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Transformations of Heun's equation and its integral relations

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Using the transformation theory for the Heun equation, we find substitutions of variables which preserve the form of the equation for the kernels of integral relations among solutions of the Heun equation. These transformations lead to new kernels for the Heun equation, given by single hypergeometric functions (Lambe-Ward-type kernels) and by products of two hypergeometric functions (Erdélyi-type). Such kernels, by a limiting process, afford new kernels for the confluent Heun equation as well.

Keywords: Heun's equation; Integral relations.

1. Introductory remarks

The group of transformations of variables which changes Heun's equation into another version of itself was initially established by the very Heun in 1889 [1] and fully accounted in 2007 by Maier [2], who tabulated the 192 substitutions in detail by writing explicitly the transformations of each parameter and variable of the equation. Firstly, we show that it is possible to construct a similar table for the kernels of integral relations among solutions of the equation.

In the second place, we show that some of these transformations generate new kernels given by hypergeometric functions when applied to the kernels found by Lambe and Ward in 1934 [3], and new kernels in terms of products of hypergeometric functions when applied to the kernels found by Erdélyi in 1942 [4]. Finally, by means of a limiting procedure we get new kernels also for the confluent Heun equation (CHE) .

The transformations of the Heun equation and its integral relations suppose the usual algebraic form for the equation [1, 2, 5], namely,

$$\frac{d^2H}{dx^2}+\left[\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x-a}\right]\frac{dH}{dx}+\left[\frac{\alpha\beta x-q}{x(x-1)(x-a)}\right]H=0,\quad [\epsilon=\alpha+\beta+1-\gamma-\delta] \tag{1}$$

where $a\in\mathbb{C}\setminus\{0,1\}$ and $x=0,1,a,\infty$ are regular singular points with indicial exponents given by $\{0,1-\gamma\}$, $\{0,1-\delta\}$, $\{0,1-\epsilon\}$ and $\{\alpha,\beta\}$, respectively. The constants a, α, β, γ and δ are called singularity parameters, whereas q is called accessory parameter since it is not associated with the singular points or their indicial exponents.

By defining the operator M_x as

$$M_x=x(x-1)(x-a)\frac{\partial^2}{\partial x^2}+\big[\gamma(x-1)(x-a)+\delta x(x-a)+\epsilon x(x-1)\big]\frac{\partial}{\partial x}+\alpha\beta x \tag{2}$$

and, by interpreting this as an ordinary derivative operator, the equation reads

$$\big[M_x-q\big]H(x)=0,\qquad [a\neq 0 \text{ or } 1], \tag{3}$$

The invariance of the equation with respect to the replacement of α by β does not imply that its solutions are symmetric in α and β; it simply means that the substitution of α for β leads to another solution. The values $a=0$ and $a=1$ are excluded because in these cases there are only three singular points and then the equation may be reduced to the Gauss hypergeometric equation

$$u(1-u)\frac{d^2F}{du^2}+\big[\mathrm{c}-(\mathrm{a}+\mathrm{b}+1)u\big]\frac{dF}{du}-\mathrm{ab}F=0, \tag{4}$$

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where $u = 0, 1, \infty$ are regular singular points with indicial exponents $\{0, 1-c\}$, $\{0, \mathrm{c}-\mathrm{a}-\mathrm{b}\}$ and $\{a, b\}$, respectively.

By keeping α, γ and δ fixed, the confluence procedure is given by the limits

$$a,\ \beta,\ q \to \infty \quad \text{such that} \quad \frac{\beta}{a} \to \frac{\epsilon}{a} \to -\rho, \quad \frac{q}{a} \to -\sigma, \tag{5}$$

where ρ and σ are constants. This yields the CHE [5], or generalised spheroidal wave equation [6],

$$x(x-1)\frac{d^2H}{dx^2} + \big[-\gamma + (\gamma+\delta)x + \rho x(x-1)\big]\frac{dH}{dx} + [\alpha\rho x - \sigma]H = 0, \tag{6}$$

where $x = 0$ and $x = 1$ are regular singularities, whereas $x = \infty$ is an irregular singularity.

On the other hand, $\mathcal{H}(x)$ is defined by [3–5]

$$\mathcal{H}(x) = \int_{y_1}^{y_2} w(x,y)\mathsf{G}(x,y)H(y)dy = \int_{y_1}^{y_2} y^{\gamma-1}(1-y)^{\delta-1}\left(1-\frac{y}{a}\right)^{\epsilon-1}\mathsf{G}(x,y)H(y)dy, \tag{7}$$

where $H(x)$ represents a solution of equation (1). Then, $\mathcal{H}(x)$ will be a solution of the Heun equation if: (i) the kernel $\mathsf{G}(x,y)$ is solution of the partial differential equation

$$[M_x - M_y]\,\mathsf{G}(x,y) = 0, \tag{8}$$

where M_y is obtained by setting $x = y$ in the expression for M_x, (ii) the integral (7) exists and (iii) the limits of integration are so chosen that the bilinear concomitant $\mathsf{P}(x,y)$, given by

$$\mathsf{P}(x,y) = y^{\gamma}(1-y)^{\delta}\left(1-\frac{y}{a}\right)^{\epsilon}\left[H(y)\frac{\partial\mathsf{G}(x,y)}{\partial y} - \mathsf{G}(x,y)\frac{dH(y)}{dy}\right], \tag{9}$$

fulfills the condition $\mathsf{P}(x,y_1) = \mathsf{P}(x,y_2)$. In Appendix A we show how these equations are obtained from the general theory of integral relations [7].

By the choice given in Eq. (7) for the weight function $w(x,y)$, equation (8) for the kernels is expressed in terms of the operator M_x which appears in the Heun equation (1) and in terms of the functionally identical operator M_y obtained by setting $x = y$ in M_x. Then, in order to establish the transformations of the kernels it is sufficient to demand that M_x and M_y transform in the same way. For the Heun equation these transformations will be inferred from the Maier transformations for the Heun equation.

By using suitable weight functions the above result holds also for the other equations of the Heun family, that is, for confluent, double-confluent, biconfluent and triconfluent Heun equations. Then, the transformation for the kernels may be inferred from the known transformations of each equation [8]. It seems that this connection has not been explored as yet [9–11].

However, the transformations become effective only if we know an initial kernel. For the Heun equation, new kernels in terms of single hypergeometric functions will be generated from the kernels found by Lambe and Ward [3], while kernels given by products of two hypergeometric functions will arise from the ones found by Erdélyi [4]. These afford initial kernels for the CHE by the limiting process (5). In addition to kernels given by confluent hypergeometric functions, we find kernels given by hypergeometric functions, products of two confluent hypergeometric functions, and products of one confluent hypergeometric function and one hypergeometric function.

In section 2, firstly we present the 8 so-called index or homotopic transformations which do not change the independent variable x, and the 24 Möbius or homographic transformations which result from linear fractional substitutions of the independent variable. Composition of such substitutions gives the group of 192 transformations. After this, the transformations are extended to the kernels of the equation, and these are used to generalise the kernels of Lambe-Ward and Erdélyi.

In order to generate the full group by composition of homotopic and homographic transformations, it is necessary to use the index transformations in Maier's form. This remark is important for avoiding incorrect results. For example, the forms given in Refs. [5] and [12] are inappropriate as we shall explain in section 2.1.

The kernels for the CHE are obtained in section 3, where we introduce as well the transformations of Eq. (6) and its kernels. In section 4 we point out that even for the double-confluent Heun equation (DCHE) it is possible to determine new kernels by using again a limiting process, and discuss how to transform certain solutions of the Heun equation into solutions useful for applications. Appendix A provides a derivation of Eqs. (8) and (9), while Appendix B lists the Möbius transformations for kernels of the Heun equation.

2. Heun's equation

First we examine the transformations for the Heun equation, emphasising that it is not allowed to permute the parameters α_i and β_i in the homotopic (index) transformations. Second, we obtain a general prescription for the transformations which preserve the equation for the kernels and write, explicitly, the index transformations for the kernels. In the third and fourth subsections, respectively, we generalise the kernels given by hypergeometric functions and by products of hypergeometric functions.

2.1. Transformations of Heun's equation

There are 24 (including the identity) Möbius substitutions of the independent variable x which leave the form of Heun's equation invariant, in general after a change of the dependent variable. They are given by fractional linear transformations $x \mapsto \varrho(x) = (Ax+B)/(Cx+D)$, $AD \neq BC$, which map three of the points 0, 1, a and ∞ onto 0, 1, ∞. The expressions for $\varrho(x)$ are displayed in the matrix

$$\left[\begin{array}{cccccccccccc} x & \frac{x}{x-1} & \frac{x}{x-a} & 1-x & \frac{x-1}{x-a} & \frac{a-x}{a}; & \frac{1}{x} & \frac{x-1}{x} & \frac{x-a}{x} & \frac{1}{1-x} & \frac{x-a}{x-1} & \frac{a}{a-x} \\ \frac{x}{a} & \frac{(a-1)x}{a(x-1)} & \frac{(1-a)x}{x-a} & \frac{1-x}{1-a} & \frac{a(x-1)}{x-a} & \frac{a-x}{a-1}; & \frac{a}{x} & \frac{a(x-1)}{(a-1)x} & \frac{x-a}{(1-a)x} & \frac{1-a}{1-x} & \frac{x-a}{a(x-1)} & \frac{a-1}{a-x} \end{array}\right] \quad (10)$$

where the elements in each column are proportional to one another and, in each row, the elements after the semicolon are the inverses of the elements before semicolon. For the identity, and for $(x-a)/(x-1)$, $a(x-1)/(x-a)$ and a/x the other singular point is mapped onto a, while for the remaining cases it changes to [2, 5]

$$\frac{1}{a}, \quad 1-a, \quad \frac{1}{1-a}, \quad \frac{a}{a-1}, \quad \frac{a-1}{a}. \quad (11)$$

Sometimes solutions for the Heun equation are denoted by $H(x) = Hl(a,q;\alpha,\beta,\gamma,\delta;x)$, where Hl means 'Heun-local', that is, a solution which converges in a region containing only one of the four singular points [2, 5]. For brevity, we drop the letter l, writing $H(x) = H(a,q;\alpha,\beta,\gamma,\delta;x)$. Then, the Möbius substitutions permit onto map a solution $H(x)$ into new solutions according to

$$H(a,q;\alpha,\beta,\gamma,\delta;x) \mapsto f(x)H\left[\tilde{a},\tilde{q};\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\varrho(x)\right], \quad (12)$$

where the prefactor $f(x)$ symbolises the transformation, if any, of the dependent variable which brings the differential equation with the variable $\varrho(x)$ into a Heun equation having parameters $\tilde{a}$, $\tilde{q}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$. Depending on the transformation considered, we have

$$f(x) = 1, \qquad x^{-\alpha}, \qquad (1-x)^{-\alpha}, \qquad (1-x/a)^{-\alpha}, \text{ or} \quad (13)$$

$$f(x) = 1, \qquad x^{-\beta}, \qquad (1-x)^{-\beta}, \qquad (1-x/a)^{-\beta}, \quad (14)$$

up to a multiplicative constant. The prefactor $f(x) = 1$ corresponds to the linear transformations, namely: $\varrho(x) = 1-x$, $(a-x)/a$, x/a, $(1-x)/(1-a)$ and $(a-x)/(a-1)$. The first form (13) is the one that will be adopted in the present article.

