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The $N = 2$ Heavenly Equation*†

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Abstract

This talk is a report of a joint work with Z. Popowicz, appeared in J. Phys. A, concerning the construction of a manifestly $N = 2$ supersymmetric heavenly equation in $2 + 1$ dimensions. Its integrability properties have been derived from an $N = 2$ supersymmetric Lax pair based on the $n \to \infty$ limit of the $sl(n|n+1)$ superalgebra series. The superhydrodynamical type of restrictions to $1 + 1$ dimensions have been analyzed.

Key-words: Supersymmetry, superalgebras, integrable models.

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1 Introduction

In the middle of the seventies Plebański produced two papers [1], [2] centered on self-dual gravity. In [2] he introduced the notion of the "heavenly equation", describing the solutions of complex Einstein equations and expressing the condition of Ricci flatness of a self-dual manifold.

A special case of this equation arises as $SU(\infty)$ limit of a Toda system, known as Boyer-Finley equation (see [3] and [4]).

More recently [5] the heavenly equation was shown describing a string theory with $N = 2$ supersymmetry in the world-sheet. It was further proven in [6] the existence of an $N = 2$ string whose target space effective geometry consists of selfdual gravity coupled to self-dual Yang-Mills theory. Un updated reference concerning the relation between $N = 2$ string geometry and the heavenly equation can be found in [7].

These considerations give strong physical motivations to investigate the heavenly equation and their supersymmetric extensions, especially the ones with extended supersymmetries.

The heavenly equation, on the other hand, was heavily investigated in the framework of integrable systems, due to its remarkable integrability properties [8]. A supersymmetric version was also produced in [9], arising as a continuum limit of a system of equations known as "supersymmetric Toda lattice hierarchy", introduced in [10]. Such system of equations and their hidden $N = 2$ supersymmetric structures have been vastly investigated [11]. In [12] we proved that an $N = 2$ manifestly supersymmetric heavenly equation arises as an $n \to \infty$ limit of a class of $N = 2$ supersymmetric Toda equations constructed in terms of Lax pairs based on the $sl(n|n + 1)$ superalgebra series. Such superalgebras admit a complex structure and therefore allow the construction of super-Toda systems based on $N = 2$ superfields, according to the scheme of [13].

In [12] we further investigated the properties of the dimensional reduction of the above system to $1 + 1$ dimensions and found that the reduced system of equations can be recasted in a superhydrodynamical form, once expressed in terms of a supergeometry involving superfields and fermionic derivatives. In the main section we explicitly construct the $N = 2$ heavenly equation, whose integrability properties are automatically guaranteed by the existence of a Lax pair formalism.

2 The $N = 2$ Superheavenly equation.

The construction of a continuum limit (for $n \to \infty$) of a discretized superToda system requires a presentation of the system in terms of a specific Cartan matrix. In the case of the series of the $sl(n|n + 1)$ superalgebras, both a symmetric or an antisymmetric presentation for their Cartan matrices are available [14]. For our purposes, it turns out that the antisymmetric presentation is the one which works. The Cartan matrix a_{ij} of $sl(n|n + 1)$ will therefore be chosen to be antisymmetric with the only non-vanishing entries given by $a_{ij} = \delta_{i,i+1} - \delta_{i,i-1}$.

The Cartan generators H_i and the fermionic simple roots $F_{\pm i}$ satisfy

$$\begin{aligned} [H_i, F_{\pm j}] &= \pm a_{ij} F_{\pm j}, \\ \{F_i, F_{-j}\} &= \delta_{ij} H_j. \end{aligned} \tag{1}$$

The continuum limit of the [9] construction could have been performed for any superalgebra admitting an $n \to \infty$ limit, such as $sl(n|n)$, etc. On the other hand, the superalgebras of the series $sl(n|n+1)$ are special because they admit a complex structure and therefore the possibility of defining an $N = 2$ manifestly supersymmetric Toda system, following the prescription of [13]. This is the content of the present section.

At first we introduce the $N = 2$ fermionic derivatives $D_\pm$, $\overline{D}_\pm$, acting on the $x_\pm$ $2D$ spacetime ($\theta_\pm$ and $\overline{\theta}_\pm$ are Grassmann coordinates). The $2D$ spacetime can be either Euclidean ($x_\pm = x \pm t$) or Minkowskian ($x_\pm = x \pm it$).

We have

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial\theta_\pm} - i\overline{\theta}_\pm\partial_\pm, \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}_\pm} + i\theta_\pm\partial_\pm. \end{aligned} \tag{2}$$

They satisfy the anticommutator algebra

$$\{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm \tag{3}$$

and are vanishing otherwise.

