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Second Hopf map and supersymmetric mechanics with Yang monopole

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Abstract

We propose to use the second Hopf map for the reduction (via $SU(2)$ group action) of the eight-dimensional $\mathcal{N} = 8$ supersymmetric mechanics to five-dimensional supersymmetric systems specified by the presence of an $SU(2)$ Yang monopole. For our purpose we develop the relevant Lagrangian reduction procedure. The reduced system is characterized by its invariance under the $\mathcal{N} = 5$ or $\mathcal{N} = 4$ supersymmetry generators (with or without an additional conserved BRST charge operator) which commute with the $su(2)$ generators.

Key-words: Yang monopoles, supersymmetric quantum mechanics.

1 Introduction

Recently, in a serie of papers, new non-linear one-dimensional supermultiplets have been suggested [1, 2, 3]. They were used to construct new models of two- and three-dimensional $\mathcal{N} = 4$ supersymmetric mechanics. An important peculiarity of these models is the appearance of external magnetic fields preserving the supersymmetry of the system [4]. The models with nonlinear supermultiplets contain, as particular cases, important systems like the $\mathcal{N} = 4$ supersymmetric Landau model [2] and the $\mathcal{N} = 4$ supersymmetric multi-center MICZ-Kepler systems, both conventional [5] and spherical [6][1]. Some unexpected phenomenon has been observed: it was found that in the two-dimensional case the nonlinear (chiral) supermultiplet provides a wide freedom in the construction of supersymmetric extensions of given bosonic systems, parameterized by an arbitrary holomorphic function ("$\lambda(z)$-freedom")[7].

It was shown in [10] and [11] that all linear one-dimensional $\mathcal{N} = 4$ multiplets are related and can be derived from the so-called $\mathcal{N} = 4$ "root multiplet" or "minimal length multiplet" (i.e. the multiplet possessing no bosonic auxiliary degrees of freedom). Developing this idea, an important step in understanding the nature of nonlinear multiplets has been done in [12]; there, the nonlinear multiplets were extracted from the $\mathcal{N} = 4$ "root multiplet" by some (Lagrangian) reduction procedure. Notice that the nonlinear chiral multiplet used in the construction of two-dimensional supersymmetric mechanics possesses the $(2, 4, 2)$ components content[2], while the three-dimensional systems are built with a multiplet possessing $(3, 4, 1)$ components content. The minimal length multiplet from which nonlinear multiplets are obtained possesses a $(4, 4, 0)$ fields content. Looking at the construction of [12] one can observe that the reduction is related with the first Hopf map $S^3/S^1 = S^2$ and with, respectively, the Kustaanheimo-Stiefel transformation [13]. The relation of the mentioned procedures with the Hopf map becomes especially transparent after their reformulation in the Hamiltonian language [14]. It is therefore not surprising that the reduced three-dimensional system is specified by the presence of a Dirac monopole field, while the two-dimensional one is specified by the presence of a constant electric field. We further notice that the performed reductions do not change the number of fermionic degrees of freedom, i.e. they are straightforward extensions of the purely bosonic reduction procedures of supersymmetric systems. The linear chiral multiplets correspond to the zero value of the $S^1 = U(1)$ symmetry generator. Fixing a non-zero value for this generator gives a system with magnetic monopole; it corresponds precisely to the system obtained in terms of the non-linear chiral supermultiplet.

Different supersymmetric extensions (for various values of $\mathcal{N}$) admit unique minimal length multiplets with a given number of bosonic and fermionic degrees of freedom. The relevant cases here are, for $\mathcal{N} = 2$, the $(2, 2, 0)$ root

[1]The MICZ-Kepler system is the generalization of the Kepler system specified by the presence of a Dirac monopole and inherits the hidden symmetry of the Kepler system. It was invented independently by Zwanziger and by McIntosh and Cisneros in Refs. [8, 9].

[2]We follow the nowadays standard convention in the literature of denoting with $(k, n, n - k)$ the supermultiplets with k physical bosons, n physical fermions and $n - k$ auxiliary bosons.

supermultiplet, for $\mathcal{N} = 8$ the $(8, 8, 0)$ supermultiplet [11]. There is no doubt that the first supermultiplet can be related with the zero-th Hopf map $S^1/S^0 = S^1$, while the latter is related with the second Hopf map $S^7/S^3 = S^4$. Since $S^0 = \mathbf{Z}_2$, the reduction associated with the zero-th Hopf map does not change the number of physical degrees of freedom; at the classical level it corresponds to a plain coordinate transformation even if, at the quantum-mechanical level, it yields the presence of magnetic fluxes generating spin 1/2 [15]. Looking at the number of components of the $(8, 8, 0)$ multiplet, one could naively expect the existing $(4, 8, 4)$ and $(5, 8, 3)$ $\mathcal{N} = 8$ linear multiplets being obtained from $(8, 8, 0)$ via a second Hopf map reduction. On the other hand, the systems produced by these linear multiplets do not contain any external gauge field.

It is more likely that a second Hopf map reduction applied to the system with $(8, 8, 0)$ multiplet (the Hamiltonian reduction is assumed via the action of the $S^3 = SU(2)$ group), would produce a five-dimensional supersymmetric mechanics model with Yang monopole and (upon a further fixation of the radius) a four-dimensional supersymmetric mechanics system with BPST instanton. Indeed, when involving only the bosonic part of the system, the $SU(2)$ reduction produces a five-dimensional model in the presence of a Yang monopole; in [16], such reduction was used for constructing the five-dimensional MICZ-Kepler system ($SU(2)$-Kepler system) from an eight-dimensional system.

The construction of the supersymmetric extensions is clearly an important task. To our knowledge, no $\mathcal{N} \geq 4$ supersymmetric mechanical model with a non-Abelian gauge field has been realized. As mentioned before, systems of this type are important not only from a purely field-theoretical context, but also in applications to condensed matter, e.g. in the theory of the four-dimensional Hall effect (which is formulated on the ground of a four-dimensional Landau problem, namely a particle on a four dimensional sphere moving in the presence of a BPST instanton field generated by the Yang monopole located at the center of the sphere) [17]. Therefore, with the supersymmetric four-dimensional Landau problem at hand, one can develop the theory of the four-dimensional quantum Hall effect, in the spirit of [18].

However, the extension of the reduction procedure of the $(4, 4, 0)$ multiplet to the $(8, 8, 0)$ (which supposes the transition from the first Hopf map to the second one) and the construction of the associated nonlinear supermultiplets, is not a trivial task. In contrast with the reduction of $(4, 4, 0)$ by the $U(1)$ group action, the $(8, 8, 0)$ multiplet must be reduced by the non-Abelian $SU(2)$ group action. Such a reduction implies the "elimination" of the three external bosonic degrees of freedom only in a limiting case (when the values of $SU(2)$ generators are equal to zero). In a general position part of the initial degrees of freedom results in internal degrees of freedom of the isospin particle interacting with a Yang monopole. In the "supermultiplet language" this means that the auxiliary fields of the resulting nonlinear supermultiplet should contain some "emergent dynamics"; indeed, they are not "auxiliary" in a strong sense. Some other points need to be clarified: performing the reduction of the $(4, 4, 0)$ multiplet to the nonlinear ones, the authors of [12] added to the initial system, by hands, a Fayet-Iliopoulos extra-term. It has the two aims of providing the final system with a nonlinearity property and with the presence of an external magnetic field. Naively, it would seem that the relation of the mentioned supermultiplets is not so straightforward. From the above construction it is not clear which sort of Fayet-Iliopoulos term should be added to the system with $(8, 8, 0)$ multiplet for producing a lower-dimensional system with Yang monopole. Finally, one can suppose, from group-theoretical considerations, that it would not be possible to reduce all initial $\mathcal{N} = 8$ supersymmetries to low dimensions.

The goal of the present paper is to clarify the listed questions and, consequently, develop the necessary tools for the reduction of the $\mathcal{N} = 8$ supersymmetric mechanics with $(8, 8, 0)$ to five (four)-dimensional mechanics in presence of Yang monopoles (BPST instantons) which possess the extended supersymmetry.

For this purpose we formulate at first the *Lagrangian reduction* procedures associated with the first and second Hopf maps. We show, that there is no need to add the Fayet-Iliopoulos-like term to the initial system: the full time-derivative term arises naturally within a *consistent Lagrangian reduction procedure*. Also, we propose a geometric construction of the transmutation of the "seemingly auxiliary" degrees of freedom in isospin degrees of freedom. Let us mention that we formulate the reduction associated with the second Hopf map by using the quaternionic language.

The simpler case related with the first Hopf map can be easily recovered by the obvious replacement of the quaternionic quantities with complex numbers. An algebraic understanding of the nature of the Hopf maps leaves to no surprise that important differences are encountered between the first and the second Hopf map. We consider the consequences of these reductions for supersymmetric mechanics. For the first Hopf map, the whole $\mathcal{N} = 4$ supersymmetry algebra of the initial four-dimensional supersymmetric mechanics can be reduced to three and two dimensions. Hence, upon an appropriate reduction, we obtain $\mathcal{N} = 4$ supersymmetric two and three dimensional systems with a Dirac monopole magnetic field. The reason lies in the fact that the initial $\mathcal{N} = 4$ superalgebra commutes with the generator of the $S^1 = U(1)$ symmetry (the defining bundle in the first Hopf map), by whose action the reduction is performed. In the second Hopf map one must reduce the $\mathcal{N}=8$ supersymmetric mechanics constructed with the $(8,8,0)$ supermultiplet in terms of the action of the $SU(2)=S^3$ group (the defining bundle in the second Hopf map). These generators do not commute with the whole set of $\mathcal{N}=8$ supersymmetry algebra, but at most with its $\mathcal{N}=5$ subalgebra. The reduced system, in the presence of a Yang monopole, is fully characterized by its invariance under the $\mathcal{N}=5$ $SU(2)$-invariant supersymmetry generators. It is even possible, under some condition on the initial eight-dimensional system, to combine the fifth supersymmetry generator with a conserved pseudosupersymmetry operator and produce a reduced $\mathcal{N}=4$ supersymmetric quantum mechanical model and an additional odd nilpotent (BRST-type) symmetry. We restrict ourselves to the presentation of the general procedure and the listed statements, postponing a detailed analysis for forthcoming publications.