On the other side, the index transformations do not change the independent variable. They are given by 8 elementary power transformations of the dependent variable [2, 5], namely,

$$H(a,q;\alpha,\beta,\gamma,\delta;x) \mapsto x^{\tau_1}(1-x)^{\tau_2}(1-x/a)^{\tau_3}\, H(a,\tilde{q};\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};x), \quad (15)$$

where τ_1, τ_2 and τ_3 are the indicial exponents at 0, 1 and a, respectively, namely: $\tau_1 = 0$ or $1-\gamma$, $\tau_2 = 0$ or $1-\delta$, and $\tau_3 = 0$ or $1-\epsilon$. Since there is no change of the independent variable, the positions of the singular points remain fixed, in contrast with the fractional transformations. For this reason, they are also called homotopic transformations.

The composition of these two types of transformations (elementary powers and fractional) generates the group containing the 192 transformations given in Maier's table [2]. We refer to such transformations by M_i $(i = 1, 2, \cdots, 192)$ following the order in which they appear in the table, M_1 being the identity transformation. Thence, by regarding the M_i as operators, the effects of both transformations on a solution $H(x)$ are represented by

$$M_iH(x) = M_iH(a,q;\alpha,\beta,\gamma,\delta;x) = f_i(x)H\left[a_i,q_i;\alpha_i,\beta_i,\gamma_i,\delta_i;\varrho_i(x)\right], \tag{16}$$

where a_i is given by a or one of the expressions written in (11), whereas $\varrho_i(x)$ is one of the elements of the matrix (10).

In Maier's table the elementary power transformations appear in the entries $M_1 - M_4$ and $M_{25} - M_{28}$, but here they are denoted by T_i $(i = 1, \cdots, 8)$ according to the correspondence

$$\begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ M_1 & M_{25} & M_2 & M_{26} & M_3 & M_{27} & M_4 & M_{28} \end{bmatrix}, \tag{17}$$

where T_1 is the identity: $T_1H(x){=}H(a,q;\alpha,\beta,\gamma;\delta;x)$. The transformations T_2, T_3 and T_5 are given by

$$\begin{aligned} &T_2H(x) = x^{1-\gamma}H\big[a,q-(\gamma-1)(\delta a+\epsilon);\beta-\gamma+1,\alpha-\gamma+1,2-\gamma,\delta;x\big],\\ &T_3H(x) = (1-x)^{1-\delta}H\big[a,q-(\delta-1)\gamma a;\beta-\delta+1,\alpha-\delta+1,\gamma,2-\delta;x\big],\\ &T_5H(x) = \big[1-(x/a)\big]^{1-\epsilon}H\big[a,q-\gamma(\alpha+\beta-\gamma-\delta);-\alpha+\gamma+\delta,-\beta+\gamma+\delta,\gamma,\delta;x\big]. \end{aligned} \tag{18}$$

and are the generators of the other T_i. In these transformations we cannot change the order of the parameters α_i and β_i, that is, we must read

$$\begin{aligned} &\text{for } T_2: \quad \alpha_2 = \beta-\gamma+1, \qquad \beta_2 = \alpha-\gamma+1;\\ &\text{for } T_3: \quad \alpha_3 = \beta-\delta+1, \qquad \beta_3 = \alpha-\delta+1;\\ &\text{for } T_5: \quad \alpha_5 = -\alpha+\gamma+\delta, \qquad \beta_5 = -\beta+\gamma+\delta. \end{aligned} \tag{19}$$

In effect, it is possible to obtain the 192 transformations only if T_2, T_3 and T_5 transform the prefactors $x^{-\alpha}$, $(1-x)^{-\alpha}$ and $(1-x/a)^{-\alpha}$ of the Möbius transformations into $x^{-\beta}$, $(1-x)^{-\beta}$ and $(1-x/a)^{-\beta}$, and *vice-versa*. For the other transformations, the positions of α_i and β_i result from the compositions

$$T_4H(x) = T_2T_3H(x), \quad T_6H(x) = T_2T_5H, \quad T_7H(x) = T_3T_5H(x), \quad T_8H = T_2T_3T_5H(x), \tag{20}$$

where the order of the operators T_i is irrelevant on the right-hand side since $T_iT_j{=}T_jT_i$. In spite of this, for the three transformations Sleeman and Kuznetsov [12] writes α_i and β_i in the inverse order, while Arscott [5] inverts the order in T_2 and T_3.

The above order for α_i and β_i is important regardless the composition among the two types of transformations. This becomes apparent by considering an Erdélyi solution in series of hypergeometric functions $F(\mathrm{a},\mathrm{b};\mathrm{c};x)$, given by [13]

$$H_1(x) = \sum_{n=0}^{\infty} b_n^{(1)}\ F\left(n+\alpha,-n-\alpha-1+\gamma+\delta;\gamma;x\right), \tag{21}$$

where the coefficients $b_n^{(1)}$ satisfy three-term recurrence relations. From this we can generate a subgroup constituted by 8 solutions by writing $H_i(x) = T_iH_1(x)$. In particular, the solution with α in the place of β is

$$\begin{aligned} H_3(x) = T_3H_1(x) &= (1-x)^{1-\delta}\sum_{n=0}^{\infty} b_n^{(3)}\ F\left(n+\beta+1-\delta,-n-\beta+\gamma;\gamma;x\right)\\ &= \sum_{n=0}^{\infty} b_n^{(3)}\ F\left(n+\beta,-n-\beta-1+\gamma+\delta;\gamma;x\right), \end{aligned} \tag{22}$$

where the last equality follows from Eq. (32) written later on. By interchanging α_3 and β_3 in T_3, we would obtain the identity $H_3(x) = H_1(x)$, that is, we would miss one solution at least.

As aforementioned, we take the Möbius transformations with prefactors given in Eqs. (13) as the basic ones. Then, the substitutions (10) correspond to the following entries in Maier's table:

$$\begin{bmatrix} M_1 & M_5 & M_{13} & M_{49} & M_{57} & M_{101}; & M_{145} & M_{53} & M_{97} & M_{149} & M_{105} & M_{157} \\ M_9 & M_{21} & M_{17} & M_{61} & M_{65} & M_{109}; & M_{153} & M_{69} & M_{117} & M_{161} & M_{113} & M_{165} \end{bmatrix}, \tag{23}$$

where each column presents the same prefactor.

2.2. Transformations of kernel equation and notations

Since Eq. (8) is independent of the parameter q, a general kernel will be denoted by $\mathsf{G}(x,y) = \mathsf{G}(a;\alpha,\beta,\gamma,\delta;x,y)$. Then, the transformations which take the place of previous Möbius and index transformations are given, respectively, by the mappings (symmetrical in x and y)

$$\begin{aligned}
&\mathsf{G}\left[a;\alpha,\beta,\gamma,\delta;x,y\right] \mapsto f(x)f(y)\mathsf{G}\left[\tilde{a};\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};\varrho(x),\varrho(y)\right],\\
&\mathsf{G}\left[a;\alpha,\beta,\gamma,\delta;x,y\right] \mapsto [xy]^{\tau_1}\left[(1-x)(1-y)\right]^{\tau_2}\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{\tau_3}\mathsf{G}\left[a;\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta};x,y\right],
\end{aligned}$$

where the prefactors $f(x)$ and $f(y)$, as well as the fractional transformations $\varrho(x)$ and $\varrho(y)$, are formally the same which occur in the transformations of the Heun equation. These substitutions preserve the form of equation $[M_x - M_y]\mathsf{G} = 0$ for the kernels because all the parameters of the operators M_x and M_y transform as in the Heun equation, and constant terms corresponding to the transformations of q cancel out. In terms of operators we rewrite these transformations as

$$\mathcal{K}_i\mathsf{G}(x,y) = \mathcal{K}_i\mathsf{G}(a;\alpha,\beta,\gamma,\delta;x,y) = f_i(x)f_i(y)\mathsf{G}\big[a_i;\alpha_i,\beta_i,\gamma_i,\delta_i;\varrho_i(x),\varrho_i(y)\big], \tag{24}$$

where $\mathcal{K}_i$ is obtained from the corresponding M_i of Eq. (16).

We split the operators $\mathcal{K}_i$ in two subgroups: the operators denoted by N_i correspond to the 8 index transformations T_i, and the operators denoted by K_i correspond to the 24 Möbius transformations. Thus, the homotopic transformations N_i for the kernels are

$$N_1\mathsf{G}(x,y) = \mathsf{G}(x,y) = \mathsf{G}\left[a;\alpha,\beta,\gamma,\delta;x,y\right] \qquad \text{[Identity]},$$

$$N_2\mathsf{G}(x,y) = (xy)^{1-\gamma}\mathsf{G}\big[a;\beta-\gamma+1,\alpha-\gamma+1,2-\gamma,\delta;x,y\big].$$

$$N_3\mathsf{G}(x,y) = [(1-x)(1-y)]^{1-\delta}\mathsf{G}\big[a;\beta-\delta+1,\alpha-\delta+1,\gamma,2-\delta;x,y\big],$$

$$\begin{aligned}
N_4\mathsf{G}(x,y) &= (xy)^{1-\gamma}[(1-x)(1-y]^{1-\delta}\\
&\times\ \mathsf{G}\big[a;\alpha-\gamma-\delta+2,\beta-\gamma-\delta+2,2-\gamma,2-\delta;x,y\big],
\end{aligned}$$

$$N_5\mathsf{G}(x,y) = \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{1-\epsilon}\mathsf{G}\big[a;-\alpha+\gamma+\delta,-\beta+\gamma+\delta,\gamma,\delta;x,y\big],$$

$$\begin{aligned}
N_6\mathsf{G}(x,y) &= (xy)^{1-\gamma}\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{1-\epsilon}\\
&\times\ \mathsf{G}\big[a;-\beta+\delta+1,-\alpha+\delta+1,2-\gamma,\delta;x;y\big],
\end{aligned}$$

$$\begin{aligned}
N_7\mathsf{G}(x,y) &= [(1-x)(1-y)]^{1-\delta}\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{1-\epsilon}\\
&\times\ \mathsf{G}\big[a;-\beta+\gamma+1,-\alpha+\gamma+1,\gamma,2-\delta;x,y\big],
\end{aligned}$$

$$\begin{aligned}
N_8\mathsf{G}(x,y) &= (xy)^{1-\gamma}[(1-x)(1-y)]^{1-\delta}\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{1-\epsilon}\\
&\times\ \mathsf{G}\big[a;2-\alpha,2-\beta,2-\gamma,2-\delta;x,y\big].
\end{aligned}$$

Notice that in fact N_1 is the identity only if $\alpha \mapsto \alpha$ and $\beta \mapsto \beta$. As in T_i, we cannot permute α_i and β_i in N_2, N_3 and N_5 because such transformations must change the exponents α of the prefactors (27) (for fractional transformations) into β, and *vice-versa*. For the other transformations, the above positions of α_i and β_i result from the compositions

$$\begin{aligned}
&N_4\mathsf{G}(x,y) = N_2N_3\mathsf{G}(x,y), \qquad N_6\mathsf{G}(x,y) = N_2N_5\mathsf{G}(x,y),\\
&N_7\mathsf{G}(x,y) = N_3N_5\mathsf{G}(x,y), \qquad N_8\mathsf{G}(x,y) = N_2N_3N_5\mathsf{G}(x,y),
\end{aligned} \tag{25}$$

where the order of the operators N_i is irrelevant on the right-hand side. Thus, N_2, N_3 and N_5 are the generators of the index transformations for the kernels.

