x) is hermitian, it is not, however, an observable.

The anticommutation rule (210), with a right hand side different from zero is welcome indeed. Otherwise, $M^2(x)$ would vanish, and in the Majorana representation, in which M(x) is hermitian, M(x) would be zero, because

$$(\Psi,\ M^2(x)\,\Psi) = (M\ (x)\,\Psi,\ M(x)\,\Psi).$$

The reason for the assumption of the fundamental rules (209) can be visualized in the following way. Let $\Psi(x)$ be a Dirac spinor field and form the field:

$$M(x) = \overset{(+)}{\psi}\ (x) + C\ \overset{(-)}{\overline{\psi}}{}^{T}(x) \tag{212}$$

where:

$$\overset{(+)}{\psi}(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2p^0}}\ \sum_r\ a(r,\vec{p})\,u\,(r,\vec{p})e^{-ipx}$$

$$C\ \overset{(-)}{\psi}{}^{T}(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2p^0}}\ \sum_r\ a^\dagger(r,\vec{p})\,C\overline{u}^{\,T}(r,\vec{p})e^{ipx} \tag{213}$$

are the positive-frequency part of $\psi(x)$ and the negative-frequency part of $C\ \overline{\psi}^{T}(x)$, according to (47).

From the anticommutation rules:

$$\left\{\overset{(+)}{\psi}(x),\overset{(-)}{\overline{\psi}}(y)\right\} = -i\ \overset{(+)}{S}\ (x-y),\quad \left\{\overset{(-)}{\psi}(x),\ \overset{(+)}{\overline{\psi}}(y)\right\} = -i\ \overset{(-)}{S}(x-y)$$

all other combinations anticommuting, you will find, by noting that:

$$C^{-1} \overset{(+)}{S}(x-y)C = - \overset{(-)}{S}{}^{T}(y-x),$$

that (212) satisfies (210). Here:

$$\left(\overset{(+)}{S}(x);\ \overset{(-)}{S}(x)\right) = -i(\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + m)(\overset{(+)}{\Delta}(x);\ \overset{(-)}{\Delta}(x)$$

$$\Delta(x) = \overset{(+)}{\Delta}(x) + \overset{(-)}{\Delta}(x) .$$

V,4. <u>Possible interactions</u>

The fact that: $:\bar{M}(x)\gamma^{\mu}M(x):$ vanishes indentically means that the charge operator $Q = \int d\sigma_{\mu}:\bar{M}(x)\gamma^{\mu} M(x): \equiv 0$. <u>A Majorana fermion is a totally neutral particle</u>: electric charge, nucleon number, lepton number, etc, are all zero for it. From the vanishing of $:\bar{M}(x)[\gamma^{\mu}, \gamma^{\nu}] M(x):$ it follows also that such a particle can have no magnetic moment.

The table (204) shows that, in principle, a Majorana-Fermi field can interact only with a scalar, a pseudoscalar and a pseudo vector field. It would be of interest to see whether such interactions are consistent with the anticommutation rule which follows from (210) for equal times.

If we now assume as a <u>general principle</u> that all couplings must conserve the operator Q - conservation of electric charge, of baryon number and lepton number - it is clear that in any reaction the same number of Majorana neutral fermions has to be in the initial and final states. It would thus be possible to have a heavy Majorana fermion decaying into a pair of baryons $Y, \bar{Y}$, according to:

$$M \rightarrow Y + \overline{Y} + M' \qquad (214)$$

and the pair be attributed to a neutral boson. Such a decay could occur via a weak pseudovector coupling of the type of that which describes beta-decay. If one wishes to attribute such a Fermi weak coupling to an intermediate coupling with hypothetical heavy bosons, we see that (214) needs a neutral vector or pseudovector boson.

Thus the postulate of the existence of such a neutral heavy boson, introduced by the author recently, would allow decays (214) to be described by a Feynman - Gell - Mann type of theory.

V,5. Parity of Majorana fermions

From (212), we see that M(x), transforms, under P, like $\psi(x) + C\,\overline{\psi}^T(x)$. By (145) and (146) we obtain:

$$P\,M(x)P^{-1} = \eta\,\gamma^{o}\,\overset{(+)}{\psi}(-\vec{x},x^{o}) - \eta^{*}\,\gamma^{o}C\,\overset{(-)}{\overline{\psi}}{}^{T}(-\vec{x},x^{o}) \qquad (215)$$

thus in order that M (x) transform under P like a spinor we must have:

$$\eta = -\eta^{*} \qquad (216)$$

which gives:

$$P\,M(x)\,P^{-1} = \eta\,\gamma^{o}\,M(-\vec{x},x^{o}) \,. \qquad (217)$$

From (216), η can only be +i or -i . Thus a Majorana Fermi field belongs to the "imaginary"-family of Yang and Tiomno.

The superselection rule forbids the transition:

2 Dirac neutral fermions $\longrightarrow$ 1 Dirac neutral fermion + 1 Majorana fermion as seen in (167). We now visualize here why this must happen, if we assume conservation of lepton or baryon number .

CHAPTER VI

TIME REVERSAL

VI, 1. Time reversal in classical physics

In classical mechanics, time reversal is the operation of inversion of the direction of motion. It changes the sign of t and of all odd functions of the velocity or momentum. The motions described by even lagrangeans of the velocities (even hamiltonians of the momenta) will obey the same laws as the inverted motions. This operation is, however, not a canonical transformation. It changes the sign of the Poisson brackets of coordinates with momenta and for this reason it may be called an anticanonical transformation.

In classical electrodynamics, the convention that the charge density is unchanged under time reversal:

$$\rho'(\vec{x},t) = \rho(\vec{x},-t) \tag{218}$$

leads to:

$$\vec{j}'(\vec{x},t) = -\vec{j}(\vec{x},-t)$$

for the conservation of charge in the reversed motion.

It then follows that:

$$\begin{aligned} \vec{E}'(\vec{x},t) &= \vec{E}(\vec{x},-t) \\ \mathcal{H}'(\vec{x},t) &= -\mathcal{H}(\vec{x},-t) \\ \vec{A}'(\vec{x},t) &= -\vec{A}(\vec{x},-t) \\ A'_0(\vec{x},t) &= A_0(\vec{x},-t) \end{aligned} \tag{219}$$

for the invariance of Maxwell's equations.