The paper is arranged as follows.

In the Second Section we present an explicit description of the first and second Hopf maps in terms needed for our purposes.

In the Third section we employ the Hopf maps to reduce the four-/eight-dimensional bosonic systems to lower dimensional systems with magnetic/SU(2) monopoles.

In the Fourth Section we apply these reduction procedures to the supersymmetric mechanics constructed in terms of, respectively, the $(4,4,0)$ and $(8,8,0)$ minimal length supermultiplets and discuss the associated resulting supermultiplets of the reduced systems.

2 Hopf maps

The Hopf maps (or Hopf fibrations) are the fibrations of the sphere over a sphere, $S^{2p-1}/S^{p-1}=S^p$, $p=1,2,4,8$. These fibrations reflect the existence of real ($p=1$), complex ($p=2$), quaternionic ($p=4$) and octonionic ($p=8$) numbers.

We are interested in the so-called first and second Hopf maps:

$$S^3/S^1=S^2 \quad \text{(first Hopf map)}, \qquad S^7/S^3=S^4 \quad \text{(second Hopf map)}. \tag{2.1}$$

Let us describe them in explicit terms. For this purpose, we consider the functions $\mathbf{x}(u_\alpha,\bar{u}_\alpha), x_{p+1}(u_\alpha,\bar{u}_\alpha)$

$$\mathbf{x}=2\bar{\mathbf{u}}_1\mathbf{u}_2, \quad x_{p+1}=\bar{\mathbf{u}}_1\mathbf{u}_1-\bar{\mathbf{u}}_2\mathbf{u}_2, \tag{2.2}$$

where $\mathbf{u}_1,\mathbf{u}_2$ are complex numbers for $p=2$ case (first Hopf map) and quaternionic numbers for the $p=4$ case (second Hopf map). One can consider them as coordinates of the $2p$-dimensional space $\mathrm{I\!R}^{2\mathrm{p}}$ ($p=2$ for $\mathbf{u}_{1,2}$ complex numbers; $p=4$ for $\mathbf{u}_{1,2}$ quaternionic numbers). In all cases x_{p+1} is a real number while $\mathbf{x}$ is, respectively, a complex number ($p=2$) or a quaternionic one ($p=4$),

$$\mathbf{x}\equiv x_p+\sum_{k=1,\ldots,p-1}\mathbf{e}_k x_k, \tag{2.3}$$

where $\mathbf{e}_k=\mathbf{i}$, $\mathbf{i}^2=-1$ for $p=2$, and $\mathbf{e}_k=(\mathbf{i},\mathbf{j},\mathbf{k})$, $\mathbf{e}_i\mathbf{e}_j=-\delta_{ij}+\varepsilon_{ijk}\mathbf{e}_k$ for $p=4$.

Hence, $(x_{p+1},\mathbf{x})$ parameterize the $(p+1)$-dimensional space $\mathrm{I\!R}^{\mathrm{p}+1}$.

The functions $\mathbf{x},x_{p+1}$ remain invariant under the transformations

$$\mathbf{u}_\alpha\to\mathbf{G}\mathbf{u}_\alpha, \quad \text{where} \quad \bar{\mathbf{G}}\mathbf{G}=1\Rightarrow\left\{\begin{array}{lll}\mathbf{G}=\lambda_1+\mathbf{i}\lambda_2 & |\lambda_1|^2+|\lambda_2|^2=1 & \text{for} \quad p=2\\ \mathbf{G}=\lambda_1+\mathbf{i}\lambda_2+\mathbf{j}\lambda_3+\mathbf{k}\lambda_4 & |\lambda_1|^2+\ldots+|\lambda_4|^2=1 & \text{for} \quad p=4.\end{array}\right. \tag{2.4}$$

Therefore, $\mathbf{G}$ parameterizes the spheres S^{p-1} of unit radius. Taking into account the isomorphism between these spheres and the groups, $S^1=U(1)$, $S^3=SU(2)$, we get that (2.2) is invariant under $G-$group transformations (where $G=U(1)$ for $p=2$, and $G=SU(2)$ for $p=4$), and that it defines the fibrations

$$\mathrm{I\!R}^4/\mathrm{S}^1=\mathrm{I\!R}^3, \quad \mathrm{I\!R}^8/\mathrm{S}^3=\mathrm{I\!R}^5. \tag{2.5}$$

One could immediately check that the following equation holds:

$$r^2\equiv\bar{\mathbf{x}}\mathbf{x}+x_{p+1}^2=(\bar{\mathbf{u}}_1\mathbf{u}_1+\bar{\mathbf{u}}_2\mathbf{u}_2)^2\equiv R^4. \tag{2.6}$$

Thus, defining the $(2p-1)$-dimensional sphere in $\mathbb{R}^{2p}$ of radius R, $\bar{\mathbf{u}}_\alpha \mathbf{u}_\alpha = R^2$, we will get the p-dimensional sphere in $\mathbb{R}^{p+1}$ with radius $r = R^2$, i.e. we obtain the Hopf maps (2.1).

The expressions (2.2) can be easily inverted by the use of

$$\mathbf{u}_\alpha = \mathbf{g} r_\alpha, \qquad \text{where} \quad r_1 = \sqrt{\frac{r + x_{p+1}}{2}}, \quad r_2 \equiv r_+ = \frac{\mathbf{x}}{\sqrt{2(r + x_{p+1})}}, \quad , \quad \bar{\mathbf{g}}\mathbf{g} = 1. \tag{2.7}$$

It follows from the last equation in (2.7) that $\mathbf{g}$ parameterizes the $(p-1)$-dimensional sphere of unit radius. Let us give the description of first and second Hopf maps in internal terms, using the decomposition $\mathbb{R}^{2p} = \mathbb{R}^1 \times S^{2p-1}$, $\mathbb{R}^{p+1} = \mathbb{R}^1 \times S^p$, and paramerizing S^p by inhomogeneous projective coordinates

$$z = \frac{\bar{\mathbf{u}}_1 \mathbf{u}_2}{\bar{\mathbf{u}}_1 \mathbf{u}_1}, \quad \Rightarrow \quad |\mathbf{u}_1|^2 = \frac{r}{1 + \bar{z}z}. \tag{2.8}$$

Hence, we get

$$\mathbf{u}_1 = \frac{\mathbf{g}\sqrt{r}}{\sqrt{1 + \bar{z}z}}, \qquad \mathbf{u}_2 = \mathbf{u}_1 z = \frac{\mathbf{g}\sqrt{r}z}{\sqrt{1 + \bar{z}z}} \tag{2.9}$$

For $r = \mathrm{const}$ we get the description of S^{2p-1} in terms of the coordinates of the base manifold S^p and of the fiber coordinates $\mathbf{g}$. The internal coordinate z of the sphere S^p is related with the Cartesian coordinates of the ambient space $\mathbb{R}^{p+1}$ (2.2) as follows

$$\mathbf{x} = r\mathbf{h}_+, \quad x_{p+1} = r h_{p+1}, \qquad \mathbf{h}_+ = \frac{2z}{1 + \bar{z}z}, \quad h_{p+1} = \frac{1 - \bar{z}z}{1 + \bar{z}z}. \tag{2.10}$$

For S^1 the group element and the corresponding left-invariant one-form can be presented as follows

$$S^1 \ : \quad \mathbf{g} = \mathrm{e}^{\mathbf{i}\varphi}, \quad \bar{\mathbf{g}} d\mathbf{g} = \mathbf{i} d\varphi, \quad \varphi \in [0, 2\pi) \tag{2.11}$$

Hence, the ambient coordinates of the S^3 sphere of unit radius are related with the internal coordinates of S^1 and S^2 by (2.9), where we put $r = 1$ and $\mathbf{g} = \mathrm{e}^{\mathbf{i}\varphi}$.

In quaternionic case we get the following expressions for the $SU(2)$ group element and its left-invariant form

$$S^3 \ : \quad \mathbf{g} = \mathrm{e}^{\mathbf{i}\gamma} \frac{1 + \mathbf{j}z}{\sqrt{1 + z\bar{z}}}, \quad \bar{\mathbf{g}} d\mathbf{g} = \Lambda_3 \mathbf{i} + \Lambda_+ \mathbf{j}, \qquad \Lambda_+ = (\Lambda_2 + \mathbf{i}\Lambda_1), \tag{2.12}$$

where

$$\Lambda_3 = h_3 d\gamma + \frac{\mathbf{i}}{2} \frac{\bar{z}dz - zd\bar{z}}{1 + z\bar{z}} \qquad \Lambda_+ = \mathbf{i}\mathbf{h}_+ d\gamma + \frac{d\bar{z}}{1 + z\bar{z}} \qquad i, j, k = 1, 2, 3. \tag{2.13}$$

Here $h_3, \mathbf{h}_\pm$ are the Euclidean coordinates of the ambient space $\mathbb{R}^3$ given by (2.10): simultaneously they play the role of Killing potentials of the Kähler structure on S^2.