On the other side, each of the 24 Möbius transformations M_i given in the matrix (23) is associated to a kernel transformation denoted by K_j ($j = 1, \cdots, 12$ for the first row and $j = 13, \cdots, 24$ for the second) as

$$\begin{bmatrix} K_1 & K_2 & K_3 & K_4 & K_5 & K_6; & K_7 & K_8 & K_9 & K_{10} & K_{11} & K_{12} \\ K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18}; & K_{19} & K_{20} & K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix}. \tag{26}$$

As in the homotopic transformations, first we must find the transformation M_i for $H(x)$ in Maier's table and, then, write the kernel transformation K_i by using the prescription given in Eq. (24). The 24 expressions for K_i are written down in Appendix B and will be used in following subsections. The prefactors for those transformations are

$$1, \qquad (zy)^{-\alpha}, \qquad [(1-x)(1-y)]^{-\alpha}, \qquad \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}. \tag{27}$$

Transformations having prefactors with exponent β are generated from the ones of Appendix B by applying N_2 when the prefactor is $(zy)^{-\alpha}$, N_3 when $[(1-x)(1-y)]^{-\alpha}$ and N_5 when $[(1-x/a)(1-y/a)]^{-\alpha}$.

We could denote an initial kernel by $\mathsf{G}_1^1(x,y)$, and by $\mathsf{G}_i^j(x,y)$ the kernels obtained from $\mathsf{G}_1^1(x,y)$ by using N_i and K_j, where

$$\text{in } \mathsf{G}_i^j(x,y) : \begin{cases} j \text{ indicates Möbius transformation, } K_j, \\ i \text{ indicates index transformation, } N_i, \end{cases}$$

being necessary to specify the transformation applied in the first place since in general index and Möbius transformations do not commute. However, this notation is not sufficient because we will consider a set of initial kernels rather than a single kernel. For the Lambe-Ward case, we start with six kernels given by distinct hypergeometric functions and use the notation $\mathsf{G}_1^{1(k)}$ where k runs from 1 to 6; for the Erdélyi case we take 36 products of hypergeometric functions and then the initial set is denoted by $\mathsf{G}_1^{1(k,l)}$. Thence, the actual notation will be

$$\mathsf{G}_i^{j(k)}(x,y) \text{ for Lambe-Ward-type kernels ;} \quad \mathsf{G}_i^{j(k,l)}(x,y) \text{ for Erdélyi-type,} \tag{28}$$

where the indices inside parentheses are not affected by the application of the transformations N_i and K_j.

The Lambe-Ward kernels $\mathsf{G}_1^{1(k)}(x,y)$ defined in Eq. (34) and Erdélyi kernels $\mathsf{G}_1^{1(k,l)}(x,y)$ defined in Eq. (47) are the initial kernels which are obtained by solving directly the kernel equation. We will find that K_j with $j = 2, \cdots, 6$ are the only effective Möbius transformations. These lead to five additional sets of Lambe-Ward-type kernels, denoted and obtained as

$$\begin{aligned} &\mathsf{G}_1^{2(k)} = K_2\mathsf{G}_1^{1(k)}, \qquad \mathsf{G}_1^{3(k)} = K_3\mathsf{G}_1^{1(k)}, \qquad \mathsf{G}_1^{4(k)} = K_4\mathsf{G}_1^{1(k)}, \\ &\mathsf{G}_1^{5(k)} = K_5\mathsf{G}_1^{1(k)}, \qquad \mathsf{G}_1^{6(k)} = K_6\mathsf{G}_1^{1(k)}, \qquad [k = 1, 2, \cdots, 6]. \end{aligned} \tag{29}$$

Similarly, the new Erdélyi-type kernels are

$$\begin{aligned} &\mathsf{G}_1^{2(k,l)} = K_2\mathsf{G}_1^{1(k,l)}, \qquad \mathsf{G}_1^{3(k,l)} = K_3\mathsf{G}_1^{1(k,l)}, \qquad \mathsf{G}_1^{4(k,l)} = K_4\mathsf{G}_1^{1(k,l)}, \\ &\mathsf{G}_1^{5(k,l)} = K_5\mathsf{G}_1^{1(k,l)}, \qquad \mathsf{G}_1^{6(k,l)} = K_6\mathsf{G}_1^{1(k,l)}, \qquad [k, l = 1, 2, \cdots, 6]. \end{aligned} \tag{30}$$

In each case, the subscript could assume eight values when we apply the homotopic transformations N_i. Nevertheless, we will find that only three of the N_i are effective due to fact that one of the generators N_2, N_3, N_5 becomes equivalent to the identity or two of them are equivalent to each other. Thus, there are only four values for the subscripts.

2.3. Generalisation of Lambe-Ward's kernels

The Lambe-Ward as well as the Erdélyi kernels are given by hypergeometric functions $F(\mathrm{a},\mathrm{b};\mathrm{c};u) = F(\mathrm{b},\mathrm{a};\mathrm{c};u)$. In fact, in the vicinity of the singular points 0, 1 and ∞, the formal solutions for the hypergeometric equation (4) are [14], respectively,

$$F^{(1)}(u) = F(\mathrm{a},\mathrm{b};\mathrm{c};u), \qquad F^{(2)}(u) = u^{1-\mathrm{c}}F(\mathrm{a}+1-\mathrm{c},\mathrm{b}+1-\mathrm{c};2-\mathrm{c};u); \tag{31a}$$

$$\begin{aligned} F^{(3)}(u) &= F\left(\mathrm{a},\mathrm{b};\mathrm{a}+\mathrm{b}+1-\mathrm{c};1-u\right), \\ F^{(4)}(u) &= (1-u)^{\mathrm{c}-\mathrm{a}-\mathrm{b}}\,F\left(\mathrm{c}-\mathrm{a},\mathrm{c}-\mathrm{b};1+\mathrm{c}-\mathrm{a}-\mathrm{b};1-u\right); \end{aligned} \tag{31b}$$

$$\begin{aligned} F^{(5)}(u) &= u^{-\mathrm{a}}F\left(\mathrm{a},\mathrm{a}+1-\mathrm{c};\mathrm{a}+1-\mathrm{b};1/u\right), \\ F^{(6)}(u) &= u^{-\mathrm{b}}F\left(\mathrm{b}+1-\mathrm{c},\mathrm{b};\mathrm{b}+1-\mathrm{a};1/u\right). \end{aligned} \tag{31c}$$

Each of these functions may be written in four forms by using the relations

$$F(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{u}) = (1-u)^{\mathrm{c}-\mathrm{a}-\mathrm{b}}F(\mathrm{c}-\mathrm{a},\mathrm{c}-\mathrm{b};\mathrm{c};\mathrm{u}), \tag{32}$$

$$F(\mathrm{a},\mathrm{b};\mathrm{c};\mathrm{u}) = (1-u)^{-\mathrm{a}}F\left[\mathrm{a},\mathrm{c}-\mathrm{b};\mathrm{c};\mathrm{u}/(\mathrm{u}-1)\right]. \tag{33}$$

The Lambe and Ward kernels [3] in terms of single hypergeometric functions are obtained by taking $u = xy/a$ and $\mathsf{G}(x,y) = F(u)$
in $[M_x - M_y]\,\mathsf{G}(x,y) = 0$. Hence, $F(u)$ satisfies equation (4) with $\mathrm{a} = \alpha$, $\mathrm{b} = \beta$ and $\mathrm{c} = \gamma$. Thus, the six formal kernels have the form

$$\mathsf{G}_1^{1(i)}(x,y) = F^{(i)}\left(\frac{xy}{a}\right), \quad [\mathrm{a}=\alpha,\ \mathrm{b}=\beta,\ \mathrm{c}=\gamma] \tag{34}$$

In this manner, up to a multiplicative constant, the initial set of kernels is given by

$$\begin{aligned} \mathsf{G}_1^{1(1)}(x,y) &= F\left(\alpha,\beta;\gamma;\frac{xy}{a}\right), \\ \mathsf{G}_1^{1(2)}(x,y) &= (xy)^{1-\gamma}F\left(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;\frac{xy}{a}\right); \end{aligned} \tag{35a}$$

$$\begin{aligned} \mathsf{G}_1^{1(3)}(x,y) &= F\left(\alpha,\beta;\alpha+\beta+1-\gamma;1-\frac{xy}{a}\right), \\ \mathsf{G}_1^{1(4)}(x,y) &= \left(1-\frac{xy}{a}\right)^{\gamma-\alpha-\beta}F\left(\gamma-\alpha,\gamma-\beta;1+\gamma-\alpha-\beta;1-\frac{xy}{a}\right); \end{aligned} \tag{35b}$$

$$\begin{aligned} \mathsf{G}_1^{1(5)}(x,y) &= (xy)^{-\alpha}F\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{a}{xy}\right), \\ \mathsf{G}_1^{1(6)}(x,y) &= (xy)^{-\beta}F\left(\beta+1-\gamma,\beta;\beta+1-\alpha;\frac{a}{xy}\right). \end{aligned} \tag{35c}$$

By using this set of initial kernels, some of the kernel transformations become superfluous. In effect, by the transformations N_i we could obtain a subgroup containing eight sets. However, for the present case N_2 is ineffective since

$$N_2\left(\mathsf{G}_1^{1(1)},\mathsf{G}_1^{1(2)},\mathsf{G}_1^{1(3)},\mathsf{G}_1^{1(4)},\mathsf{G}_1^{1(5)},\mathsf{G}_1^{1(6)}\right) = \left(\mathsf{G}_1^{1(2)},\mathsf{G}_1^{1(1)},\mathsf{G}_1^{1(3)},\mathsf{G}_1^{1(4)},\mathsf{G}_1^{1(6)},\mathsf{G}_1^{1(5)}\right),$$

that is, N_2 simply rearranges in a different order the previous kernels. In this sense the generator N_2 is equivalent to the identity N_1 and, so, the index transformations can generate only three additional sets due to composition relations (25). In fact we find that

$$N_3\mathsf{G}_1^{1(i)} \Leftrightarrow N_4\mathsf{G}_1^{1(i)}, \qquad N_5\mathsf{G}_1^{1(i)} \Leftrightarrow N_6\mathsf{G}_1^{1(i)}, \qquad N_7\mathsf{G}_1^{1(i)} \Leftrightarrow N_8\mathsf{G}_1^{1(i)}.$$

