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PROPERTIES OF NUCLEAR MATTER*

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ABSTRACT

An account is given of recent theories by Brueckner, Eden, Swiatecki, Bethe and co-workers about the properties of nuclear matter, which is identical in content, but different in its method of presentation. This presentation tries to clarify the reasons for the applicability of the independent-particle model and to determine what particular features of the nuclear forces are responsible for the vali-

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dity of this model.

I. INTRODUCTION

It is the intention of this paper to give a simple theoretical account of certain general properties of nuclear matter. This problem has recently attracted great attention and has been treated in detail, mainly in the papers by Brueckner, Eden and co-workers[1] and by Bethe and Goldstone[2] and by Swiatecki[3]. The presentation which will be given here does not contain any new approach and does not lead to new results. Solely the methods of presentation are quite different from the ones found in the recent papers about this subject. It is attempted to use methods which are conceptually simpler and intuitively easier to understand. Naturally this can only be done at the expense of some logical rigor. It is hoped that our approach will be helpful to clarify the concepts and to shed some light upon the reasons for the strange and unexpected properties of nuclear matter.

The concept of nuclear matter is a generalization and an idealization from certain facts relating to the structure of nuclei. It is believed that a large number A of Z protons and (A-Z) neutrons, $Z \backsim A/2$, in their lowest state of energy, would form a stable configuration, if the Coulomb repulsion between the protons did not exist. This hypothetical configuration is called nuclear matter; its properties are supposed to be independent of the number of constituents if A is so large that surface effects can be neglected.

This configuration does not exist in reality. The Coulomb effects increase with the square of Z, and they become important before the number A is large enough for the neglect of the surface effects. In fact, the Coulomb force prevents the formation of any stable or metastable nucleus when A is considerably higher than 200. However,

it is possible to understand some of the important properties of complex nuclei by assuming that they consist in first approximation of this hypothetical nuclear matter and by introducing the effects of the surface and the Coulomb field as a subsequent step. The very center of complex nuclei is a good sample of nuclear matter.

The properties of nuclear matter can be expressed as follows: The particle density ρ and the energy per particle ϵ are both independent of A and are approximately:

$$\rho = 2.21 \times 10^{38} \text{ particles/cm}^3$$

$$\epsilon = -15 \text{ Mev}$$

These numbers are the values for Z = A/2. For different values of Z the energy per particle is higher and the particle density probably lower. The number for ρ is taken from the actual nuclear density at the center of complex nuclei. Hence, the above value for ρ is probably slightly too low, since, in actual nuclei, both the Coulomb effect and the deviations of Z from A/2 decrease the density. The value for ϵ is taken from the volume term of the semi-empirical mass formula and includes, therefore, the corrections to the actual binding energy (which is about 8 Mev) caused by the absence of the Coulomb field, by the surface effects and by the reduction to the case Z = A/2.

Most striking are the dynamical properties of nuclear matter. They can be described approximately by considering it as a system of independent particles enclosed in the volume which it occupies. This means that, in first approximation, the system can be considered as an assembly of A particles, each moving freely in a potential which is independent of the position of the other particles. This dynamical picture has proved its validity as a first approximation by the successes of the independent-particle model of the nucleus.

One of the fundamental problems of nuclear physics is the explanation of the properties of nuclear matter on the basis of the forces between the constituents. This problem is of old standing. One of its difficulties comes from the fact that the forces are not well known. All one knows about them is that they are very short ranged, rather "strong", and, on the whole, attractive since they give rise to binding effects. Just these qualities seem at first sight to be in contradiction with the properties of nuclear matter. "Strong", short-ranged forces do not seem to be compatible with the independent particle aspect. In fact, the development of nuclear physics until the advent of the shell model was under the impact of the idea that such forces would give rise to a situation in which energy and momentum are rapidly exchanged between the constituents and which would be quite different from the actual one as described by the independent particle model.

It was also recognized early in the history of this problem that the stability of nuclear matter excludes the possibility of completely attractive ordinary forces between pairs of nucleons. If such existed and could be approximated by a square well potential of range b and depth V_o, a system of A nucleons would collapse to a sphere of radius b/2 and the energy per particle would be proportional to A for large values of A. The collapsed state has such low energy because the potential energy in the collapsed state is proportional to the number of pairs: $A(A-1)/2$. This is not compensated by the kinetic energy since the exclusion principle forces the kinetic energy at the constant radius b/2 to rise only with the power $A^{5/3}$. Hence, a system of A nucleons with purely attractive pair forces would not give rise to nuclear matter. There must be a repulsive element some-

there.

Several years ago the forces between nucleons were largely unknown except for their overall effect at low energies. The low energy experiments give information only about the forces in relative S-states (singlet and triplet) and can only be used to give certain integral properties of these forces. In recent years, however, scattering experiments at higher energies have given some more detailed information about the interaction between nucleons. We are still far from a comprehensive knowledge of the nuclear forces; however, it seems probable that, at the energies relevant for the nuclear problem, the interaction can be represented by a velocity-independent, but spin-dependent, potential and a short-range spin-orbit force. It contains two elements of repulsion. First, there seem to exist a strongly repulsive core which prevents nucleons from coming nearer to each other than about 0.4 or 0.5 f.[4] Second, the potential has exchange character, i.e., it depends not only upon the relative spin orientation, but also upon the symmetry of the quantum state in which the two nucleons are in respect to each other. In particular, repulsive forces were found between particles in odd states[5] for certain spin orientations. It will be seen how these two repulsive elements could give rise to the saturation effects.

True enough, scattering experiments can only furnish information about the forces between an isolated pair of nucleons. It is by no means obvious that the forces are the same when the nucleons are imbedded in nuclear matter. Any reasonable meson theoretic speculation leads to the prediction of some change in the nuclear interaction. There is today no positive or negative indication as to such

many particle influences. In view of the absence of any definite information, we will assume, however, that these effects do not play an important role in our problem. This assumption is supported by the fact that the properties of nuclear matter seem to be explainable without taking many-body forces into account.

It will be shown in this paper how the nuclear forces, as we know them to exist between pairs, could give rise to the required properties of nuclear matter.

II. ASSUMPTIONS ABOUT NUCLEAR FORCES

In order to reduce mathematical operations to a minimum, we will describe the forces between a pair of nucleons in very simple terms, which represent them with rather low accuracy and only within the energy region which is of importance in the nucleus. We do not expect to get quantitatively correct results from this starting point, but we will be able to study the important qualitative features of the approach.

We first assume that the forces are "Serber forces." That means that their exchange character is such that there is no force in the states of odd relative angular momentum. Actually the observations indicate that there are forces also in the odd states, especially in P-states. They are small enough, however, to permit our simplifying assumption, especially when averaged over the different spin directions.

The forces in the states of even angular momentum will be described as follows: They consist of a central force part and a tensor force part. The central force is assumed to be spin independent; it is therefore the same for triplet and singlet states. The observed difference between the forces in these two states is assumed to be due

entirely to the tensor force. We describe the central force by the following potential V(r):

$$V(r) = \infty \quad r < c$$
$$V(r) = -V_o \quad c < r < b + c \tag{1}$$
$$V(r) = 0 \quad c + b < r$$

This is a square well ptoential with a repulsive core. In order to give rise to the correct singlet scattering at low energy, we must put

$$2c + b = r_{os} = 2.7f$$
$$V_o = \frac{\hbar^2\pi^2}{4mb^2} \tag{1a}$$

The last relation expresses the fact that the singlet force gives rise to a stationary state of zero binding energy. The recent interpretations of the high-energy scattering data indicate a repulsive core of $c = 0.4$ f. Hence we choose: $c = 0.4$ f, $b = 1.9$ f.

The tensor force, although quite strong, will be neglected in our treatment. The reason is partly simplicity and partly the fact that, in nuclear matter, the tensor force has much less effect upon the wave functions and, therefore, on the energy, than in two-body problems. The reason for this fact will be understood at a later stage. We must be prepared, however, that we underestimate the binding energy of nuclear matter by neglecting tensor force.

III. ASSUMPTIONS MADE IN CALCULATING THE ENERGY OF NUCLEAR MATTER

We now proceed to calculate the energy of nuclear matter and the wave functions describing it on the basis of the nuclear forces as assumed in Section II. We consider a large piece of nuclear matter consisting of A particles, A/2 neutrons and A/2 protons. Since we assume that the nuclear forces are spin independent, we can consider the system as consisting of four different species of particles: A/4 neu-

trons of either spin, A/4 protons of either spin. Only pair forces are acting between the nucleons and, hence, the total energy of a system of A particles will be given by

$$E = (\Psi \mid \sum_{i=1}^{A} - \frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} \sum_{i=1}^{A} \sum_{k=1}^{A} v(ik) \mid \Psi) \qquad (2)$$

Here $\frac{-\hbar^2}{2m} \nabla_i^2$ is the operator of the kinetic energy of the ith particle and v(ik) is the operator of the potential energy between the pair i,k. E is the expectation value of the Hamiltonian in the ground state of the system, given by the wave function Ψ. It is this wave function Ψ which we will study in some detail. We start out with the simplest assumption in regard to the structure of the ground state: We first regard the A particles as completely free within a volume Ω.[6] The particles will then be found in single particle levels α which are the eigen-states in the volume Ω. These states are plane waves with a momentum k_α. In order to avoid confusion we use the term "level" for single particle states and denote these levels with greek letters α, β, etc. We will consider the proton and neutron levels of each spin as different levels even if they have the same momentum k_α. In each level α there can be not more than one particle according to the exclusion principle. The wave function for the state of the total system can be written in the form of a Slater determinant:

$$\Psi_o = \mathcal{A} \prod_{i=1}^{A} \phi_{\alpha_i} (r_i) \qquad (3)$$

Here $\phi_{\alpha_i} (r_i)$ is the wave function of the ith particle in the level α_i:

$$\phi_{\alpha_i} = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}_{\alpha_i}\cdot\vec{r}_i} \tag{4}$$

$\prod_i$ is a product over all ϕ_{α_i} ; the symbol $\mathcal{A}$ represents anti-symmetrization among equal particles. Ψ_o is the solution of our problem if there were no interactions at all, and the energy of this state would be:

$$E_o = \sum \frac{\hbar^2}{2m} k_\alpha^2 \tag{5}$$

where the sum goes over all occupied levels. The lowest energy state for the case of equal numbers of protons and neutrons is the one where the particles occupy all states up to the Fermi limit. Its energy is

$$E_o = \frac{3}{5} A \frac{\hbar^2}{2m} k_F^2 \tag{6}$$

where k_F is the "Fermi momentum", which is the highest occupied momentum in the ground state. In the case of four different kinds of particles, each kind numbering A/4, we get

$$k_F = (\frac{3}{2}\pi^2\rho)^{1/3} \ , \quad \rho = \frac{A}{\Omega} \tag{7}$$

It is interesting to express the "Fermi wave length" λ_F in terms of the average distance $d = \rho^{-1/3}$ of the particles:

$$\lambda\!\!\!^{-}_F = (\frac{2}{3\pi^2})^{1/3} d = 0.41d \tag{8}$$

In nuclear matter, therefore, we have d = 1.66f and $\lambda\!\!\!^{-}_F$ = 0.675 f, and $\epsilon_o = E_o/A$ = 27.3 Mev.

If nuclear forces are acting between the particles, the wave function (3) and the energy (5) will be altered. In the extreme independent-particle model (I.P.M.) one still maintains the wave function (3), but one expresses the effect of the interaction in the form

of a potential W, in which each particle moves independently. Because of the uniform nature of nuclear matter, this potential cannot depend upon the space coordinates of the particles; however, it may depend upon their momentum p_α . The total energy in the I.P.M. is then no longer given by (5), but by

$$E_{I.P.M.} = E_o + \sum_\alpha W(p_\alpha) \tag{9}$$

It is customary to approximate the momentum dependence of W by a quadratic function:

$$W(p_\alpha) = W_o + p^2 W_1 = W_o + \frac{1}{2}\left(\frac{m-m_{eff}}{m m_{eff}}\right)p^2 \tag{10}$$

which gives rise to the concept of effective mass[7] m_{eff}: The energy of the level α can then be written

$$\epsilon_\alpha = \frac{p_\alpha^2}{2m_{eff.}} + W_o \tag{11}$$

which leads to the form:

$$E_{I.P.M.} = A\left(\frac{3}{5}\frac{\hbar^2}{2m_{eff.}}k_F^2 + W_o\right) \tag{12}$$

In our analysis of nuclear matter we will go beyond the I.P.M. Clearly the wave function (3) is too simple an approximation for our purpose. For example, the repulsive core in the nuclear forces does not permit two particles to come closer than its radius c. However, in (3) the probability of finding two particles of different kind at any place is independent of their distance.

We therefore proceed one step further and introduce the "independent-pair-model", which we denote with I.PA.M. Here we consider the interaction between any pair of particles as exactly as possible, butin doing so we neglect the interaction of all other particles among themselves and with the pair.

The energy of the system is then written in the form

$$E = E_o + \frac{1}{2} \sum_{\alpha,\beta} U_{\alpha\beta} \qquad (13)$$

where α and β are occupied levels and $U_{\alpha\beta}$ is the energy correction coming from the interaction of a pair of particles in the levels α and β. The sum is extended over all pairs of occupied levels. Each $U_{\alpha\beta}$ is calculated by assuming that there is interaction only between the pair which occupies α and β. Hence $U_{\alpha\beta}$ is the correction to the energy E_o for the hypothetical case that interactions exist only when the particles are in the levels α and β and that the interaction is zero for any particle which is not in α or β. These levels can be identified by the assymptotic behavior of the wave function for large distances, at which they correspond to two states of well-defined momentum.[8]

The energy corrections $U_{\alpha\beta}$ can be used within the framework of the I.PA.M. for the determination of the average potential W in which the particle moves. If a particle finds itself in the state α, the interaction energy with all other particles must be $\sum_{\beta} U_{\alpha\beta}$. Hence we get:

$$W(p_\alpha) = \sum_{\beta} U_{\alpha\beta} \qquad (14)$$

It is in the nature of infinite nuclear matter that W does not depend on the position. But it will depend on the momentum p_α of the particle.

As a further step towards a better approximation, we can improve the I.PA.M. by making it self consistent. In this "self- consistent I.PA.M." we again consider the interaction between the particles in a given pair of levels by putting equal to zero any interaction of a particle in another occupied level. However, we assume

all particles to move in a momentum-dependent one-particle potential W(p). This assumption will have an influence upon the values of $U_{\alpha\beta}$. We will try to choose W(p) in a self-consistent way; namely, that the resulting $U_{\alpha\beta}$ reproduce the original choice with equation (14).

The next section contains, in greater detail the calculations of the wave functions and of the energies on the basis of the I.PA.M. The justification of the model will be discussed in Section V.

IV. TWO PARTICLE INTERACTIONS IN A FERMI GAS

We now come to the problem which is fundamental to the I.PA.M. It is the treatment of the interaction of a pair of particles in a Fermi gas. In the spirit of the independent pair approximation, we intend to treat each interaction separately. Hence, we introduce a force which acts between a pair only when the particles are in the levels α and β of the Fermi distribution; if one or both particles are in any other of the occupied levels, there is no interaction.[9] Hence, all levels $\gamma \neq \alpha, \beta$ are completely undisturbed by the interaction, and their wave functions are plane waves. The particles in the levels α, β, however, are subjected to the nuclear forces and the wave function of such a pair, $\Psi_{\alpha\beta}(r_1, r_2)$ will not be a product of two plane waves any more; it is our task to calculate this wave function and also the energy correction $U_{\alpha\beta}$ which this interaction produces.