In the classical electron theory, one may require that time reversal keeps the energy of a free electron positive. As this is $m \frac{dx_o}{ds}$, where m is the rest mass and s is the proper time, this means that we should define as time reversal:

$$\vec{x}' = \vec{x}, \quad x^{o\prime} = -x^o, \quad s' = -s. \tag{220}$$

The equations of the Lorentz electron theory are:

$$m \frac{d^2 Z_\mu}{ds^2} = e\left[F_{\mu\nu}(Z) + F^o_{\mu\nu}(Z)\right]\frac{dZ_\nu}{ds} \; ; \tag{221}$$

$$\Box A^\mu(x) = e\int_{-\infty}^{\infty} \frac{dZ_\mu}{ds}\,\delta\big(x - Z(s)\big)\,ds\,;$$

$$\Box A^o_\mu(x) = 0 \quad ; \quad \lim_{x^o \to -\infty} A^\mu(x) = 0 \, .$$

The limiting condition defines $F_{\mu\nu}(Z)$ as the retarded field, $F^o_{\mu\nu}(Z)$ is an external field.

Time reversal changes the boundary condition into:

$$\lim_{x^o \to +\infty} A^\mu(x) = 0 \tag{222}$$

and thus the field goes over into the advanced one. In this sense, the equations of motion are invariant under (220).

VI,2. Time reversal in non-relativistic quantum mechanics.

This was first investigated by Wigner, in 1932.

Consider a particle which moves with a defined momentum $\vec{p}$ and positive energy E. It will be described by a plane wave:

$$\psi(x) = Ae^{i(\vec{p}\cdot\vec{x} - Et)} \, . \tag{223}$$

Our observations of the progress of this particle in time are such that the time intervals between any two of them are positive, $\Delta t > 0$.

With this convention, the momentum and the energy in (223) and the eigenvalues of $-i\vec{\nabla}$ and $i\frac{\partial}{\partial t}$, respectively:

$$-i\vec{\nabla}\psi(x) = \vec{p}\,\psi(x)$$
$$i\frac{\partial}{\partial t}\psi(x) = E\,\psi(x), \quad E > 0. \tag{224}$$

We now want to have, under time reversal, a particle moving with momentum $-\vec{p}$ and positive energy E. In this new frame of reference, the time intervals between two observations are the negative of those of the original frame.

The Schrödinger type of time reversal in Hilbert space, transforms the wave functions but not the operators. Call T this transformation. We want to have

$$-i\vec{\nabla}\left(T\,\psi(x)\right) = -\vec{p}\left(T\,\psi(x)\right)$$
$$i\frac{\partial}{\partial t}\left(T\;\psi(x)\right) = E\left(T\,\psi(x)\right), \; E > 0\;. \tag{225}$$

This will be achieved by setting:

$$T\,\psi(x) = A^{*}\,e^{-i(\vec{p}\cdot\vec{x}\,-\,Et)} \tag{226}$$

and, in addition, having in mind that the time intervals are now negative of those of $\psi(x)$. Thus

$$\frac{\partial}{\partial t}\,T\,\psi(t) = -\lim_{\Delta t\to 0}\frac{T\,\psi(t+\Delta t) - T\,\psi(t)}{|\Delta t|} \tag{227}$$

if

$$\frac{\partial\psi(t)}{\partial t} = \lim_{\Delta t\to 0}\frac{\psi(t+\Delta t)-\psi(t)}{|\Delta t|}$$

and (225) are satisfied by (226). We see that in Hilbert space, time reversal is here the operation of complex conjugation.

In the Heisenberg type of time reversal, only the operators are transformed, not the wave functions, with the same convention on the time running backwards in the new frame of reference.

The two types of transformation must be equivalent for the observer of this frame.

How are we to express this equivalence?

We first remark that the condition (3), of the equality of the expectation values in the two types of description, is rather too strong. What physics requires is the equality of the transition probabilities:

$$|(\Psi, O'(x)\Phi)|^2 = |(\Psi', O(x)\Phi')|^2 \tag{228}$$

which, of course, is fulfilled by the equality of the transition amplitudes:

$$(\Psi, O'(x)\Phi) = (\Psi', O(x)\Phi'). \tag{229}$$

But I could also have:

$$(\Psi, O'(x)\Phi) = (\Psi', O(x)\Phi')^* \tag{230}$$

which satisfies (228) as well.

Call K the operation of complex conjugation. This was the operation of time reversal as defined by (225) and (226):

$$T = K . \tag{231}$$

This obviously does not conserve the internal product in Hilbert space:

$$(K\psi, K\varphi) = (\psi^*, \varphi^*) = (\psi, \varphi)^* . \tag{232}$$

We are thus led to adopt (230) for time reversal, which shows that, in the case (231):

$$O'(x) = O^*(x), \qquad \Delta t < 0 . \qquad (232)'$$

Thus:

$$(-i\vec{\nabla})' = K(-i\vec{\nabla})K = (-i\vec{\nabla})^* = i\vec{\nabla}$$

$$(i\frac{\partial}{\partial t})' = K(i\frac{\partial}{\partial t})K = (i\frac{\partial}{\partial t})^* = -i\frac{\partial}{\partial t}\,. \qquad (233)$$

These must be applied, in the Heisenberg picture, to the unchanged ψ, (223), with the convention $\Delta t < 0$.

Usually, one takes this convention into account by stating that the transformation is complex conjugation in Hilbert space, and $t \rightarrow -t$ (as was analogously employed in the space-reflection and corresponding parity transformation):

$$\begin{aligned} T\psi(x) &= \psi^*(\vec{x},-t) \\ T^{-1}(-i\vec{\nabla})T &= i\vec{\nabla} \\ T^{-1}(i\frac{\partial}{\partial t})T &= i\frac{\partial}{\partial t}\,. \end{aligned} \qquad (234)$$

The hamiltonian $H(\vec{x},-i\vec{\nabla})$ will be invariant if it is an even function of the momentum operator, so that $H^* = H$.

VI,3. Time reversal of the Pauli spinor function

The non-relativistic spin $\frac{1}{2}$-particle is described, in the c-number theory, by the 2-component Pauli spinor.

Let $\mathcal{H}(x)$ be an external magnetic field. The hamiltonian:

$$H = -\frac{1}{2m}\vec{\nabla}^2 + V(r) + \frac{e}{2m}(\vec{\sigma}\cdot\vec{\mathcal{H}}) \qquad (235)$$

will transform, under (232)', into:

$$H^* = -\frac{1}{2m}\vec{\nabla}^2 + V(r) - \frac{e}{2m}(\vec{\sigma}^*\cdot\vec{\mathcal{H}}) \qquad (236)$$

because, according to (219), the convention $\Delta t < 0$ in the transformed frame entails $\mathcal{H} \rightarrow -\vec{\mathcal{H}}$. For H, (235), to be invariant under time reversal, one needs, besides complex conjugation, a unitary operator such that:

$$U^{-1} H^{*} U = H$$

i.e,

$$U^{-1} \vec{\sigma}^{*} U = -\vec{\sigma} ; U \vec{x} = \vec{x} U ; U \vec{p} = \vec{p} U .$$

In the usual representation of the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we see that:

$$U = \sigma_y .$$

Therefore, now T is given by:

$$T = \sigma_y K \quad . \tag{237}$$

VI,4. <u>Square of time reversal on the Schrödinger and the Pauli wave functions</u>.