Therefore, it is sufficient to use the transformations N_3, N_5 and N_7 to produce three additional sets, namely: $\mathsf{G}_3^{1(k)}$, $\mathsf{G}_5^{1(k)}$ and $\mathsf{G}_7^{1(k)}$. The eight kernels $\mathsf{G}_i^{1(1)}$ and $\mathsf{G}_i^{1(2)}$ $(i = 1,3,5,7)$ coincide with the ones given by Lambe and Ward. Now, we regard the generalisations arising from the Möbius transformations K_j. By one side we find

$$K_{13}\left(\mathsf{G}_1^{1(1)},\mathsf{G}_1^{1(2)},\mathsf{G}_1^{1(3)},\mathsf{G}_1^{1(4)},\mathsf{G}_1^{1(5)},\mathsf{G}_1^{1(6)}\right) = \left(\mathsf{G}_1^{1(1)},\mathsf{G}_1^{1(2)},\mathsf{G}_1^{1(3)},\mathsf{G}_1^{1(4)},\mathsf{G}_1^{1(5)},\mathsf{G}_1^{1(6)}\right),$$

that is, the transformations K_1 (identity) and K_{13} of the first column of (26) are equivalent, a fact that holds for the transformations of any column. Thus we have to take into account only the twelve transformations of the first row. However, we find as well that

$$K_7\left(\mathsf{G}_1^{1(1)},\mathsf{G}_1^{1(2)},\mathsf{G}_1^{1(3)},\mathsf{G}_1^{1(4)},\mathsf{G}_1^{1(5)},\mathsf{G}_1^{1(6)}\right) = \left(\mathsf{G}_1^{1(5)},\mathsf{G}_1^{1(6)},\mathsf{G}_1^{1(3)},\mathsf{G}_1^{1(4)},\mathsf{G}_1^{1(1)},\mathsf{G}_1^{1(2)}\right)$$

that is, K_7 rearranges the initial kernels. This could be expected because $K_1 \mapsto K_7$ corresponds to the inversions $(x,y) \mapsto (1/x, 1/y)$ which are already incorporated into the original set. This is the same for the other transformations corresponding to inverted mappings, that is, $K_i \Leftrightarrow K_{i+6}$ $(i = 1, \cdots, 6)$. Therefore, we may apply only the transformations K_2, K_3, K_4, K_5 and K_6 on $\mathsf{G}_1^{1(k)}$ according to Eq. (29) in order to find new sets, each one containing six hypergeometric functions. To determine which are the transformations N_i that are suitable to generate new kernels, it is sufficient to regard the pair $\left(\mathsf{G}_1^{j(1)}, \mathsf{G}_1^{j(2)}\right)$; the other pairs may be obtained by replacing the $F(\mathrm{a}, \mathrm{b}; \mathrm{c}; u)$ which appears in $\mathsf{G}_1^{j(1)}$ by the hypergeometric functions written in Eqs. (31b-c).

Thus, from K_2 we find

$$
\begin{aligned}
\mathsf{G}_1^{2(1)}(x,y) &= [(1-x)(1-y)]^{-\alpha} \, F\left[\alpha, 1+\alpha-\delta; \gamma; \frac{(a-1)xy}{a(x-1)(y-1)}\right], \\
\mathsf{G}_1^{2(2)}(x,y) &= [(1-x)(1-y)]^{\gamma-\alpha-1} \, (xy)^{1-\gamma} \times \\
&\qquad F\left[\alpha+1-\gamma, 2+\alpha-\gamma-\delta; 2-\gamma; \tfrac{(a-1)xy}{a(x-1)(y-1)}\right],
\end{aligned} \tag{36}
$$

together with the two pairs generated by using the hypergeometric functions (31b-c), as explained above. For this case, the generators N_2 and N_3 are equivalent to each other and, consequently, the N_i afford only three additional sets. We find

$$
N_1 \mathsf{G}_1^{2(i)} \Leftrightarrow N_4 \mathsf{G}_1^{2(i)}, \quad N_2 \mathsf{G}_1^{2(i)} \Leftrightarrow N_3 \mathsf{G}_1^{2(i)}, \quad N_5 \mathsf{G}_1^{2(i)} \Leftrightarrow N_8 \mathsf{G}_2^{1(i)}, \quad N_6 \mathsf{G}_1^{1(i)} \Leftrightarrow N_7 \mathsf{G}_1^{1(i)}.
$$

Thence we may use only the transformations N_2, N_5 and N_6. On the other hand, by using K_3, we get

$$
\begin{aligned}
\mathsf{G}_1^{3(1)}(x,y) &= \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha} F\left[\alpha, \gamma+\delta-\beta; \gamma; \frac{(1-a)xy}{(a-x)(a-y)}\right], \\
\mathsf{G}_1^{3(2)}(x,y) &= [xy]^{1-\gamma} \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{\gamma-1-\alpha} \times \\
&\qquad F\left[\alpha+1-\gamma, \delta+1-\beta; 2-\gamma; \frac{(1-a)xy}{(a-x)(a-y)}\right].
\end{aligned} \tag{37}
$$

This time $N_1 \Leftrightarrow N_6$, $N_2 \Leftrightarrow N_5$, $N_3 \Leftrightarrow N_8$ and $N_4 \Leftrightarrow N_7$. Thus, the transformations N_2, N_3 and N_4 are sufficient. The transformation K_4 yields

$$
\begin{aligned}
\mathsf{G}_1^{4(1)}(x,y) &= F\left[\alpha, \beta; \delta; \frac{(x-1)(y-1)}{1-a}\right], \\
\mathsf{G}_1^{4(2)}(x,y) &= [(1-x)(1-y)]^{1-\delta} \, F\left[\alpha+1-\delta, \beta+1-\delta; 2-\delta; \frac{(x-1)(y-1)}{1-a}\right].
\end{aligned} \tag{38}
$$

Since $N_1 \Leftrightarrow N_3$, $N_2 \Leftrightarrow N_4$, $N_5 \Leftrightarrow N_7$ and $N_6 \Leftrightarrow N_8$, we can choose only N_2, N_5 and N_6. By K_6 we get

$$
\begin{aligned}
\mathsf{G}_1^{5(1)}(x,y) &= \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha} F\left[\alpha, \gamma+\delta-\beta; \delta; \frac{a(x-1)(y-1)}{(x-a)(y-a)}\right], \\
\mathsf{G}_1^{5(2)}(x,y) &= [(1-x)(1-y)]^{1-\delta} \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{\delta-1-\alpha} \times \\
&\qquad F\left[\alpha+1-\delta, \gamma+1-\beta; 2-\delta; \frac{a(x-1)(y-1)}{(x-a)(y-a)}\right],
\end{aligned} \tag{39}
$$

together with the sets obtained by applying N_2, N_3 and N_4, because $N_1 \Leftrightarrow N_7$, $N_2 \Leftrightarrow N_8$, $N_3 \Leftrightarrow N_5$ and $N_4 \Leftrightarrow N_6$ for this subgroup. Finally, the transformation K_6 leads to

$$
\begin{aligned}
\mathsf{G}_1^{6(1)}(x,y) &= F\left[\alpha, \beta; \epsilon; \frac{(x-a)(y-a)}{a(a-1)}\right], \\
\mathsf{G}_1^{6(2)}(x,y) &= \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{1-\epsilon} F\left[\gamma+\delta-\alpha, \gamma+\delta-\beta; 2-\epsilon; \frac{(x-a)(y-a)}{a(a-1)}\right],
\end{aligned} \tag{40}
$$

and the sets resulting from this by N_2, N_3 and N_4 since now $N_1 \Leftrightarrow N_5$, $N_2 \Leftrightarrow N_6$, $N_3 \Leftrightarrow N_7$ and $N_4 \Leftrightarrow N_8$.

In summary, the substitutions of the independent variables that convert the kernel equation (8) into a hypergeometric equation are $(x,y) \mapsto u(x,y)$, where $u(x,y)$ is given by

$$\frac{xy}{a}, \ \frac{(a-1)xy}{a(x-1)(y-1)}, \ \frac{(1-a)xy}{(a-x)(a-y)}, \ \frac{(x-1)(y-1)}{1-a}, \ \frac{a(x-1)(y-1)}{(x-a)(y-a)}, \ \frac{(x-a)(y-a)}{a(a-1)}. \quad (41)$$

In general it is necessary to perform a substitution of the dependent variable as well.

2.4. Generalisation of Erdélyi's kernels

These are given by products of hypergeometric functions containing an arbitrary separation constant λ. When this constant is appropriately chosen, we recover the Lambe-Ward-type kernels. The Erdélyi kernels are obtained by rewriting equation (8) in terms of the independent variables [4]

$$\xi = \frac{xy}{a}, \qquad \zeta = \frac{(x-a)(y-a)}{(1-a)(xy-a)}, \quad (42)$$

and, then, by accomplishing separation of variables in the resulting equation. It turns out that

$$\mathsf{G}(x,y) = \left(1-\xi\right)^{-\lambda} P(\xi)\ Q(\zeta), \quad (43)$$

where $P(\xi)$ and $Q(\zeta)$ satisfy the hypergeometric equation (4) with the following sets of parameters:

$$\begin{aligned} &P(\xi): \ \mathrm{a} = \alpha - \lambda, && \mathrm{b} = \beta - \lambda, && \mathrm{c} = \gamma; \\ &Q(\zeta): \ \mathrm{a} = \lambda, && \mathrm{b} = \alpha + \beta - \gamma - \lambda, && \mathrm{c} = \epsilon. \end{aligned} \quad (44)$$

To show this, firstly we perform the substitutions (42) in Eq. (4). We get the equation

$$\begin{aligned} &(1-\xi)\left\{\xi(1-\xi)\frac{\partial^2 \mathsf{G}}{\partial \xi^2} + \Big[\gamma - (\alpha+\beta+1)\xi\Big]\frac{\partial \mathsf{G}}{\partial \xi} - \alpha\beta\mathsf{G}\right\} + \\ &\zeta(1-\zeta)\frac{\partial^2 \mathsf{G}}{\partial \zeta^2} + \Big[\epsilon - (\delta+\epsilon)\zeta\Big]\frac{\partial \mathsf{G}}{\partial \zeta} = 0, \end{aligned}$$

which, by the separation of variables $\mathsf{G}(\xi,\zeta) = \bar{P}(\xi)Q(\zeta)$, becomes

$$\begin{aligned} &\frac{(1-\xi)}{\bar{P}}\left\{\xi(1-\xi)\frac{d^2\bar{P}}{d\xi^2} + \Big[\gamma - (\gamma+\delta+\epsilon)\xi\Big]\frac{d\bar{P}}{d\xi} - \alpha\beta\bar{P}\right\} + \\ &\frac{1}{Q}\left\{\zeta(1-\zeta)\frac{d^2 Q}{d\zeta^2} + \Big[\epsilon - (\delta+\epsilon)\zeta\Big]\frac{dQ}{d\zeta}\right\} = 0. \end{aligned}$$

