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POLARIZATION OF SPIN ONE PARTICLES BY NUCLEAR SCATTERING*

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Polarized deuteron beams have been recently abtained from nuclear scattering experiments (O. Chamberlain et al., 1954). In these experiments a high energy deuteron beam, emerging unpolarized from the cyclotron, becomes partially polarized when scattered by a first target. The state of polarization is investigated by mean of another scattering by a second target which plays the role of analyser. It has been found that the angular distribution in the double scattering experiment has the form (R.Tripp et al.1955):

$$\sigma = \sigma_0 + a + b\cos\varphi + c\cos 2\varphi\ , \qquad (1)$$

where σ_0 is the cross section for single scattering, and φ is the azimuthal angle of the direction of the second scattering. The

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magnitude of <u>a</u> and <u>c</u> resulted in all experiments of the order of experimental errors.

Theoretical analysis of this problem have been made on quite general grounds (L. Wolfenstein and J. Ashkin, 1952) and formula (1) has been proved (W. Lakin, 1955). However general arguments doesn't exist which, in all cases, account for the smallness of <u>a</u> and <u>c</u> .

We conceived that the method of phase shifts for the determination of the scattering matrix might give more information on this respect.

In the present paper this method is developed for the scattering of spin one particles by spin zero nuclei. The scattering matrix is obtained by an extension of Lepore's method for spin 1/2 particles (L. V. Lepore, 1950). Cross sections for single and double scattering are abtained and arguments are given to justify the smallness of the above mentioned terms.

DETERMINATION OF THE SACATTERING MATRIX

We look for a solution of the scattering problem, prooceding exactly in the same way as Lepore does, by writing the wave function for the particle in the assymptotic form:

$$\Psi = \sum_{\ell=0}^{\infty}{}' \, (2\ell + 1)^{1/2} \; i^{\ell} \sum_{s}{}' A_{\ell}^{s} \frac{U_{\ell}^{s}}{kr} \Gamma_{\ell}^{s} \, Y_{\ell}^{0} \chi_{inc} \quad .$$

This is the analogous to formula (10) of his paper. Here χ_{inc} is a three component vector which represents the spin state of the incident beam and Π_ℓ^s are the following projection operators:

$$\Pi_\ell^1 = \frac{\vec{S}\cdot\vec{L}+1}{2\ell+1}\left(\frac{\vec{S}\cdot\vec{L}}{\ell+1}+1\right),$$

$$\Pi_\ell^0 = 1 - \frac{\vec{S}\cdot\vec{L}(\vec{S}\cdot\vec{L}+1)}{\ell(\ell+1)}, \qquad (2)$$

$$\Pi_\ell^{-1} = \frac{\vec{S}\cdot\vec{L}+1}{2\ell+1}\left(\frac{\vec{S}\cdot\vec{L}}{\ell}-1\right).$$

When applied to a function $Y_\ell^0 \chi_{inc}$ they have the property of selecting only the state with $j=\ell+s$. They also satisfy the condition:

$$\sum_s \Pi_\ell^s = 1 \qquad (3)$$

The radial functions U_ℓ^s are:

$$U_\ell^s(kr) = \sin\left(kr - \ell\frac{\pi}{2} - \alpha \ln 2kr + \delta_\ell^s\right).$$

The coefficients A_ℓ^s must be so determined that the total wave function represents assymptoticaly an incident plane wave plus an outgoing spherical wave.

This condition requires that:

$$\sum_s A_\ell^s \, e^{-i\delta_\ell^s} \, \Pi_\ell^s = 1$$

From this equation and identity (3) we get:

$$A_\ell^s = e^{i\,\delta_\ell^s}$$

Therefore the assymptotic scattered wave is:

$$\psi_{sc} \sim \frac{e^{i(kr - \alpha \ln 2kr)}}{r} \; f(\theta) \chi_{inc} ,$$

where

$$f(\theta) = \frac{\sqrt{2}}{k} \sum_{\ell} (2\ell + 1)^{1/2} \sum_{s} \frac{e^{2i\,\delta_\ell^s} - 1}{2i} \Pi_\ell^s \, Y_\ell^0 \quad , \qquad (4)$$

is the scattering matrix.