Of course, both (231) and (237) could have an arbitrary phase factor ϵ :

$$\epsilon \epsilon^{*} = 1 .$$

If:

$$T = \epsilon K$$

we get:

$$T^2 = \epsilon K \ \epsilon K = \epsilon\epsilon^{*} = I . \tag{238}$$

If:

$$T = \epsilon \sigma_y K$$

then:

$$T^2 = \epsilon\sigma_y K \epsilon \sigma_y K = \sigma_y \sigma_y^* = - I \,. \tag{239}$$

It will be shown later that $T^2 = I$ for all states with an integral total angular momentum and $T^2 = - I$ for all states with a half-integral total angular momentum.

VI,5. Time reversal is an antiunitary operation

A linear and unitary operator O defined on a space of functions Ψ is one which satisfies the two conditions:

$$O\,(\alpha_1 \Psi_1 + \alpha_2 \Psi_2) = \alpha_1 O \Psi_1 + \alpha_2 O \Psi_2 \tag{240}$$

$$(O\psi_1, O\psi_2) = (\psi_1, \psi_2) \,, \tag{241}$$

where α_1 and α_2 are complex numbers.

The operation of complex conjugation K:

$$K\psi = \psi^*$$

does not fulfill (240) nor (241). It satisfies the following two relations:

$$K\,(\alpha_1 \psi_1 + \alpha_2 \psi_2) = \alpha_1^* K \psi_1 + \alpha_2^* K \psi_2 \tag{242}$$

called antilinearity condition, and:

(232):

$$(K\psi_1 \,,\, K\psi_2) = (\psi_1 \,,\, \psi_2)^*$$

<u>called antiunitarity relation</u>.

The complex conjugate of an operator Ω is $\Omega^{*} = K \Omega K$.

Every antilinear and antiunitary operator T (simply called antiunitary) is the product of a unitary operator U by K:

$$T = U K \qquad (243)$$

(this U will not be confused with U (L) in (26)).

This follows from the factor that $K^2 = I$ and that T and K satisfy both (232) and (242).

<u>Time reversal is in general given by (243)</u>. Example: (237).

From (243) it follows that $T^2 = \pm I$. Indeed, T^2 must be cI, where c is a complex number.

Now:

$$T^2 = U K U K = U U^* = U (U^{-1})^T = cI$$

hence:

$$U = c\, U^T$$

$$U^T = c^2 U^T , \quad c = \pm 1 \qquad (244)$$

$c = +1$ for integral angular momentum states, while $c = -1$ for half-integral angular momentum states, as will be proved later.

The antiunitarity of T shows itself also in the fact that it does not conserve commutation rules which have imaginary commutators. Thus, as $x' = T^{-1} x T = x$, $p' = T^{-1} p T = -p$, we see that:

$$[x, p] = i$$

goes over into:

$$[x', p'] = -i \ . \qquad (245)$$

Also:

$$[J_1, J_2] = i\, J_3$$

goes over into:

$$[J'_1\ J'_2] = -i\ J'_3$$

where

$$J'_i = T\ J_i\ T^{-1} = -J_i$$

VI,6. Time reversal of Dirac's spinor functions

We are still in the c-number theory, where $\psi(x)$ is a Dirac spinor function, not quantized. Consider the Dirac equation of a particle in a classical eletromagnetic field:

$$\left(\gamma^K \left(i \frac{\partial}{\partial x^K} - e\ A_K\right) - m\right) \psi(x) = 0 \qquad (246)$$

$$\left(\gamma^{K^T}\left(i \frac{\partial}{\partial x^K} + e\ A_K\right) + m\right) \bar{\psi}^T (x) = 0 \qquad (247)$$

where T as an upper index means transposition in spinor space.

We want the time reversed spinor $\psi'(x')$ to satisfy Dirac's equations in the time reversed frame:

$$\left(\gamma^K \left(i \frac{\partial}{\partial x^{K'}} - e\ A'_K\right) - m\right) \psi'(x') = 0 \qquad (248)$$

$$\left(\gamma^{K^T}\left(i \frac{\partial}{\partial x^{K'}} + e\ A'_K\right) + m\right) \bar{\psi}'^T (x') = 0 \qquad (249)$$

where:

$$\vec{x}' = \vec{x} \ , \ x^{o\,\prime} = -x^{o} \ . \qquad (250)$$

Take (250) and (219) into (248). This will be identical to (247) if:

$$\psi'(x') = \epsilon \, B \, \bar{\psi}^{T}(x) \qquad (251)$$

where ϵ is an arbitrary phase factor and:

$$B^{-1} \gamma^{o} B = (\gamma^{o})^{T} \qquad (252)$$

$$B^{-1} \vec{\gamma} B = -(\vec{\gamma})^{T}$$

or:

$$B^{-1} \gamma^{\mu} B = (\gamma^{o} \gamma^{\mu} \gamma^{o})^{T} \ . \qquad (253)$$

From this, you prove, by taking the hermitian conjugate, that B is unitary:

$$B^{\dagger} = B^{-1} \ .$$

Now, take the transpose of (251), you will show that $B^{T} \ B^{-1}$ commutes with γ^{μ}, hence Schur's lemma:

$$B^{T} = k \, B, \ \ k \text{ being a number } (254) .$$

Now:

$$B = k \, B^{T} = k^{2} B$$

$$k = \pm 1 \ .$$

Which of these two values must we take ? By means of (253), you will see that the following relations hold for the 16 matrices:

$$B^T = k\ B\ ;$$

$$(\gamma^o\ B)^T = k\ (\gamma^o\ B)\ ;$$

$$(\vec{\gamma}\, B)^T = -k\ (\vec{\gamma}\ B)\ ;$$

$$(\gamma^o \vec{\gamma}\, B)^T = k\ (\gamma^o\ \vec{\gamma}\ B)\ ;$$

$$(\gamma^i \gamma^k B)^T = -k\ (\gamma^i \gamma^k\ B),\ i \neq k;\ i,k = 1,\ 2,\ 3;$$

$$(\gamma^o\ \gamma^5 B)^T = \ k\ (\gamma^o \gamma^5\ B)$$

$$(\gamma^i\ \gamma^5 B)^T = -k\ (\gamma^i \gamma^5\ B),\ i = 1,\ 2,\ 3$$

$$(\gamma^5 B)^T = -k\ (\gamma^5\ B)\ .$$

Only for $k = -1$, do we get 6 antisymmetric matrices and 10 symmetric matrices, as it must be.