The vector fields dual to the above one-forms look as follows

$$\mathbf{V}_3 = \frac{\partial}{\partial \gamma} + 2\mathbf{i}\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right), \qquad \mathbf{V}_+ = \frac{\partial}{\partial \bar{z}} + z^2 \frac{\partial}{\partial z} - \mathbf{i}\frac{z}{2}\frac{\partial}{\partial \gamma}, \qquad \mathbf{V}_- = \overline{\mathbf{V}}_+ \, : \tag{2.14}$$

$$\Lambda_3(\mathbf{V}_3) = \Lambda_\pm(\mathbf{V}_\pm) = 1, \qquad \Lambda_\pm(\mathbf{V}_\mp) = \Lambda_\pm(\mathbf{V}_3) = \Lambda_3(\mathbf{V}_\pm) = 0. \tag{2.15}$$

Let us also write down the following expressions

$$-\left(\bar{\mathbf{g}} d\mathbf{g}\right)^2 = \Lambda_i \Lambda_i = \left(d\gamma - \frac{\mathbf{i}}{2}\frac{\bar{z}dz - zd\bar{z}}{1 + z\bar{z}}\right)^2 + \frac{dzd\bar{z}}{(1 + z\bar{z})^2}. \tag{2.16}$$

We also need another $SU(2)$ group element parameterizing the sphere S^3 and "commuting" with (2.12):

$$\widetilde{\mathbf{g}} = \frac{1 + \mathbf{j}z}{\sqrt{1 + z\bar{z}}} \mathrm{e}^{-\mathbf{i}\gamma}, \qquad \widetilde{\mathbf{g}}\bar{\widetilde{\mathbf{g}}}\mathbf{g}\widetilde{\mathbf{g}} = 1. \tag{2.17}$$

The corresponding left-invariant forms are given by the expressions

$$\bar{\tilde{\mathbf{g}}}d\tilde{\mathbf{g}} = \widetilde{\Lambda}_3\mathbf{i} + \widetilde{\Lambda}_+\mathbf{j}, \qquad \widetilde{\Lambda}_+ = \widetilde{\Lambda}_2 + \mathbf{i}\widetilde{\Lambda}_1, \qquad \widetilde{\Lambda}_3 = d\gamma + \frac{\mathbf{i}}{2}\frac{zd\bar{z} - \bar{z}dz}{1+z\bar{z}} \qquad \widetilde{\Lambda}_+ = \frac{\mathrm{e}^{2\mathbf{i}\gamma}d\bar{z}}{1+z\bar{z}}, \tag{2.18}$$

while the vector fields dual to these forms look as follows:

$$\mathbf{U}_3 = -\frac{\partial}{\partial\gamma}, \qquad \mathbf{U}_+ = \mathrm{e}^{-2\mathbf{i}\gamma}\left((1+z\bar{z})\frac{\partial}{\partial\bar{z}} + \frac{\mathbf{i}z}{2}\frac{\partial}{\partial\gamma}\right), \qquad \mathbf{U}_- = \overline{\mathbf{U}}_+ : \tag{2.19}$$

$$\widetilde{\Lambda}_3(\mathbf{U}_3) = \widetilde{\Lambda}_\pm(\mathbf{U}_\pm) = 1, \qquad \widetilde{\Lambda}_\pm(\mathbf{U}_\mp) = \widetilde{\Lambda}_\pm(\mathbf{U}_3) = \widetilde{\Lambda}_3(\mathbf{U}_\pm) = 0. \tag{2.20}$$

From the second expression in (2.18) follows the commutativity of the $\mathbf{V}_a$ and $\mathbf{U}_a$ fields. This pair forms the the $so(4) = so(3)\times so(3)$ algebra of isometries of the S^3 sphere.

$$[\mathbf{V}_i, \mathbf{V}_j] = 2\varepsilon_{ijk}\mathbf{V}_k, \qquad [\mathbf{U}_i, \mathbf{U}_j] = 2\varepsilon_{ijk}\mathbf{U}_k, \qquad [\mathbf{V}_i, \mathbf{U}_j] = 0, \qquad i,j,k = 1,2,3. \tag{2.21}$$

The commutativity of $\mathbf{V}_i$ and $\mathbf{U}_i$ plays a key role in our further considerations. Notice also that we can pass from the parametrization (2.18) to (2.12) via the $z \to \tilde{z}\mathrm{e}^{-2\mathbf{i}\tilde{\gamma}}$, $\gamma = -\tilde{\gamma}$ transformation.

For our further considerations this is all we need to know from the Hopf maps.

3 Reduction

Let us consider a free particle on the $2p$-dimensional space equipped with the G-invariant conformal flat metric. Taking into account the expressions (2.7) we can represent its Lagrangian as follows

$$\mathcal{L}_{2p} = g(\bar{\mathbf{u}}\cdot\mathbf{u})\dot{\bar{\mathbf{u}}}_\alpha\dot{\mathbf{u}}_\alpha =$$

$$= g(r_\pm, r_1)\left(\dot{r}_+\dot{r}_- + \dot{r}_1^2 - r\left(\bar{\mathbf{g}}\dot{\mathbf{g}}\mathcal{A} + \mathcal{A}\bar{\mathbf{g}}\dot{\mathbf{g}}\right) - r(\bar{\mathbf{g}}\dot{\mathbf{g}})^2\right) = g\left(\dot{r}_+\dot{r}_- + \dot{r}_1^2\right) - gr\Lambda_i A_i + gr\Lambda_i\Lambda_i, \tag{3.1}$$

Here and in the following Λ_i are defined by (2.11) for $p=2$, and by (2.13) for $p=4$, with the differentials replaced by the time derivatives, while

$$\mathcal{A} = A_i\mathbf{e}_i \equiv \frac{\dot{r}_+ r_- - r_+\dot{r}_-}{r} = \frac{\mathbf{x}\dot{\bar{\mathbf{x}}} - \dot{\mathbf{x}}\bar{\mathbf{x}}}{2r(r + x_{p+1})}. \tag{3.2}$$

We have used the identity $r_+ r_- + r_1^2 = r$ and the notations $r_- = \overline{r}_+$, $\bar{\mathbf{u}}\cdot\mathbf{u} \equiv \bar{\mathbf{u}}_\alpha\cdot\mathbf{u}_\beta$.

One can see, for the $p=2$ case (the complex numbers) that $\mathcal{A}$ defines a Dirac monopole potential

$$\mathcal{A} = \mathbf{i}A_D = \mathbf{i}\frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r(r+x_3)}. \tag{3.3}$$

In the $p=4$ case (the quaternionic numbers) A_i defines the potential of the the $SU(2)$ Yang monopole. The explicit formulae for A_i in terms of the real coordinates $x_1, \ldots, x_5$ (where $\mathbf{x} = x_4 + \mathbf{e}_i x_i$, x_5) look as follows:

$$A_i = \frac{\eta^i_{ab}x_a\dot{x}_b}{r\,(r+x_5)}, \qquad \eta^i_{ab} = \delta_{ia}\delta_{4b} - \delta_{4a}\delta_{ib} - \varepsilon_{iab4},$$

where η^i_{ab} is the t'Hooft symbol, and $a,b = 1,2,3,4$.

The Lagrangian (3.1) is manifestly invariant under the $G-$group action.

In the $p=2$ case the generator of the $G = U(1)$ group is given by the vector field $\mathbf{V} = \partial/\partial\varphi$: indeed, taking into account (2.11), one can see that, for $p=2$, φ is a cyclic variable in (3.1).

In the $p=4$ case the generators of the $G = SU(2)$ group are given by the vector fields $\mathbf{U}_i$ (2.19).

By making use of the Noether constants of motion we can decrease the dimensionality of the system.

In the $p=2$ case we have a single Noether constant of motion defined by the vector field dual to the left-invariant form $\Lambda = \dot{\varphi}$; this is precisely the momentum conjugated to φ, which appears in the Lagrangian (3.1) as a cyclic variable. Hence, excluding this variable, we shall get, for $p=2$, a three-dimensional system.

On the other hand, in the $p = 4$ case, thanks to the non-Abelian nature of the $G = SU(2)$ group, only the γ variable is a cyclic one, even if $z, \bar{z}$ appear in the Lagrangian (3.1) without time-derivatives too. It is therefore expected that in this second case the reduction procedure would be more complicated. In contrast with the Hamiltonian reduction procedure, the Lagrangian reduction is a less common, or at least a less developed, procedure which deserves being done with care.

For this reason, we shall describe the Lagrangian counterparts of the Hamiltonian reduction procedures separately for both the $p = 2$ and the $p = 4$ cases.

3.1 The $U(1)$ reduction

Let us consider the reduction of the four-dimensional particle given by the Lagrangian (3.1) to a three-dimensional system. Taking into account the expression (2.11), we can re-write the Lagrangian as follows:

$$\mathcal{L} = g\left(\dot{r}_+\dot{r}_- + \dot{r}_1^2 - r\dot{\varphi}\mathcal{A}_D + r\dot{\varphi}^2\right). \tag{3.4}$$

Since φ is a cyclic variable, its conjugated momentum is a conserved quantity

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -rg\mathcal{A}_D + 2gr\dot{\varphi} \quad \Leftrightarrow \quad \dot{\varphi} = \frac{1}{2}\left(\frac{p_\varphi}{gr} + \mathcal{A}_D\right). \tag{3.5}$$

Naively one could expect that the reduction would require fixing the value of the Noether constant and substituting corresponding expression for $\dot{\varphi}$ in the Lagrangian (3.4). However, acting in this way, we shall get a three-dimensional Lagrangian without a linear term in the velocities, i.e. without a magnetic field (of the Dirac monopole). This is in *obvious contradiction* with the result of the Hamiltonian reduction of the four-dimensional system via the $U(1)$ group action, i.e. the suggested Lagrangian reduction procedure is incorrect! The correct Lagrangian reduction procedure should looks as follows. At first we have to replace the Lagrangian (3.4) by the following, variationally equivalent, one (obtained by performing the Legendre transformation for $\dot{\varphi}$):

$$\widetilde{\mathcal{L}} = p_\varphi\dot{\varphi} - \frac{p_\varphi}{2}\mathcal{A}_D - \frac{p_\varphi^2}{4rg} - \frac{gr}{4}A_D^2 + g\left(\dot{r}_+\dot{r}_- + \dot{r}_1^2\right). \tag{3.6}$$

Indeed, varying the independent variable p_φ, we shall arrive to the initial Lagrangian.

The isometry of the Lagrangian (3.6), corresponding to the $U(1)$-generator $\mathbf{V} = \frac{\partial}{\partial\varphi}$, is given by the same vector field. It defines the Noether constant of motion p_φ.

Upon fixing the value of the Noether constant

$$p_\varphi = 2s, \tag{3.7}$$

the first term of the new Lagrangian transforms as a full time derivative and can therefore be ignored.