In spite of the fact that the other particles are assumed not to interact with the pair, they still have an important influence on the interaction of the pair because of the Pauli principle. If the pair would be isolated, it would perform a scattering process; starting with the momenta $\vec{k}_\alpha$ and $\vec{k}_\beta$, it would end up with different mo-

menta $\vec{k}_\gamma$ and $\vec{k}_\delta$. Just this scattering is made impossible by the fact that the levels with the momenta $\vec{k}_\gamma$ and $\vec{k}_\delta$ are occupied by other particles. Hence, a real scattering cannot take place; at large distances the particles must have always the well-defined momenta $\vec{k}_\alpha$ and $\vec{k}_\beta$. However, at small distances the Pauli principle does not prevent the wave function from being distorted by the interaction.

The mathematical formulation of this interaction has been given by Bethe and Goldstone[2] and can be expressed in simple terms. Let us start with the ordinary wave equation governing the scattering of an isolated pair of particles;

$$(\nabla_1^2 + \nabla_2^2 + k_\alpha^2 + k_\beta^2)\,\Psi(\vec{r}_1\vec{r}_2) = J(\vec{r}_1\,\vec{r}_2)$$

$$J(\vec{r}_1\vec{r}_2) = \frac{2m}{\hbar^2}\, v(\vec{r}_1\vec{r}_2)\,\Psi(\vec{r}_1\vec{r}_2) \qquad (15)$$

Here $\vec{k}_\alpha$, $\vec{k}_\beta$ are the asymptotic wave vectors of the two particles, and $v(r_1, r_2)$ is the potential of the forces between the two particles, as given by (1). The solution of this equation expresses the scattering of the two particles by the potential v. In particular, the cross section for the scattering into the end states $\vec{k}_\gamma$ and $\vec{k}_\delta$ is proportional to the square of the matrix element:

$$\sigma(\vec{k}_\alpha\,\vec{k}_\beta\,;\,\vec{k}_\gamma\,\vec{k}_\delta) = \text{const.}\left|\int \exp\left\{-i\left|\vec{k}\,\vec{r}_1 + \vec{k}_\delta\vec{r}_2\right|\right\} J(\vec{r}_1\vec{r}_2)\,d\vec{r}_1\,d\vec{r}_2\right|^2 \qquad (16)$$

This matrix element is the Fourier component of the right-hand side of (15) in respect to the final wave numbers $\vec{k}_\gamma$ and $\vec{k}_\delta$.

If the pair is imbedded in a Fermi distribution, the wave equation (15) is no longer valid. Any scattering into occupied states

must be excluded. According to Bethe and Goldstone[2] we replace (15) by another equation, which contains this restriction. It is not very different from (15). All that is done[10] is the elimination, from the inhomogeneity J in (15), of the Fourier components which correspond to the levels occupied by the other particles. Instead of (15) we then get the "Bethe-Goldstone wave equation"[11] (B.-G. equation), which we write in the following form:

$$\left[\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)+\epsilon_{\alpha\beta}\right]\Psi_{\alpha\beta} = Q^F_{\alpha\beta}(V\,\Psi_{\alpha\beta}) \qquad (17)$$

whose solution $\Psi_{\alpha\beta}$ we call the B.-G. wave function. $\epsilon_{\alpha\beta}$ is the eigenvalue and $Q^F_{\alpha\beta}$ is a projection operator which rejects all Fourier components that are not outside the Fermi distribution, except the ones corresponding to the levels α and β. Such an operator obviously is defined as follows: If $\phi(r_1 r_2)$ is an arbitrary function of $r_1 r_2$, we have[12]

$$Q^F_{\alpha\beta}\phi(\vec{r}_1\vec{r}_2) = \sum_{k_\gamma > k_\beta}\sum_{k_\delta > k_F}\phi_\gamma(\vec{r}_1)\phi_\delta(\vec{r}_2)(\gamma\delta|\,\phi) + \phi_\alpha(1)\phi_\beta(2)(\alpha\beta|\,\phi) \qquad (18)$$

with

$$(\gamma\delta|\,\phi) = \int \phi^*_\gamma(\vec{r}_1)\,\phi^*_\delta(\vec{r}_2)\,\phi(\vec{r}_1\vec{r}_2)\,d\vec{r}_1\,d\vec{r}_2$$

Here $\phi_\alpha(r)$ are the normalized plane wave eigenfunctions in the volume Ω:

$$\phi_\alpha(\vec{r}) = \frac{1}{\sqrt{\Omega}}\,e^{i\vec{k}_\alpha\cdot\vec{r}}$$

and the sums are taken over all eigenstates of the volume Ω with $|\vec{k}_{\gamma,\delta}| > k_F$. The right-hand side of (17) does not have any Fourier components corresponding to any energetically allowed final scattering states, different from the initial state, and, therefore, all

scattering matrix elements (16) vanish. Thus, all phase shifts are zero, and asymptotically, for $|r_1-r_2|\to\infty$, the solution of (17) must be equal to the unperturbed (v=0) solution.

Hence, we get for $|r_1-r_2|\to\infty$:[13]

$$\lim_{|\vec{r}_1-\vec{r}_2|\to\infty} \psi_{\alpha\beta}(\vec{r}_1\vec{r}_2)\to\phi_\alpha(\vec{r}_1)\,\phi_\beta(\vec{r}_2) \qquad \text{if } \alpha,\beta \text{ are levels of unequal particles}[14]$$

$$\to\frac{1}{\sqrt{2}}\left(\phi_\alpha(\vec{r}_1)\,\phi_\beta(\vec{r}_2)-\phi_\alpha(\vec{r}_2)\phi_\beta(\vec{r}_1)\right) \qquad \text{if } \alpha,\beta \text{ are levels of equal particles}[14] \tag{19}$$

For small distances, however, the solution $\psi_{\alpha\beta}$ is different from (19). In fact, for very small distances ($|r_1-r_2|\ll\lambda_F$) $\psi_{\alpha\beta}$ is equal to the solution ψ of the scattering problem of the isolated pair since, in that region, only the high Fourier components above the Fermi distribution are relevant. Hence the Fermi distribution limits the effects of the interaction. It allows only a modulation of the unperturbed wave function at small distances. At large distances the wave function assumes its unperturbed form and therefore has no scattered wave.

Let us introduce the difference

$$g(\vec{r}_1\vec{r}_2)\begin{cases}=\psi_{\alpha\beta}-\phi_\alpha(\vec{r}_1)\,\phi_\beta(\vec{r}_2) & \alpha,\beta \text{ unequal particles}\\ =\psi_{\alpha\beta}-\frac{1}{\sqrt{2}}\left(\phi_\alpha(\vec{r}_1)\phi_\beta(\vec{r}_2)-\phi_\alpha(\vec{r}_2)\,\phi_\beta(\vec{r}_1)\right) & \alpha,\beta \text{ equal particles}\end{cases} \tag{20}$$

The distance $|r_1-r_2|$ beyond which $g(r_1r_2)$ becomes negligible compared to $\psi_{\alpha\beta}$ will be called the "healing distance". We can picture the effect of the interaction as a "wound" in the unperturbed wave (19), which must heal at larger distances; whereas, in the case of the isolated pair, the wound has permanent effects at large distances; the scattered wave. Obviously, the healing distance will be of the

order of λ_F. The actual size of it is of great importance for the validity of the I.PA. model and will be discussed in Section V.

The eigenvalue $\epsilon_{\alpha\beta}$ of the B.-G. equation determines the energy change which the interaction produces. It can be written in the form:

$$\epsilon_{\alpha\beta} = \left(\Psi_{\alpha\beta} \left| \frac{-\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) + Q^F_{\alpha\beta} v(12) \right| \Psi_{\alpha\beta} \right) \quad (21)$$

when the solutions $\psi_{\alpha\beta}$ are assumed to be normalized to unity. If there were no interaction (v(12) = 0), we would get the asymptotic forms (19) as solutions and

$$\epsilon_{\alpha\beta} = \frac{\hbar^2}{2m} (k_\alpha^2 + k_\beta^2) \qquad \text{for } v = 0.$$

Hence, the energy correction, due to interaction, is given by

$$U_{\alpha\beta} = \epsilon_{\alpha\beta} - \frac{\hbar^2}{2m} (k_\alpha^2 + k_\beta^2) \quad (22)$$

it should be mentioned that $\psi_{\alpha\beta}$ has no Fourier components corresponding to momenta within the Fermi distribution, except $\vec{k}_\alpha \vec{k}_\beta$. Hence $Q^F_{\alpha\beta} \psi_{\alpha\beta} = \psi_{\alpha\beta}$, and we get:

$$\epsilon_{\alpha\beta} = \left(\Psi_{\alpha\beta} \left| -\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) + v(1,2) \right| \Psi_{\alpha\beta} \right) \quad (23)$$

In the spirit of the I.PA.M., we assume that each nucleon pair acts independently from all others, and, therefore, in this model the total energy of nuclear matter can be expressed in the form (13), with $U_{\alpha\beta}$ determined by equations (22), (23).

We now improve our calculation by applying the "self-consistent I.PA.M." Here we assume that all particles move in a momentum-dependent one-particle potential W(p). Hence the B.-G. equation will be instead of (17):

$$\left(\frac{-\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) + W(p_1) + W(p_2) - \epsilon_{\alpha\beta}\right) \Psi_{\alpha\beta} = -Q^F_{\alpha\beta} v \Psi_{\alpha\beta} \quad (24)$$

where p_1 and p_2 are the momentum operators of the particles 1 and 2. For the sake of simplicity we approximate the p-dependence of W by the quadratic function (10) and get

$$\left(-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)+2W_o-\epsilon_{\alpha\beta}\right)\Psi_{\alpha\beta}=-Q^F_{\alpha\beta}\,v\,\Psi_{\alpha\beta} \tag{25}$$

which differs from (17 only by the effective mass; obviously the constant $2W_o$ is irrelevant. The eigenvalue $\epsilon_{\alpha\beta}$ is given by

$$\epsilon_{\alpha\beta}=\left(\Psi_{\alpha\beta}\left|-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)+W(p_1)+W(p_2)+v\right|\Psi_{\alpha\beta}\right)$$

which, after introduction of an effective mass, becomes

$$\epsilon_{\alpha\beta}=\left(\Psi_{\alpha\beta}\left|-\frac{\hbar^2}{2m^*}\left(\nabla_1^2+\nabla_2^2\right)+2W_o+v\right|\Psi_{\alpha\beta}\right) \tag{26}$$

The energy correction $U_{\alpha\beta}$ caused by the interaction is the difference between $\epsilon_{\alpha\beta}$ and the value for v = 0:

$$\begin{aligned}U_{\alpha\beta}&=\epsilon_{\alpha\beta}-W(p_\alpha)-W(p_\beta)-\frac{\hbar^2}{2m}\left(k_\alpha^2+k_\beta^2\right)\\&=\left(\Psi_{\alpha\beta}\left|-\frac{\hbar^2}{2m^*}\left(\nabla_1^2+\nabla_2^2+k_\alpha^2+k_\beta^2\right)+v\right|\Psi_{\alpha\beta}\right)\end{aligned} \tag{27}$$

This expression enables us to determine the one-particle potential $W'(p_\alpha)$, which would result from this calculation

$$W'(p_\alpha)=\sum_\beta U_{\alpha\beta} \tag{28}$$

We assure self-consistency by trying to choose W(p) such that the resulting W'(p) is as close to it as possible. This is especially simple when one uses the effective mass approximation for W(p) and solves (25) for a given value of m*. One then obtains from (28) a potential $W'(m^*,p)$, which is a function of p and of the value m^* used in (24), (25). (26). In general, $W'(m^*,p)$ will not be a quadratic function in p. One can, however, try to approximate $W'(m^*,p)$ by the best parabola and then equate m^* with the value one would get from this parabola. This leads to the following equation for m^*:

$$\frac{1}{p^*}\left(\frac{\partial W'(m^*,p)}{\partial p}\right)_{p=p^*}=\frac{1}{m^*}-\frac{1}{m} \tag{29}$$

If W' would be a quadratic function of p, the left side would be

independent of p. In the general case one must choose a suitable value of p, namely p^*, for the determination of m^* by (29). Since the deviation of $\psi_{\alpha\beta}$ from the unperturbed form (19) comes from Fourier components above the Fermi momentum (but not too highly above), we will choose $p^* = p_F$. Hence (29), with $p^* = p_F$ allows us to determine the self-consistent value of m^*.

V. THE SOLUTIONS OF THE BETHE-GOLDSTONE EQUATION

The solving of the Bethe-Goldstone equation (17) is more complicated than the solving of the problem of an isolated pair (Eq. (15)) on two accounts. One is the fact that the term $Q^F_{\alpha\beta}(v\psi)$ contains a non-local potential operator $Q^F_{\alpha\beta}\, v$, and the second is the fact that $Q^F_{\alpha\beta} v$ depends upon the momentum $\vec{P} = \vec{p}_\alpha + \vec{p}_\beta$ of the center-of-mass motion of the two particles. From the center of mass of a pair of particles, the Fermi distribution does not look isotropic in momentum space when $\vec{P} \neq 0$. Hence, for $\vec{P} \neq 0$, the relative angular momentum is not a constant of motion, and the problem does not decompose into separate sub-states with a definite angular momentum.

The present discussion, however, will be restricted to the case $\vec{P} = 0$, where such separation is possible. A more detailed calculation of cases with $\vec{P} \neq 0$ has shown that the main features of the solutions are the same as in the case $\vec{P} = 0$. Hence, we only discuss the case $\vec{P} = 0$ and assume that the solutions for the general case have similar properties. For the case $\vec{P} = 0$ only the first complication remains. The equation (17) can then be expressed completely in the relative coordinates $\vec{r} = \vec{r}_1 - \vec{r}_2$, and we get

$$\left(\frac{\hbar^2}{m^*} \nabla^2 + \epsilon \right) \psi(r) = Q^F_\alpha \left(v(r)\, \psi(r) \right) \qquad (30)$$

Here Q_α^F is a projection operator, which removes from $v(r)\psi(r)$ those Fourier components whose wave number is less than k_F[15] and unequal to $\vec{k}_\alpha$ and $-\vec{k}_\alpha$. These are the components which correspond to the relative momenta of all other pairs of particles within the Fermi distribution, as long as $\vec{P} = \vec{k}_1 + \vec{k}_2 = 0$.

The equation (30) can be separated according to the relative angular momentum quantum number. Here we will only discuss the $\ell = 0$ part. As long as we are only studying the $\ell = 0$ part, we have automatically excluded pairs of equal particles. The latter pairs do not have a relative S-wave because of the requirement of antisymmetry. For unequal pairs the S-wave equation becomes

$$\left(\frac{\hbar^2}{m^*}\frac{d^2}{dr^2} + \epsilon\right) u(r) = \tilde{Q}_\alpha^F \left(v(r)\, u(r)\right) \tag{31}$$

with $u(r)/r$ being the $\ell = 0$ component of $\psi(r)$ and $\tilde{Q}_{\alpha\beta}^F$ the corresponding projection operator. We note here without further proof[16] that the following relation holds:

$$\tilde{Q}_\alpha^F(v(r)\, u(r)) = v(r)\, u(r) - \int_0^\infty f(r,r')V(r')u(r')dr'$$

$$f(rr') = \frac{1}{\pi}\left(\frac{\sin k_F(r-r')}{k_F(r-r')} - \frac{\sin k_F(r+r')}{k_F(r+r')}\right) \tag{32}$$

The solution we are looking for has the asymptotic form of the unperturbed solutions. The $\ell = 0$ part of the unperturbed relative motion is

$$\lim_{r\to\infty} = \sqrt{\frac{2}{R}}\ \sin kr \tag{33}$$

where k is the relative momentum of the pair under consideration. The normalization is chosen such that the square integral of the

function is unity in the normalization volume which we now choose as a big sphere with a radius R.