So:

$$B^T = -\ B\ . \tag{255}$$

A choice of B is:

$$B = \gamma^o\ \gamma^5\ C \tag{255'}$$

where C is defined by (186).

If one now takes (250) and (219) into (249), one finds:

$$\bar{\psi}'(x') = \ \eta\, \psi^T\ (x)\ B^{-1}\ . \tag{256}$$

η is another arbitrary phase factor, which will be fixed in terms of ϵ of (251) by a condition on the transformation of $\bar{\Psi}(x)\ \Psi(x)$.

VI,7. Time reversal in quantum field theory.

In the c-number theory, the hermitian conjugate $\psi^\dagger\ (x)$

of a spinor means this operation in spin space. When $\Psi(x)$ is an operator, we have used, in (47), $\Psi^+(x)$ to designate the hermitian conjugate of Ψ in both spin and Hilbert space.

Thus formulae (251) and (256) are well expressed in the c-number theory. In quantum field theory, however, we must take complex conjugation in Hilbert space. Denote by a <u>superscript H the operation of transposition in Hilbert space</u>. If we still keep the convention (47), that Ψ^+ is the hermitian conjugate of Ψ in Hilbert space, I must rewrite the transformed of Ψ and $\bar{\Psi}$ as operators, as follows:

$$T^{-1}\Psi(x)\,T = \epsilon\,B\,\bar{\Psi}^{TH}(\vec{x}, -x^o)$$
$$T^{-1}\bar{\Psi}(x)\,T = \eta\,\Psi^{TH}(\vec{x}, -x^o)\,B^{-1}. \tag{257}$$

Let us give the general definition of time reversal in quantum field theory.

We assume (230) and (243):

$$(\Psi, O'(x)\,\Phi) = (\Psi', O(x)\,\Phi')^* \tag{258}$$

$$\Psi' = T\Psi = UK\Psi,$$
$$UU^+ = U^+U = I$$

we obtain:

$$(\Psi, O'(x)\,\Phi) = (K\Psi, U^{-1}\,O(x)\,U\,K\Phi)^*$$

thus:

$$O'(x) = (U^{-1}\,O(x)\,U)^*. \tag{259}$$

Call:

$$V = U^*, \quad VV^+ = V^+ V = I .$$

Then:

$$O' (x) = V^{-1} O^* (x) V . \tag{260}$$

Therefore, if a physicist calculates a transition amplitude of an operator O (x) between two states, the initial Φ and the final Ψ, $(\Psi, O(x)\Phi)$, the physicist of the time reversed frame of reference will compute the transition amplitude $(\Psi, O'(x)\Phi)$ with O' (x) given by (260). The transformed of the former amplitude is thus:

$$(\Psi, V^{-1} O^* (x) V \Phi) . \tag{262}$$

This may also be written:

$$(\Psi, V^{-1} O^{+H} (x) V \Phi) = (U \Phi^*, O^+ (x) U \Psi^*) . \tag{263}$$

The last expression gives rise to a rule: <u>to transform a transition amplitude</u> $(\Psi, O(x)\Phi)$ <u>between an initial and a final state, calculate the amplitude of</u> $O^+(x)$ <u>between the transformed of the original final state as the new initial state and the transformed of the original initial state as the new final state</u>:

$$(\Psi, O(x)\Phi) \longrightarrow (T\Phi, O^+ (x) T\Psi), \tag{263}$$

where O^+ is hermitian conjugate of O in Hilbert space.

Consider the product of two or more operators $O_1 (x_1)$ $O_2 (x_2)$.

We have:

$$[\, O_1 (x_1) \;\; O_2 (x_2) \,]' = V^{-1} \, O_1^* (x_1) \;\; O_2^* (x_2) \, V \, . \qquad (264)$$

I can also write:

$$(\Psi, V^{-1} \, O_1^* (x_1) \, O_2^* (x_2) \, V \, \Phi) = (\Psi, V^{-1} \, O_1^{+H} (x_1) \, O_2^{+H} (x_2) V \, \Phi) =$$

$$= (\Psi, V^{-1} (O_2^+ (x_2) \, O_1^+ (x_1))^H \, V \Phi)$$

$$= (U \, \Phi^*, O_2^+ (x_2) \, O_1^+ (x_1) \, U \Psi^*) \, . \qquad (265)$$

We thus see that the transformed amplitude of:

$$(\Psi, O_1 (x_1) \, O_2 (x_2) \, \Phi) \text{ is } (T \, \Phi, O_2^+ (x_2) \, O_1^+ (x_1) \, T \, \Psi). \qquad (266)$$

Hence the following rule, due to Schwinger and Pauli (Niels Bohr and the development of Physics, Mac Graw Hill Book Company, 1955, page 30) : the transition amplitudes of the product of hermitian operators transform, under time reversal, into the transition amplitudes of the inverted product of these operators. This is the case of all observables. Another way of saying is: if you read the matrix element of the product of hermitian operators from left to right, read the time reverted matrix element from right to left.

VI,8. Time reversal transformation of the spinor bilinear forms.

This is now obvious with the result (266). The operator O (x) are now the spinors which transform as given in (257). O' (x), in (260), is now of the form B $O^*(x)$ where B acts on the spinor indices. $\bar{O}'(x)$ will be of the form $\bar{O}^{TH}(x)\,B^{-1}$. Thus, instead of (266), we will now have:

$$(\Psi\,,\ \bar{O}_1(x_1)\,\Gamma\,O_2(x_2)\,\Phi) \rightarrow (T\,\Phi,\ \bar{O}_2(x_2)\,(B^{-1}\,\Gamma B)^T O_1(x_1)\,T\Psi) \tag{267}$$

where Γ is one of the 16 γ - matrices, and B satisfies (253). The transposition H in Hilbert space to arrive at (266) will entail the transposition T in spinor space. If we call $\bar{O}_2(x_2)\,(B^{-1}\,\Gamma B)^T$ $O_1(x_1)$ the transformed of $\bar{O}_1(x_1)\,\Gamma\,O_2(x_2)$ we shall obtain the following table:

	Bilinear form $\bar{O}_1(x_1)\,\Gamma\,O_2(x_2)$	Time reverted $\bar{O}_2(x_2)\,(B^{-1}\gamma^\mu B)^T\,O_1(x_1)$ in (267)
S	$\bar{\Psi}_1(x_1)\ \psi_2(x_2)$	$\bar{\Psi}_2(x_2)\ \psi_1(x_1)$
V	$\bar{\Psi}_1(x_1)\,\gamma^\mu\psi_2(x_2)$	$\bar{\Psi}_2(x_2)\,\gamma^0\gamma^\mu\gamma^0\psi_1(x_1)$
T	$\frac{1}{2}\,\bar{\Psi}_1(x_1)[\gamma^\mu,\gamma^\nu]\,\psi_2(x_2)$	$-\frac{1}{2}\,\bar{\Psi}_2(x_2)\,\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\psi_1(x_1)$
A	$\bar{\Psi}_1(x_1)\gamma^\mu\gamma^5\psi_2(x_2)$	$\bar{\Psi}_2(x_2)\,\gamma^0\gamma^\mu\gamma^0\gamma^5\psi_1(x_1)$
P	$i\,\bar{\Psi}_1(x_1)\,\gamma^5\,\psi_2(x_2)$	$-\,i\,\bar{\Psi}_2(x_2)\gamma^5\psi_1(x_1)$
		$\times\ \epsilon_1^*\,\epsilon_2$

In (257), we take $\eta = \epsilon^*$ so as to have no phase factor in the transformed of bilinear form of one spinor field.

VI,9. Conditions for invariance of the Fermi beta-decay interaction under time reversal.

These follows immediately from the transformations (268) and the requirement of invariance of $\mathcal{L}$ given by (93).

One obtains:

$$\epsilon^* (p)\, \epsilon (n)\, \epsilon^* (e)\, \epsilon (\nu) = 1$$

and:

$$C_i = C_i^*, \quad C'_i = C'^*_i, \quad i = S, V, T, A, P. \qquad (269)$$

Time reversal invariance of the β-decay processes imposes that the coupling constants in (93) be real.

It may be appropriate to emphasize that in the transformation (263), the hermitian conjugation of $O^+ (x)$ does not change i into $-i$. This is because all numbers contained in $O (x)$ can be incorporated as a factor of Φ. On the second-hand side of (263) this factor will be the same because of the double complex conjugation once in $T\, \Phi$, the other in the transposition of $T\, \bar{\Phi}$ to the dual Hilbert space. With this remark in mind, table (268) and conditions (269) follow immediately from (263).

VI,10. Time reversal of emission and absorption operators of fermions.

We leave it to the reader to verify, with the help of

(257), (47), (255)' and the tables (150), that:

$$
\begin{aligned}
T^{-1}\, a(r, \vec{p})\, T &= \epsilon\, a^{*}\, (r, -\vec{p}) \\
T^{-1}\, b^{+}\, (r, \vec{p})\, T &= -\epsilon\, b^{H}\, (r, -\vec{p}) \\
T^{-1}\, a^{+}\, (r, \vec{p})\, T &= \epsilon^{*}\, a^{H}\, (r, -\vec{p}) \\
T^{-1}\, b(r, \vec{p})\, T &= -\epsilon^{*}\, b^{*}\, (r, -\vec{p})\,.
\end{aligned}
\qquad (270)
$$

If we employ the rule (263), we will have hermitian conjugation instead of complex conjugation, and no transposition H, in the second-hand side of (270). In this case, one has to keep in mind that initial and final states are switched in the same operation.

VI,11. Time reversal of Bose fields

It will be left to the reader to find the transformations for spinless and vector fields.

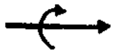
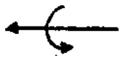
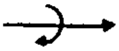
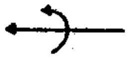
For the electromagnetic field, one imposes that (260) lead to transformations like the classical ones (219).

One than obtains for photons

$$
\begin{aligned}
T^{-1}\, a\, (R, \vec{k})\, T &= -a^{*}\, (R, -\vec{k}) \\
T^{-1}\, a\, (L, \vec{k})\, T &= -a^{*}\, (L, -\vec{k}) \\
T^{-1}\, a^{+}(R, \vec{k})\, T &= -a^{H}\, (R, -\vec{k}) \\
T^{-1}\, a^{+}(L, \vec{k})\, T &= -a^{H}\, (L, -\vec{k})\,.
\end{aligned}
\qquad (271)
$$

Here, apply the same remark of the end of last paragraph.

Thus time reversal keeps the polarization of photons (as expected because both spin and momentum change sign):

Photon	Time image
	
	

(272)

VI,12. Reciprocal theorem. Principle of detailed balancing.

Consider a reaction which transforms a state with n particles with momenta $\vec{p}_1, \dots \vec{p}_n$, at $t = -\infty$, into a state with k particles with momenta $\vec{p}'_1, \dots \vec{p}'_k$. Let $r_1, \dots r_n$ and $r'_1 \dots r'_k$ be the corresponding polarization variables.

The transition amplitude of the reaction is:

$$\Lambda = (a^\dagger (r'_1, \vec{p}'_1) \dots b^\dagger (r'_k, \vec{p}'_k) \Psi_o, \ S a^\dagger (r_1, \vec{p}_1) \dots b^\dagger (r_n, \vec{p}_n) \Psi_o)$$

where S is the S-matrix. This can also be written:

$$\Lambda = (\Psi_o, b (r'_k, \vec{p}'_k) \dots a (r'_1, \vec{p}'_1) \ S \ a^\dagger (r_1, \vec{p}_1) \dots b^\dagger (r_n, \vec{p}_n) \Psi_o).$$

Invariance of the transition amplitude under time reversal gives:

$$\Lambda = \epsilon \, (\Psi_o, b(r_n, -\vec{p}_n) \dots a (r_1, -\vec{p}_1) \ S \ a^\dagger (r'_1, -\vec{p}'_1) \dots b^\dagger (r'_k, -\vec{p}'_k) \Psi_o)$$

where ϵ is a phase factor.

Thus such an invariance entails the equality of the follow ing reaction probabilities:

$$\left| \left(\Psi[r'_1, \vec{p}'_1, \ldots r'_k, \vec{p}'_k], S\Psi[r_1, \vec{p}_1, \ldots r_n, \vec{p}_n] \right) \right|^2 =$$

$$= \left| \left(\Psi[r_1, -\vec{p}_1, \ldots r_n, -\vec{p}_n], S\Psi[r'_1, -\vec{p}'_1, \ldots r'_k, -\vec{p}'_k] \right) \right|^2. \tag{273}$$

The principle of detailed balancing is a particular case of last relation by omitting the negative sign of the momenta. It is valid only if the interaction is invariant under both parity transformation and time reversal, T P.

VI,13. The square of time reversal is -1 for half-integral angular momentum states.