As a result, we shall get the following three-dimensional Lagrangian

$$\mathcal{L}_3 = g\left(\dot{r}_+\dot{r}_- + \dot{r}_1^2\right) - s\mathcal{A}_D - \frac{gr}{4}\mathcal{A}_D^2 - \frac{s^2}{rg} = \frac{\widetilde{g}\dot{x}_\mu\dot{x}_\mu}{2} - s\mathcal{A}_D - \frac{s^2}{2r^2\widetilde{g}}, \qquad \widetilde{g} \equiv \frac{g}{2r}. \qquad \mu = 1, 2, 3. \tag{3.8}$$

Clearly, it describes the motion of a particle moving in a three-dimensional space equipped by the metric $\widetilde{g}_{\mu\nu} = \frac{g}{2r}\delta_{\mu\nu}$ in presence of a Dirac monopole generating a magnetic field with strength

$$\vec{B} = \frac{s\vec{x}}{\widetilde{g}x^3}. \tag{3.9}$$

Let us notice the appearance, in the reduced system, of the specific centrifugal term $s^2/2r^2\widetilde{g}$. For spherically symmetric systems this term provides a minor modification of the solutions of the initial system (without monopole) after incorporating the Dirac monopole: at the classical level it yields only the rotation of the orbital plane to the $\arccos s/J$ angle [19] and, at the quantum level, the shift of the validity range of the orbital momentum J from $[0, \infty)$ $[|s|, \infty)$ [20]. Schwinger [21] incorporated by hands, for the first time such a term in planar systems ($\widetilde{g} = 1$) with Dirac monopole.

The above construction corresponds to the bosonic part of the reduction of the four-dimensional $\mathcal{N}=4$ supersymmetric mechanics to a three-dimensional $\mathcal{N}=4$ supersymmetric mechanics considered in [12]. In that paper the authors corrected the wrong procedure, described in the beginning of this Subsection, by the addition of an ad-hoc extra-term, the specific Fayet-Iliopoulos term. As a matter of fact, the Fayet Iliopoulos term (which is a full-time derivative) corresponds to $p_\varphi \dot{\varphi}$ in (3.6) when restricted to the level surface $p_\varphi = 2s$.

A further reduction of the system to two dimensions corresponds to a system with a nonlinear chiral multiplet $(2,4,2)$, obtained by fixing the "radius" $r = const$. Since the Dirac monopole potential $\mathcal{A}_D$ does not depend on r, we shall get a two-dimensional system moving in the same magnetic field. It applies in particular to the particle on the sphere moving in a constant magnetic field (the Dirac monopole is located at the center of the sphere), i.e. the Landau problem on sphere.

Let us also mention the serie of papers [22], where the $U(1)$ reduction procedure of the supersymmetric Lagrangian mechanics has been performed by the use of a specific "gauging" procedure, which seemingly could be reduced, in the bosonic sector, to the above presented one.

3.2 The $SU(2)$ reduction

In the case of the second Hopf map we have to reduce the Lagrangian (3.1) with $p = 4$ via the action of the $SU(2)$ group expressed by the vector fields (2.19). Due to the non-Abelian nature of the $SU(2)$ group the system will be reduced to a five (or higher)-dimensional one.

For a correct reduction procedure we have to replace the initial Lagrangian by one which is variationally equivalent, extending the initial configuration space with new variables, π, $\bar{\pi}$, p_γ, playing the role of conjugate momenta to z, $\bar{z}$, γ. In other words, we will replace the sphere S^3 (parameterized by z, $\bar{z}$, γ) by its cotangent bundle T^*S^3 parameterized by the coordinate z, $\bar{z}$, γ, π, $\bar{\pi}$, p_γ. Let us further define, on T^*S^3, the Poisson brackets given by the relations

$$\{\pi, z\} = 1, \qquad \{\bar{\pi}, \bar{z}\} = 1, \qquad \{p_\gamma, \gamma\} = 1. \tag{3.10}$$

We introduce the Hamiltonian generators P_a corresponding to the vector fields (2.14) (replacing the derivatives entering the vector fields $\mathbf{V}_a$ by the half of corresponding momenta)

$$P_+ = \frac{P_2 - \mathbf{i}P_1}{2} = \frac{\pi + \bar{z}^2\bar{\pi}}{2} - \mathbf{i}\bar{z}\frac{p_\gamma}{4}, \quad P_- = \bar{P}_+, \quad P_3 = \frac{p_\gamma}{2} - \mathbf{i}\left(z\pi - \bar{z}\bar{\pi}\right). \tag{3.11}$$

In the same way we introduce the Hamiltonian generators I_a corresponding to the vector fields (2.19):

$$I_3 = -\frac{p_\gamma}{2}, \quad I_+ = \frac{I_2 - \mathbf{i}I_1}{2} = \frac{\mathbf{i}p_\gamma z + 2\bar{\pi}\left(1 + z\bar{z}\right)}{4} e^{-2\mathbf{i}\gamma}, \quad I_- = \bar{I}_+. \tag{3.12}$$

These quantities define, with respect to the Poisson bracket (3.10), the $so(4) = so(3) \times so(3)$ algebra

$$\{P_i, P_j\} = \varepsilon_{ijk}P_c, \quad \{I_i, I_j\} = \varepsilon_{ijk}I_k, \quad \{I_i, P_j\} = 0. \tag{3.13}$$

The functions P_i, I_i obey the following equality, important for our considerations

$$I_k I_k = P_k P_k. \tag{3.14}$$

At this point we replace the initial Lagrangian (3.1) by the following one, which is variationally equivalent

$$\mathcal{L}_{int} = 2\left(P_+\Lambda_+ + P_-\Lambda_- + P_3\Lambda_3\right) - P_i A_i - \frac{P_i P_i}{gr} - \frac{gr A_i A_i}{4} + g\left(\dot{r}_+\dot{r}_- + \dot{r}_1^2\right). \tag{3.15}$$

The isometries of this modified Lagrangian corresponding to (2.19) are defined by the vector fields

$$\widetilde{\mathbf{U}}_i \equiv \{I_i, \ \}, \tag{3.16}$$

where I_i are given by (3.12) and the Poisson brackets are given by (3.10). The quantities I_i entering (3.16) are the Noether constants of motion of the modified Lagrangian (3.15). This can be easily seen taking into account the following equality

$$2\left(P_+\Lambda_+ + P_-\Lambda_- + P_3\Lambda_3\right) = p_\gamma\dot{\gamma} + \pi\dot{z} + \bar{\pi}\dot{\bar{z}}. \tag{3.17}$$

We have now to perform the reduction via the action of the $SU(2)$ group given by the vector fields (3.16). For this purpose we have to fix the Noether constants of motion (3.12), setting

$$I_k = const, \qquad I_k I_k \equiv s^2.$$

Since the constants of motion I_k do not dependent on the $r_\pm, r_5$ coordinates we can perform an orthogonal rotation so that only the third component of this se, I_3, assumes a value different from zero. Equating I_+ and I_- with zero we obtain:

$$-I_3 = \frac{p_\gamma}{2} = s, \qquad \bar\pi = \mathbf{i}s\frac{z}{1+z\bar z}, \quad \pi = -\mathbf{i}s\frac{\bar z}{1+z\bar z}. \tag{3.18}$$

Hence,

$$P_+ = -\mathbf{i}s\frac{\bar z}{1+z\bar z}, \quad P_- = \mathbf{i}s\frac{z}{1+z\bar z}, \quad P_3 = -s\frac{1-z\bar z}{1+z\bar z}. \tag{3.19}$$

Therefore P_k coincide with the Killing potentials of the S^2 sphere! This is by no means an occasional coincidence.

Taking in mind the equality (3.17) we can conclude that the third term entering (3.15) can be ignored because it is a full time derivative. Besides that, taking into account (3.14), we can rewrite the Lagrangian as follows:

$$\mathcal{L}_{red} = \frac{\widetilde{g}\dot x_\mu \dot x_\mu}{2} - \mathbf{i}s\frac{\bar z\dot z - z\dot{\bar z}}{1+z\bar z} - sh_k(z,\bar z)A_k - \frac{s^2}{2r^2\widetilde{g}}, \qquad \widetilde{g} \equiv \frac{g}{2r}, \qquad \mu = 1,\dots,5, \tag{3.20}$$

where we have used the identity

$$-\frac{1}{4}grA_iA_i + g\left(\dot r_+\dot r_- + \dot r_1^2\right) = g\frac{\dot x_\mu \dot x_\mu}{4r}.$$

The second term in the above reduced Hamiltonian is the one-form defining the symplectic (and Kähler) structure on S^2, while h_k given in (2.10) are the Killing potentials defining the isometries of the Kähler structure. We have in this way obtained the Lagrangian describing the motion of a five-dimensional isospin particle in the field of an $SU(2)$ Yang monopole. The metric of the configuration space is defined by the expressions $\widetilde{g}_{\mu\nu} = \frac{g}{2r}\delta_{\mu\nu}$. For a detailed description of the dynamics of the isospin particle we refer to [23].

Similarly to the $U(1)$ case, the reduced system is specified by the presence of a centrifugal potential $s^2/2\widetilde{g}r^2$, which essentially cancels the impact of the monopole in the classical and quantum solutions of the system. Particularly, for spherically symmetric systems (including those with extra potential terms), the impact of the Yang monopole on the spectrum implies a change in the validity range of the orbital momentum [20]. In supersymmetric systems, on the other hand, the presence of a monopole can change essentially the supersymmetric properties.

It therefore follows that the Noether constants of motion do not allow us to exclude the $z, \bar z$ variables. However, their time derivatives appear in the Lagrangian in a linear way only and define the internal degrees of freedom of the five-dimensional isospin particle interacting with a Yang monopole. As a consequence, the dimensionality of the phase space of the reduced system is $2\cdot 5 + 2 = 12$. Only for the particular case $s = 0$, corresponding to the absence of the Yang monopole, we obtain a five-dimensional system. This means that *locally* the Lagrangian of the system can be formulated in a six-dimensional space. Such a representation seems, however, useless, in contrast with the one presented here.

The further reduction of the constructed $(5+\ldots)$-dimensional system to a $(4+\ldots)$-dimensional one would be completely similar to the $U(1)$ case: it requires fixing the radial variable r. The resulted system describes the isospin particle moving in a four-dimensional space and interacting with the BPST instanton.

In this Section we have considered the Lagrangian reduction procedures, restricting ourselves to $2p$-dimensional systems with *conformal flat metrics* only. From our considerations it is however clear that similar reductions can be performed also for particles moving on other G-invariant $2p$-dimensional spaces (not necessarily conformally flat), in presence of a G-invariant potential. The modifications do not yield any qualitative difference with the proposed reduction procedures and will be reflected in more complicated forms of the resulting Lagrangians. The possibility of adding to the initial system G-invariant potentials is obvious.