The differential equation (31) can be solved by different approximation methods, which will not be treated in this paper. We will only describe and discuss the solutions for a few characteristic potentials. We begin with a repulsive core alone and set:

$$\begin{aligned} v(r) &= \infty \quad r < c \\ V(r) &= 0 \quad r > c \end{aligned} \tag{34}$$

In particular, we choose $c = 0.4 \times 10^{-13}$ cm, which is the probable value for the repulsive core in the nuclear interaction. Figures 1 and 2 show the solutions of (31) for two values of k, k= 0 and $k = \frac{1}{2}k_F$. The figures allow us to compare the actual solution u(r) with two other functions. One is the unperturbed free particle wave function $u_o(r)$, which is the solution of (31) for v = 0:

$$u_o(r) = \sqrt{\frac{2}{R}} \sin kr \tag{35}$$

The other is the solution $u_1(r)$ for the corresponding problem of an isolated pair;

$$\frac{\hbar^2}{m}\left(\frac{d^2}{dr^2} + k^2\right) u_1(r) = v(r)\, u_1(r) \tag{36}$$

For the potential (34) we have:

$$u_1(r) = \sqrt{\frac{2}{R}} \sin k(r-c)$$

The solution u(r) starts out at r = c, just like $u_1(r)$. It then gets closer to $u_o(r)$ and, after oscillating around $u_o(r)$, becomes asymptotically identical with $u_o(r)$, as required by (33). Here we have the graphical representation of the "healing" of the wave function. The potential changes the unperturbed wave function. In the case

$u_i(r)$ of an isolated pair this change is maintained until infinite distance in the form of a phase shift. In the B.-G. solution $u(r)$, the change is quickly "healed," and the function approaches its unperturbed form within a "healing distance" of the order of λ_F.

We now discuss the solutions for the nuclear potential as assumed in (1), which differs from (34) by the attractive part outside the repulsive core. As seen in Figures 3,4,5, the wave functions $u(r)$ again get close to the unperturbed wave function within a distance λ_F from the core. In fact, they are almost identical with the wave functions of the hard core problem (36). Why does the attractive part of the potential not cause any appreciable modulation of the wave function? It certainly would do so in the case of an isolated pair according to Eq. (36). The reason lies in the fact that the modulation of the wave function would contain mostly Fourier components with wave numbers below the Fermi distribution[17], and these components are not admitted in the B.-G. equation (they are occupied by other particles).

It will be important for the understanding of nuclear dynamics to realize that the wave function $u(r)$ of two interacting particles is almost the same as if only the repulsive core were present and also almost the same as the free particle wave function $u_o(r)$ except for the immediate neighborhood of the core. This is caused by a "stiffness" of the wave function against the influence of the potential, a stiffness which comes from the effect of the other particles via the exclusion principle. It prevents any change of the wave function except into Fourier components which are not occupied. In order to illustrate this effect, we have plotted in Figures 6 and 7 the difference $g(r) = u(r) - u_o(r)$ between the B.-O. wave function and

the unperturbed one and, as a comparison, also the difference $g_1(r) = u_1(r) - u_o(r)$ between the wave function $u_1(r)$ of the isolated pair and $u_o(r)$. Obviously we get:

$$g_1(r) = 2 \sin \frac{\delta}{2} \cos \left(kr + \frac{\delta}{2}\right) \qquad (37)$$

It is visible how much smaller g(r) is in comparison to $g_1(r)$ and also how g(r) decreases with the distance.

VI. THE REASONS FOR THE VALIDITY OF THE INDEPENDENT PAIR MODEL AND THE SHELL MODEL

The study of the shape of the B.-G. wave functions allows one to understand directly and simply what problems are involved in the question of the validity of the independent pair model and, in the last instance, also of the surprisingly good validity of the shell model. We do not intend to discuss the question of convergence of the corrections to these models, a question which must be clarified before the validity can be considered as established. All we intend to do is to give some reasons why such validity might perhaps be plausible.

The main point arises from the fact that the B.-G. wave function does not differ much from the free-particle wave function. It differs only when the distance of the particle is much less than the average distance in the nucleus. This latter distance $d = \rho^{-1/3} = (3\pi^2/2)^{1/3} \hbar_F = 1.60f$ is marked upon the abscissas of Figures 3-8, and it is seen that the wave function u(r) does no longer deviate much from the free-particle solution $u_o(r)$ at that point and beyond. This fact can be interpreted as follows: when one of the partners of the pair makes a close encounter with a third particle, the distance between the original pair is in the average of the order d. At that distance the wave function of the relative motion is

already "well healed"; it practically has assumed the value of the free-particle solutions; thus most of the collisions with a third particle take place under conditions as if the original pair had no interaction at all. This is just the assumption of the I.PA.M., and, hence, it is plausible that this model represents a good approximation.

The "healing" process is even better visible if one plots the actual value of the $\ell = 0$ part of the wave function $\Psi_s = u(r)/kr$ and compares it with the free-particle value $\Psi_{os} = \sin kr/kr$. This is done[18] in Figures 8 and 9, and it is easily seen that the "wound" which the potential (mainly the core) inflicts upon the wave functions disappears quickly and that its effect is negligible at $r = d$. It is important to realize that, in a degenerate Fermi gas, multiple scattering is impossible. No scattered wave can ever be formed after the interaction of a pair, since the corresponding quantum states are all occupied. Hence, all that the potential can cause is a modulation of the wave function at small distances. The I.PA.M. has a chance to be a convergent method if the regions of strong modulation are very small. The absence of scattering in a Fermi gas has been early recognized as an essential reason for the validity of the I.PA.M.[19]. The B.-G; wave equation is the quantitative formulation of this quantative idea.

We have so far only discussed the S-part ($\ell = 0$) of the wave function $\Psi_{B.-G.}$. The higher angular momenta are not of great importance for our discussion here. The P-state wave function is not influenced at all since we have assumed that no forces act in odd states. The nuclear forces are fully acting in D-states, but the shielding effect of the centrifugal term is so strong that the in-

fluence of the forces on the wave function is negligible anyway, even before considering the effect of the exclusion principle. Hence, both in the P and D states, the wave function is practically equal to the unperturbed one even for an isolated pair. The interaction in angular momenta states higher than D can be neglected completely.

The same property of the wave functions also provides a plausibility argument for the validity of the shell model. It shows that, in spite of the nuclear interaction, the wave functions are almost identical with the free particle solutions, apart from those regions in the configuration space where two particles are very close together at distances much smaller than the average distance d. Therefore, it is a good approximation to represent the nucleons in the nucleus by free-particle wave functions except for those problems in which the close encounter of two particles play an essential role.[20]

The validity of the independent-particle approximation in nuclear matter is such an unexpected result that it might be useful at this place to enumerate the factors that contribute to it on the basis of our present approach. The main cause for the approximate free-particle shape of the wave functions is their "stiffness" against deformation, which is caused by the exclusion principle. It is difficult for the wave function to change its form under the influence of the forces because all adjacent Fourier components are occupied and, therefore, unavailable for the change. However, this stiffness is not unlimited. If the interaction forces are strong enough for breaking it, the I.PA.M. would no longer be valid, and the characteristic independent particle aspects would disappear. After all, we know of systems of Fermi particles which do not show

these properties. A typical example is solid nitrogen at zero temperature, which behaves rather like a crystal than a Fermi gas. Nevertheless, it represents the lowest state of an assembly of Fermi particles (the nitrogen atoms). Therefore, we are justified to ask what are the particular features of the nuclear interaction which are responsible for the properties of nuclear matter.

It appears from our analysis that three factors play an essential role: the first is the existence of a repulsive core. It not only prevents the nucleus from collapsing, as will be shown in the next section, it is also small enough for allowing the B.-G. wave function to "heal" well within the distance d. The second factor is the nature of the attractive potential. It contributes to the validity of the free-particle approximation mainly because of its comparative weakness and long range. This statement appears paradoxical, since the nuclear forces are often regarded as particularly strong forces. But even the simplest example of nuclear binding, the deuteron, shows that the nuclear force is just about the weakest one that would give rise to a bound state. The weakness has two effects in the problem of nuclear matter. First, it causes the nuclear density to be rather low, such that d is much larger than the core radius. Here the absence of attraction in the P-states is of importance. This is necessary for the requirement that the "healing" distance be smaller than d. A larger attractive potential would lead to higher equilibrium density. Second, the weakness together with the relatively long range has the following effect: The modulation of the isolated pair-wave function caused by the attractive part of the potential contains only rather low Fourier components whose wave numbers are mainly less than k_F. Therefore, this modulation is greatly reduced

and essentially eliminated by the exclusion principle. If the forces were more strongly attractive, they would modulate on the wave function noticeably even within the Fermi distribution, and, hence, the wave function would not be healed at the distance d.

The interaction between nitrogen atoms is an example of forces which do not give rise at zero temperature to a state similar to nuclear matter. There is also a repulsive core at small distances and an attractive potential beyond the core, but the attraction is so strong that the average distance at the equilibrium density is not much larger than the radius of the repulsive core. The relatively greater strength of attraction also manifests itself in the fact that the N_2 molecule posesses many rotational states, in contrast to the nuclear situation in the deuteron. Hence, we do not expect the wave function of a pair to be "healed" at the average distance. The corrections to the I.P.A.M. will be large, and the dynamics of nitrogen at low temperature will be of the "strong interaction" type, as we know it in liquids or solids.

It is instructive to express the relevant "strength" of interaction in the following way: The wave number $\tilde{k}$ which the depth V_o of the attractive part of the potential impresses upon the wave function of isolated particles is of the order of

$$\tilde{k} \sim \sqrt{mV_o/\hbar^2}$$

If this wave number is smaller than the Fermi wave number k_F, the attractive potential will not be able to change the wave function appreciably within a Fermi distribution. We introduce a strength factor f by the following relation (d is the average distance of the

particles)

$$V_o = f \frac{\hbar^2}{m} k_F^2 = f\left(\frac{3\pi^2}{2}\right)^{2/3} \frac{\hbar^2}{md^2}$$

and we can express the condition $\tilde{k} < k_F$, which is necessary for the validity of the I.PA.M. in the form $f < 1$. For nuclear matter we obtain $f \sim \frac{1}{3}$; for solid nitrogen we put approximately $d \sim 10^{-8}$ cm, $V_o \sim 7$ ev (dissociation energy of N_2), and get $f \sim 250$. These values illustrate clearly the "strength" of the binding forces between atoms and the "weakness" of the nuclear forces.

VIII. THE DETERMINATION OF THE NUCLEAR DENSITY AND BINDING ENERGY

Once the B.-G. wave function is determined, the nuclear binding energy can be calculated quite straightforwardly. Equation (27) allows us to calculate $U_{\alpha\beta}$, from which the one-particle potential $W'(p)$ is found with (28) and the self-consistent value of the effective mass determined with (29). Then Equation (13) gives the total energy of nuclear matter in terms of the $U_{\alpha\beta}$. All magnitudes involved are functions of the Fermi momentum and, therefore, a function of the density. This enables us to find the density ρ for which the energy E has a minimum. These two values then are the density and energy of nuclear matter. The determination of these expressions can be simplified by making use of some of the properties of $\psi_{\alpha\beta}$, which we have discussed in Section V. We have found that the $\psi_{\alpha\beta}$ are not very different from the free-particle wave functions, and, furthermore, that they are almost identical with the solutions of the problem in which $v(r)$ contains the repulsive core only. The latter property makes it easy to determine the contribution of the core to the energy. Let us call v_c the part of the potential $v(r)$ which contains

the core only and v_A the attractive part.

$$v = v_c + v_A \tag{40}$$

$$v_c = \infty \qquad \text{for } r < c$$

$$v_c = 0 \qquad \text{for } r > c$$

$$v_A = \frac{1}{2}(1 + P_M)V_o \qquad \text{for } c < r < b$$

$$v_A = 0 \qquad \text{for } r > b$$

P_M is the Majorana exchange operator. We then can separate $U_{\alpha\beta}$ in a core part and the rest. The part of $U_{\alpha\beta}$ in (27) which responds to the core is

$$U^c_{\alpha\beta} = (\psi_{\alpha\beta} \mid \frac{-\hbar^2}{2m^*}(\nabla_1^2 + \nabla_2^2 + k_\alpha^2 + k_\beta^2) + v_c \mid \psi_{\alpha\beta}) \tag{41}$$

We call U_c the contribution to the energy which comes from this part:

$$U_c = \frac{1}{2}\sum_{\alpha,\beta} U^c_{\alpha\beta} \tag{42}$$

In view of what was said before about $\psi_{\alpha\beta}$, U_c is almost identical with the energy correction of a nuclear Fermi gas of particles with mass m^* and with repulsive cores only, since $\psi_{\alpha\beta}$ is almost the correct wave function for that case. This energy can be calculated from Equation (41) and is plotted in Figure 11. One can expand U_c as a power series in k_F, which starts with a cubic term as has been done by Huang[21], Yang and Lee[22], and Martin and deDominicis[23]. The result is[24]

$$\frac{1}{A}U_c = \frac{\hbar^2 k_F^2}{2m^*}\left(\frac{2}{\pi}ck_F + \frac{12}{35\pi^2}(11-2\ln 2)c^2k_F^2 + 0{,}26c^3k_F^3 + \ldots\right) \tag{43}$$

In the relevant region, the terms beyond k_F^5 are small, and this expression gives a very good approximation to the above curve.

It should be noted that U_c is purely a kinetic energy since the potential energy term in (41) vanishes because $\psi_{\alpha\beta} = 0$ at and inside the core. The kinetic energy $U^c_{\alpha\beta}$ comes from the additional curvature above its unperturbed form which is forced upon $\psi_{\alpha\beta}$ near the core. (See Figures 1, 2, 3). It shows an increase of the average kinetic energy, over and above the well-known quadratic expression, $\frac{3}{5}\frac{\hbar^2}{2m}k_F^2$, of the free Fermi gas. Physically speaking, the quadratic dependence comes from the fact that the volume available for each particle decreases in proportion with ρ^{-1}; the additional term U_c is an expression of the fact that the volume available for each particle decreases more rapidly than ρ^{-1} when there are repulsive cores around the particles; in fact, the available volume vanishes and U_c reaches infinity when ρ^{-1} becomes c^3.

The part of $U_{\alpha\beta}$ which comes from the attractive potential v_A and which is not included in (43) is

$$U^A_{\alpha\beta} = (\psi_{\alpha\beta} \mid v_A \mid \psi_{\alpha\beta}) \tag{44}$$

Since ψ_{BG} is almost equal to the free-particle wave function, it is possible to evaluate $U^A_{\alpha\beta}$ by the Born approximation;

$$U^A_{\alpha\beta} \cong (\psi_o \mid v_A \mid \psi_o) \tag{45}$$

where ψ_o is the free particle wave function of the pair:

$$\psi_o = \frac{\sqrt{2}}{\Omega} \sin(\vec{k}\cdot\vec{r}) \cdot \exp\left[i(\vec{k}_\alpha + \vec{k}_\beta)\cdot\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)\right]$$

if α, β are levels of equal particles

$$\psi_o = \frac{1}{\Omega}\exp\left[i\,\vec{k}\cdot\vec{r}\,\right]\cdot\exp\left[i(\vec{k}_\alpha + \vec{k}_\beta)\cdot\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)\right] \tag{46}$$

if α, β are levels of unequal particles and $\vec{k} = (\vec{k}_\alpha - \vec{k}_\beta)/2$ is the relative momentum.