Denoting the separation constant by $\lambda(\alpha+\beta-\gamma-\lambda)$, we obtain

$$\begin{aligned} &\zeta(1-\zeta)\frac{d^2 Q}{d\zeta^2} + \Big[\epsilon - (\delta+\epsilon)\zeta\Big]\frac{dQ}{d\zeta} - \lambda(\alpha+\beta-\gamma-\lambda)Q = 0, \\ &\xi(1-\xi)\frac{d^2\bar{P}}{d\xi^2} + \Big[\gamma - (\gamma+\beta+1)\xi\Big]\frac{d\bar{P}}{d\xi} - \left[\alpha\beta - \frac{\lambda(\alpha+\beta-\gamma-\lambda)}{1-\xi}\right]\bar{P} = 0. \end{aligned} \quad (45)$$

The additional substitution $\bar{P} = (1-\xi)^{-\lambda}P$ leads to

$$\xi(1-\xi)\frac{d^2 P}{d\xi^2} + \Big[\gamma - (\alpha+\beta+1-2\lambda)\xi\Big]\frac{dP}{d\xi} - (\alpha-\lambda)(\beta-\lambda)P = 0. \quad (46)$$

In this manner, $P(\xi)$ and $Q(\zeta)$ satisfy hypergeometric equations with the parameters given in (44). If $\lambda = 0$, we can take $Q(\zeta) =$ constant in order to recover the Lambe-Ward kernels, $\mathsf{G}_1^{1(i)}$. On the other hand, for $\lambda = \alpha$ and $P(\xi) =$ constant, the resulting kernels belong to the generalised Lambe-Ward kernels $\mathsf{G}_i^{6(j)}(x,y)$ which accompany the kernels

(39), since by relation (33) the arguments of the hypergeometric functions take the form $u = (x-a)(y-a)/[a(x-1)(y-1)]$, $1-u$ or $1/u$.

For $P(\xi)$ we select two hypergeometric functions in the neighbourhood of each singular point, as in the Lambe-Ward case. Since for $Q(\zeta)$ there are six possibilities as well, we write the initial set as

$$\mathsf{G}_1^{1(k,l)}(x,y) = \left(1-\xi\right)^{-\lambda} P^{(k)}(\xi)\; Q^{(l)}(\zeta), \tag{47}$$

where $P^{(k)}(\xi)$ and $Q^{(l)}(\zeta)$ are obtained from the six hypergeometric functions (31a-c), having parameters specified in (44). Explicitly, we find

$$\begin{aligned}
\mathsf{G}_1^{1(1,l)}(x,y) &= \left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\alpha-\lambda,\beta-\lambda;\gamma;\frac{xy}{a}\right] Q^{(l)}(\zeta),\\
\mathsf{G}_1^{1(2,l)}(x,y) &= (xy)^{1-\gamma}\left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\alpha+1-\gamma-\lambda,\beta+1-\gamma-\lambda;2-\gamma;\frac{xy}{a}\right] Q^{(l)}(\zeta);
\end{aligned} \tag{48a}$$

$$\begin{aligned}
\mathsf{G}_1^{1(3,l)}(x,y) &= \left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\alpha-\lambda,\beta-\lambda;1+\alpha+\beta-\gamma-2\lambda;1-\frac{xy}{a}\right] Q^{(l)}(\zeta),\\
\mathsf{G}_1^{1(4,l)}(x,y) &= \left(1-\frac{xy}{a}\right)^{\gamma-\alpha-\beta+\lambda}\times\\
&\quad F\left[\gamma-\alpha+\lambda,\gamma-\beta+\lambda;1+\gamma-\alpha-\beta+2\lambda;1-\frac{xy}{a}\right] Q^{(l)}(\zeta);
\end{aligned} \tag{48b}$$

$$\begin{aligned}
\mathsf{G}_1^{1(5,l)}(x,y) &= (xy)^{\lambda-\alpha}\left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\alpha-\lambda,\alpha+1-\gamma-\lambda;1+\alpha-\beta;\frac{a}{xy}\right] Q^{(l)}(\zeta),\\
\mathsf{G}_1^{1(6,l)}(x,y) &= (xy)^{\lambda-\beta}\left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\beta-\lambda,\beta+1-\gamma-\lambda;1+\beta-\alpha;\frac{a}{xy}\right] Q^{(l)}(\zeta),
\end{aligned} \tag{48c}$$

where

$$Q^{(1)}(\zeta) = F\left[\lambda,\alpha+\beta-\gamma-\lambda;\epsilon;\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right],$$

$$Q^{(2)}(\zeta) = \left[\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right]^{1-\epsilon} F\left[\lambda+1-\epsilon,\delta-\lambda;2-\epsilon;\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right],$$

$$Q^{(3)}(\zeta) = F\left[\lambda,\alpha+\beta-\gamma-\lambda;\delta;1-\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right],$$

$$Q^{(4)}(\zeta) = \left[1-\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right]^{1-\delta} F\left[\epsilon-\lambda,1+\lambda-\delta;2-\delta;1-\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right],$$

$$Q^{(5)}(\zeta) = \left[\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right]^{-\lambda} F\left[\lambda,1+\lambda-\epsilon;1+2\lambda-\alpha-\beta;\frac{(1-a)(xy-a)}{(x-a)(y-a)}\right],$$

$$\begin{aligned}
Q^{(6)}(\zeta) \;&=\; \left[\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right]^{\lambda+\gamma-\alpha-\delta}\\
&\quad\times\; F\left[\delta-\lambda,\alpha+\beta-\gamma+\lambda;1-2\lambda+\alpha+\beta;\frac{(1-a)(xy-a)}{(x-a)(y-a)}\right].
\end{aligned} \tag{48d}$$

We can generate additional kernels with the same arguments for the hypergeometric functions by applying the index transformations N_i. However, as in the case of the Lambe-Ward kernels, we find that

$$N_1 G_1^{1(k,l)} \Leftrightarrow N_2 G_1^{1(k,l)},\; N_3 G_1^{1(k,l)} \Leftrightarrow N_4 G_1^{1(k,l)},\; N_5 G_1^{1(k,l)} \Leftrightarrow N_6 G_1^{1(k,l)},\; N_7 G_1^{1(k,l)} \Leftrightarrow N_8 G_1^{1(k,l)}.$$

Thence, again it is sufficient to use the transformations N_3, N_5 and N_7 in order to generate the first subgroup of kernels.

The next step refers to the generalisation of the Erdélyi kernels by means of the Möbius transformations K_i, which now change as well the arguments of the two hypergeometric functions. We transform only the kernel $G_1^{1(1,1)}$ whose explicit form is

$$\begin{aligned} \mathsf{G}_1^{1(1,1)}(x,y) &= \left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\alpha-\lambda,\beta-\lambda;\gamma;\frac{xy}{a}\right] \\ &\times\ F\left[\lambda,\alpha+\beta-\gamma-\lambda;\epsilon;\frac{(x-a)(y-a)}{(1-a)(xy-a)}\right]. \end{aligned} \tag{49}$$

The 36 kernels are obtained by replacing each hypergeometric function by the other expressions given in Eqs. (31a-c), all of them with the same λ.

Again we can use only the transformations K_2, K_3, K_4, K_5 and K_6, the same ones employed in the Lambe-Ward case. In effect, K_{13} gives

$$\begin{aligned} K_{13}\mathsf{G}_1^{1(1,1)}(x,y) &= \left(1-\frac{xy}{a}\right)^{-\lambda} F\left[\alpha-\lambda,\beta-\lambda;\gamma;\frac{xy}{a}\right] \\ &\times\ F\left[\lambda,\alpha+\beta-\gamma-\lambda;\delta;\frac{a(x-1)(y-1)}{(a-1)(xy-a)}\right] = \mathsf{G}_1^{1(1,3)}(x,y). \end{aligned}$$

Considering the other kernels, we conclude that K_{13} is equivalent to the identity K_1 and, consequently, the kernels corresponding to the transformations of every column of matrix (26) are equivalent to each another. In addition, as in Lambe-Ward case, we find $K_i \Leftrightarrow K_{i+6}$ $(i=1,\cdots,6)$.

Thus, applying K_2, K_3, K_4, K_5 and K_6 on $\mathsf{G}_1^{1(1,1)}$, we find the kernels

$$\begin{aligned} \mathsf{G}_1^{2(1,1)}(x,y) &= [(1-x)(1-y)]^{-\alpha}\left[1-\frac{(a-1)xy}{a(x-1)(y-1)}\right]^{-\lambda} \\ &\times\ F\left[\alpha-\lambda,1+\alpha-\delta-\lambda;\gamma;\frac{(a-1)xy}{a(x-1)(y-1)}\right] \\ &\times\ F\left[\lambda,1+2\alpha-\gamma-\delta-\lambda;\epsilon;\frac{(x-a)(y-a)}{a(1-x-y)+xy}\right], \end{aligned} \tag{50}$$

$$\begin{aligned} \mathsf{G}_1^{3(1,1)}(x,y) &= \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\left[1-\frac{(1-a)xy}{(x-a)(y-a)}\right]^{-\lambda} \\ &\times\ F\left[\alpha-\lambda,\gamma+\delta-\beta-\lambda;\gamma;\frac{(1-a)xy}{(a-x)(a-y)}\right] \\ &\times\ F\left[\lambda,\alpha-\beta+\delta-\lambda;\delta;\frac{(x-1)(y-1)}{a+xy-x-y}\right], \end{aligned} \tag{51}$$

$$\begin{aligned} \mathsf{G}_1^{4(1,1)}(x,y) &= \left[1-\frac{(x-1)(y-1)}{1-a}\right]^{-\lambda} F\left[\alpha-\lambda,\beta-\lambda;\gamma;\frac{(x-1)(y-1)}{1-a}\right] \\ &\times\ F\left[\lambda,\alpha-\beta-\delta-\lambda;\epsilon;\frac{(x-a)(y-a)}{a(a+xy-x-y)}\right], \end{aligned} \tag{52}$$

$$\begin{aligned} \mathsf{G}_1^{5(1,1)}(x,y) &= \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\left[1-\frac{a(x-1)(y-1)}{(x-a)(y-a)}\right]^{-\lambda} \\ &\times\ F\left[\alpha-\lambda,\gamma+\delta-\beta-\lambda;\delta;\frac{a(x-1)(y-1)}{(x-a)(y-a)}\right] \\ &\times\ F\left[\lambda,\alpha-\beta+\gamma-\lambda;\gamma;\frac{xy}{xy-a}\right], \end{aligned} \tag{53}$$

$$\begin{aligned} \mathsf{G}_1^{6(1,1)}(x,y) &= \left[1-\frac{(x-a)(y-a)}{a(a-1)}\right]^{-\lambda} F\left[\alpha-\lambda,\beta-\lambda;\epsilon;\frac{(x-a)(y-a)}{a(a-1)}\right] \\ &\times\ F\left[\lambda,\gamma+\delta-1-\lambda;\delta;\frac{a(x-1)(y-1)}{a(1-x-y)+xy}\right]. \end{aligned} \tag{54}$$

The index transformations suitable for each set are: N_2, N_5 and N_6 for $\mathsf{G}_1^{2(k,l)}$ and $\mathsf{G}_1^{4(k,l)}$; N_2, N_3 and N_4 for $\mathsf{G}_1^{3(k,l)}$, $\mathsf{G}_1^{5(k,l)}$ and $\mathsf{G}_1^{6(k,l)}$. The previous kernels show that, in addition to the Erdélyi substitutions (42) for the independent variables, there are five other choices for (ξ, ζ).