4 Supersymmetry

We discuss now the supersymmetric extensions, both for $p = 2$ and $p = 4$, of the bosonic constructions we have dealt so far. For our purposes we have to ensure the compatibility of the supersymmetry transformations acting on the "root" or "minimal length" supermultiplets $(2p, 2p, 0)$, with the bilinear transformations

$$x_\mu = u^T \gamma_\mu u, \tag{4.1}$$

where, for $p = 2$, $\mu = 1, 2, 3$ and the γ_μ's are the generators of the Euclidean Clifford algebra $Cl(3, 0)$ while,for $p = 4$, $\mu = 1, 2, 3, 4, 5$, the γ_μ's are the generators of the Euclidean Clifford algebra $Cl(5, 0)$.

In the $p = 2$ case we can choose

$$\gamma_1 = \mathbf{1}_2 \otimes \tau_1, \quad \gamma_2 = \mathbf{1}_2 \otimes \tau_2, \quad \gamma_3 = \tau_A \otimes \tau_A, \tag{4.2}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.3}$$

Due to the Schur's lemma [24], the three gamma matrices in (4.2) commute with a single matrix

$$\sigma_3 = \tau_A \otimes \mathbf{1}_2 \tag{4.4}$$

($\sigma_3{}^2 = \mathbf{1}_4$) which defines the complex structure in $Cl(3, 0)$.

For the $p = 5$ case the γ-matrices look as follows

$$\begin{array}{l} \gamma_1 = \tau_A \otimes \tau_1 \otimes \tau_A, \\ \gamma_2 = \tau_A \otimes \tau_2 \otimes \tau_A, \\ \gamma_3 = \tau_A \otimes \tau_A \otimes \mathbf{1}_2, \;. \\ \gamma_4 = \tau_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, \\ \gamma_5 = \tau_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, \end{array} \tag{4.5}$$

where the matrices τ_1, τ_2, τ_A are defined in (4.3).

The real coordinates u_a, $a = 1, \ldots 2p$, are related with the complex/quaternionic coordinates $\mathbf{u}_\alpha, \overline{\mathbf{u}}_\alpha$ considered in the previous Sections, by the expressions

$$\mathbf{u}_1 = u_4 + \mathbf{e}_i u_i, \qquad \mathbf{u}_2 = u_8 + \mathbf{e}_i u_{4+i}, \qquad i = 1, 2, 3. \tag{4.6}$$

The $\mathbf{V} = \partial\varphi$ vector field defining, in the $p = 2$case, the $U(1)$ isometry, therefore looks

$$\mathbf{V} = u^T \sigma_3 \frac{\partial}{\partial u}. \tag{4.7}$$

In the $p = 4$ case, the $\mathbf{U}_i$ vector fields defining the $SU(2)$ isometries are given by the expressions

$$\mathbf{U}_i = u\Sigma_i \frac{\partial}{\partial u}, \qquad \Sigma_1 = \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \quad \Sigma_2 = \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \quad \Sigma_3 = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \tau_A. \tag{4.8}$$

It is easily proven that the $su(2)$ matrix generators Σ_i commute with the Gamma-matrices Γ_μ ($[\Sigma_i, \Gamma_\mu] = 0$). This is in agreement with the fact that $\mathbf{U}_i$ define the isometries of the eight-dimensional Lagrangian (3.1).

The relation pointed out in [11] between Clifford algebra and the associated supersymmetric root multiplets has a consequence that the Schur's lemma induces real, complex or quaternionic structures, see [25] and [26], on the minimal length multiplets.

For $p = 2$, the $(4, 4, 0)$ root multiplet is an $\mathcal{N} = 4$ quaternionic multiplet, since the supersymmetry algebra

$$Q_a Q_b + Q_b Q_a = \delta_{ab} \mathbf{1} \partial_t, \qquad Q_a H - H Q_a = 0, \qquad H \equiv \mathbf{1}\partial_t, \quad a, b = 1, \ldots, \mathcal{N} = 4 \tag{4.9}$$

is realized through the supermatrices acting on the $(u_1, u_2, u_3, u_4; \psi_1, \psi_2, \psi_3, \psi_4)$ multiplet, given by

$$Q_4 = \begin{pmatrix} 0 & \mathbf{1}_4 \\ \mathbf{1}_4 \partial_t & 0 \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & \widehat{\gamma}_i \\ -\widehat{\gamma}_i \partial_t & 0 \end{pmatrix}, \qquad i = 1, 2, 3, \tag{4.10}$$

where

$$\widehat{\gamma}_1 = \tau_A \otimes \tau_1, \quad \widehat{\gamma}_2 = \tau_A \otimes \tau_2, \quad \widehat{\gamma}_3 = \mathbf{1}_2 \otimes \tau_A \tag{4.11}$$

and Q_i, Q_4 all commute with the three matrices $\widetilde{\Sigma}_j = \sigma_j \otimes \sigma_j$, $j = 1,2,3$, ($\sigma_1 = \tau_1 \otimes \tau_A$, $\sigma_2 = \tau_2 \otimes \tau_A$, while σ_3 is given by (4.4)). Notice that $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2$ (contrary to $\widetilde{\Sigma}_3$) do not leave invariant the coordinates x_1, x_2, x_3 entering, for $p = 2$, (4.1).

For $p = 4$ the situation is as follows. According to the supersymmetric extension of the Schur's lemma, [26] and [27], there are at most $\mathcal{N} = 5$ supersymmetry generators commuting with the $su(2)$ generators $\widetilde{\Sigma}_j$ (now $\widetilde{\Sigma}_j = \Sigma_j \oplus \Sigma_j$, with Σ_j given in (4.8)) and acting on the $(8,8,0)$ root multiplet[3].

The $\mathcal{N} = 8$ supersymmetry transformations acting on the root multiplet with fields $(u_a; \psi_b)$, $(a, b = 1, 2, \ldots, 8)$ are given by

$$Q_k = \begin{pmatrix} 0 & \gamma_k \\ -\gamma_k \cdot H & 0 \end{pmatrix}, \qquad Q_8 = \begin{pmatrix} 0 & \mathbf{1}_8 \\ \mathbf{1}_8 \cdot H & 0 \end{pmatrix}, \qquad k = 1, 2, \ldots, 7, \tag{4.12}$$

where

$$\begin{array}{llll} \gamma_1 = \tau_1 \otimes \tau_A \otimes \mathbf{1}_2, & \gamma_2 = \tau_2 \otimes \tau_A \otimes \mathbf{1}_2, & \gamma_3 = \tau_A \otimes \mathbf{1}_2 \otimes \tau_1, & \gamma_4 = \tau_A \otimes \mathbf{1}_2 \otimes \tau_2, \\ \gamma_5 = \mathbf{1}_2 \otimes \tau_1 \otimes \tau_A, & \gamma_6 = \mathbf{1}_2 \otimes \tau_2 \otimes \tau_A, & \gamma_7 = \tau_A \otimes \tau_A \otimes \tau_A. & \end{array} \tag{4.13}$$

The subset of $\mathcal{N} = 5$ supersymmetry transformations commuting with the above specified $su(2)$ generators $\widetilde{\Sigma}_j$,

$$[Q_I, \widetilde{\Sigma}_j] = 0, \tag{4.14}$$

is explicitly given by Q_1, Q_2, Q_5, Q_6, Q_8.

In accordance with the above results, the reduced Lagrangians, invariant under the extended supersymmetry algebra and compatible with the G-group action, where $G = U(1)$ for $p = 2$ and $G = SU(2)$ for $p = 4$, can be recast in a complex and, respectively, quaternionic formalism. We will discuss them separately in the next subsections.

4.1 The $U(1)$ reduction

We discuss the reduction of the $\mathcal{N} = 4$ supersymmetric systems with a $(4,4,0)$ supermultiplet.

The three $U(1)$-invariant fields x_1, x_2, x_3 constructed in (4.1), for $p = 2$, as bilinear combinations of the four u_i fields, transform under $\mathcal{N} = 4$ supersymmetry with transformations induced by (4.10). It is easily verified that the induced supersymmetry closes linearly and the resulting supermultiplet corresponds to the $\mathcal{N} = 4$ $(3,4,1)$ fields content where, in addition to the three x_μ, we have 4 fermions and an auxiliary bosonic field. All the fields belonging to this multiplet are $U(1)$-invariant and given by bilinear combinations of the u_i and ψ_i fields entering the original $(4,4,0)$ supermultiplet.

The commutativity of the $\mathcal{N} = 4$ supersymmetry algebra with the $U(1)$ generator makes possible to use an alternative description, more suitable in describing the $\mathcal{N} = 4$ supersymmetric quantum mechanical system in presence of a monopole. It makes use of the complex coordinates (bosonic and, respectively, fermionic) $\mathbf{u}_\alpha$, ψ_α and the "chiral supercharge" generators $Q_k^{\pm} = Q_k \pm \imath Q_{k+2}$ $(k = 1, 2)$. The supersymmetry transformations can therefore be re-expressed as

$$\begin{array}{llll} Q_1^+ \mathbf{u}_\alpha = \psi_\alpha, & Q_1^+ \psi_\alpha = \dot{\mathbf{u}}_\alpha, & Q_1^- \overline{\mathbf{u}}_\alpha = \overline{\psi}_\alpha, & Q_1^- \overline{\psi}_\alpha = \dot{\overline{\mathbf{u}}}_\alpha, \\ Q_2^+ \mathbf{u}_\alpha = \epsilon_{\alpha\beta}\psi_\beta, & Q_2^+ \psi_\alpha = \epsilon_{\alpha\beta}\dot{\mathbf{u}}_\beta, & Q_2^- \overline{\mathbf{u}}_\alpha = \epsilon_{\alpha\beta}\overline{\psi}_\beta, & Q_2^- \overline{\psi}_\alpha = \epsilon_{\alpha\beta}\dot{\overline{\mathbf{u}}}_\beta, \\ Q_k^+ \overline{\mathbf{u}}_\alpha = 0, & Q_k^+ \overline{\psi}_\alpha = 0. & & \end{array} \tag{4.15}$$

The $U(1)$ group acts on the complex variables $(\mathbf{u}_\alpha, \psi_\alpha)$ as follows

$$\mathbf{u}_\alpha \to \mathrm{e}^{\imath\kappa}\mathbf{u}_\alpha, \quad \overline{\mathbf{u}}_\alpha \to \mathrm{e}^{-\imath\kappa}\overline{\mathbf{u}}_\alpha, \quad \psi_\alpha \to \mathrm{e}^{\imath\kappa}\psi_\alpha, \quad \overline{\psi}_\alpha \to \mathrm{e}^{-\imath\kappa}\overline{\psi}_\alpha, \tag{4.16}$$

where κ is arbitrary real parameter.