It should be emphasized here that the validity of the Born approximation for the evaluation of the attractive part of the interaction energy is solely due to the influence of the fermi distribution. In the case of an isolated pair, the expression (45) would be completely wrong.

We find then the contribution to the energy:

$$U_A = \frac{1}{2} \sum_{\alpha,\beta} U^A_{\alpha\beta} \qquad (47)$$

which is easy to calculate from (44) or (45). The Figure 11 gives U_A as a function of k_F. The total energy of nuclear matter is then

$$E = E_o + U_c + U_A \,, \quad E_o = \frac{3}{5} \cdot \frac{\hbar^2}{2m} k_F^2$$

which is plotted in Figure 12. Here the self-consistent value of m^* is used, the determination of which will be mentioned later. It turns out that $m^* = 0.68$ m for the equilibrium density.[25] One observes a minimum for $k_F = 1.35 \times 10^{13} cm^{-1}$, which is very near to the correct value. The energy at that density is F/A = -6.9 Mev and consists of the following parts;

$$E_o = 22.8 \text{ Mev}$$
$$\frac{1}{2}U_c = 27.7 \text{ Mev}$$
$$\frac{1}{2}U_A = -57.4 \text{ Mev}$$

The value of E/A is considerably less than the actual value of 15 Mev. However, the breakdown shows that the difference is only 14% of the contribution of the attractive potential energy. It very probably comes from our leaving out the influence of the tensor force. This neglecting of that force is based upon our finding that the wave functions are almost equal to the unperturbed ones. In this case the contribution of the tensor force is zero, as it must be for

any Born approximation. We cannot expect, however, that our calculations be as good as to exclude a 14% change of the potential energy value.

The situation is somewhat better when we consider the values at the observed density which corresponds to a value $k_F = 1.48\ f^{-1}$. Here we get

$$E_o = 27.3 \text{ Mev}$$

$$\frac{1}{A}V_c = 38.4 \text{ Mev}$$

$$\frac{1}{A}U_A = -72.1 \text{ Mev}$$

Hence the error made neglecting the tensor force is only 12% of the potential energy.

The density dependence of U_c and U_A gives some answers to the fundamental question: What keeps the nucleus together and what keeps it from collapsing? The expectation value U_A of the attractive potential increases its magnitude (it is negative) so steeply with increasing density (with k_F^3 for very high densities) that it would overcome the free kinetic energy E_o, and they cause a collapse if it were not for U_c. This is in agreement with the well-known theorem which says that attractive Serber forces do not saturate.[26] The additional kinetic energy U_c coming from the volume reduction of the repulsive core, however, rises sufficiently fast with density so as to prevent this collapse. Hence, the repulsive core is the expanding element in nuclear matter. The attractive potential U_A keeps it from flying apart, but it is weak enough for allowing the equilibrium to be at rather low density, at a density for which the average distance d is large compared to the core radius ($d/c \sim 5$). The

large ratio d/c in turn makes it possible that the "healing" distance of the B.-G. solution is small compared to d and, thus, is responsible for the validity of the independent particle approximation.

We do not report here the detailed calculations of U_c and U_A. Suffice it to say that the evaluation of (41) and (45) gives rise to expressions which, to a very good approximation, depend only upon the relative momentum of the pair $k = \frac{1}{2} | \vec{k}_\alpha - \vec{k}_\beta |$ and upon k_F. Hence, we can write

$$U_{\alpha\beta}^{c,A} = \begin{cases} \frac{1}{A} U_e^{c,A}(k,k_F) & \text{if } \alpha,\beta \text{ are levels for particles of equal type} \\ \frac{1}{A} U_u^{c,A}(k,k_F) & \text{if } \alpha,\beta \text{ are levels for particles of unequal type} \end{cases} \tag{48}$$

The distinction between equal or unequal pairs comes in because of the fact that $\psi_{\alpha\beta}$ must be antisymmetric in the first case and, hence, has no even orbital angular momentum parts. The factor A^{-1} is introduced in order to make the functions $U^{c,A}(k,k_F)$ finite for $A \to \infty$. The summation over the levels in (42) can then be performed by means of an integral. For this purpose one needs the function $n(k)dk$, which is the number of pairs in a degenerate Fermi gas of $\bar{N}$ equal particles whose relative momentum is between k and k + dk:

$$n(k)\,dk = \bar{N}^2 \; 12 \left(1 - \frac{3}{2} \cdot \frac{k}{k_F} + \frac{1}{2} \left(\frac{k}{k_F}\right)^3\right) \frac{k^2 dk}{k_F^3} \tag{49}$$

We then get

$$\frac{1}{2} U^{c,A} = \frac{1}{\bar{N}A} \int_0^{k_F} \left[U_e^{c,A}(k,k_F) + 3\, U_u^{c,A}(k,k_F) \right] n(k)dk$$

The form (48) of $U_{\alpha\beta}^{c,A}$ also lends itself well for the evaluation of the one-particle potential $W(k_\alpha)$ and its dependence on the momen-

tum which is needed for the determination of m^*. We introduce another function $N(k_\alpha,k)dk$, which is the number of particles in a degenerate Fermi gas of $\bar{N}$ equal particles which have a relative momentum between k and k + dk with a given particle of fixed momentum k_α:

$$N(k_\alpha,k)dk = \bar{N}\ 12\ (1 + \eta(k_\alpha,k))\ \frac{k^2\ dk}{k_F^3}$$

with

$$\eta(k_\alpha k) = 1 \qquad \text{for } k < \tfrac{1}{2}\ (k_F - k_\alpha)$$

$$\eta(k_\alpha k) = \frac{k^2_F - k_\alpha - 4k^2}{4kk_\alpha} \qquad \text{for } \tfrac{1}{2}(k_F - k_\alpha) < k < \tfrac{1}{2}(k_F + k_\alpha)$$

Evidently n(k)dk as given in (49) can be obtained from $N(k_\alpha,k)$ by integrating k_α over all occupied values. With the help of this function we obtain in line with (28)

$$W'(k_\alpha) = \frac{1}{A}\int_0^{k_F} \left[U_e(k', k_F) + 3\ U_u(k'\ k_F)\right] N\ (k_\alpha k')dk'$$

where $\quad U_{e,u} = U^c{}_{e,u} + U^A_{e,u}$

This allows us to determine the effective mass m^* by means of (29).

APPENDIX I

Consider a system of A particles, four groups of A/4 identical particles, we first consider the case in which there is no interaction between the particles. In this case the particles occupy single-particle levels $\alpha, \beta, \ldots$, which are described by normalized wave functions $\varphi_\alpha(r)$. The lowest state of the system is characterized by the fact that, for each type of particle, the first A/4 levels are occupied. The wave function of this state is given by (3). We introduce the following symbol: In the lowest state of this system, $\alpha < F$ means α is occupied, $\alpha > F$ means α is not occupied.

We now introduce a two-particle interaction $V_{\alpha\beta}$ in the following form;[27]

$$V_{\alpha,\beta} = \sum_{i,k} v(\vec{r}_i, \vec{r}_k)\, q_\alpha^F(i)\, q_\beta^F(k) \qquad (A1)$$

Here $v(r_i, r_k)$ is the two-particle nuclear potential, as discussed in Section II, between the particles i and k. The operators $q_\alpha^F(i)$ are projection operators. They remove from every function to which it is applied those Fourier components in the coordinate r_i which correspond to one-particle states within the Fermi distribution except for the levels α_i

$$q_\alpha^F(i)\psi = \sum_{\gamma>F} (\varphi_\gamma(i), \psi)\, \varphi_\gamma(\vec{r}_i) + (\varphi_\alpha(i), \psi)\, \varphi_\alpha(\vec{r}_i) \qquad (A2)$$

where

$$(\varphi_\gamma(i), \psi) = \int \varphi_\gamma^*(\vec{r}_i)\, \psi\, d\vec{r}_i$$

We maintain that the ground state of the system with the interaction (A1) is a state in which only particles in the levels α and β interact. This can be seen in the following way: The interaction (A1) is zero whenever a particle is in a level different from α and β. The levels α, β themselves are influenced by the interaction, and their wave function is changed. We therefore write the wave function of the ground state in the form

$$\Psi = \mathcal{A}\left[\psi_{\alpha\beta}(\vec{r}_1,\vec{r}_2)\,\varphi_\gamma(\vec{r}_3)\,\varphi_\delta(\vec{r}_4)\ldots\varphi_\omega(\vec{r}_A)\right] \qquad (A3)$$

where the synbol $\mathcal{A}$ means antisymmetrization in respect to the coordinates of equal particles. This form expresses the fact that the levels $\gamma \neq \alpha, \beta$ are the same as without interaction, but the levels α, β are changed. Let us note that, because of the antisymmetrization, $\psi_{\alpha\beta}$ cannot contain any Fourier components within the Fermi distribution except φ_α and φ_β:

$$(\varphi_\gamma, \psi_{\alpha\beta}) = 0 \quad \text{for } \gamma < F \text{ and } \gamma \neq \alpha, \beta \qquad (A4)$$

As a consequence $\psi_{\alpha\beta}$ differs from $\varphi_\alpha(r_1)\,\varphi_\beta(r_2)$ by Fourier components which are <u>not</u> contained in the Fermi distribution. Hence,

$$q_\alpha^F(1)\psi_{\alpha\beta} = \psi_{\alpha\beta}\,; \quad q_\beta^F(2)\psi_{\alpha\beta} = \psi_{\alpha\beta} \qquad (A5)$$

This relation shows that the interaction (A1) is equivalent to the ordinary nuclear interaction for the levels α, β, but vanishes for all other levels, as far as the ground state of the system is concerned. Hence, the interaction (A1) fulfills the requirements which we have asked for in the I.PA.M.

We now derive a wave equation for $\psi_{\alpha\beta}(r_1, r_2)$. We take the equation for the total wave function (A3)

$$\left(\sum_i t_i + V_{\alpha\beta} - E \right) \Psi = 0$$

where $t_i = -(\hbar^2/2m)\nabla_i^2$ is the operator of the kinetic energy of the i-th particle. We then multiply this equation with ϕ^*, which is the complex conjugate of

$$\phi = \varphi_\gamma(\vec{r}_3)\, \varphi_\delta(\vec{r}_4) \cdots \varphi_\omega(\vec{r}_A)$$

and integrate over all coordinates except r_1, r_2.

We then get:

$$\begin{aligned}(t_1 + t_2 - \epsilon_{\alpha\beta})\, \psi_{\alpha\beta}(1,2) = &- v(1,2)\, \psi_{\alpha\beta}(1,2) \\ &+ \sum_\delta{}' \varphi_\delta(2) \int \varphi_\delta^*(3)\, v(1,3)\, \psi_{\alpha\beta}(1,3)\, d\vec{r}_3 \\ &+ \sum_\gamma{}' \varphi_\gamma(1) \int \varphi_\gamma^*(3)\, v(3,2)\, \psi_{\alpha\beta}(3,2)\, d\vec{r}_3 \\ &- \sum_\gamma{}' \sum_\delta{}' \varphi_\gamma(1)\, \varphi_\delta(2)(\gamma,\delta|v\psi_{\alpha\beta})\end{aligned} \tag{A6}$$

Here the summation $\sum_\gamma{}'$ or $\sum_\delta{}'$ means a sum over all occupied levels $\gamma, \delta < F$ except $\gamma, \delta = \alpha$ and $\gamma, \delta = \beta$; furthermore, we have

$$\epsilon_{\alpha\beta} = E - \sum_\gamma{}' \frac{\hbar^2}{2m} k_\gamma^2 \tag{A7}$$

$$(\gamma\delta \mid v\psi) = \int \varphi_\gamma^*(3)\, \varphi_\delta^*(4)\, v(3,4)\, \psi(3,4)\, d\vec{r}_3\, d\vec{r}_4 \tag{A8}$$

and the numbers 1, 2, 3, 4 stand for the coordinates r_1, r_2, r_3, r_4. We now use the following identities:

$$v(1,2)\, \psi(1,2) = \sum_\gamma \sum_\delta \varphi_\gamma(1) \varphi_\delta(2)\, (\gamma\delta|v\psi)$$

$$\int \varphi_\gamma^*(3)\, v(3,2)\, \psi(3,2)\, d\vec{r}_3 = \sum_\delta \varphi_\delta(2)(\gamma\delta \mid v\psi)$$

$$\int \varphi_\delta^*(3)\, v(1,3)\, \psi(1,3)\, d\vec{r}_3 = \sum_\gamma \varphi_\gamma(1)(\gamma\delta \mid v\psi)$$

where the summation $\sum_\gamma$ without prime signifies a complete sum over all levels. Then equation (A6) can be written simply as follows:

$$(t_1 + t_2 - \epsilon_{\alpha\beta})\,\psi_{\alpha\beta}(1,2) = - \sum_\gamma'' \sum_\delta'' \varphi_\gamma(3)\varphi_\delta(4)(\gamma\delta|v\,\psi_{\alpha\beta}) \tag{A9}$$

$$= -q_\alpha^F(1)\; q_\beta^F(2)\; v(1,2)\,\psi_{\alpha\beta}(1,2)$$

where $\sum_\delta'' \sum_\gamma''$ signifies the summation over all levels $\delta, \gamma > F$ and also over $\gamma = \alpha$, $\delta = \beta$. This is the wave equation for the two-particle interaction imbedded in a Fermi gas and is identical with equation (17). Evidently the operator $Q_{\alpha\beta}^F$ defined in (18) is equivalent to the product:

$$Q_{\alpha\beta}^F = q_\alpha^F(1)\; q_\beta^F(2)$$

The eigenvalue $\epsilon_{\alpha\beta}$ is given by the expression:

$$\epsilon_{\alpha\beta} = (\psi_{\alpha\beta}|\; t_1 + t_2 + v|\,\psi_{\alpha\beta}) \tag{A10}$$

if $\psi_{\alpha\beta}$ is normalized. The projection operators are omitted in this expression because of (A5).

An alternative way to derive the equation (A9) is as follows: We prove that the wave function (A3) is an eigenfunction of the Hamiltonian

$$H = \sum_i h_i + V_{\alpha\beta} \quad , \quad h_i = \frac{-\hbar^2}{2m}\,\nabla_i^2$$

when $\psi(r_1 r_2)$ fulfills the equation (A9). For this purpose we write

$$H\Psi = E\,\Psi \tag{A11}$$

and observe that

$$\sum_i h_i \Psi = \mathcal{A}\left[(h_1 + h_2)\psi(1,2)\,\varphi_\gamma(3)\,\ldots\right] + \frac{\hbar^2}{2m}\,\sum{}' k_\gamma^2\,\Psi \tag{A12}$$

We find

$$V_{\alpha\beta}\Psi = \mathcal{A}[v(1,2)\psi(1,2)\,\varphi_\gamma(3)\ldots]$$
$$= \mathcal{A}\left[\left\{q_\alpha^F(1)\; q_\beta^F(2)\; v(1,2)\psi(1,2)\right\}\varphi_\gamma(3)\ldots\right] \qquad (A13)$$

where the second identity comes from the fact that the antisymmetrization removes any part of $v(12)\psi(12)$ which is not orthogonal to the other occupied levels. We then can write (A11) in the following form, in view of (A7):

$$\mathcal{A}\left[\left\{h_1 + h_2 + q_\alpha^F(1)\; q_\beta^F(2)\; v(1,2)\psi(1,2)\right\}\varphi_\gamma(3)\ldots\right]$$
$$= \epsilon_{\alpha\beta}\mathcal{A}[\psi(1,2)\;\varphi_\gamma(3)\ldots]$$

It is obvious that this equation is fulfilled if $\psi(12)$ is a solution of (A9).