Chiral (Φ) and antichiral ($\overline{\Phi}$) $N = 2$ superfields are respectively constrained to fulfill the conditions

$$\begin{aligned} \overline{D}_\pm\Phi &= 0, \\ D_\pm\overline{\Phi} &= 0. \end{aligned} \tag{4}$$

Accordingly, a generic chiral superfield Φ is expanded in its bosonic φ, F (the latter is auxiliary) and fermionic component fields (ψ_+, ψ_-) as

$$\Phi(\hat{x}_\pm, \theta_\pm) = \varphi + \theta_+\psi_+ + \theta_-\psi_- + \theta_+\theta_- F, \tag{5}$$

with φ, $\psi_\pm$ and F evaluated in $\hat{x}_\pm = x_\pm + i\overline{\theta}_\pm\theta_\pm$.

Similarly, the antichiral superfield $\overline{\Phi}$ is expanded as

$$\overline{\Phi}(\overline{x}_\pm, \overline{\theta}_\pm) = \overline{\varphi} + \overline{\theta}_+\overline{\psi}_+ + \overline{\theta}_-\overline{\psi}_- + \overline{\theta}_+\overline{\theta}_-\overline{F}, \tag{6}$$

with all component fields evaluated in $\overline{x}_\pm = x_\pm - i\overline{\theta}_\pm\theta_\pm$.

Due to the complex structure of $sl(n|n+1)$, its Cartan and its simple (positive and negative) root sector can be split into its conjugated parts

$$\begin{array}{ll} \mathcal{H} \equiv \{H_{2k-1}\}, & \overline{\mathcal{H}} \equiv \{H_{2k}\}, \\ \mathcal{F}_+ \equiv \{F_{2k-1}\}, & \mathcal{F}_- \equiv \{F_{-(2k-1)}\}, \\ \overline{\mathcal{F}}_+ \equiv \{F_{-2k}\}, & \overline{\mathcal{F}}_- \equiv \{F_{2k}\}, \end{array} \tag{7}$$

for $k = 1, 2, \ldots, n$.

Following [13], we can introduce the $sl(n|n+1)$ $N = 2$ superToda dynamics, defined for the Cartan-valued chiral ($\mathbf{\Phi}$) and antichiral ($\overline{\mathbf{\Phi}}$) $N = 2$ superfields,

$$\begin{aligned} \mathbf{\Phi} &= \sum_{k=1}^{n} \Phi_k H_{2k-1}, \\ \overline{\mathbf{\Phi}} &= \sum_{k=1}^{n} \overline{\Phi}_k H_{2k}, \end{aligned} \tag{8}$$

through the Lax operators $\mathcal{L}_\pm$ and $\overline{\mathcal{L}}_\pm$, given by

$$\begin{aligned}\mathcal{L}_+ &= D_+\mathbf{\Phi} + e^{\overline{\mathbf{\Phi}}}F_+e^{-\overline{\mathbf{\Phi}}},\\ \mathcal{L}_- &= -F_-\end{aligned}\tag{9}$$

and

$$\begin{aligned}\overline{\mathcal{L}}_+ &= \overline{F}_+,\\ \overline{\mathcal{L}}_- &= \overline{D}_-\overline{\mathbf{\Phi}} + e^{\mathbf{\Phi}}\overline{F}_-e^{-\mathbf{\Phi}},\end{aligned}\tag{10}$$

where

$$\begin{array}{ll} F_+ = \sum_k F_{2k-1}, & F_- = \sum_k F_{-(2k-1)},\\ \overline{F}_+ = \sum_k F_{-(2k)}, & \overline{F}_- = \sum_k F_{2k},\end{array}\tag{11}$$

(as before, the sum is over the positive integers up to n).

Explicitly, we have

$$\begin{aligned}\mathcal{L}_+ &= \sum_k (D_+\Phi_k H_{2k-1} + e^{\overline{\Phi}_{k-1}-\overline{\Phi}_k}F_{2k-1}),\\ \mathcal{L}_- &= -\sum_k F_{-(2k-1)},\\ \overline{\mathcal{L}}_+ &= \sum_k F_{-2k},\\ \overline{\mathcal{L}}_- &= \sum_k (\overline{D}_-\overline{\Phi}_k H_{2k} + e^{\Phi_k-\Phi_{k+1}}F_{2k}).\end{aligned}\tag{12}$$

Please notice that, in order to have a more compact notation, here and in the following we have formally introduced the superfields $\overline{\Phi}_0$ and Φ_{n+1}, both set equal to zero ($\overline{\Phi}_0 \equiv 0 \equiv \Phi_{n+1}$).

The zero-curvature equations are given by

$$\begin{aligned}D_+\mathcal{L}_- + D_-\mathcal{L}_+ + \{\mathcal{L}_+,\mathcal{L}_-\} &= 0,\\ \overline{D}_+\overline{\mathcal{L}}_- + \overline{D}_-\overline{\mathcal{L}}_+ + \{\overline{\mathcal{L}}_+,\overline{\mathcal{L}}_-\} &= 0,\end{aligned}\tag{13}$$

so that the following set of equations for the constrained (anti)chiral $N = 2$ superfields is obtained

$$\begin{aligned}D_+D_-\Phi_k &= -e^{\overline{\Phi}_{k-1}-\overline{\Phi}_k},\\ \overline{D}_+\overline{D}_-\overline{\Phi}_k &= -e^{\Phi_k-\Phi_{k+1}},\end{aligned}\tag{14}$$

for the positive integers $k = 1, 2, \ldots, n$.