By reducing the $\mathcal{N} = 4$ $(4,4,0)$ supersymmetric system above via the $U(1)$ group action we obtain a system still possessing the $\mathcal{N} = 4$ supersymmetry. This is reached by choosing, in complete analogy with the bosonic case, besides the three $U(1)$-invariant bosonic coordinates (4.1), four $U(1)$ invariant fermionic coordinates χ_α given below and an extra-bosonic field $2\varphi = \imath \log \mathbf{u}_1/\overline{\mathbf{u}}_1$. The whole set of coordinates of the reduced system are [1]

$$\mathbf{x} = 2\bar{\mathbf{u}}_1\mathbf{u}_2, \quad x_3 = \bar{\mathbf{u}}_1\mathbf{u}_1 - \bar{\mathbf{u}}_2\mathbf{u}_2, \quad \chi_\alpha = \mathrm{e}^{-\imath\phi}\psi_\alpha, \quad \overline{\chi}_\alpha = \mathrm{e}^{\imath\phi}\overline{\psi}_\alpha. \tag{4.17}$$

[3] An extra pseudosupersymmetry operator, $\widetilde{Q}$, such that $\widetilde{Q}^2 = -H$, is allowed.

The general $\mathcal{N}=4$ Lagrangian constructed with the $(4,4,0)$ supermultiplet is given by (see, e.g., [12])

$$\mathcal{L}_4^{SUSY}=\mathcal{L}_4+\frac{\imath g(u,\overline{u})}{2}\left(\overline{\psi}\cdot D_t\psi-D_t\overline{\psi}\cdot\psi\right)-\mathcal{R}(\psi\cdot\overline{\psi})(\psi\cdot\overline{\psi}),\quad D_t\psi\equiv\dot{\psi}+\Gamma\psi\dot{u}, \tag{4.18}$$

where $D\psi$ is defined by the connection of the metric $ds^2=g du\cdot d\overline{u}$, $\mathcal{R}$ is the curvature of this connection and $\mathcal{L}_4$ is the bosonic Lagrangian given in (3.1). Therefore, for a $U(1)$ invariant metric, the supersymmetric Lagrangian also possesses an $U(1)$ invariance.

When re-writing the initial system in terms of $\mathbf{r}_\pm, r_1, \chi_\alpha, \overline{\chi}_\alpha, \varphi$, we recover that φ is a cyclic variable. Excluding it, in analogy with the bosonic case, we obtain an $\mathcal{N}=4$ supersymmetric system with 3 bosonic dimensions. The presence of the fermionic degrees of freedom does not yield qualitative changes in the reduction procedure. The fields content of this system is $(3,4,1)$. On this supermultiplet the $\mathcal{N}=4$ supersymmetry is realized non-linearly. This is however a "fake" non-linearity. Indeed, it is not difficult to verify that the fields entering (4.17) can be expressed as non-linear combinations of the $U(1)$-invariant bilinear fields entering the *linear* $(3,4,1)$ supermultiplet discussed above.

Concerning the reduction, we refer to the paper [12] for its detailed description and make some extra comment. Performing the four-dimensional Lagrangian reduction the authors of [12] added "by hands" the special "Fayet-Iliopoulos term" which generates a reduced system in the presence of a Dirac monopole. This obviously artificial action was required to improve the naive and incorrect Lagrangian reduction procedure, described in Subsection 3.1. In this work we have proposed a consistent Lagrangian reduction procedure which can be implemented in the supersymmetric case as well.

4.2 The $SU(2)$ reduction

We discuss now the reductions of the $(8,8,0)$ supersymmetric multiplet via the $SU(2)$ group action. In contrast with the previous case, the $su(2)$ algebra does not commute which the whole set of the $\mathcal{N}=8$ supersymmetry generators (4.12). For that reason the reduced system cannot inherit the whole $\mathcal{N}=8$ supersymmetry, but only its $\mathcal{N}=5$ subalgebra (we recall that an explicit presentation of the supersymmetry tranformations is given by Q_1, Q_2, Q_5, Q_6, Q_8 entering (4.12)).

It is worth mentioning that there are $\mathcal{N}=6$ supersymmetry generators commuting with the $U(1)$ group action defined, e.g., by $\widetilde{\Sigma}_3$ (the extra supersymmetry generator closing $\mathcal{N}=6$ corresponds to Q_7). As a consequence, the $U(1)$ reduction of the $(8,8)$ supermultiplet produces an $\mathcal{N}=6$ supersymmetric mechanics on CP^3 in presence of a constant magnetic field. The reduction by the whole $SU(2)$ group yields further restrictions on the number of supersymmetries since at most $\mathcal{N}=5$ supersymmetry generators commute with the $su(2)$ generators which define the quaternionic structure.

In order to exploit the quaternionic properties it is convenient to redefine the $(8,8)$ variables as follows

$$\begin{array}{llllllll} u_1\to v_0, & u_2\to v_2, & u_3\to v_3, & u_4\to v_1, & u_5\to \bar{v}_0, & u_6\to \bar{v}_2, & u_7\to \bar{v}_3, & u_8\to \bar{v}_1 \\ \psi_1\to \overline{\lambda}_0, & \psi_2\to \overline{\lambda}_2, & \psi_3\to \overline{\lambda}_3, & \psi_4\to \overline{\lambda}_1, & \psi_5\to \lambda_0, & \psi_6\to \lambda_2, & \psi_7\to \lambda_3, & \psi_8\to \lambda_1. \end{array} \tag{4.19}$$

After this redefinition the $\mathcal{N}=5$ supersymmetry transformations take the following form.
The Q_i $(i=1,2,3)$ transformations are $(\epsilon_{123}=1)$:

$$\begin{array}{llll} Q_i v_0=\lambda_i, & Q_i v_j=-(\delta_{ij}\lambda_0+\epsilon_{ijk}\lambda_k), & Q_i\overline{v}_0=-\overline{\lambda}_i, & Q_i\overline{v}_j=\delta_{ij}\overline{\lambda}_0+\epsilon_{ijk}\overline{\lambda}_k, \\ Q_i\lambda_0=-\dot{v}_i, & Q_i\lambda_j=\delta_{ij}\dot{v}_0+\epsilon_{ijk}\dot{v}_k, & Q_i\overline{\lambda}_0=\dot{\overline{v}}_i, & Q_i\overline{\lambda}_j=-(\delta_{ij}\dot{\overline{v}}_0+\epsilon_{ijk}\dot{\overline{v}}_k). \end{array} \tag{4.20}$$

The Q_4 transformation is

$$\begin{array}{llll} Q_4 v_0=\lambda_0, & Q_4 v_j=\lambda_j, & Q_4\overline{v}_0=\overline{\lambda}_0, & Q_4\overline{v}_j=\overline{\lambda}_j, \\ Q_4\lambda_0=\dot{v}_0, & Q_4\lambda_j=\dot{v}_j, & Q_4\overline{\lambda}_0=\dot{\overline{v}}_0, & Q_4\overline{\lambda}_j=\dot{\overline{v}}_j. \end{array} \tag{4.21}$$

The Q_5 transformation is

$$\begin{array}{llll} Q_5 v_0=\overline{\lambda}_0, & Q_5 v_j=\overline{\lambda}_j, & Q_5\overline{v}_0=-\lambda_0, & Q_5\overline{v}_j=-\lambda_j, \\ Q_5\lambda_0=-\dot{\overline{v}}_0, & Q_5\lambda_j=-\dot{\overline{v}}_j, & Q_5\overline{\lambda}_0=\dot{v}_0, & Q_5\overline{\lambda}_j=\dot{v}_j. \end{array} \tag{4.22}$$

The $\widetilde{Q}$ pseudosupersymmetry operator ($\widetilde{Q}^2 = -H$) which commutes with the $su(2)$ generators is given by

$$\begin{array}{llll} \widetilde{Q}v_0 = \overline{\lambda}_0, & \widetilde{Q}v_j = \overline{\lambda}_j, & \widetilde{Q}\overline{v}_0 = \lambda_0, & \widetilde{Q}\overline{v}_j = \lambda_j, \\ \widetilde{Q}\lambda_0 = -\dot{\overline{v}}_0, & \widetilde{Q}\lambda_j = -\dot{\overline{v}}_j, & \widetilde{Q}\overline{\lambda}_0 = -\dot{v}_0, & \widetilde{Q}\overline{\lambda}_j = -\dot{v}_j. \end{array} \quad (4.23)$$

Notice that the pseudosupersymmetry operator $\widetilde{Q}$, together with Q_5, can be used to define a BRST-type transformation Q_{BRST} ($Q^2_{BRST} = 0$) given by $Q_{BRST} = \frac{1}{2}(Q_5 + \widetilde{Q})$, such that

$$\begin{array}{llll} Q_{BRST}v_0 = \overline{\lambda}_0, & Q_{BRST}v_j = \overline{\lambda}_j, & Q_{BRST}\overline{v}_0 = 0, & Q_{BRST}\overline{v}_j = 0, \\ Q_{BRST}\lambda_0 = -\dot{\overline{v}}_0, & Q_{BRST}\lambda_j = -\dot{\overline{v}}_j, & Q_{BRST}\overline{\lambda}_0 = 0, & Q_{BRST}\overline{\lambda}_j = 0. \end{array} \quad (4.24)$$

The BRST-operator Q_{BRST} commutes with the $su(2)$ generators and anticommutes with the remaining $\mathcal{N} = 4$ $su(2)$-invariant supercharges.

The most general $su(2)$-invariant $\mathcal{N} = 4, 5$ actions for the $(8, 8)$ multiplet can be computed with the construction of [26] (further developed in [28]). A manifestly $\mathcal{N} = 4$ invariant action is obtained from the lagrangian

$$\mathcal{L} = Q_1Q_2Q_3Q_4 f(v, \bar{v}), \quad (4.25)$$

where the supercharges $Q_1, \ldots, Q_4$ are given by (4.20), (4.21) and f is an *unconstrained* function of the bosonic coordinates $v_0, v_1, v_2, v_3, \bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3$. The explicit expression for $\mathcal{L}$, obtained with the help of a package for Maple 11 and written in terms of the quaternionic structure constants, is reported for completeness in the Appendix.