In this derivation one might think that (A11) is fulfilled also if $\psi(12)$ obeys an equation different from (A9). We could have chosen the first form for $V_{\alpha\beta}\Psi$ in (A12), and this choice would have given equation (15) for ψ, which is the equation for isolated pairs. The latter choice, however, is not admissible since the solutions of (15) do not fulfill (A4) and, therefore, do not lead to the relation (A12) for the kinetic energy. The second form of $V_{\alpha\beta}\Psi$ in (A13) is the only one which leads to an equation for ψ fulfilling the conditions (A4).

We now discuss the asymptotic behavior of $\psi_{\alpha\beta}$. First we write (A9) in the form

$$(t_1 + t_2 - \epsilon_{\alpha\beta})\psi_{\alpha\beta} = -Q^F\, v\psi - Q_{\alpha\beta}\, v\psi$$

where Q^F is the projection operator which excludes all states within the Fermi distribution without exception, and $Q_{\alpha\beta}$ is the projection

operator which singles out the levels α,β only for both particles. Evidently:

$$Q^F_{\alpha\beta} = Q^F + Q_{\alpha\beta}$$

We then can write (A9) in the form:

$$(t_1 + t_2 - \epsilon_{\alpha\beta})\psi_{\alpha\beta} = - Q^F(v\psi_{\alpha\beta}) - (\alpha\beta|\ v\psi_{\alpha\beta})\varphi_\alpha(1)\ \varphi_\beta(2) \qquad (A14)$$

We can express the first term on the right side in the following way:

$$Q^F(v\psi_{\alpha\beta}) = v\psi_{\alpha\beta} - \int G(\vec{r}_1 - \vec{r}_1')\ G\ (\vec{r}_2 - \vec{r}_2')\ v\ (\vec{r}_1'\ \vec{r}_2')\psi_{\alpha\beta}(\vec{r}_1'\ \vec{r}_2')d\vec{r}_1'\ d\vec{r}_2'$$

where

$$G(\vec{r} - \vec{r}') = \sum_{\alpha<F} \varphi_\alpha^*(\vec{r})\varphi_\alpha(\vec{r}')$$

This function is a delta function of finite width with a width of the order λ_F and vanishes at $(r - r') \to \infty$. Hence $Q^F(v\psi_{\alpha\beta})$ vanishes at $(r - r') \to \infty$, when v(12) is short ranged. One recognizes then easily from (A14) that the asymptotic form of $\psi_{\alpha\beta}$ is[27]

$$\psi_{\alpha\beta}(1,2) \longrightarrow \varphi_\alpha(1)\ \varphi_\beta(2) \quad \text{for} \quad (\vec{r}_1 - \vec{r}_2) \to \infty$$

and one gets from (A14), taken at $(r_1 - r_2) \to \infty$:

$$\epsilon_{\alpha\beta} = (\alpha\beta\,|v\psi_{\alpha\beta}) = \frac{\hbar^2}{2m}\ (k_\alpha^2 + k_\beta^2)$$

It is then more convenient to write the equation (A14) in the form

$$(t_1 + t_2 - k_\alpha^2 - k_\beta^2)\psi_{\alpha\beta} = -Q^F(v\psi_{\alpha\beta}) + (\alpha\beta|\ v\psi_{\alpha\beta})(\psi_{\alpha\beta} - \varphi_\alpha\varphi_\beta)$$

The second term on the right side vanishes for large distances $(r_1 - r_2)$. We can also neglect it for small distances in our determination of $\psi_{\alpha\beta}$, since $(\alpha\beta|v\psi_{\alpha\beta})$ is infinitesimally small for a large volume Ω of nuclear matter. In fact, it is of the order $v_o b^3/\Omega$, where v_o and b are the depth and range of the potential. When this

term is neglected, the B. - G. equation (17) goes over into the form

$$(t_1 + t_2 - k_\alpha^2 - k_\beta^2)\psi_{\alpha\beta} = -Q^F(v\psi_{\alpha\beta}) \qquad (A15)$$

which is the form found in Reference (5).

One can use (A15) or (17) for the determination of $\psi_{\alpha\beta}$. It is, of course, more convenient to work with (A15).

1. Brueckner, Levinson and Mahmoud, Phys. Rev. 95, 217 (1954).

K. A. Brueckner, Phys. Rev. 96, 508 (1954).

K. A. Brueckner, Phys. Rev. 97, 1353 (1955).

K. A. Brueckner and C. A. Levinson, Phys. Rev. 97, 1344 (1955).

K. A. Brueckner, Phys. Rev. 100, 36 (1955).

K. A. Brueckner and W. W. Wada, Phys. Rev. 103, 1008 (1956).

Brueckner, Eden and Francis, Phys. Rev. 98, 1445 (1955).

Brueckner, Eden and Francis, Phys. Rev. 99, 76 (1955).

Brueckner, Eden and Francis, Phys. Rev. 100, 891 (1955).

R. J. Eden and N. C. Francis, Phys. Rev. 97, 1366 (1955).

R. J. Eden, Phys. Rev. 99, 1418 (1955).

R. J. Eden, Proc. Roy. Soc. A 235, 408 (1956).

2. H. A. Bethe and J. Goldstone, Proc. Roy. Soc. A 238, 551 (1957).

H. A. Bethe, Phys. Rev. 103, 1353 (1956).

3. W. J. Swiatecki, Phys. Rev. 101, 1321 (1956); Phys. Rev. 103, 265 (1956).

4. One fermi (f) is 10^{-13} cm.

5. Odd states are states in which the relative angular momentum quantum number is an odd number.

6. We introduce periodic boundary conditions at the surface of this volume. This volume serves only for purposes of normalization.

7. From purely energetic considerations, one can derive a relation between m_{eff} and E, which is valid for the I.P.M. $E \rightarrow E_{IPM}$:

$(m/m^*) = \frac{3}{2}(1+(E_{IPM}/E_o))$ See V. F. Weisskopf, Nucl. Phys. 3, 423 (1957).

8. The mathematical formulation of this hypothetical case is discussed in Appendix I.

9. The mathematical definition of the operator which corresponds to this force will be given in the Appendix.

10. A derivation of the Bethe-Goldstone equation is found in Appendix I.

11. Actually the equation (17) differs slightly from the original Bethe-Goldstone equation as derived in Reference 5. The difference is negligible, as shown in Appendix I.

12. The above formula is valid if α and β are levels belonging to different

particles. If the two particles are identical, the second term on the right-hand side of (18) should be:

$$\frac{1}{\sqrt{2}} (\phi_\alpha(1)\, \phi_\beta(2) - \phi_\alpha(2)\, \phi_\beta(1))(\alpha\beta|\, \bar{Q})$$

13. As to details, see Appendix I.

14. We remind the reader that α, β denote one-particle levels and that the notation is chosen such that a level which contains a proton has always a different symbol from a level which contains a neutron, in spite of the fact that they may have the same momentum. Hence, a given level is characterized by its momentum and by the kind of particles it shelters.

15. One first might think that this limit should be $2k_F$. This is not correct, as one can see as follows: The maximum relative velocity of a pair in the Fermi distribution is $2v_F$, where v_F is the maximum velocity

$$v_F = \frac{\hbar k_F}{m} .$$

However, the relative momentum is $k_F = \mu v_F$, where $\mu = m/2$ is the reduced mass.

16. The function $f(r,r')$ is derived in H. A. Bethe and J. Goldstone, Proc. Roy. Soc. A238, 551 (1957).

17. The wave number of $u_i(r)$ in the attractive part of (1) is

$$K = (k_o^2 + k^2)^{1/2}, \text{ with } k_o = (m^* V)^{1/2}/\hbar = \frac{\sqrt{\pi}}{2} \left(\frac{m^*}{m}\right)^{1/2} b^{-1} .$$

Hence, we get

$$\left(\frac{K}{k_F}\right)^2 = (k_o/k_F)^2 + (k/k_F)^2 .$$ With the values $k_F = 1.4$ f, $b = 1.9$ f, $m^* \simeq \frac{1}{2} m$, we get $\left(\frac{K}{k_F}\right)^2 = 0.18 + (k/k_F)^2$. Hence, $K < k_F$ for values $k < 0.9 k_F$. This includes all the important relative momenta in nuclear matter, since the distribution function for relative momenta is heavily weighted in favor of the low values. (See Eq. (49)).

18. The same wave functions also have been computed by Brueckner and Gammel (to appear shortly in the Phys. Rev.) who used more exact expressions for the nuclear forces. Their result, however, is practically the same as ours. We are grateful to the authors for the communication of their results before publication.

19. V. F. Weisskopf, Helv. Phys. Acta 23, 187 (1950); Science 113, 101 (1951).

20. Some of those problems are e.g. the photonuclear effect at very high energies, the π- or μ-meson capture and some of the high-energy pick-up processes. For the treatment of these cases, the B.-G wave function might be of use.

21. K. Huang and C. N. Yang, Phys. Rev. 105, 767 (1957).

22. C. N. Yang and T. D. Lee, Phys. Rev. 105, 1119 (1957).

23. P. Martin and C. DeDominicis, Phys. Rev. 105, 1417 (1957).

24. We have not included the contribution $\frac{h^2k_F^2}{2m^*}\left(\frac{1}{\pi}c^3k_F^3\right)$ of the P-state since we have assumed no forces at all in that state; we have also omitted the contribution $\frac{h^2k_F^2}{2m^*}(0.20c^3k_F^3)$, which is due to higher cluster effects (P. Martin, private communication) since we are only considering two-body correlations.

25. We define our m^* according to (28) so as to fit the function W(k) at the top of the Fermi distribution only. For an overall fit the value of m^* would be $m^* = 0.59m$. The latter value should be compared with m^* as used in the literature.

26. See, for example, J. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics, Chapter III, John Wiley and Sons, 1952.

27. We treat here the case only in which the two levels α, β are levels belonging to two different types of particles (see footnote 11, page F1). If α, β belong to the same type, the derivation is very similar. One only must take care of the antisymmetrization.

FIGURE CAPTIONS

Figure 1. The wave functions of the relative motion in the s-state of the two particles for the case of no interaction ($u_o(r)$), for the case of a repulsive core interaction for an isolated pair ($u_i(r)$) and for a pair embedded in a Fermi distribution with $k_F = (u(r))$. The relative momentum $k = 0$.

Figure 2. The wave functions of the relative motion in the s-state of the two particles for the case of no interaction ($u_o(r)$), for the case of a repulsive core interaction for an isolated pair ($u_i(r)$ and for a pair embedded in a Fermi distribution with $k_F = (u(r))$. The relative momentum $k = 0.5\ k_F$.

Figure 3. The wave function of the relative motion in the s-state of two particles for the case of no interaction ($u_o(r)$) and for the case of the nuclear interaction ($u(r)$) for a pair embedded in a Fermi distribution $k_F = 1.48\ f^{-1}$. The relative momentum is $k = 0$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 4. The wave function of the relative motion in the s-state of two particles for the case of no interaction ($u_o(r)$) and for the case of the nuclear interaction ($u(r)$ for a pair embedded in a Fermi distribution $k_F = 1.48\ f^{-1}$. The relative momentum is $k = 0.3\ k_F$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 5. The wave function of the relative motion in the s-state of two particles for the case of no interaction ($u_o(r)$) and for the case of the nuclear interaction ($u(r)$) for a pair embedded in a Fermi distribution $k_F = 1.48\ f^{-1}$. The relative momentum is $k = 0.6\ k_F$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 6. The difference between the wave functions with and without interaction (hard core) for an isolated and for an imbedded pair. The relative momentum is $k = 0.3\ k_F$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 7. The difference between the wave functions with and without interaction (hard core) for an isolated and for an embedded pair. The relative momentum is $k = 0.6\ k_F$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 8. Full value $u(r)/kr$ of the s-state wave function for a pair embedded in a Fermi distribution ($k_F = 1.48$ f) without and with interaction. Relative momentum $k = 0.3\ k_F$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 9. Full value $u(r)/kr$ of the s-state wave function for a pair embedded in a Fermi distribution ($k_F = 1.48$ f) without and

with interaction. Relative momentum $k = 0.6\ k_F$. The heavy line indicates the nuclear potential as a function of r. The average distance d of the next neighbor is also indicated.

Figure 10. The energy per particle due to the repulsive core as a function of the density measured by k_F. This is the correction to the kinetic energy.

Figure 11. The energy per particle due to the attractive part of the nuclear potential as a function of the density measured by k_F. This is the potential energy.

Figure 12. Total energy per particle as a function of the density measured by k_F

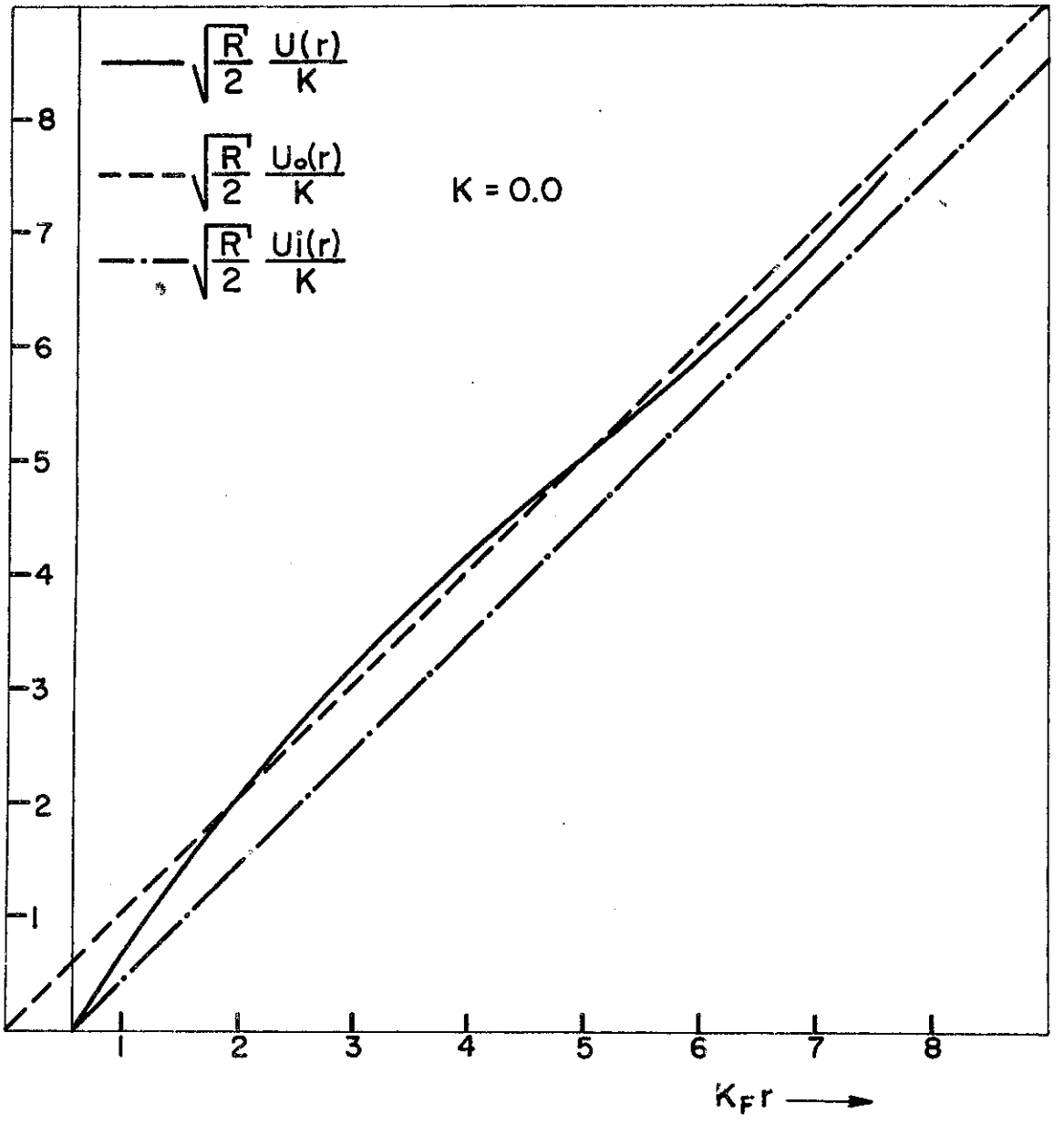
$\sqrt{\frac{R}{2}}\,\frac{U(r)}{K}$
$\sqrt{\frac{R}{2}}\,\frac{U_o(r)}{K}$
$\sqrt{\frac{R}{2}}\,\frac{U_i(r)}{K}$
K = 0.0
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
$K_F r$