3. Confluent Heun equation

In this section we show that each transformation of the confluent Heun equation (CHE) is associated with a transformation of the equation for its kernels. We also show that some kernels of Heun's equation lead to kernels for the CHE through the limits (5). For this end we rewrite the CHE (6) as

$$[M_x - \sigma]\, H(x) = 0, \quad M_x = x(x-1)\frac{\partial^2}{\partial x^2} + \big[-\gamma + (\gamma+\delta)x + \rho x(x-1)\big]\frac{\partial}{\partial x} + \alpha\rho x, \quad (55)$$

where in the previous equation M_x is an ordinary differential operator. For the limits (5), the integral (7) becomes

$$\mathcal{H}(x) = \int_{y_1}^{y_2} e^{\rho y}\; y^{\gamma-1}(1-y)^{\delta-1}\mathsf{G}(x,y)H(y)dy, \quad (56)$$

where the exponential results from the limit of $(1-y/a)^{\epsilon-1}$. Besides this, Eq. (8) for $\mathsf{G}(x,y)$ takes the form

$$[M_x - M_y]\,\mathsf{G}(x,y) = 0, \quad (57)$$

while the expression (9) for the bilinear concomitant becomes

$$\mathsf{P}(x,y) = e^{\rho y}\; y^{\gamma}(1-y)^{\delta}\left[H(y)\frac{\partial \mathsf{G}(x,y)}{\partial y} - \mathsf{G}(x,y)\frac{dH(y)}{dy}\right]. \quad (58)$$

First we discuss the transformations for the CHE (55) and its kernels (57) and, after this, we explain how to get kernels for the CHE from the ones of the general equation (1).

3.1. Transformation of the confluent equation and its kernels

There are 16 variables substitutions which preserve the form of the CHE [8]. If $H(x) = H(\sigma;\rho,\alpha,\gamma,\delta;x)$ denotes one solution of the CHE (55), these transformations are summarised in the rules T_1, T_2, T_3 and T_4 that operate as

$$\begin{aligned}
&\mathrm{T}_1 H(x) = (1-x)^{1-\delta}\; H\big[\sigma - \gamma(1-\delta);\rho,\alpha+1-\delta,\gamma,2-\delta;x\big],\\
&\mathrm{T}_2 H(x) = x^{1-\gamma}\; H\big[\sigma + (1-\gamma)(\rho-\delta);\rho,\alpha+1-\gamma,2-\gamma,\delta;x\big],\\
&\mathrm{T}_3 H(x) = e^{-\rho x}\; H[\sigma - \gamma\rho; -\rho, \gamma+\delta-\alpha,\gamma,\delta;x],\\
&\mathrm{T}_4 H(x) = H\big[\sigma - \rho\alpha; -\rho,\alpha,\delta,\gamma;1-x\big].
\end{aligned} \quad (59)$$

Compositions of these give the group having 16 elements.

On the other side, equation (57) for the kernels is written in terms of the same differential operator which appears in the CHE (6). Consequently, proceeding as in the case of the general Heun equation, the corresponding rules K_i for transforming a given kernel $\mathsf{G}(x,y) = \mathsf{G}(\rho,\alpha,\gamma,\delta;x,y))$ are

$$\begin{aligned}
&\mathrm{K}_1\mathsf{G}(x,y) = \big[(1-x)(1-y)\big]^{1-\delta}\; \mathsf{G}\big[\rho,\alpha+1-\delta,\gamma,2-\delta;x,y\big],\\
&\mathrm{K}_2\mathsf{G}(x,y) = (xy)^{1-\gamma}\; \mathsf{G}\big[\rho,\alpha+1-\gamma,2-\gamma,\delta;x,y\big],\\
&\mathrm{K}_3\mathsf{G}(x,y) = e^{-\rho(x+y)}\; \mathsf{G}[-\rho,\gamma+\delta-\alpha,\gamma,\delta;x,y],\\
&\mathrm{K}_4\mathsf{G}(x,y) = \mathsf{G}\big[-\rho,\alpha,\delta,\gamma;1-x,1-y\big].
\end{aligned} \quad (60)$$

These rules can be verified by substitutions of variables. However, they are useful to produce new kernels when one knows an initial kernel for the CHE. In the following we obtain initial kernels as limits of kernels for the Heun equation. Any kernel which can be generated by the above transformations is omitted.

3.2. "Lambe-Ward-type" kernels

The limits (5) applied to kernels of Sec. 2.3 give three kinds of kernels for the CHE, two in terms of confluent hypergeometric functions and one in terms of Gauss hypergeometric functions. The regular and irregular confluent hypergeometric functions, denoted by $\Phi(\mathrm{a},\mathrm{c};y)$ and $\Psi(\mathrm{a},\mathrm{c};y)$, respectively, are solutions of the equation

$$u\frac{d^2\varphi}{du^2}+(\mathrm{c}-u)\,\frac{d\varphi}{du}-\mathrm{a}\varphi=0. \tag{61a}$$

The following types of solutions for Eq. (61a)

$$\begin{aligned}&\varphi^{(1)}(u)=\Phi(\mathrm{a},\mathrm{c};u), &&\varphi^{(2)}(u)=e^{u}u^{1-\mathrm{c}}\Phi(1-\mathrm{a},2-\mathrm{c};-u),\\ &\varphi^{(3)}(u)=\Psi(\mathrm{a},\mathrm{c};u), &&\varphi^{(4)}(u)=e^{u}u^{1-\mathrm{c}}\Psi(1-\mathrm{a},2-\mathrm{c};-u),\end{aligned} \tag{61b}$$

are all defined and distinct only if c is not an integer [14]. Alternative forms for these solutions follow from the relations

$$\Phi(\mathrm{a},\mathrm{c};u)=e^{u}\Phi(\mathrm{c}-\mathrm{a},\mathrm{c};-u),\qquad \Psi(\mathrm{a},\mathrm{c};u)=u^{1-\mathrm{c}}\Psi(1+\mathrm{a}-\mathrm{c},2-\mathrm{c};u). \tag{61c}$$

In the present context, the above confluent hypergeometric functions result from the limits [14]

$$\begin{aligned}&\lim_{\mathrm{c}\to\infty}F\left(\mathrm{a},\mathrm{b};\mathrm{c};1-\frac{\mathrm{c}}{u}\right)=\lim_{\mathrm{c}\to\infty}F\left(\mathrm{a},\mathrm{b};\mathrm{c};-\frac{\mathrm{c}}{u}\right)=u^{\mathrm{a}}\Psi(\mathrm{a},\mathrm{a}+1-\mathrm{b};u),\\ &\lim_{\mathrm{b}\to\infty}F\left(\mathrm{a},\mathrm{b};\mathrm{c};\frac{u}{\mathrm{b}}\right)=\Phi(\mathrm{a},\mathrm{c};u).\end{aligned} \tag{62}$$

Sometimes, before applying these limits, it is necessary to use the relations (32) and/or (33) and, in addition, multiply the kernels by suitable constants depending on the parameter a. Furthermore, note that both a and β tend to infinity but such that $\beta=-\rho a$, where ρ is constant. For this reason we can write, for example,

$$F\left(\alpha,\beta;\gamma;\frac{xy}{a}\right)=F\left(\alpha,\beta;\gamma;\frac{-\rho xy}{\beta}\right),\quad \left(1-\frac{xy}{a}\right)^{-\beta}=\left(1+\frac{\rho xy}{\beta}\right)^{-\beta}$$

and, thence, accomplish the limit $\beta\to\infty$ by keeping ρxy fixed.

Thus, the first set of section 2.3, $\mathsf{G}_1^{1(i)}(x,y)$, gives the limits

$$\begin{aligned}&\mathsf{G}_1^{1(1)}(x,y)\to\Phi\left(\alpha,\gamma;-\rho xy\right),\\ &\mathsf{G}_1^{1(2)}(x,y)\to e^{-\rho xy}\left(xy\right)^{1-\gamma}\Phi\left(1-\alpha,2-\gamma;\rho xy\right),\\ &\mathsf{G}_1^{1(3)}(x,y)\text{ and }\mathsf{G}_1^{1(5)}(x,y)\to\Psi\left(\alpha,\gamma;-\rho xy\right),\\ &\mathsf{G}_1^{1(4)}(x,y)\text{ and }\mathsf{G}_1^{1(6)}(x,y)\to e^{-\rho xy}(xy)^{1-\gamma}\ \Psi\left(1-\alpha,2-\gamma;\rho xy\right),\end{aligned}$$

which, by means of (61b), can be written as

$$G_1^{(i)}(x,y)=\varphi^{(i)}(u),\quad\text{with}\quad u=-\rho xy,\quad \mathrm{a}=\alpha,\quad \mathrm{c}=\gamma\quad [i=1,2,3,4]. \tag{63}$$

This initial set of kernels is really due to Lambe and Ward because it is given by confluent hypergeometric functions whose arguments depend on the product xy as in [3]. The previous transformations K_i produce new solutions of this type.

The limits of the set $\mathsf{G}_1^{2(i)}(x,y)$, Sec. 2.3, gives six kernels in terms of Gauss hypergeometric functions (31a-c), written as

$$\tilde{G}_1^{(i)}(x,y)=\left[(1-x)(1-y)\right]^{-\alpha}F^{(i)}(u),\qquad [i=1,\cdots,6] \tag{64}$$

$$\text{with}\quad u=\frac{xy}{(x-1)(y-1)},\quad \mathrm{a}=\alpha,\quad \mathrm{b}=1+\alpha-\delta,\quad \mathrm{a}=\gamma.$$

We can also apply the transformations K_j in order to generate a group of kernels. Notice that this group results from a generalisation of the Lambe-Ward kernels and, as far as we know, it is new.

Finally, another set of kernels is given by confluent hypergeometric functions (61b) whose arguments are proportional to $x+y-1$. It is obtained as limits of the kernels (40) and reads

$$\hat{G}_1^{(i)} = \varphi^{(i)}(u), \quad \text{with } u = -\rho(x+y-1), \quad \mathrm{a} = \alpha, \quad \mathrm{c} = \gamma+\delta \quad [i = 1,2,3,4]. \tag{65}$$

Other kernels follow from the transformations K_i. This group is also a result of a generalisation of the Lambe-Ward kernels, but kernels having this form are already known in the literature [9, 11].