The operators Π_ℓ^s contain terms in $\vec{S}.\vec{L}$ and $(\vec{S}.\vec{L})^2$ which, when applied to Legendre polinomials, may be replaced by the following expressions:

$$\vec{S}.\vec{L} = -i(\vec{S} . \vec{a}_1 \wedge \vec{a}_o) \frac{\partial}{\partial \cos\theta} = -i(\vec{S} . \vec{n}) \sin\theta \; \frac{\partial}{\partial \cos\theta} \quad , \qquad (5)$$

$$(\vec{S}.\vec{L})^2 = -(\vec{S} . \vec{n})^2 \frac{\partial^2}{\partial \cos\theta^2} + (\vec{S} \wedge \vec{a}_1).(\vec{S} \wedge \vec{a}_o) \frac{\partial}{\partial \cos\theta} \quad , \qquad (6)$$

where $\vec{a}_o$ and $\vec{a}_1$ are the unit vectors on the directions of $\vec{k}_o$ and $\vec{k}_1$ the incident and scattered wave vectors, and $\vec{n}$ is the normal to the plane of scattering:

$$\vec{n} \sin\Theta = \vec{a}_1 \wedge \vec{a}_o \, . \tag{7}$$

We now introduce the symmetric second rank tensor:

$$Q_{ij} = \frac{1}{2} (S_i \; S_j + S_j \; S_i) - \frac{2}{3} \delta_{ij} \, , \tag{8}$$

so defined that:

$$Q_{ij} \, \delta^{ij} = 0 \, . \tag{9}$$

Then we can write:

$$(\vec{S} . \vec{L})^2 = - (Q_{ij} \, n^i n^j + \frac{2}{3}) \sin^2 \Theta \frac{\partial^2}{\partial \cos\Theta^2} +$$

$$+ \left[2 \cos\Theta - (\vec{S} . \vec{a}_o)(\vec{S} . \vec{a}_1) \right] \frac{\partial}{\partial \cos\Theta} =$$

$$= - \frac{2}{3} \frac{\partial}{\partial \cos\Theta} \sin^2\Theta \frac{\partial}{\partial \cos\Theta} - Q_{ij} \left(n^i n^j \sin^2\Theta \frac{\partial^2}{\partial \cos\Theta^2} + \right.$$

$$\left. + \; a_1^i a_o^j \frac{\partial}{\partial \cos\Theta} \right) + \frac{i}{2} (\vec{S} . \vec{n}) \sin\Theta \frac{\partial}{\partial \cos\Theta} \tag{10}$$

The scattering matrix may now be written:

$$f(\Theta) = A - \frac{2}{3}\frac{\partial}{\partial\cos\Theta}\left(C\sin^2\Theta\right) - i\,(\vec{S}\cdot\vec{n})\sin\Theta\left(B - \frac{C}{2}\right) -$$

$$- Q_{ij}\left(n^i n^j \sin^2\Theta \frac{\partial C}{\partial\cos\Theta} + a_o^i a_1^j C\right). \qquad (11)$$

Here the coefficients A, B, C are the following linearly independent functions of $\cos\Theta$:

$$A = \frac{1}{k\sqrt{2\pi}}\sum_\ell \left[e^{i\delta_\ell^1} \sin\delta_\ell^1 + (2\ell+1)\, e^{i\delta_\ell^0} \sin\delta_\ell^0 - e^{i\delta_\ell^{-1}} \sin\delta_\ell^{-1}\right] P_\ell(\cos\Theta),$$

$$B = \frac{1}{k\sqrt{2\pi}}\sum_\ell \left[\left(1+\frac{1}{\ell+1}\right) e^{i\delta_\ell^1}\sin\delta_\ell^1 - \frac{2\ell+1}{\ell(\ell+1)} e^{i\delta_\ell^0}\sin\delta_\ell^0 - \left(1-\frac{1}{\ell}\right) e^{i\delta_\ell^{-1}}\sin\delta_\ell^{-1}\right] \frac{\partial P_\ell}{\partial\cos\Theta}, \qquad (12)$$

$$C = \frac{1}{k\sqrt{2\pi}}\sum_\ell \left[\frac{1}{\ell+1} e^{i\delta_\ell^1}\sin\delta_\ell^1 - \frac{2\ell+1}{\ell(\ell+1)} e^{i\delta_\ell^0}\sin\delta_\ell^0 + \frac{1}{\ell} e^{i\delta_\ell^{-1}}\sin\delta_\ell^{-1}\right] \frac{\partial P_\ell}{\partial\cos\Theta}.$$

For what follows it is more convenient to write the scattering matrix in a different form by separating a factor $e^{i\theta(\vec{S}\cdot\vec{n})}$ which makes the state vector to rotate through an angle θ, about the normal axis to the plane of scattering.