FIG. 1

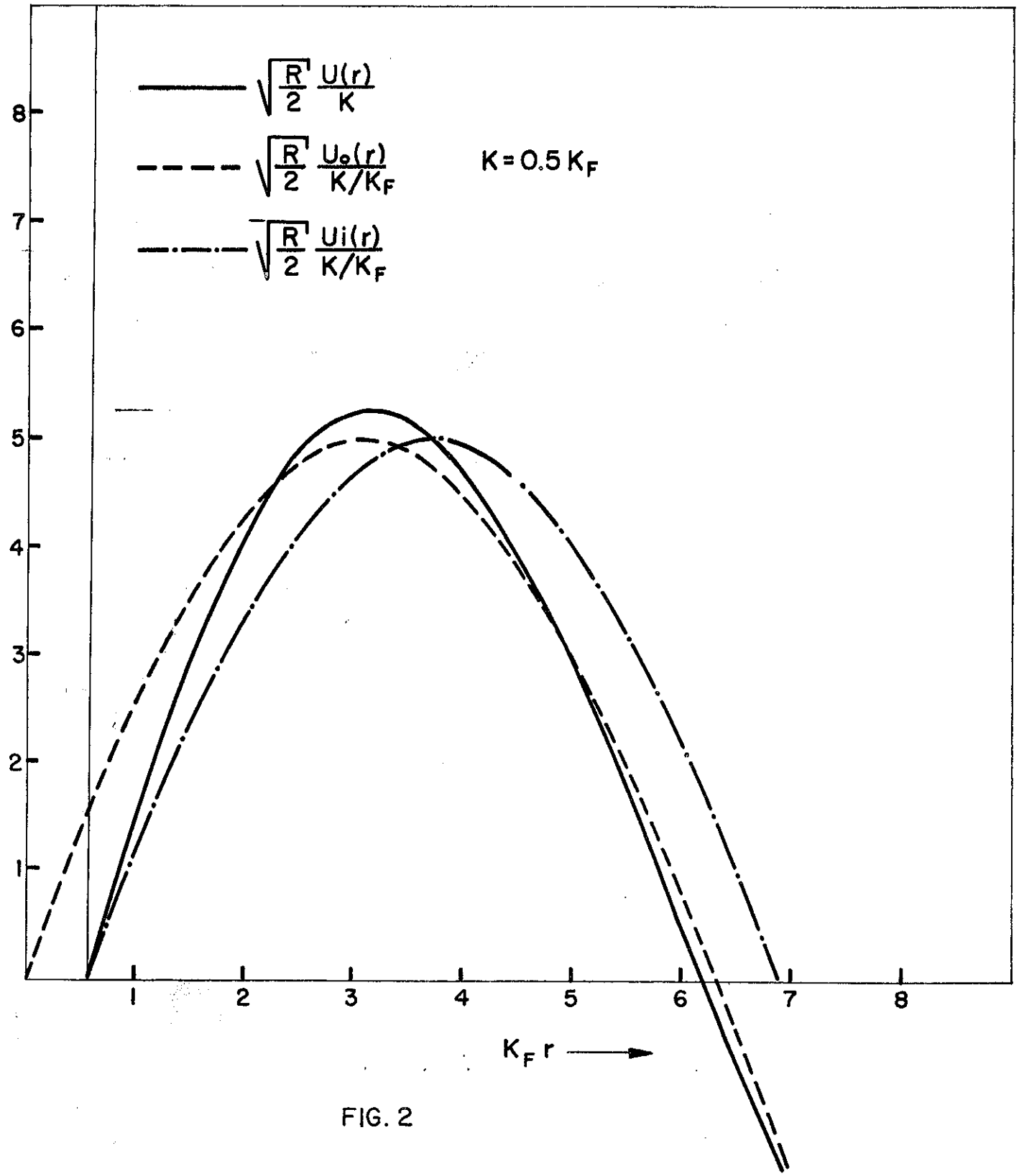
$\sqrt{\frac{R}{2}}\frac{U(r)}{K}$
$\sqrt{\frac{R}{2}}\frac{U_e(r)}{K/K_F}$
$\sqrt{\frac{R}{2}}\frac{U_i(r)}{K/K_F}$
$K = 0.5\,K_F$
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
$K_F\,r$

FIG. 2

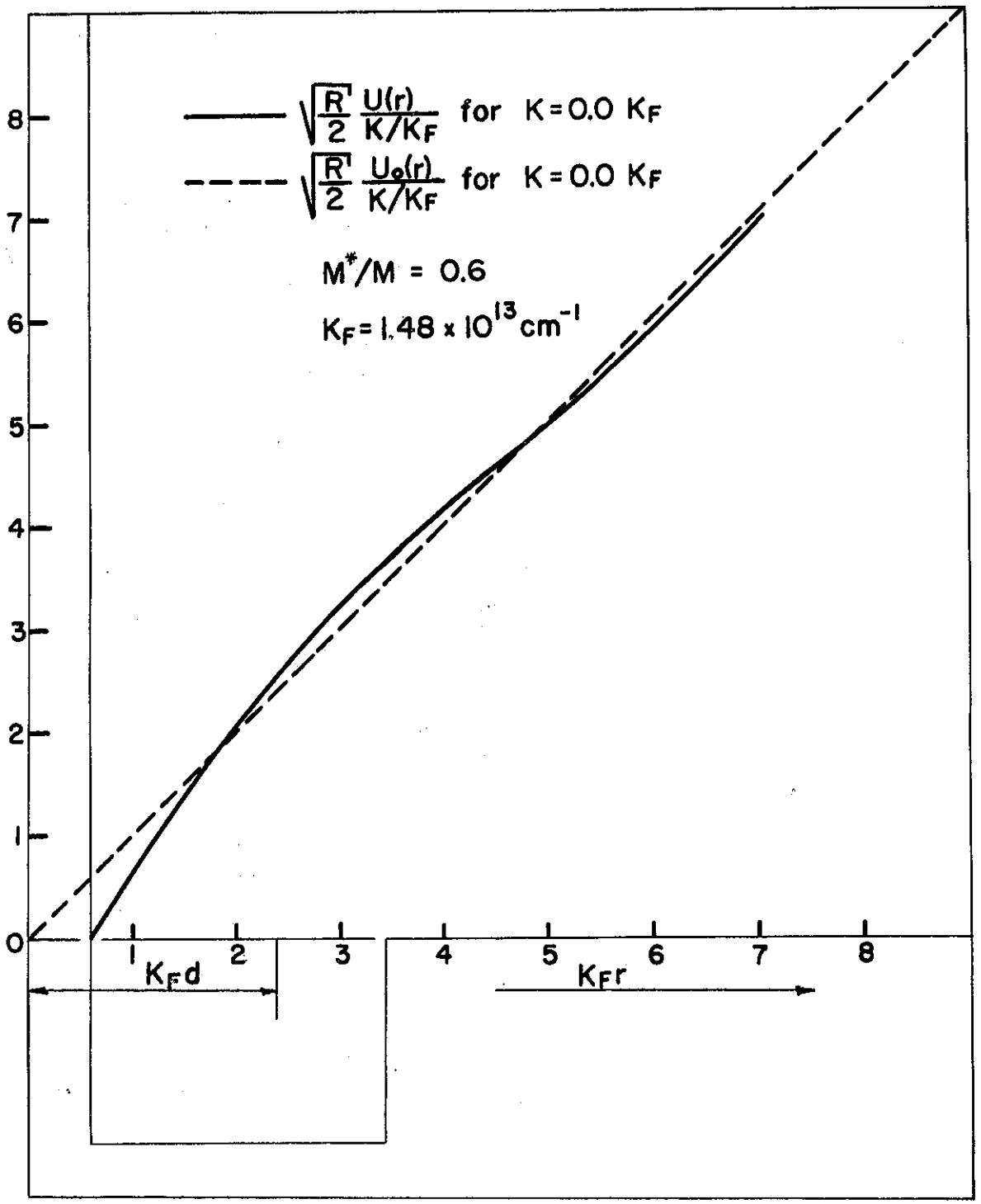
$\sqrt{\frac{R'}{2}}\frac{U(r)}{K/K_F}$ for K = 0.0 K_F
$\sqrt{\frac{R'}{2}}\frac{U_o(r)}{K/K_F}$ for K = 0.0 K_F
$M^*/M = 0.6$
$K_F = 1.48 \times 10^{13} cm^{-1}$
0
1
2
3
4
5
6
7
8
$K_F d$
$K_F r$

FIG. 3

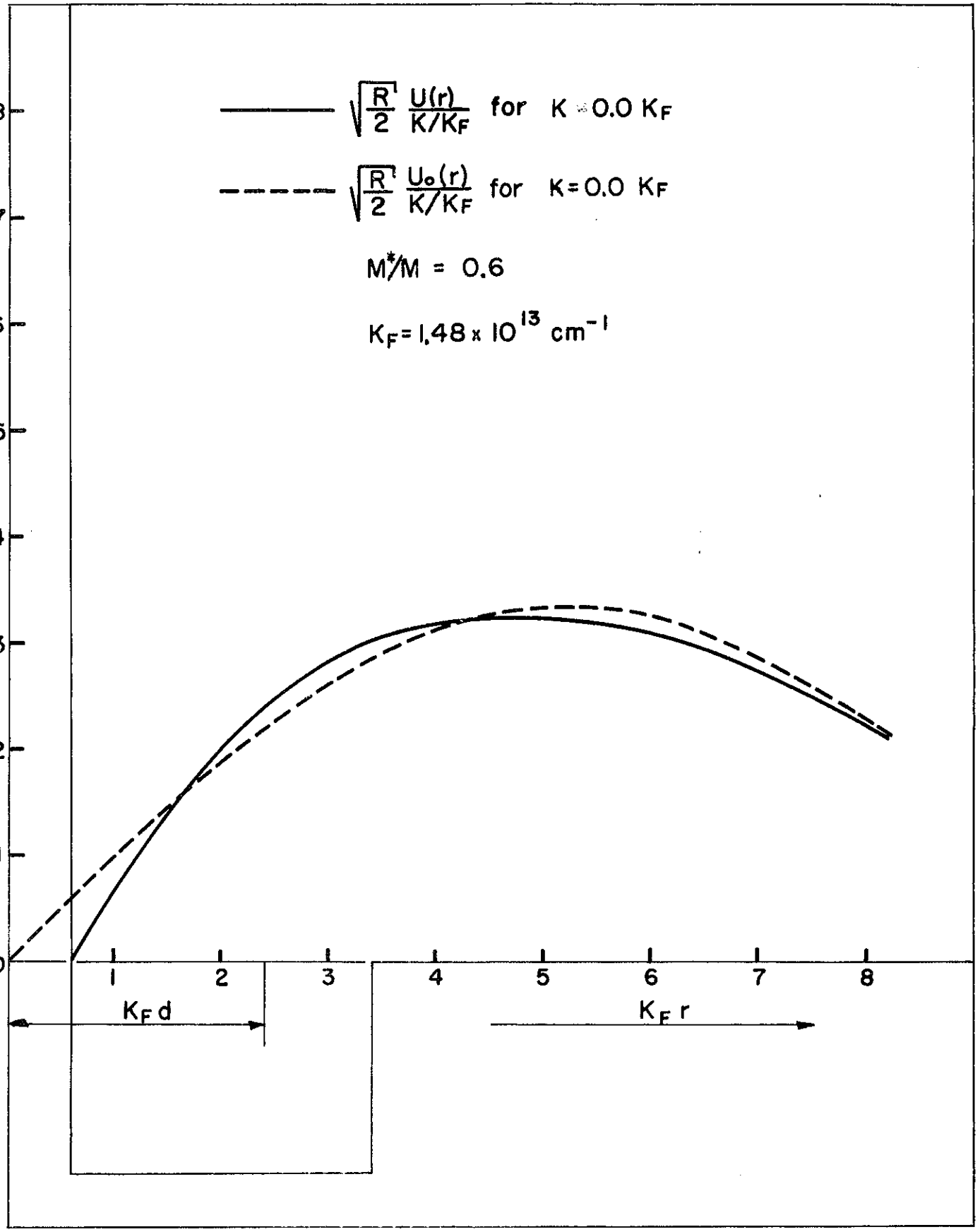
$\sqrt{\frac{R'}{2}}\frac{U(r)}{K/K_F}$ for $K = 0.0\ K_F$
$\sqrt{\frac{R'}{2}}\frac{U_o(r)}{K/K_F}$ for $K = 0.0\ K_F$
$M^*/M = 0.6$
$K_F = 1.48 \times 10^{13}\ cm^{-1}$
0
1
2
3
4
5
6
7
8
$K_F d$
$K_F r$