By setting,

$$B_k = \Phi_k - \Phi_{k+1}, \qquad \overline{B}_k = \overline{\Phi}_k - \overline{\Phi}_{k+1},\tag{15}$$

we get the systems of equations

$$D_+D_-B_k = e^{\overline{B}_k} - e^{\overline{B}_{k-1}}, \qquad \overline{D}_+\overline{D}_-\overline{B}_k = e^{B_{k+1}} - e^{B_k},\tag{16}$$

for k restricted to $k = 1, 2, \ldots, n-1$, together with

$$D_+ D_- B_n = -e^{\overline{B}_{n-1}}, \qquad \overline{D}_+ \overline{D}_- \overline{B}_0 = e^{B_1}. \tag{17}$$

By identifying k as a discretized extra time-like variable τ we obtain, in the continuum limit for $n \to \infty$,

$$D_+ D_- B = \partial_\tau e^{\overline{B}}, \qquad \overline{D}_+ \overline{D}_- \overline{B} = \partial_\tau e^{B}, \tag{18}$$

which corresponds to the $N = 2$ extension of the superheavenly equation.

Indeed, the presence in the previous equations of the first derivative in τ is an artifact of the $N = 2$ superfield formalism. Once solved the equations at the level of the component fields and eliminated the auxiliary fields in terms of the equations of motion, we are left with a system of second-order equations.

We have, in terms of the component fields,

$$\begin{aligned} B &= \Big(1 + i\overline{\theta}_+ \theta_+ \partial_+ + i\overline{\theta}_- \theta_- \partial_- - \overline{\theta}_+ \theta_+ \overline{\theta}_- \theta_- \partial_+ \partial_-) C, \\ \overline{B} &= \Big(1 - i\overline{\theta}_+ \theta_+ \partial_+ - i\overline{\theta}_- \theta_- \partial_- - \overline{\theta}_+ \theta_+ \overline{\theta}_- \theta_- \partial_+ \partial_-\Big) \overline{C}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} C &= \Big(b + \theta_+ \psi_+ + \theta_- \psi_- + \theta_+ \theta_- a\Big), \\ \overline{C} &= \Big(\overline{b} + \overline{\theta}_+ \overline{\psi}_+ + \overline{\theta}_- \overline{\psi}_- + \overline{\theta}_+ \overline{\theta}_- \overline{a}\Big), \end{aligned} \tag{20}$$

with a, $\overline{a}$ bosonic auxiliary fields. All component fields are evaluated in $x_\pm$ only.

The equations of motion of the $N = 2$ superheavenly equation for the component fields, after eliminating the auxiliary fields, are explicitly given by

$$\begin{aligned} 2i\partial_- \psi_+ &= (\overline{\psi}_- e^{\overline{b}})_\tau, \\ -2i\partial_+ \psi_- &= (\overline{\psi}_+ e^{\overline{b}})_\tau, \\ 4\partial_+ \partial_- b &= \Big((e^b)_\tau e^{\overline{b}} + \overline{\psi}_+ \overline{\psi}_- e^{\overline{b}}\Big)_\tau, \end{aligned} \tag{21}$$

and

$$\begin{aligned} 2i\partial_- \overline{\psi}_+ &= (\psi_- e^{b})_\tau, \\ -2i\partial_+ \overline{\psi}_- &= (\psi_+ e^{b})_\tau, \\ 4\partial_+ \partial_- \overline{b} &= \Big((e^{\overline{b}})_\tau e^{b} + \psi_+ \psi_- e^{b}\Big)_\tau. \end{aligned} \tag{22}$$

The bosonic component fields $b, \overline{b}$, as well as the fermionic ones $\psi_\pm$, $\overline{\psi}_\pm$, are all independent. The equations (21, 22) are a manifestly $N = 2$ supersymmetric extension of the system introduced in [9].

3 Conclusions

The construction of the heavenly equation and other integrable models with extended number of supersymmetries, besides being important for the physical applications recalled in the Introduction, is also relevant in the classification of integrable systems. As pointed out in [15], bosonic hierarchies are produced in association with the bosonic sector of superalgebras. A natural extension of the method and results here presented would amount to the construction of the $N = 4$ supersymmetric extension of the heavenly equation. Due to the relation between extended supersymmetries and division algebras, see [16], one could expect it to be obtained as a continuum limit of an infinite series of quaternionic superalgebras (this problem is currently under investigation).

Let us conclude by pointing out an open problem concerning the $1+1$-dimensional reduction of the (21, 22) system, leading to a superhydrodynamical type of equation. Despite being integrable, a supersymmetric dispersionless Lax operator expressing the τ dynamics is not yet available, while in the bosonic sector such an operator exists. This problem has already been encountered in related contexts, see [17].

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