It follows from (255) and (257) that for a spinor field $\psi(x)$:

$$T^{-2}\psi(x)\, T^2 = -\psi(x)\,. \tag{274}$$

Therefore for any state Ψ with an odd number of fermions:

$$\Psi_B = \psi_1(x_1) \ldots \psi_n(x_n)\, \Psi_o$$

one obtains:

$$T^{-2}\,\Psi_B = -\psi_1(x_1) \ldots \psi_n(x_n)\, T^{-2}\,\Psi_o\,.$$

The natural assumption:

$$T^{-2}\Psi_o = \Psi_o$$

gives:

$$T^{-2}\,\Psi_B = -\Psi_B$$

or:

$$T^2\,\Psi_B = -\Psi_B \,. \qquad (275)$$

For integral angular momentum states:

$$T^2\,\Psi_A = \Psi_A \,. \qquad (276)$$

CHAPTER VII

STRONG REFLECTION

VII,1. Definition of strong reflection

It is the change of particles into antiparticles accompanied of space reflection and time reversal.

The classical formulae (218), (219) are here replaced by:

$$\begin{aligned} \rho'(x) &= -\rho(-x) \\ \vec{j}'(x) &= -\vec{j}(-x) \\ \vec{E}'(x) &= \vec{E}(-x) \\ \vec{\mathcal{H}}'(x) &= \vec{\mathcal{H}}(-x) \\ A^{\mu'}(x) &= -A^{\mu}(-x) \; . \end{aligned} \tag{278}$$

In non-relativistic quantum mechanics, one can easily derive the transformation S in the space of wave functions which corresponds to a strong reflection. (223) is transformed, in the Schrödinger type of strong reflection, into:

$$S\,\psi(x) = \psi^*(x) \tag{279}$$

<u>with the convention that the intervals $\Delta t, \Delta X_i$, i = 1, 2, 3 in the measurements in state $S\psi$ are the negative of those in the measurements in state ψ</u>. One requires that momenta and energies do not change, which is satisfied by this transformation.

The abbreviated form, analogous to (234) is:

$$S\,\Psi(x) = \psi^{*}(-x)$$
$$S^{-1}(-i\,\vec{\nabla})\,S = -i\,\vec{\nabla} \qquad (280)$$
$$S^{-1}\left(i\,\frac{\partial}{\partial t}\right)S = i\,\frac{\partial}{\partial t}\,.$$

As the charge now changes its sign, the transformation

$$S = \sigma_y\,K \qquad (281)$$

represents also strong reflection for the invariance of (235).

VII,2. Strong reflection of field operators.

S may be represented by the product $\mathbb{C}\,P\,T$. Now $\mathbb{C}\,P$ is a unitary operator because $\mathbb{C}$ and P are unitary. T, we saw, is the product of a unitary operator U by complex conjugation. Thus we may also write:

$$S = W\,K \qquad (282)$$

where W is unitary.

Thus, extending the definitions (258) for S, (282), in place of T, we obtain:

$$0'(x) = (W^{-1}\,0(x)\,W)^{*}. \qquad (283)$$

Now the operator W, which is essentially $\mathbb{C}\,P$, changes 0 (x) into its hermitian conjugate in Hilbert space. Thus, we obtain:

$$0'(x) = M^{-1}\,0^{H}(x)\,M \qquad (284)$$

where H is the transposed in Hilbert space and M is a unitary operator.

A transition amplitude $(\Psi, O(x)\,\Phi)$ transforms, under strong reflection, into:

$$(\Psi, O(x)\,\Phi) \rightarrow (\Psi, M^{-1}\,O^{H}(x)\,M\,\Phi) = (M^{*}\,\Phi^{*}, O(x)\,M^{*}\,\Psi^{*}). \quad (285)$$

The rule is: <u>if you interchange initial and final states, the transformed amplitude will be that of O (x) with the new states</u>.

Also:

$$(\Psi, O_1(x_1)\,O_2(x_2)\,\Phi) \quad (M^{*}\,\Phi^{*}, O_2(x_2)\,O_1(x_1)M^{*}\Psi^{*}) \quad (286)$$

with an obvious rule (compare with that following (266)).

The following are the transformation laws under strong reflection:

Operator F (x) in $(\Psi, F(x)\, \Phi)$	Strong reflected $F'(x)$ in $(M^* \Phi^*, F'(x)\, M^* \Psi^*)$
Spinor $\Psi(x)$	$\epsilon \gamma^5 \Psi(-x)$
Adjoint spinot $\bar{\Psi}(x)$	$-\epsilon^* \bar{\Psi}(-x)\gamma^5$
Fermi annihilation operator $a(r, \vec{p})$	$\epsilon\, b^+(r, \vec{p})$
Creation operator $a^+(r, \vec{p})$	$\epsilon^*\, b\ (r, \vec{p})$
$b(r, \vec{p})$	$\epsilon^*\, a^+(r, \vec{p})$
$b^+(r, \vec{p})$	$\epsilon\, a\ (r, \vec{p})$
$:\bar{\Psi}_1(x_1)\ \Psi_2(x_2):$	$:\bar{\Psi}_1(-x_1)\ \Psi_2(-x_2):$
$:\bar{\Psi}_1(x_1)\gamma^\mu\Psi_2(x_2):$	$-:\bar{\Psi}_1(-x_1)\gamma^\mu\Psi_2(-x_2):$
$:\bar{\Psi}_1(x_1)\frac{i}{2}[\gamma^\mu, \gamma^\nu]\ \Psi_2(x_2):$	$:\bar{\Psi}_1(-x_1)\frac{i}{2}[\gamma^\mu, \gamma^\nu]\ \Psi_2(-x_2):$
$:\bar{\Psi}_1(x_1)\gamma^\mu\gamma^5\Psi_2(x_2):$	$-:\bar{\Psi}_1(-x_1)\gamma^\mu\gamma^5\Psi_2(-x_2):$
$i:\bar{\Psi}_1(x_1)\gamma^5\Psi_2(x_2):$	$i:\bar{\Psi}_1(-x_1)\gamma^5\Psi_2(-x_2):$

You will note that for the bilinear spinor forms the transformation laws given above refer to the normal products. <u>They do not hold for the forms as ordinary products</u>. Thus:

$$\bar{\Psi}_2(x_2)\gamma^5\Psi_1(x_1) \rightarrow -\Psi_1^T(-x_1)\gamma^{5^T}\bar{\Psi}_2^T(-x_2). \tag{288}$$

You must also note that from the table one should not conclude that $a^+a \rightarrow bb^+$. The true transformation is, without changing initial into final state, $a^+a \rightarrow b^H\, b^{+H} = (b^+b)^H$. When you now switch initial and final states, as assumed in the above table, then: $a^+a \rightarrow b^+b$.

This shows that the energy-momentum vector P^ν and the angular momentum tensor $\mathcal{M}^{\mu\nu}$, transform as:

$$P^\nu \longrightarrow P^\nu$$
$$\mathcal{M}^{\mu\nu} \longrightarrow -\mathcal{M}^{\mu\nu} \qquad (289)$$

as it is intuitive from the representation of S as $\mathbb{C}$ P T.