FIG. 4

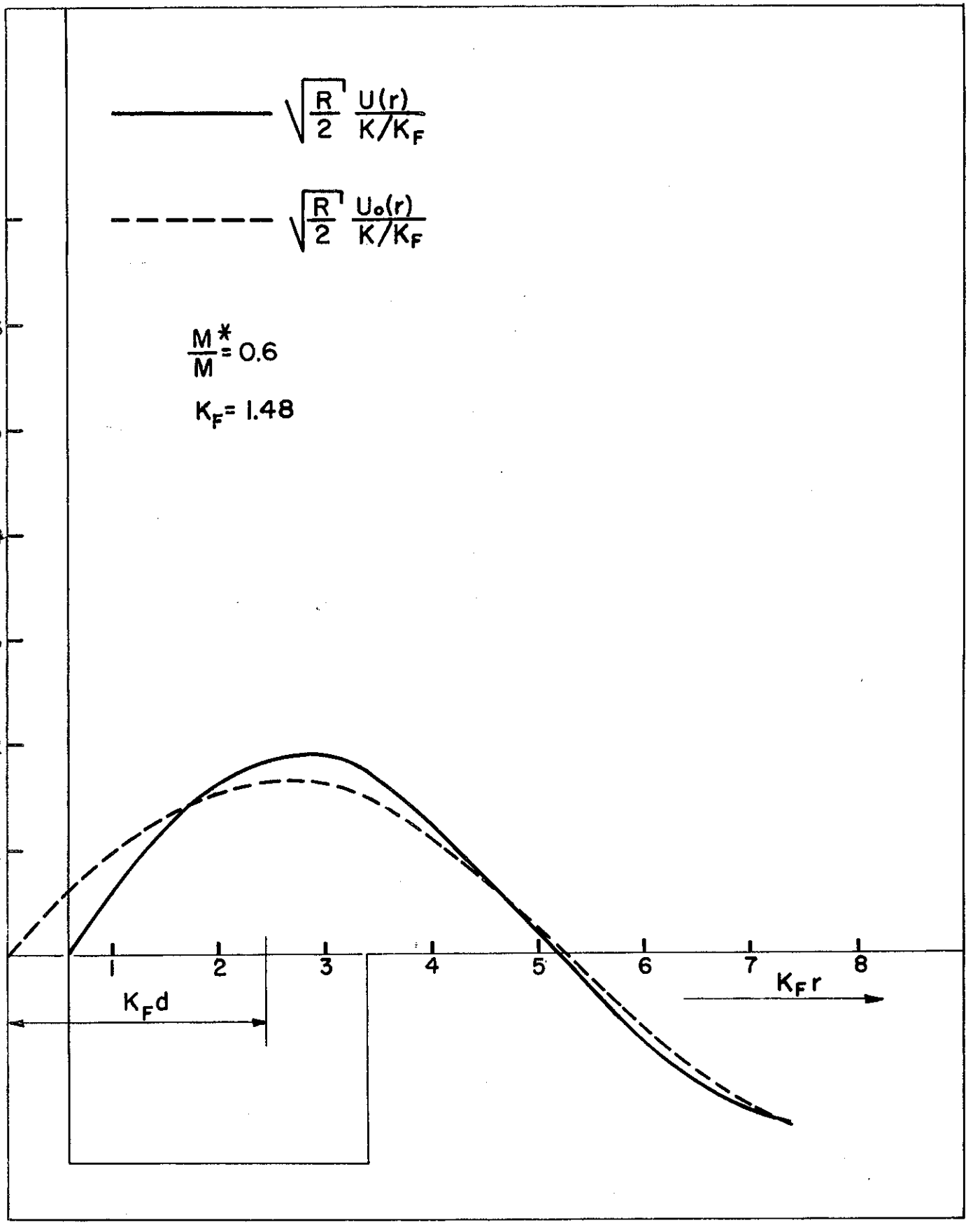
$\sqrt{\frac{R}{2}}\frac{U(r)}{K/K_F}$
$\sqrt{\frac{R}{2}}\frac{U_0(r)}{K/K_F}$
$\frac{M^*}{M}=0.6$
$K_F=1.48$
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
$K_F r$
$K_F d$

FIG. 5

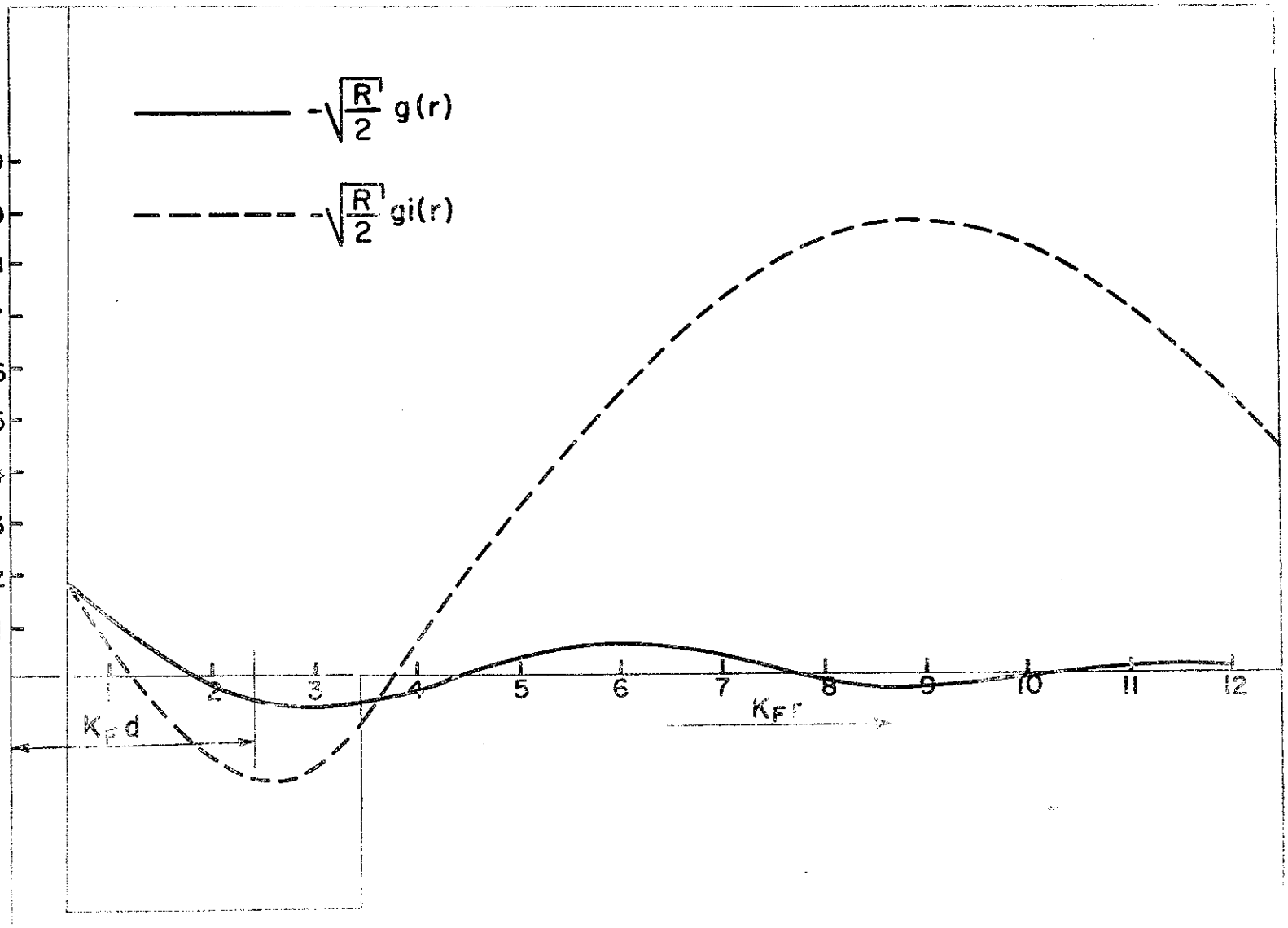
$-\sqrt{\frac{R}{2}}\, g(r)$
$-\sqrt{\frac{R}{2}}\, g_i(r)$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
2
3
4
5
6
7
8
9
10
11
12
$K_F d$
$K_F r$

FIG. 6

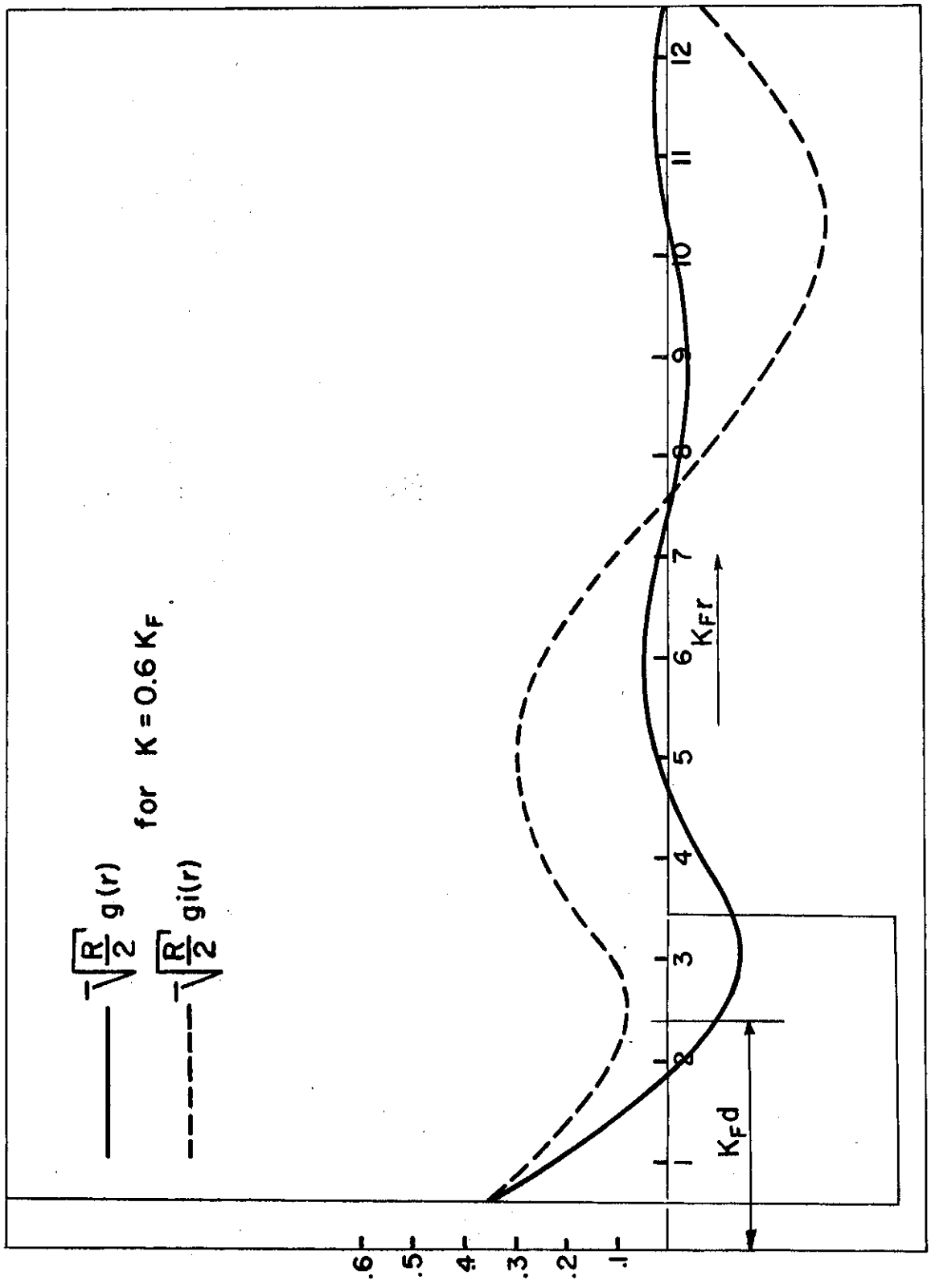
$\sqrt{\frac{R}{2}}\, g(r)$
$\sqrt{\frac{R}{2}}\, gi(r)$
for $K = 0.6\, K_F$
$K_F r$
$K_F d$
1
2
3
4
5
6
7
8
9
10
11
12
.6
.5
.4
.3
.2
.1

FIG. 7

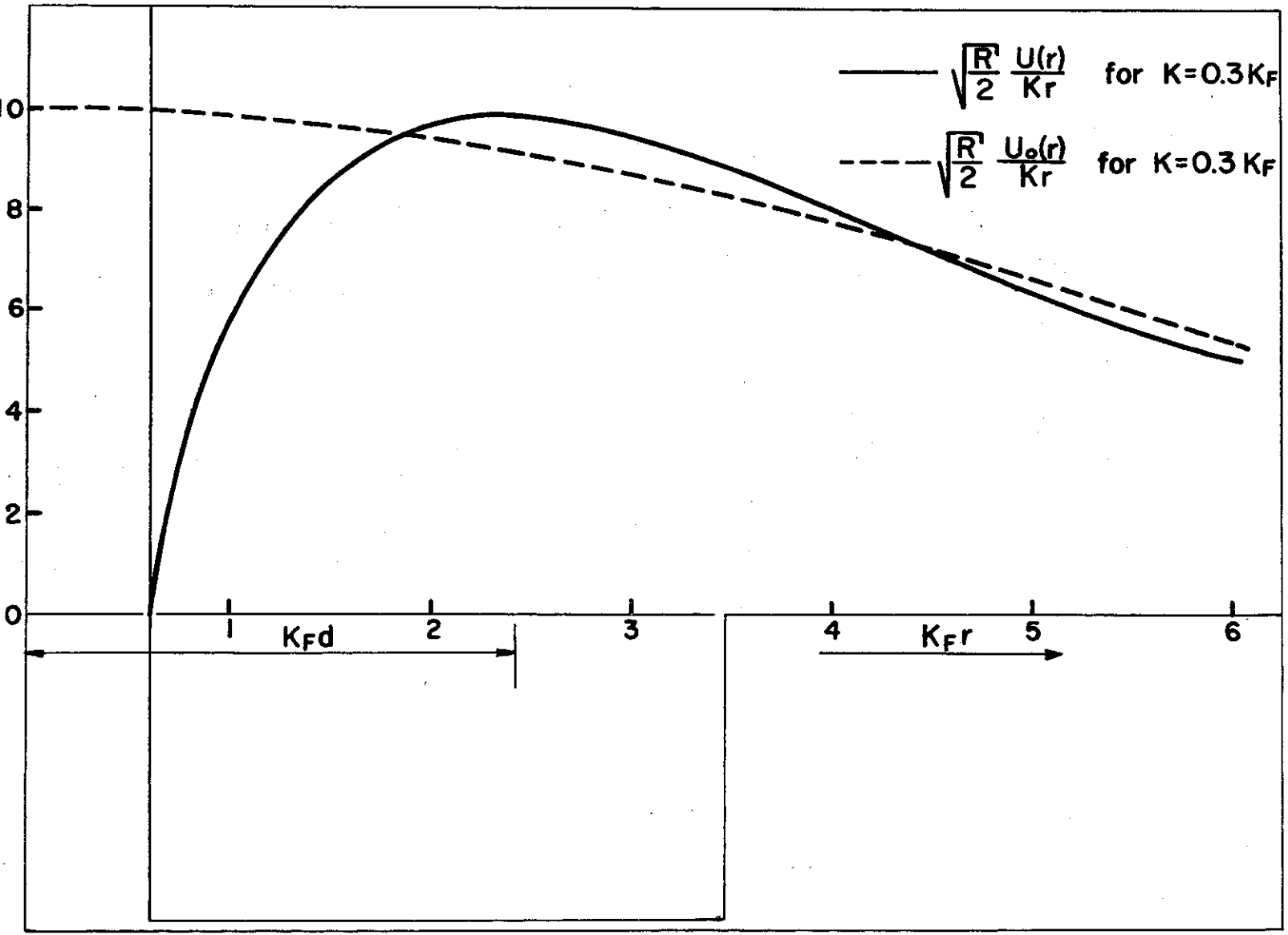
$\sqrt{\frac{R'}{2}}\frac{U(r)}{Kr}$ for $K=0.3K_F$
$\sqrt{\frac{R'}{2}}\frac{U_o(r)}{Kr}$ for $K=0.3K_F$
10
8
6
4
2
0
1
K_Fd
2
3
4
K_Fr
5
6