The $\mathcal{N} = 5$ invariance is obtained by a constraint, induced by the fifth $su(2)$-invariant supersymmetry transformation Q_5, which requires $Q_5\mathcal{L}$ be a total time-derivative. The $\mathcal{N} = 5$ requirement implies that f must satisfy the equation

$$\Delta_8 f \equiv f_{\mu\mu} + f_{\bar{\mu}\bar{\mu}} = 0, \quad (4.26)$$

where $\mu = 0, 1, 2, 3$ and $f_\mu \equiv \partial f/\partial v_\mu$, $f_{\bar{\mu}} \equiv \partial f/\partial \bar{v}_\mu$.

An alternative constraint is obtained by requiring both the $\mathcal{N} = 4$ invariance *and* the Q_{BRST} invariance. In this case f must satisfy

$$\Delta_4 f \equiv f_{\mu\mu} = 0. \quad (4.27)$$

In order to have an $su(2)$-invariant action, an $su(2)$-invariant constraint has to be imposed on f. This constraint can be explicitly solved by expressing f not directly in terms of v_μ, $\overline{v}_\mu$ (or $u_1, \ldots, u_8$), but through the $su(2)$-invariant "bilinear coordinates" x_μ (now $\mu = 1, 2, 3, 4, 5$) entering (4.1). We obtain as a result an $su(2)$-invariant, $\mathcal{N} = 5$ supersymmetric lagrangian for a 5-dimensional system (given by the x_μ coordinates).

In analogy with the case discussed in the previous subsection, we can compute the supermultiplet generated by the 5 $su(2)$-invariant bilinear fields x_μ. Its fields content is given [27] by $(5, 11, 10, 5, 1)$. This supermultiplet corresponds to a $(1, 5, 10, 10, 5, 1) \rightarrow (0, 5, 11, 10, 5, 1)$ dressing of the $\mathcal{N} = 5$ "enveloping multiplet" (see [26]), whose fields content is given by Netwon's binomials. All fields entering $(5, 11, 10, 5, 1)$ are $su(2)$-invariant and given by bilinear combinations of the original u_i, ψ_i fields. This multiplet contains twice as many fields entering a minimal (irreducible, in physicists' language) $\mathcal{N} = 5$ multiplet. It is a reducible, but indecomposable, multiplet which can be better described in the basis of the irreducible $(5, 8, 3, 0, 0)$ and $(0, 3, 5, 5, 1)$ (see [26]) supermultiplets. Just as in the previous case, the $\mathcal{N} = 5$ supersymmetry is realized linearly on $(5, 11, 10, 5, 1)$.

An important comment has to be made. In [28] it has been explicitly proven that requiring the $\mathcal{N} = 5$ invariance for an off-shell action based on the $(2, 8, 6)$ multiplet, automatically induces a full $\mathcal{N} = 8$ invariance. Similarly, the $\mathcal{N} = 5$ invariance constraint (4.26) for the $(8, 8, 0)$ multiplet automatically guarantees an $\mathcal{N} = 8$ invariance. This is in agreement with the result of the first paper in [3], where the same constraint was derived by requiring the whole $\mathcal{N} = 8$ invariance, and with [29], where the general superfield and component actions of this multiplet were explicitly given. It was also proven there that the 8-dimensional harmonicity condition for the Lagrangian is a necessary and sufficient condition to have an $\mathcal{N} = 8$ supersymmetry. Therefore, combining (4.26) and the $SU(2)$ constraint (expressed by the fact that f is function of the five bilinear coordinates entering (4.1)) produces an $\mathcal{N} = 8$ $SU(2)$-invariant system. On the other hand, the three extra supersymmetry generators (the ones which do not commute with the $su(2)$ algebra generators) are not essential to derive the symmetries of the action. They also close on a much larger multiplet than $(5, 11, 10, 5, 1)$, containing fields which are not $SU(2)$-invariant. We recall that the $SU(2)$ group acts on the fields entering $(5, 11, 10, 5, 1)$ as the identity operator. Furthermore, a quaternionic structure is only available for the $\mathcal{N} = 5$ subalgebra.

The supersymmetry transformations (4.20), (4.21), (4.22) preserve the quaternionic structure. We can therefore express the $\mathcal{N} = 5$ $(8, 8, 0)$ component fields in a quaternionic framework, in such a way that the $SU(2)$ group action is expressed through

$$\mathbf{u}_\alpha \to \mathbf{G}\mathbf{u}_\alpha, \Psi_\alpha \to \mathbf{G}\Psi_\alpha, \quad \text{where} \quad \mathbf{G}\bar{\mathbf{G}} = 1, \quad \mathbf{G},\ \mathbf{u}_\alpha,\ \Psi_\alpha \in \mathbb{H}, \qquad \alpha = 1, 2. \tag{4.28}$$

In this language the 5 bilinear coordinates x_μ and 8 $SU(2)$-invariant fermions can be expressed as follows

$$\mathbf{x} = 2\bar{\mathbf{u}}_1\mathbf{u}_2, \quad x_5 = \bar{\mathbf{u}}_1\mathbf{u}_1 - \bar{\mathbf{u}}_2\mathbf{u}_2, \quad \chi_\alpha = \bar{\mathbf{g}}\Psi_\alpha. \tag{4.29}$$

These positions mimic, in the $SU(2)$ reduction case, what happens in the $U(1)$ case. They suggest the existence of a supersymmetric description of a five-dimensional system with a Yang monopole realizing the $\mathcal{N} = 5$ supersymmetry non-linearly on a $(5, 8, 3)$ field content. The main difference with respect to the $U(1)$ case is the fact that the supersymmetric $SU(2)$-invariant multiplet realized with bilinear combinations of the $(8, 8, 0)$ fields contain twice as many fields as the ones entering $(5, 8, 3)$. A possible strategy consists in extracting the linear $(5, 8, 3)$ multiplet entering $(5, 11, 10, 5, 1)$ by setting equal to zero the fields entering its $(0, 3, 7, 5, 1)$ submultiplet. A non-linear transformation allows to re-express the $(5, 8, 3)$ fields entering the bilinear basis with the $(5, 8, 3)$ fields entering (4.29). This issue will be detailed in a forthcoming publication.

5 Summary and Discussion

Let us briefly summarize our results. We investigated the properties of the supersymmetric mechanics associated with the second Hopf map. We found that the reduction via the $SU(2)$ group action of the $(8, 8, 0)$ multiplet generates a five-dimensional supersymmetric multiplet induced by the $\mathcal{N} = 5$ supersymmetry generators acting on $(8, 8, 0)$ and commuting with the $su(2)$ algebra generators. The resulting supermultiplet is a reducible, but indecomposable length-5 multiplet with fields content $(5, 11, 10, 5, 1)$. The $SU(2)$ action on this field coincides with the identity operator. The resulting invariant action has been explicitly computed. We proved that it admits both $SU(2)$ invariance and an $\mathcal{N} = 8$ invariance. The invariance under the three extra supersymmetry operators is less important for two reasons. The first one is that it is automatically induced by the invariance under the $\mathcal{N} = 5$ $SU(2)$-invariant operators. The second one is that the $N = 8$ action closes on a much larger multiplet than $(5, 11, 10, 5, 1)$ and the extra-fields are inessential to derive the invariant action.

We further pointed out that an extra, BRST-like, symmetry can be imposed on the reduced system. Constraining the 5 $SU(2)$-invariant coordinates living on the surface of the $S^4 \subset \mathbb{R}^5$ sphere, produces an $\mathcal{N} = 5$ non-linear multiplet generated by the 4 angular coordinates of the sphere. The description of a system in presence of an $SU(2)$ Yang monopole/BPST instanton requires further work. At first the $(5, 11, 10, 5, 1)$ linear multiplet should be decomposed into its two basic irreducible constituents $(5, 8, 3, 0, 0)$ and $(0, 3, 5, 7, 1)$ (the latter is a length-4 $\mathcal{N} = 5$ multiplet first described in [26]); next the fields entering the $(0, 3, 5, 7, 1)$ multiplet should be consistently set to zero. The fields entering $(5, 8, 3, 0, 0)$ can now be equated, through non-linear transformations, with the $(5, 8, 3)$ $SU(2)$-invariant fields describing the Yang monopole and introduced in (4.29). Due to the non-linearity of the transformation, the $\mathcal{N} = 5$ supersymmetry is realized non-linearly in this new basis. This procedure corresponds to its simpler $U(1)$ counterpart concerning the reduction of $\mathcal{N} = 4$ $(4, 4, 0)$ into $\mathcal{N} = 4$ $(3, 4, 1)$. It is worth pointing out that, in contrast with the $U(1)$ reduction case, for the non-abelian $SU(2)$ reduction the auxiliary fields cannot be completely removed from the Lagrangian. Indeed, they "partially" transmute into isospin degrees of freedom. This difference between the two reduction procedures was expected from the beginning, since it has a purely bosonic origin. Much less expected are the subtle issues concerning the supersymmetric reductions. For the $U(1)$ reduction, the whole set of $\mathcal{N} = 4$ extended supersymmetries is $U(1)$ invariant while, for $SU(2)$, only $\mathcal{N} = 4$ or $\mathcal{N} = 5$ of the original $\mathcal{N} = 8$ supersymmetries are $SU(2)$-invariant.

In this work we prepared the ground for further developments, clarifying the general features of the supersymmetric reductions and postponing to forthcoming papers the detailed descriptions.