3.3. "Erdélyi-type" kernels

By taking the limits of the Erdélyi-type kernels we find two groups of kernels for the CHE. The group given by products of confluent hypergeometric functions has already appeared in the literature [15], whereas the group given by products of hypergeometric and confluent hypergeometric functions seems to be new. In the limit process we suppose that λ is kept fixed, that is, we assume that λ does not depend on the parameters β and a of the Heun equation.

Thus, by taking the limit of the kernels $\mathsf{G}_1^{1(k,l)}(x,y)$ given in (47), we get an initial set of kernels given by

$$G_1^{(i,j)} = \varphi^{(i)}(\xi)\bar{\varphi}^{(j)}(\zeta), \qquad [i,j = 1,2,3,4] \tag{66}$$

where $\varphi^{(i)}(\xi)$ and $\bar{\varphi}^{(j)}(\zeta)$ are the solutions (61b) for the confluent hypergeometric equation, having the following arguments and parameters :

$$\varphi^{i}(\xi): \quad \xi = -\rho xy,\ \mathrm{a} = \alpha-\lambda,\ \mathrm{c} = \gamma; \quad \bar{\varphi}^{j}(\zeta): \quad \zeta = \rho(x-1)(y-1),\ \mathrm{a} = \lambda,\ \mathrm{c} = \delta. \tag{67}$$

The four kernels given by products of regular functions Φ are

$$G_1^{(1,1)}(x,y) = \Phi\left[\alpha-\lambda,\gamma;-\rho xy\right]\Phi\left[\lambda,\delta;\rho(x-1)(y-1)\right],$$

$$\begin{aligned} G_1^{(1,2)}(x,y) &= e^{\rho(x-1)(y-1)}\left[(x-1)(y-1)\right]^{1-\delta}\Phi\left[\alpha-\lambda,\gamma;-\rho xy\right] \\ &\times \Phi\left[1-\lambda,2-\delta;-\rho(x-1)(y-1)\right], \end{aligned}$$

$$G_1^{(2,1)}(x,y) = e^{-\rho xy}(xy)^{1-\gamma}\Phi\left[\lambda+1-\alpha,2-\gamma;\rho xy\right]\Phi\left[\lambda,\delta;\rho(x-1)(y-1)\right],$$

$$\begin{aligned} G_1^{(2,2)}(x,y) &= e^{-\rho(x+y)}(xy)^{1-\gamma}\left[(x-1)(y-1)\right]^{1-\delta}\Phi\left[\lambda+1-\alpha,2-\gamma;\rho xy\right] \\ &\times \Phi\left[1-\lambda,2-\delta;-\rho(x-1)(y-1)\right]. \end{aligned}$$

Replacing one or both Φ by Ψ we get the set with 16 kernels. The other sets result from this by the transformations K_i.

In the second place, the kernel $\mathsf{G}_1^{6(1,1)}(x,y)$, Eq. (54), yields a kernel $\tilde{G}_1^{(2,1)}(x,y)$ constituted by a product of hypergeometric and confluent hypergeometric functions, namely,

$$\begin{aligned} \tilde{G}_1^{(2,1)}(x,y) &= (1-x-y)^{-\lambda}\,\Psi\left[\alpha-\lambda,\gamma+\delta-2\lambda;\rho(1-x-y)\right] \\ &\times F\left[\lambda,\gamma+\delta-1-\lambda;\delta;\frac{(x-1)(y-1)}{1-x-y}\right]. \end{aligned}$$

By considering the limits of the full set $\mathsf{G}_1^{6(k,l)}(x,y)$ we obtain the initial set

$$\tilde{G}_1^{(i,j)}(x,y) = \varphi^{(i)}(\xi)F^{(j)}(\zeta), \qquad [i = 1,\cdots,4;\ j = 1,\cdots,6] \tag{68}$$

where $\varphi^{(i)}$ are the four solutions for confluent hypergeometric equation and where $F^{(j)}$ are the six solutions for hypergeometric equation with the following arguments and parameters:

$$\begin{array}{llll} \varphi^{(i)}(\xi): & \xi = \rho(1-x-y), & \mathrm{a} = \alpha-\lambda, & & \mathrm{c} = \gamma+\delta-2\lambda \\ F^{(j)}(\zeta): & \zeta = \frac{(x-1)(y-1)}{1-x-y}, & \mathrm{a} = \lambda, & \mathrm{b} = \gamma+\delta-1-\lambda, & \mathrm{c} = \delta. \end{array} \tag{69}$$

The kernels $\mathsf{G}_1^{2(k,l)}(x,y)$, Eq.(50), also lead to kernels given by products hypergeometric and confluent hypergeometric functions, but we may show that these are connected with the previous ones by the transformation K_4.

4. Concluding remarks

We have taken the following steps to deal with kernels for integral relations among solutions of Heun equations:

- the use of an integral with a weight function $w(x,y)$ which allows to write a given Heun equation and the respective equation for the kernels in terms of operators functionally identical, say, M_x and M_y;
- the use of the known transformations of the equation in order to get the actual form for the transformations of the kernel equation;
- the generation of new kernels by applying the previous transformations to an initial kernel or set of kernels.
- the use of a limiting procedure to generate kernels for the confluent Heun equation.

For the (general) Heun equation we have used Maier's transformations, discarding as inappropriate the forms given in Refs. [5] and [12] for the homotopic transformations. As initial kernels we have employed the ones found by Lambe and Ward, Eq. (34), and by Erdélyi, Eq. (47).

In this manner, in section 2 the transformations for integral relations have afforded several new kernels for the Heun equation, given by a single hypergeometric function and by products of two hypergeometric functions. We have seen that only six of the homographic transformations for the kernels are effective, namely: K_1, K_2,$\cdots$,K_6, where K_1 is the identity. The fact these are just the six first transformations of Appendix B is a consequence of manner in which we have written the elements of matrix (10).

We have written only some of the possible kernels, but a wider list can be generated by index transformations which lead to new kernels where the hypergeometric functions possess the same argument but different parameters. In addition, from a kernel with a given argument, new kernels follow from the fact that the hypergeometric equation formally admits solutions with different arguments in the vicinity of each singular point.

In section 3, the confluence procedure (5) has led to five sets of initial kernels for the CHE, three of them arising from generalisations of the Lambe-Ward and Erdélyi kernels by means of Möbius transformations. This in association with the Leaver version for the CHE and the concept of Whittaker-Ince limit suggest new kernels also for the double-confluent Heun equation (DCHE) and for limiting cases of the CHE and DCHE. In effect, by substitutions of variables, the CHE (6) can be written in the Leaver form [6], namely,

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3-2\eta\omega(z-z_0)+\omega^2 z(z-z_0)\right]U=0,\quad [\omega\neq 0] \tag{70}$$

where B_i, η and ω are constants, and $z=0$ and $z=z_0$ are the regular singular points. When $z_0=0$ (Leaver's limit), this gives the DCHE

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left(B_3-2\eta\omega z+\omega^2z^2\right)U=0,\qquad [B_1\neq 0,\ \omega\neq 0], \tag{71}$$

where now $z=0$ and $z=\infty$ are irregular singularities. On the other side, these equations admit the limit [16]

$$\omega\to 0,\quad \eta\to\infty,\ \text{such that}\quad 2\eta\omega=-q,\qquad [\text{Whittaker-Ince limit}]$$

where q should not be confused with the parameter q of the Heun equation (1). The Whittaker-Ince limit of the CHE and DCHE are, respectively, the equations

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3+q(z-z_0)\right]U=0,\qquad [q\neq 0] \tag{72}$$

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+(B_3+qz)\,U=0,\qquad [q\neq 0,\ B_1\neq 0] \tag{73}$$

which have a different type of singularity at $x=\infty$ as compared with the original CHE and DCHE [16–18]. As the preceding limits connect the CHE with the DCHE and their respective Whittaker-Ince limits, we may expect to find kernels for each of these out of kernels arising from the Heun equation. Thus, by developing the results of section 3 we could unify the treatment of these equations. We advance that we will find that the usual kernels for the Mathieu equation [20] are particular cases of the kernels for Eq. (72).

The construction of new kernels can be envisaged as a first step for seeking new solutions for the Heun equations by means of integral relations [19]. However, as in the case of the Mathieu equation [20], it is not easy to use this technique. Thus, in the following we focus on possibility of using some of the Maier transformations to extend the solutions in series of hypergeometric functions given by Svartholm in 1939 [21] and by Erdélyi in 1944 [13]. Also in this context, the previous limits become relevant.

As a prior consideration, we note that there are three types of recurrence relations for the series coefficients – not just one as given in the original articles and repeated since then [5, 12]. The two additional relations may be found by the procedure used in Appendix A of Ref. [22]. Then, by virtue of the new relations and by means of the homotopic transformations, we may show that the Svartholm solutions include as particular cases the eight Fourier-type solutions found by Ince in 1940 for the Lamé equation [12, 23].

On the other side, both the Svartholm and the Erdélyi solutions are valid only if the parameter a satisfy the condition $\mathrm{Re}\sqrt{1-(1/a)} > 0$ which is assured by requiring that $a \notin [0,1]$. In addition, the former solution is given by a single expansion in series of hypergeometric functions and converges only over a finite region of the complex plane; however, the latter is in fact a set of expansions in terms of hypergeometric functions which, by analytical continuation, may cover the entire complex plane provided that a characteristic equation is fulfilled.

Then, Erdélyi's solutions are candidates to solve a cosmological problem formulated by Kantowski [24] because in this case the variable x extend to infinity. Nevertheless, the problem demands solutions valid also for $a \in [0,1]$. These may be derived by applying on the Erdélyi solutions one of the following linear transformations:

$$M_9 H(x) = H\left(\tfrac{1}{a}, \tfrac{q}{a}; \alpha, \beta, \gamma, \epsilon; \tfrac{x}{a}\right), \quad M_{61} H(x) = H\left(\tfrac{1}{1-a}, \tfrac{q-\alpha\beta}{a-1}; \alpha, \beta, \delta, \epsilon; \tfrac{x-1}{a-1}\right),$$
$$M_{101} H(x) = H\left(\tfrac{a-1}{a}, \tfrac{-q+\alpha\beta a}{a}; \alpha, \beta, \epsilon, \gamma; \tfrac{a-x}{a}\right),$$
$$M_{109} H(x) = H\left(\tfrac{a}{a-1}, \tfrac{-q+\alpha\beta a}{a-1}; \alpha, \beta, \epsilon, \delta; \tfrac{a-x}{a-1}\right).$$

Hence we find, respectively, the conditions: $a \notin [1,\infty)$, $a \notin (-\infty,0]$, $a \notin [1,\infty)$ and $a \notin (-\infty,0]$. However, for each case it is necessary to reexamine the domains of convergence.