FIG. 8

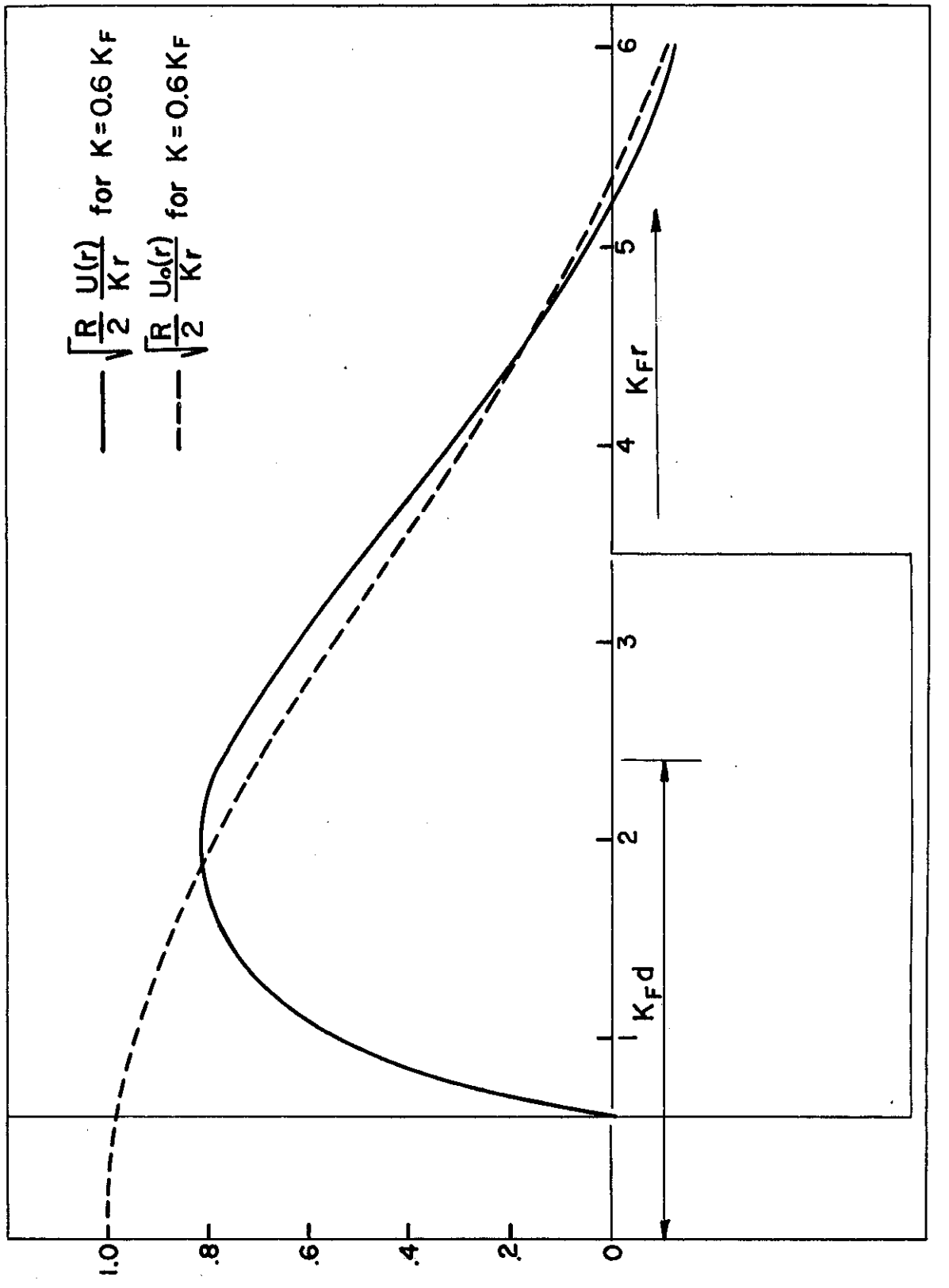
$\sqrt{\frac{R}{2}}\frac{U(r)}{Kr}$ for K=0.6 K_F
$\sqrt{\frac{R}{2}}\frac{U_o(r)}{Kr}$ for K=0.6 K_F
$K_F r$
$K_F d$
1.0
.8
.6
.4
.2
0
1
2
3
4
5
6

FIG. 9

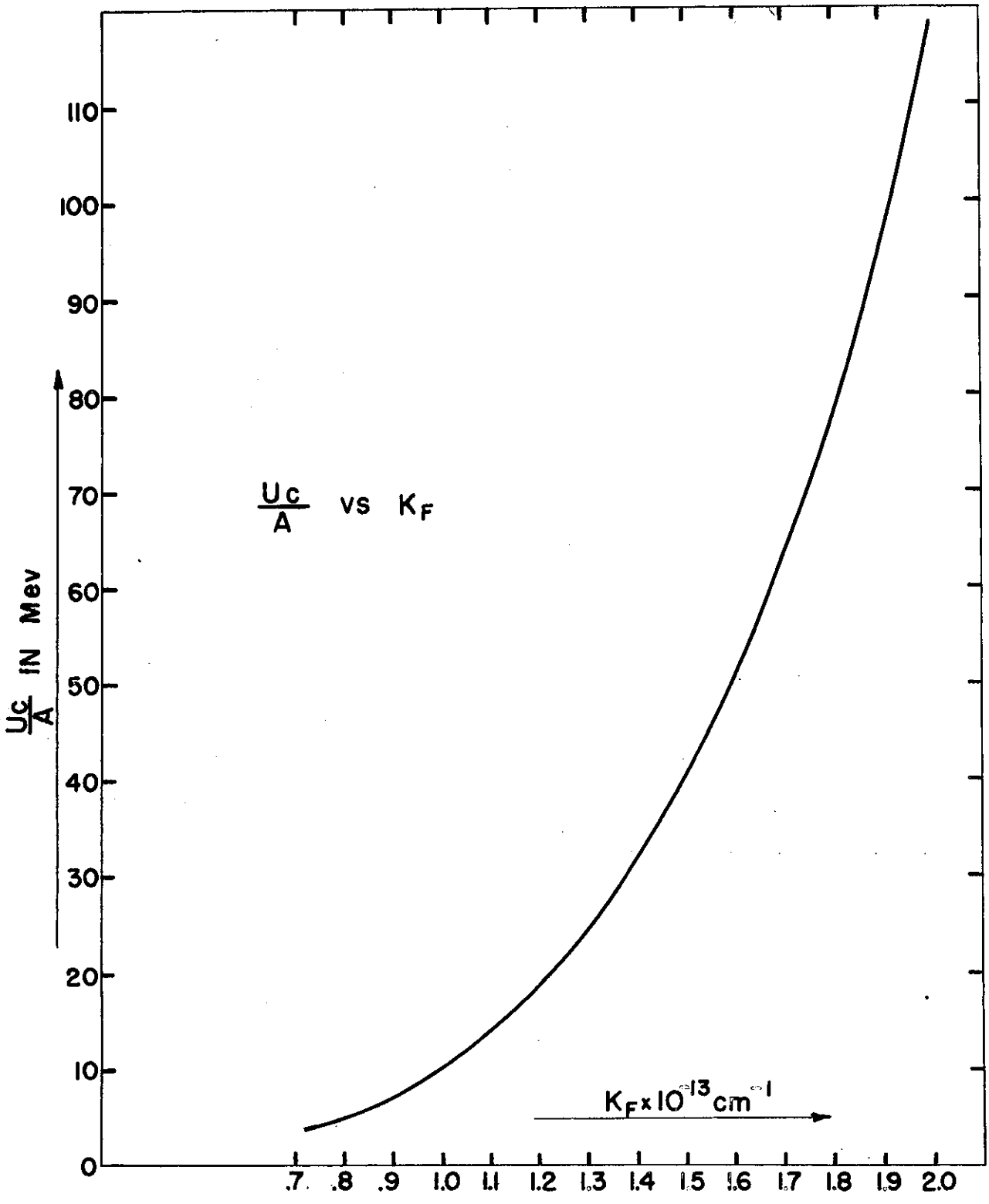
$\frac{Uc}{A}$ vs K_F
$\frac{Uc}{A}$ IN Mev
$K_F \times 10^{-13}$ cm^{-1}
0
10
20
30
40
50
60
70
80
90
100
110
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0

FIG. 10

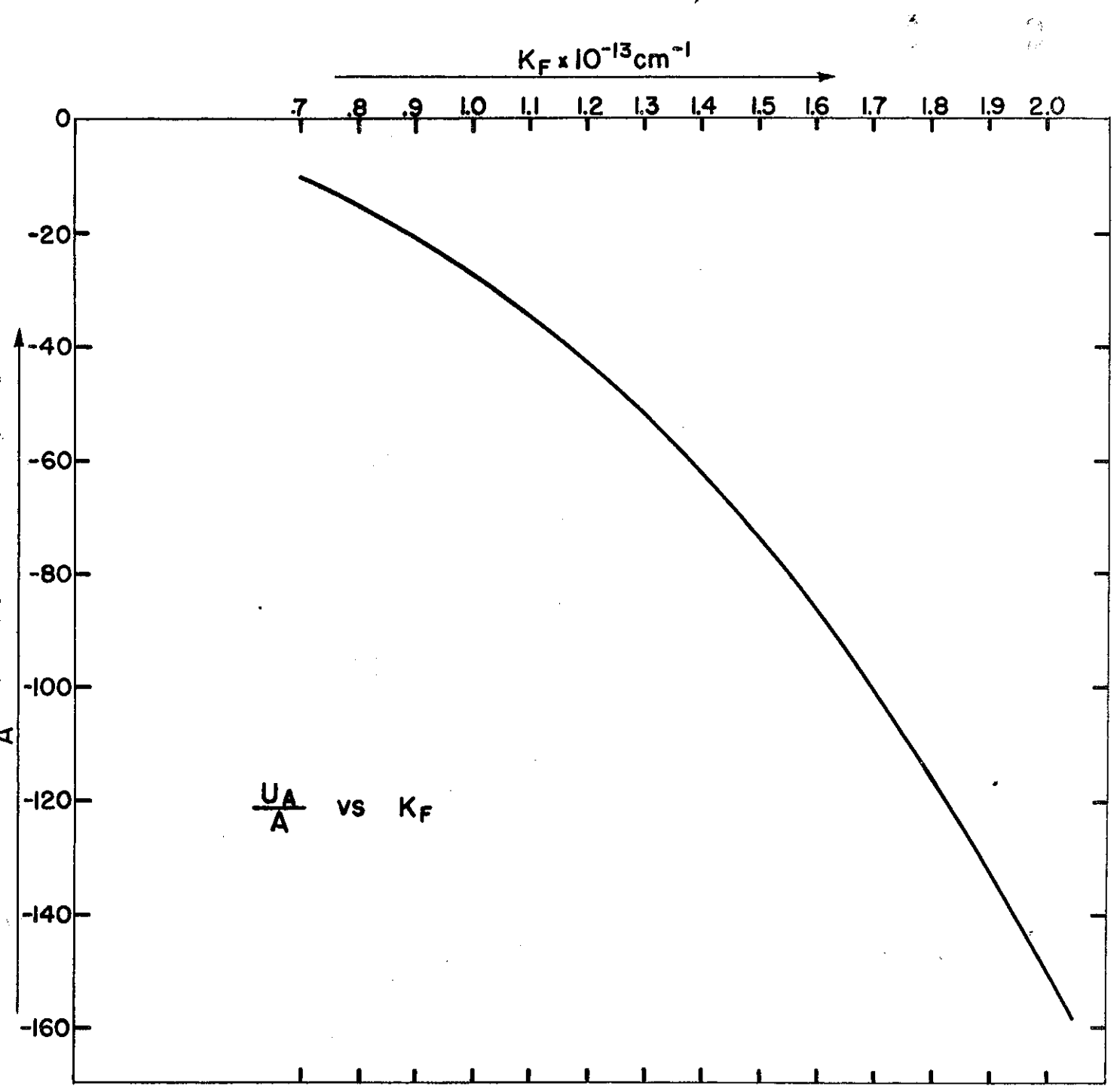
$K_F \times 10^{-13} cm^{-1}$
.7 .8 .9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0
0
-20
-40
-60
-80
-100
-120
-140
-160
A
$\frac{U_A}{A}$ vs K_F

FIG. 11

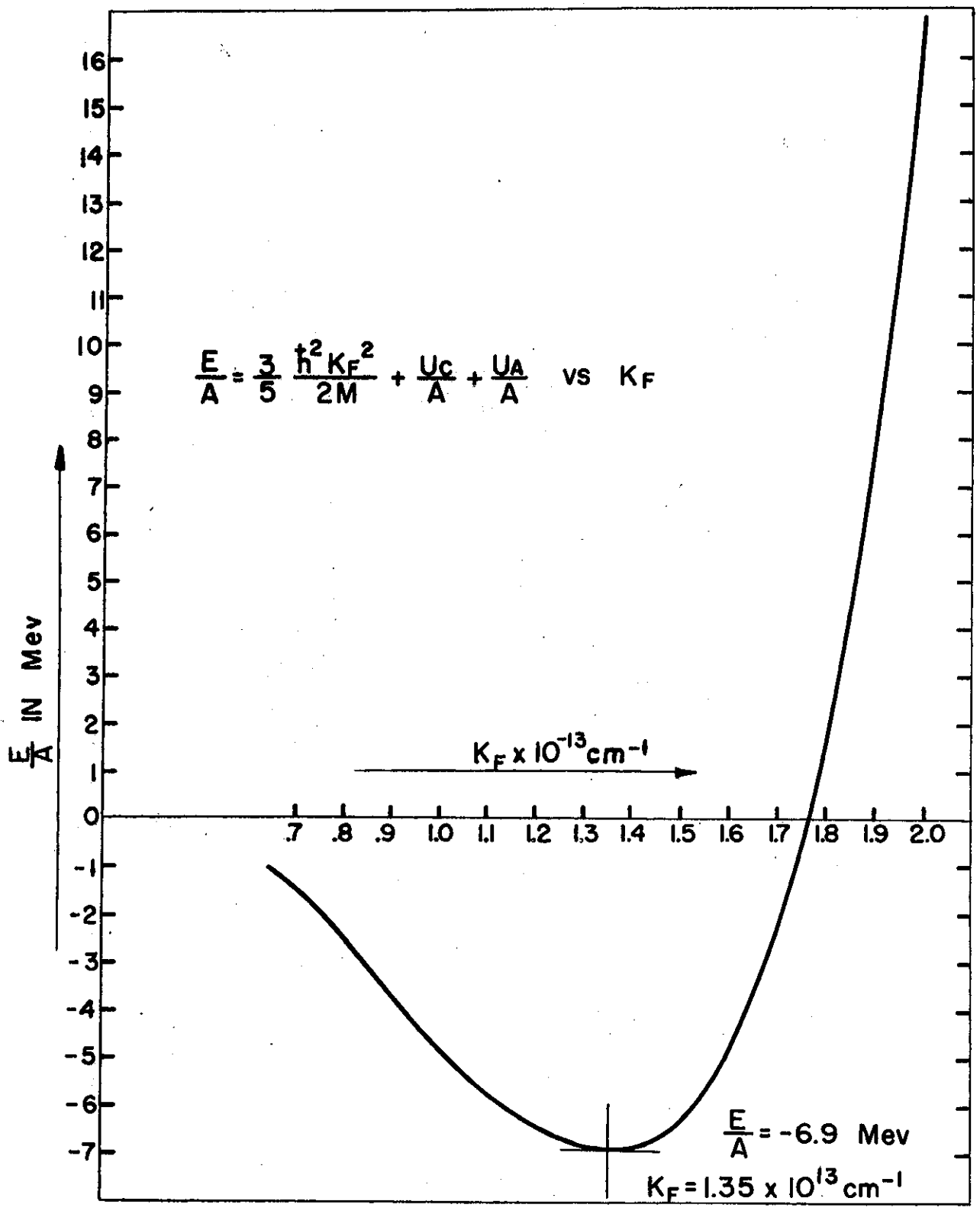
$\frac{E}{A} = \frac{3}{5}\frac{\hbar^2 K_F^2}{2M} + \frac{U_C}{A} + \frac{U_A}{A}$ vs K_F
$\frac{E}{A}$ IN Mev
$K_F \times 10^{-13}$ cm^{-1}
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
$\frac{E}{A}$ = -6.9 Mev
K_F = 1.35 x 10^{13} cm^{-1}

FIG. 12