VII,3. The $\mathbb{C}$ P T - theorem.

We note that, according to the table (287), scalars and pseudoscalars transform in the same way under strong reflection. This is also true for vectors and pseudovectors.

As a result, all interaction lagrangeans which were previously constructed by the condition of hermitianity and invariance under the proper and orthochronous Lorentz group and as normal products are automatically invariant under strong reflection.

This statement, in its general form, constitutes the so-called $\mathbb{C}$ P T - theorem. You may check this with the Fermi interaction lagrangean, by means of the table (287).

First we observe that, according to (286), the expecta tion value of normal products of operators transforms in the following way:

$$(\Psi, :O_1(x)\,O_2(x):\Phi) \rightarrow \pm\epsilon^2 (M^*\Phi^*, :O_1(-x)\,O_2(-x): M^*\Psi^*) \qquad (290)$$

where we have allowed for a phase factor ϵ in (282). The $\pm$ sign comes from the Bose or Dirac statistics which the O's obey.

This factor ϵ appears in the spinor transformation given in table (287). Pauli has restricted its value by imposing that the reality conditions which the dotted and undotted spinors obey, be preserved under strong reflection. Let ψ^λ and $\psi_{\dot\lambda}$ be that two 2-component spinors which form ψ:

$$\psi = \begin{pmatrix} \psi^\lambda \\ \psi_{\dot\lambda} \end{pmatrix} \qquad (291)$$

and choose γ^5 diagonal:

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \qquad (292)$$

Then as $\psi \rightarrow \epsilon\,\gamma^5\,\psi$ under strong reflection, we have:

$$\psi^\lambda \rightarrow \epsilon\,\psi^\lambda, \quad \psi_{\dot\lambda} \rightarrow -\epsilon\,\psi_{\dot\lambda} \qquad (293)$$

hence:

$$\psi^{\dot\lambda} \rightarrow -\epsilon\,\psi^{\dot\lambda}\,.$$

Now $\psi^{\dot\lambda}$ transforms like $(\psi^\lambda)^*$ so that we can set:

$$\psi^{\dot\lambda} = (\psi^\lambda)^*\,. \qquad (294$$

If we require that this reality condition be preserved under strong reflection we must have:

$$-\epsilon \psi^{\dot{\lambda}} = (\epsilon \psi^{\lambda})^*$$

hence:

$$\epsilon^* = -\epsilon . \tag{295}$$

Thus, under this additional requirement, ϵ must be + i or -i.

Any tensor or spinor entity can be expressed as function u (k, ℓ) of two indices k, ℓ and

$$\begin{aligned} &u\ (k,\ell\ ) \text{ is a spinor, if } 2\ (k+\ell\ ) = \text{odd integral number,} \\ &u\ (k,\ell\ ) \text{ is a tensor, if } 2\ (k+\ell\ ) = \text{even integral.} \end{aligned} \tag{296}$$

For Ψ one has for ψ^{λ} and $\psi^{\dot{\lambda}}$, u ($\frac{1}{2}$, 0) and u (0, $\frac{1}{2}$) respectively, and the transformation laws under strong reflection, (293), can be written:

$$\begin{aligned} &u\left(\tfrac{1}{2}, 0\right) \longrightarrow \epsilon u\left(\tfrac{1}{2}, 0\right) \\ &u\left(0, \tfrac{1}{2}\right) \rightarrow \epsilon^* u\left(0, \tfrac{1}{2}\right) = -\epsilon\, u\left(0, \tfrac{1}{2}\right) . \end{aligned} \tag{297}$$

These relations will be generalized for tensors and higher order spinors as follows:

$$\begin{aligned} &u\ (k,\ell\ ) \rightarrow \epsilon(-1)^{2k}\ u\ (k,\ell\ ) \text{ for } 2\ (k+\ell\ ) = \text{odd, i.e., for \underline{spinors}};\\ &u\ (k,\ell\ ) \rightarrow (-1)^{2k}\ u\ (k,\ell\ ) \text{ for } 2\ (k+\ell\ ) = \text{even, i.e., for \underline{tensors},} \end{aligned} \tag{298}$$

under strong reflections. The first relation (298) is a generaliza tion of (297) for higher order spinors. The second relation (298) is a generalization of the transformation of the coordinates, to higher order tensors. Of course, in the transformed frame one has to take into account the convention: Δx^{μ} are the negative of the Δx^{μ} in the original frame.

Now, according to (298), a product of n <u>spinors</u> transforms as:

$$u(k_1,\ell_1) \ldots u(k_n,\ell_n) \rightarrow (-\epsilon)^n (-1)^{2(k_1+\ldots+k_n)} u(k_n,\ell_n) \ldots u(k_1,\ell_1) \tag{299}$$

while a product of n <u>tensors</u>, according to (298), transforms as follows:

$$u(k_1,\ell_1) \ldots u(k_n,\ell_n) \rightarrow (-1)^{2(k_1+\ldots+k_n)} u(k_n,\ell_n) \ldots u(K_1,\ell_1) \tag{300}$$

for $2(k_1+ \ldots + k_n)$ = even.

We see that (300) transforms like a tensor, if:

$$u(k_1,\ell_1) \ldots u(k_n,\ell_n) = u(k_n,\ell_n) \ldots u(k_1,\ell_1) \, .$$

Consider, however, (299). If n is even, the product (299) is a tensor, hence one must have

$$(-\epsilon)^n (-1)^{2(k_1+\ldots+k_n)} = (-1)^{2(k_1+\ldots+k_n)} \quad \text{for } \underline{n \text{ even}}, \text{ and}$$
$$u(k_1,\ell_1) \ldots u(k_n,\ell_n) = u(k_n,\ell_n) \ldots u(k_1,\ell_1). \tag{301}$$

If n is odd, (299) is another spinor, if the u's commute hence, according to (299) and (298), one must have:

$$(-\epsilon)^n (-1)^{2(k_1+\ldots+k_n)} = -\epsilon(-1)^{2(k_1+\ldots+k_n)} \quad \text{for } \underline{n \text{ odd}}. \tag{302}$$

There is, therefore, in the c-number theory, an extra factor in (299), which is, according to (301) and (302):

$(-\epsilon)^n$, for n even;

$(-\epsilon)^{n-1}$, for n odd

or:

$(-\epsilon)^{2\nu}$, for $n = 2\nu$, ν integer;

$(-\epsilon)^{2\nu}$, for $n = 2\nu+1$, ν integer.