The result is:

$$f(\theta)=\left\{\left[\frac{1}{3}(D+F)-i\,E\sin\theta\,(\vec{S}\cdot\vec{n})+\left[\left(\frac{1}{2}(F-C)-D\right)n^i n^j -\right.\right.\right.$$
$$\left.\left.-\,C a_1^i\,a_0^j\right]Q_{ij}\right\} e^{i\theta(\vec{S}\cdot\vec{n})}, \qquad (13)$$

where:

$$D = A + C\cos\theta,$$
$$E = A + B\cos\theta - \frac{\partial}{\partial\cos}(C\sin^2\theta), \qquad (14)$$
$$F = 2(E\cos\theta - B) + C.$$

It can be noticed that the matrix f, as should be expected, is invariant under time reversal that is under the following substitutions:

$$\vec{a}_0 \longrightarrow -\vec{a}_1$$
$$\vec{a}_1 \longrightarrow -\vec{a}_0$$
$$\vec{S} \longrightarrow -\vec{S} \qquad (15)$$

CROSS SECTION AND DENSITY MATRIX

The state of polarization of the incident beam, may be described, following von Neuman's formalism, by a density matrix $\rho_0 = \sum \chi_i p_i \chi_i^+$ which is the sum of the projection operators over all the spin states of the particles, multiplied by the probability to find it in the corresponding state.

After the scattering process each state χ_i will be transformed into $f(\theta)\chi_i$ and the differential cross section is then given by:

$$\sigma = \mathrm{Tr.}\,(f\,\rho_o\,f^+) = \mathrm{Tr.}\,(\rho_o\,f^+ f) \qquad (16)$$

It is the average value of the operator $f^+ f$ in the incident beam.

The density matrix of the scattered beam is:

$$\rho = \frac{f\,\rho_o\,f^+}{\sigma}\,. \qquad (17)$$

In order to calculate σ we must obtain $f^+ f$. After some computation this is found to be:

$$f^+ f = |D|^2 + \frac{1}{3}(F,C) + \frac{2}{3}\,\Gamma\sin^2\theta - (F, i\,E\sin\theta)(\vec{S}\cdot\vec{n}) -$$

$$- \left[\frac{1}{2}(F,C)\,a_o^i\,a_o^j + (C, E\sin\theta)\,a_o^i\,b_o^j - \Gamma\sin^2\theta\,n^i\,n^j\right] Q_{ij}. \qquad (18)$$

Here we introduce the definitions:

$$\vec{b}_o = \vec{a}_o \wedge \vec{n} \qquad\qquad \vec{b}_1 = -\,\vec{a}_1 \wedge \vec{n}$$

$$(X, Y) = \frac{1}{2}(X^*\,Y + Y^*X);$$

and

$$\Gamma \sin^2\theta = \left| \frac{1}{2} (F - C) \right|^2 + | E \sin\theta |^2 - | D |^2 ,$$

the last relation being equivalent to:

$$\Gamma = - \left(2 D - \sin^2\theta \frac{\partial C}{\partial \cos\theta} , \frac{\partial C}{\partial \cos\theta} \right) + | B |^2 \quad . \tag{19}$$

Let $\langle S_i \rangle_o$ and $\langle Q_{ij} \rangle_o$ be respectively the mean values of S_i and Q_{ij} in the incident beam. The cross section will be found immediately, ~~taking~~ the average value of expression (18):

$$\sigma = | D |^2 + \frac{1}{3} (F , C) + \frac{2}{3} \Gamma \sin^2\theta - (F , i E \sin\theta) \langle \vec{S} \rangle \cdot \vec{n} -$$

$$- \left[\frac{1}{2} (F , C) a_o^i a_o^j + (C , E \sin\theta) a_o^i b_o^j - \Gamma \sin^2\theta \, n^i n^j \right] \langle Q_{ij} \rangle_o . \tag{20}$$

Let us now consider the case of unpolarized incident beam. Then $\rho_o = \frac{I}{3}$ and cross section reduces to:

$$\sigma = | D |^2 + \frac{1}{3} (F , C) + \frac{2}{3} \Gamma \sin^2\theta \, . \tag{21}$$

On the other hand the density matrix becomes:

$$\rho = \frac{f f^+}{3 \sigma} . \tag{22}$$

The expression for $f f^+$ can be deduced from that for $f^+ f$ by using the invariance under time reversal. Denoting with primes, quantities refering to time reversed states, that is, subjected to the substitutions (15) we may write:

$$f' = f \; ; \quad (f^+ f)' = f' f'^+ = f f^+ . \tag{23}$$

Therefore $f f^+$ is found from the expression (18) by introducing there, the substitutions (15).

We thus obtain for the mean values of the operators S_i and Q_{ij} in the scattered beam the following expressions:

$$\langle \vec{S} \rangle_1 = - \frac{2}{3\sigma} (F, i E \sin\theta) \, \vec{n} ,$$

$$\langle Q_{ij} \rangle_1 = - \frac{1}{3\sigma} \Big[\frac{1}{2} (F , C) (a_1^i a_1^j - \frac{1}{3} \delta^{ij}) - (E \sin\theta, C) a_1^i b_1^j - \Gamma \sin^2\theta (n^i n^j - \frac{1}{3} \delta^{ij}) \Big] . \tag{24}$$

We see that the spin vector is normal to the plane of scattering and is a principal direction of the tensor; this is a general result which may be easily deduced from invariance arguments.