After these preliminaries, we conclude by adding that some of the Svartholm and Erdélyi solutions – the solution (21), for example – lead to solutions for the CHE by means of the limits (5). To prove this it is sufficient to divide the recurrence relations by the parameter a before performing the limits. Furthermore, we can show as well that the solutions for the CHE in the form (70) supply solutions for the DCHE (71) through the Leaver limit ($z_0 \to 0$). These are additional reasons for choosing the Svartholm and Erdélyi solutions as a starting point for further investigation.

Appendix A. Equations of the first section

Equations (8) and (9) as well as the condition $\mathsf{P}(x, y_1) = \mathsf{P}(x, y_2)$ are obtained from the general theory of integral relations [7] which is established for $w(x,y) = 1$, that is, for

$$\mathcal{H}(x) = \int_{y_1}^{y_2} \mathbb{K}(x,y) H(y) dy \tag{A1}$$

where $\mathbb{K}(x,y)$ denotes the kernel. In this case the equation for $\mathbb{K}(x,y)$ is given in terms of the operators M_x and $\bar{M}_y$, where $\bar{M}_y$ is the adjoint operator [7] corresponding to M_y, that is,

$$\begin{aligned}\bar{M}_y &= y(y-1)(y-a)\frac{\partial^2}{\partial y^2} + \left[(2-\gamma)(y-1)(y-a) + (2-\delta)y(y-a) + (2-\epsilon)y(y-1)\right]\frac{\partial}{\partial y} \\ &+ \left[4 - 2(\alpha+\beta+1) + \alpha\beta\right]y + a(\gamma+\delta-2) + \epsilon + \gamma - 2.\end{aligned}$$

By applying M_x to the integral (A1)

and supposing that the integration endpoints are independent of x, we find

$$M_x \mathcal{H}(x) = \int_{y_1}^{y_2} H(y)\left[M_x - \bar{M}_y\right]\mathbb{K}(x,y)dy + \int_{y_1}^{y_2} H(y)\bar{M}_y \mathbb{K}(x,y)dy.$$

Then, by requiring that the kernel satisfies the partial differential equation

$$\left[M_x - \bar{M}_y\right]\mathbb{K}(x,y) = 0, \tag{A2}$$

the right side of of the previous integral reduces to $\int H(y)\bar{M}_y\mathbb{K}(x,y)dy$. Using the Lagrange identity

$$H(y)\bar{M}_y\mathbb{K}(x,y)=\mathbb{K}(x,y)M_yH(y)+\frac{\partial}{\partial y}\mathsf{P}(x,y)\overset{(3)}{=}q\mathbb{K}(x,y)H(y)+\frac{\partial}{\partial y}\mathsf{P}(x,y),$$

where now $\mathsf{P}(x,y)$ is given by

$$\begin{aligned}\mathsf{P}(x,y)&=y(y-1)(y-a)\left[H(y)\frac{\partial\mathbb{K}(x,y)}{\partial y}-\mathbb{K}(x,y)\frac{dH(y)}{dy}\right]\\&+\Big[(1-\gamma)(y-1)(y-a)+(1-\delta)y(y-a)+(1-\epsilon)y(y-1)\Big]H(y)\mathbb{K}(x,y),\end{aligned}\quad\text{(A3)}$$

then we find that the integral yields

$$M_x\mathcal{H}(x)=\int_{y_1}^{y_2}\left[q\mathbb{K}(x,y)H(y)+\frac{\partial\mathsf{P}(x,y)}{\partial y}\right]dy\overset{(A1)}{\Longleftrightarrow}[M_x-q]\mathcal{H}(x)=\mathsf{P}(x,y)\Big|_{y=y_1}^{y=y_2}.\quad\text{(A4)}$$

Therefore, $\mathcal{H}(x)$ will be a solution of the Heun equation if $\mathbb{K}$ is solution of (A2), if the integral (A1) exists and the limits of integration are so chosen that $\mathsf{P}(x,y_1)=\mathsf{P}(x,y_2)$. Further, by setting $\mathbb{K}(x,y)=w(x,y)\mathsf{G}(x,y)$, where $w(x,y)$ is defined in Eq. (7), we recover equations (8) and (9). Notice that Eq. (A2) is inadequate to deal with the kernel transformations because the operators M_x and $\bar{M}_y$ do not present the same functional form.

Appendix B. Möbius transformations for kernels

As in the index transformations, firstly we find the Möbius transformations M_i for $H(x)$ in Maier's table and, then, write the kernel transformation K_i in accordance with the rule (24). Thus, the 24 expressions for the K_i of matrix (26) are the ones given below.

$$K_1\mathsf{G}(x,y)=\mathsf{G}(x,y)=\mathsf{G}\left[a;\alpha,\beta,\gamma,\delta;x,y\right],\qquad\text{[Identity]}.$$

$$K_2\mathsf{G}(x,y)=[(1-x)(1-y)]^{-\alpha}\,\mathsf{G}\left[\frac{a}{a-1};\alpha,1+\alpha-\delta,\gamma,1+\alpha-\beta;\frac{x}{x-1},\frac{y}{y-1}\right].$$

$$K_3\mathsf{G}(x,y)=\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\mathsf{G}\left[\frac{1}{1-a};\alpha,\gamma+\delta-\beta,\gamma,1+\alpha-\beta;\frac{x}{x-a},\frac{y}{y-a}\right].$$

$$K_4\mathsf{G}(x,y)=\mathsf{G}\left[1-a;\alpha,\beta,\delta,\gamma;1-x,1-y\right].$$

$$K_5\mathsf{G}(x,y)=\left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\mathsf{G}\left[\frac{1}{a};\alpha,\gamma+\delta-\beta,\delta,1+\alpha-\beta;\frac{x-1}{x-a},\frac{y-1}{y-a}\right].$$

$$K_6\mathsf{G}(x,y)=\mathsf{G}\left[\frac{a-1}{a};\alpha,\beta,\epsilon,\gamma;\frac{a-x}{a},\frac{a-y}{a}\right].$$

$$K_7\mathsf{G}(x,y)=(xy)^{-\alpha}\,\mathsf{G}\left[\frac{1}{a};\alpha,1+\alpha-\gamma,1+\alpha-\beta,\delta;\frac{1}{x},\frac{1}{y}\right].$$

$$K_8\mathsf{G}(x,y)=(xy)^{-\alpha}\,\mathsf{G}\left[\frac{a-1}{a};\alpha,1+\alpha-\gamma,\delta,1+\alpha-\beta;\frac{x-1}{x},\frac{y-1}{y}\right].$$

$$K_9\mathsf{G}(x,y)=(xy)^{-\alpha}\,\mathsf{G}\left[1-a;\alpha,1+\alpha-\gamma,\epsilon,1+\alpha-\beta;\frac{x-a}{x},\frac{y-a}{y}\right].$$

$$K_{10}\mathsf{G}(x,y)=[(1-x)(1-y)]^{-\alpha}\,\mathsf{G}\left[\frac{1}{1-a};\alpha,1+\alpha-\delta,1+\alpha-\beta,\gamma;\frac{1}{1-x},\frac{1}{1-y}\right].$$

$$K_{11}\mathsf{G}(x,y) = [(1-x)\,(1-y)]^{-\alpha}\mathsf{G}\left[a;\alpha,1+\alpha-\delta,\epsilon,1+\alpha-\beta;\frac{x-a}{x-1},\frac{y-a}{y-1}\right].$$

$$K_{12}\mathsf{G}(x,y) = \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\mathsf{G}\left[\frac{a}{a-1};\alpha,\gamma+\delta-\beta,1+\alpha-\beta,\gamma;\frac{a}{a-x},\frac{a}{a-y}\right].$$

$$K_{13}\mathsf{G}(x,y) = \mathsf{G}\left[\frac{1}{a};\alpha,\beta,\gamma,\epsilon;\frac{x}{a},\frac{y}{a}\right].$$

$$K_{14}\mathsf{G}(x,y) = [(1-x)(1-y)]^{-\alpha}\mathsf{G}\left[\frac{a-1}{a};\alpha,1+\alpha-\delta,\gamma,\epsilon;\frac{(a-1)x}{a(x-1)},\frac{(a-1)y}{a(y-1)}\right].$$

$$K_{15}\mathsf{G}(x,y) = \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\mathsf{G}\left[1-a;\alpha,\gamma+\delta-\beta,\gamma,\delta;\frac{(1-a)x}{x-a},\frac{(1-a)y}{y-a}\right].$$

$$K_{16}\mathsf{G}(x,y) = \mathsf{G}\left[\frac{1}{1-a};\alpha,\beta,\delta,\epsilon;\frac{1-x}{1-a},\frac{1-y}{1-a}\right].$$

$$K_{17}\mathsf{G}(x,y) = \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\mathsf{G}\left[a;\alpha,\gamma+\delta-\beta,\delta,\gamma;\frac{a(x-1)}{x-a},\frac{a(y-1)}{y-a}\right].$$

$$K_{18}\mathsf{G}(x,y) = \mathsf{G}\left[\frac{a}{a-1};\alpha,\beta,\epsilon,\delta;\frac{a-x}{a-1},\frac{a-y}{a-1}\right].$$

$$K_{19}\mathsf{G}(x,y) = (xy)^{-\alpha}\mathsf{G}\left[a;\alpha,1+\alpha-\gamma,1+\alpha-\beta,\epsilon;\frac{a}{x},\frac{a}{y}\right].$$

$$K_{20}\mathsf{G}(x,y) = (xy)^{-\alpha}\mathsf{G}\left[\frac{a}{a-1};\alpha,1+\alpha-\gamma,\delta,\epsilon;\frac{a(x-1)}{(a-1)x},\frac{a(y-1)}{(a-1)y}\right].$$

$$K_{21}\mathsf{G}(x,y) = (xy)^{-\alpha}\mathsf{G}\left[\frac{1}{1-a};\alpha,1+\alpha-\gamma,\epsilon,\delta;\frac{x-a}{(1-a)x},\frac{y-a}{(1-a)y}\right].$$

$$K_{22}\mathsf{G}(x,y) = [(1-x)(1-y)]^{-\alpha}\mathsf{G}\left[1-a;\alpha,1+\alpha-\delta,1+\alpha-\beta,\epsilon;\frac{1-a}{1-x},\frac{1-a}{1-y}\right].$$

$$K_{23}\mathsf{G}(x,y) = [(1-x)\,(1-y)]^{-\alpha}\mathsf{G}\left[\frac{1}{a};\alpha,1+\alpha-\delta,\epsilon,\gamma;\frac{x-a}{a(x-1)},\frac{y-a}{a(y-1)}\right].$$

$$K_{24}\mathsf{G}(x,y) = \left[\left(1-\frac{x}{a}\right)\left(1-\frac{y}{a}\right)\right]^{-\alpha}\mathsf{G}\left[\frac{a-1}{a};\alpha,\gamma+\delta-\beta,1+\alpha-\beta,\delta;\frac{1-a}{x-a},\frac{1-a}{y-a}\right].$$

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