So the extra factor is: $(-\epsilon)^{2\nu} = (-1)^{\frac{n(n-1)}{2}}$, in both cases.

This extra-factor, which is needed for the consistency of the transformation formulae (298), is obtained automaticcaly in the transformation of normal products of operators. Indeed, from (299) we get:

$$: u(k_1, \ell_1) \ldots u(k_n, \ell_n) : \rightarrow (-\epsilon)^n \, (-1)^{2(k_1+\ldots+k_n)} : u(k_n, \ell_n) \ldots u(k_1, \ell_1):$$

$$= (-1)^{n\,(n-1)/2} \, (-\epsilon)^n \, (-1)^{2(k_1+\ldots+k_n)} : u(k_1, \ell_1) \ldots u(k_n, \ell_n): \tag{303}$$

where $(-1)^{n\,(n-1)/2}$ came from the anticommutation of spinors, which has to be taken into account in the normal product.

The general formula of transformation of a normal product of field operators which may contain j γ^μ matrices (distinct from γ^5) is:

$$(\Psi, : O_1(x_1) \Gamma_1 \ldots \Gamma_j \, O_n(x_n) : \Phi) \rightarrow (-1)^j \, (-1)^{n(n-1)/2} \, (-\epsilon)^n \times$$

$$\times \, (-1)^{2(k_1+\ldots+k_n)} \, (M^* \Phi^*, : O_1(x_1) \, \Gamma_1 \ldots \Gamma_j O_n(x_n) : M^* \Psi^*). \tag{304}$$

The coefficient $(-1)^j$ came from the anticommutation of Γ's with γ^5, needed becauce of the transformation formulae for Ψ in

(287).

Now, the $\mathbb{C}$ P T - theorem is actually proved by considering that for all observables $\Omega_{\alpha\beta\gamma\ldots}(x)$, the proper and orthochronous Lorentz group transforms them into:

$$U^{-1}(L)\Omega_{\alpha\beta\ldots}(x)\,U(L) = \ell_{\alpha}^{\alpha'}\ \ell_{\beta}^{\beta'}\ldots\ \Omega_{\alpha'\beta'\ldots}(L^{-1}x). \qquad (305)$$

Thus the transformation (304) is preserved under the proper orthochronous Lorentz transformations.

VII,4. Theorem:

Corresponding groups of particles and antiparticles have the same energy spectrum.

Consider n free particles. We have:

$$H_o = \sum a^{\dagger}(r, \vec{p})\, a(r, \vec{p})\, p^o$$

and their energy is:

$$(\Psi, H_o\Psi) = E_o\ .$$

Under strong reflection

$$(\Psi, a^{\dagger}(r, \vec{p})\, a(r, \vec{p})\Psi) = (\Psi, b^{\dagger}(r, \vec{p})\, b(r, \vec{p})\Psi)$$

hence:

$$E_o\ (\text{particles}) = E_o\ (\text{antiparticles}).$$

The contributions from the interaction energy are also invariant. A particular case is, thus: the masses of particles and antiparticles are equal.

APPENDIX I

The angular momentum of a spinor field

The lagrangean, the energy-momentum vector and the angular momentum tensor must be hermitian operators. Formulae such as (18) and (21), taken as normal products, must be completed with terms in the hermitian conjugate of the field so that this reality requirement is fulfilled.

The lagrangean of a free spinor field may be taken as L_2 in (77) when the hermitianity condition is not invoked. The hermitian lagrangean of this field is the following:

$$\mathcal{L} = : \frac{1}{2}\left(\bar{\psi}\gamma^\mu \frac{\partial\psi}{\partial x^\mu} - \frac{\partial\bar{\psi}}{\partial x^\mu}\gamma^\mu\psi\right) - m\ \bar{\psi}\psi :$$

The energy-momentum tensor is:

$$T_\beta^\nu = : \frac{1}{2}\, g^{\alpha\nu}\left(\bar{\psi}\gamma_\beta \frac{\partial\bar{\psi}}{\partial x\alpha} - \frac{\partial\bar{\psi}}{\partial x o}\gamma_\beta\psi\right) +$$

$$+ \delta_\beta^\nu\left(-\frac{1}{2}\left(\bar{\psi}\gamma^\mu \frac{\partial\psi}{\partial x^\mu} - \frac{\partial\bar{\psi}}{\partial x^\mu}\gamma^\mu\psi\right) + m\ \bar{\psi}\psi\right) :$$

and the angular momentum tensor density:

$$\mathcal{M}_{\alpha\beta}^{\lambda} = T_\alpha^\lambda\ x_\beta - T_\beta^\lambda\ x_\alpha + S_{\alpha\beta}^{\lambda}$$

where:

$$S_{\alpha\beta}^{\lambda} = -\frac{1}{8} :\ (\bar{\psi}\gamma^\lambda[\gamma_\alpha, \gamma_\beta]\psi + \bar{\psi}[\gamma_\alpha, \gamma_\beta]\gamma^\lambda\psi) : .$$

Thus the angular momentum of the spinor field has the form:

$$\mathcal{M}^{\mu\nu} = \int d\sigma_\lambda\ \mathcal{M}^{\lambda;\mu\nu} .$$

We see that $\mathcal{M}^{\mu\nu}$ is hermitian. The spin part is:

$$S^{\mu\nu} = -\frac{1}{8}\int d^3x : (\Psi^{+}[\gamma^\mu, \gamma^\nu]\Psi + \Psi^{+}\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\Psi):$$

which shows that $S^{ok} = 0$. The reader may verify that this is consistent with (38) by making use of the relation:

$$\gamma^\lambda[P_\lambda, \Psi(x)] = -m\Psi(x)$$

for a free field, which follows from (37).

APPENDIX II

On the proof of the superselection rule

In the left-hand side of formula (159), we could allow for a phase factor ϵ. Due to the fact that U is unitary and the observables Ω are hermitian, this factor can only be 1 or -1. The choice $\epsilon = +1$ is imposed by the assumption that observables transform like (proper) tensors under the inhomogeneous proper orthochronous Lorentz group.

When, however, U acts on a state vector there is an indeterminate phase factor ω. Allowing for such factor, the relations (162) are replaced by:

$$(\omega R \Psi, \Omega\omega' R \Psi') = \omega^{*}\omega' (\Psi, \Omega\Psi')$$

in virtue of (160); and:

$$(\omega R \Psi, \Omega\omega' R \Psi') = \omega^{*}\omega' e^{2\pi i (m'-m)} (\Psi, \Omega\Psi^{-1})$$

due to (161). Hence

$$(\Psi, \Omega\Psi') = 0$$

if $m'-m$=half-integer, independently of ω and ω'.

Literature

Many papers have been written on the subject and most are quoted in 5, 6, 7 below:

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