ANALISIS OF POLARIZATION BY DOUBLE SCATTERING.

The properties of the polarized beam may be investigated by a second scattering event. The differential cross section will not have axial symmetry about the direction of propagation; it will depend on the angle φ between the planes of the first and second scatterings. Consider single and double scattering of unpolarized beams. We denote the cross section for single scattering at an angle θ , by $\sigma(\theta)$ and that for the second event in a double scattering, by $\sigma(\theta_1, \theta_2, \varphi)$ where θ_1 is the first scattering angle and (θ_2, φ) are the polar angles of the second scattering direction. The value of $\sigma(\theta_1, \theta_2, \varphi)$ may be computed from (20), taking into account (21) and (24) and observing that the incident direction is now, the direction $\vec{a}_1$, of the scattered beam.

Choosing $\vec{a}_1$ as x-axis and $\vec{n}$ as z-axis, we find:

$$\sigma(\theta_1, \theta_2, \varphi) = \sigma(\theta_2)\Big\{1 + 3\Big[\frac{3}{2}\langle Q_{11}\rangle_1\langle Q_{11}\rangle_2 + \Big(\frac{1}{2}\langle S_3\rangle_1\langle S_3\rangle_2 - 2\langle Q_{12}\rangle_1\langle Q_{12}\rangle_2\Big)\cos\varphi + \frac{1}{2}\Big(\langle Q_{33}\rangle_1 - \langle Q_{22}\rangle_1\Big)\Big(\langle Q_{33}\rangle_2 - \langle Q_{22}\rangle_2\Big)\cos 2\varphi\Big]\Big\} \qquad (25)$$

where $\langle S_i\rangle_\lambda$ and $\langle Q_{ij}\rangle_\lambda$ are the average values that the corresponding operators acquire in a single scattering beam at an angle θ_λ .

In obtaining the cross section in the form given by (25) we used the following relations computed from (24):

$$\langle S_3 \rangle = - 2 \left(F, i E \sin\theta \right) / 3 \sigma$$

$$\langle Q_{11} \rangle = \frac{1}{3} \left[(F, C) - \Gamma \sin^2 \theta \right] / 3 \sigma \quad ,$$

$$\langle Q_{12} \rangle = \left(E \sin\theta , C \right) / 3 \sigma \quad , \qquad (26)$$

$$\langle Q_{33} \rangle - \langle Q_{22} \rangle = - \Gamma \sin^2\theta / 3 \sigma \quad .$$

Formula (25) was first derived by Lakin, but there was an error in the sign of the second term of his $\cos\varphi$ coefficient.

CONCLUDING REMARKS

There is experimental mesures of the cross section for single and double scattering in C^{12} and others nucleus ; for the second event the cross section behaves like it is predicted by formula (25), in what concerns the φ dependence, but within the experimental errors the terms $\langle Q_{11} \rangle$ and $(\langle Q_{33} \rangle - \langle Q_{12} \rangle)$ have not yet been conclusively observed. It is possible that these terms really exist but are small in the experimental conditions used untill now. Indeed, inspection of the expressions (26) shows that the second of these terms is proportional to $\sin^2\theta$; therefore its effect on the cross section will be quite insensitive, if at least one of the scattering angles is sufficiently small, as seems to be the experimental situation.

On the other hand the value of $\langle Q_{11} \rangle$ apart from a term in $\sin^2\theta$ contains another one v.g. $-\frac{1}{3}(F, C)$.

The smallness of this term may be explained, assuming an $(\vec{S} . \vec{L})$ spin-orbit coupling. The energies of the states with the same angular momentum are split, by amounts proportional to $\ell + 1$, 0, $-\ell$, corresponding respectively to the eigenfunctions of the total angular momentum with $j = \ell + 1$, $j = \ell$, $j = \ell - 1$. If the spin-orbit interaction is considered as a perturbation and the phase shifts are developed in power series of the correction term it gives rise, then we find $C = 0$ in first approximation.

This result which was also obtained using Born approximation seems to justify the smallness of $\langle Q_{11} \rangle$. The coefficient of $\cos\varphi$ is proportional to $\sin\theta_1 \sin\theta_2$ and under the above assumption the vector term predominates over the tensor one; the sign of that coefficient will decide about the correctness of this statement.

The available experimental data are not sufficient for a more detailed discussion and we hope that new experiments especially at larger angles will give more information about this subject.

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