Appendix

For completeness we are reporting the $\mathcal{N}=4$ $su(2)$-invariant lagrangian $\mathcal{L}$ for the $(8,8)$ multiplet, expressed in terms of the quaternionic structure constants. After setting $\epsilon_{123}=+1$, $\Gamma=f_{00}+f_{11}+f_{22}+f_{33}$, $\overline{\Gamma}=f_{\overline{00}}+f_{\overline{11}}+f_{\overline{22}}+f_{\overline{33}}$ ($f_\mu\equiv\partial f/\partial\upsilon_\mu$, $f_{\bar\mu}\equiv\partial f/\partial\bar\upsilon_\mu$ for $\mu=0,1,2,3$), $\mathcal{L}$ is explicitly given by

$$
\begin{aligned}
\mathcal{L} \;=\;& -\Gamma(\dot\upsilon_0^2+\sum\dot\upsilon_i^2)+\overline{\Gamma}(\dot{\bar\upsilon}_0^2+\sum\dot{\bar\upsilon}_i^2)+\\
&\Gamma(\lambda_0\dot\lambda_0+\lambda_i\dot\lambda_i)-\overline{\Gamma}(\overline{\lambda}_0\dot{\overline{\lambda}}_0+\overline{\lambda}_i\dot{\overline{\lambda}}_i)+\\
&(\epsilon_{ijk}(\overline{\Gamma}_k\dot\upsilon_j+\overline{\Gamma}_{\bar j}\dot{\bar\upsilon}_k)+(\overline{\Gamma}_0\dot\upsilon_i+\overline{\Gamma}_{\bar 0}\dot{\bar\upsilon}_i-\overline{\Gamma}_i\dot\upsilon_0-\overline{\Gamma}_{\bar i}\dot{\bar\upsilon}_0))\overline{\lambda}_0\overline{\lambda}_i+\\
&(\epsilon_{ijk}(\Gamma_k\dot\upsilon_j+\Gamma_{\bar j}\dot{\bar\upsilon}_k)+(\Gamma_0\dot\upsilon_i+\Gamma_{\bar 0}\dot{\bar\upsilon}_i-\Gamma_i\dot\upsilon_0-\Gamma_{\bar i}\dot{\bar\upsilon}_0))\lambda_i\lambda_0+\\
&(\epsilon_{ijk}(\Gamma_{\bar k}\dot\upsilon_j+\Gamma_j\dot{\bar\upsilon}_k)-(\overline{\Gamma}_0\dot{\bar\upsilon}_i+\Gamma_{\bar 0}\dot\upsilon_i+\overline{\Gamma}_i\dot{\bar\upsilon}_0+\Gamma_{\bar i}\dot\upsilon_0))\overline{\lambda}_i\lambda_0+\\
&(\epsilon_{ijk}(\overline{\Gamma}_k\dot{\bar\upsilon}_j+\overline{\Gamma}_{\bar j}\dot\upsilon_k)-(\overline{\Gamma}_0\dot{\bar\upsilon}_i+\Gamma_0\dot\upsilon_i+\overline{\Gamma}_i\dot{\bar\upsilon}_0+\Gamma_i\dot\upsilon_0))\overline{\lambda}_0\lambda_i+\\
&\frac{1}{2}(\epsilon_{ijk}(\overline{\Gamma}_i\dot\upsilon_0-\overline{\Gamma}_0\dot\upsilon_i-\overline{\Gamma}_{\bar i}\dot{\bar\upsilon}_0-\overline{\Gamma}_{\bar 0}\dot{\bar\upsilon}_i)+(\overline{\Gamma}_j\dot\upsilon_k-\overline{\Gamma}_k\dot\upsilon_j+\overline{\Gamma}_{\bar j}\dot{\bar\upsilon}_k-\overline{\Gamma}_{\bar k}\dot{\bar\upsilon}_j))\overline{\lambda}_j\overline{\lambda}_k+\\
&\frac{1}{2}(\epsilon_{ijk}(\Gamma_i\dot\upsilon_0-\Gamma_{\bar 0}\dot{\bar\upsilon}_i-\overline{\Gamma}_0\dot\upsilon_i-\Gamma_{\bar i}\dot{\bar\upsilon}_0)+(\Gamma_k\dot\upsilon_j-\Gamma_j\dot\upsilon_k-\Gamma_{\bar k}\dot{\bar\upsilon}_j-\Gamma_{\bar j}\dot{\bar\upsilon}_k))\lambda_j\lambda_k-\\
&(\Gamma_i\dot\upsilon_j+\overline{\Gamma}_i\dot{\bar\upsilon}_j)(\lambda_i\overline{\lambda}_j-\overline{\lambda}_i\lambda_j)+\\
&(\Gamma_{\bar 0}\dot\upsilon_0+\overline{\Gamma}_0\dot{\bar\upsilon}_0+\Gamma_{\bar i}\dot\upsilon_i+\overline{\Gamma}_j\dot{\bar\upsilon}_j)(\overline{\lambda}_0\lambda_0+\overline{\lambda}_k\lambda_k)+\\
&\epsilon_{ijk}(\Gamma_{\bar k}\dot\upsilon_0+\overline{\Gamma}_0\dot\upsilon_k-\Gamma_{\bar 0}\dot\upsilon_k-\overline{\Gamma}_k\dot{\bar\upsilon}_0)\overline{\lambda}_j\lambda_i+\\
&(\Gamma_{\overline{ij}}-\overline{\Gamma}_{ij})\lambda_0\overline{\lambda}_0\lambda_i\overline{\lambda}_j+\epsilon_{ijk}\lambda_0\overline{\lambda}_0(\overline{\Gamma}_{0k}\lambda_i\overline{\lambda}_j+\Gamma_{\overline{0k}}\lambda_i\overline{\lambda}_j)+\\
&\lambda_0\overline{\lambda}_0((\Gamma_{\bar ij}\lambda_i\lambda_j+\overline{\Gamma}_{i\bar j}\overline{\lambda}_i\overline{\lambda}_j)+\frac{1}{2}\epsilon_{ijk}(\overline{\Gamma}_{0\bar j}\overline{\lambda}_i\overline{\lambda}_k+\overline{\Gamma}_{j\bar 0}\overline{\lambda}_i\overline{\lambda}_k-\Gamma_{0\bar k}\lambda_i\lambda_j-\Gamma_{i\bar 0}\lambda_j\lambda_k))+\\
&\frac{1}{2}\epsilon_{ijk}((\Gamma_{0\bar 0}+\Gamma_{p\bar p})\lambda_0\lambda_i\lambda_j\overline{\lambda}_k+(\overline{\Gamma}_{0\bar 0}+\overline{\Gamma}_{p\bar p})\overline{\lambda}_0\overline{\lambda}_i\overline{\lambda}_j\lambda_k)+\\
&(\Gamma_{j\bar 0}-\Gamma_{0\bar j})\lambda_0\lambda_j\lambda_i\overline{\lambda}_i-\frac{1}{2}\epsilon_{ijk}\delta_{pq}(\Gamma_{p\bar k}+\Gamma_{k\bar p})\lambda_0\lambda_i\lambda_j\overline{\lambda}_q+\\
&(\Gamma_{\overline{j0}}-\overline{\Gamma}_{0j})\lambda_0\overline{\lambda}_j\lambda_i\overline{\lambda}_i+\frac{1}{2}\epsilon_{ijk}\delta_{pq}(\overline{\Gamma}_{kp}+\Gamma_{\bar k\bar p})\lambda_0\overline{\lambda}_j\overline{\lambda}_i\lambda_q+\\
&(\overline{\Gamma}_{0j}-\Gamma_{\overline{0j}})\overline{\lambda}_0\lambda_j\lambda_i\overline{\lambda}_i-\frac{1}{2}\epsilon_{ijk}\delta_{pq}(\overline{\Gamma}_{kp}+\Gamma_{\bar k\bar p})\overline{\lambda}_0\lambda_j\lambda_i\overline{\lambda}_q+\\
&(\overline{\Gamma}_{0\bar j}-\overline{\Gamma}_{j\bar 0})\overline{\lambda}_0\overline{\lambda}_j\lambda_i\overline{\lambda}_i-\frac{1}{2}\epsilon_{ijk}\delta_{pq}(\overline{\Gamma}_{p\bar k}+\overline{\Gamma}_{k\bar p})\overline{\lambda}_0\overline{\lambda}_j\overline{\lambda}_i\lambda_q+\\
&(\Gamma_{\overline{jk}}-\overline{\Gamma}_{jk})\lambda_k\overline{\lambda}_j\lambda_i\overline{\lambda}_i+\frac{1}{2}\epsilon_{ijk}\delta_{pq}(\overline{\Gamma}_{0p}+\Gamma_{\bar 0\bar p}\lambda_i\lambda_k\overline{\lambda}_j\overline{\lambda}_q)+\\
&\frac{1}{2}((\Gamma_{k\bar j}-\Gamma_{j\bar k})\lambda_j\lambda_k\lambda_i\overline{\lambda}_i+(\overline{\Gamma}_{k\bar j}-\overline{\Gamma}_{j\bar k})\overline{\lambda}_j\overline{\lambda}_k\overline{\lambda}_i\lambda_i)+\\
&\frac{1}{4}\epsilon_{ijk}\delta_{pq}((\Gamma_{0\bar p}+\Gamma_{p\bar 0})\lambda_i\lambda_j\lambda_k\overline{\lambda}_q-(\overline{\Gamma}_{p\bar 0}+\overline{\Gamma}_{0\bar p})\overline{\lambda}_i\overline{\lambda}_j\overline{\lambda}_k\lambda_q)+\\
&\frac{1}{6}\epsilon_{ijk}((\Gamma_{0\bar 0}-\Gamma_{p\bar p})\lambda_i\lambda_j\lambda_k\overline{\lambda}_0-(\overline{\Gamma}_{0\bar 0}-\overline{\Gamma}_{p\bar p})\overline{\lambda}_i\overline{\lambda}_j\overline{\lambda}_k\lambda_0)+\\
&(\Gamma_{\overline{00}}-\overline{\Gamma}_{00})\lambda_0\overline{\lambda}_0\lambda_i\overline{\lambda}_i+\frac{1}{2}\epsilon_{ijk}(\Gamma_{\overline{pp}}-\overline{\Gamma}_{00})(\lambda_0\lambda_i\overline{\lambda}_j\overline{\lambda}_k+\overline{\lambda}_0\lambda_i\overline{\lambda}_k\overline{\lambda}_j)+\\
&\frac{1}{6}\epsilon_{ijk}((\Gamma_{00}+\Gamma_{pp})\lambda_i\lambda_j\lambda_k\lambda_0-(\overline{\Gamma}_{\overline{00}}+\overline{\Gamma}_{\overline{pp}})\overline{\lambda}_i\overline{\lambda}_j\overline{\lambda}_k\lambda_0).
\end{aligned}
\qquad (5.30)
$$

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