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ON THE THERMODYNAMICS OF ONE-FLUID SZEKERES'-LIKE COSMOLOGIES

by

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ABSTRACT

The thermodynamic behavior of the inhomogeneous Szekeres type cosmologies with a perfect fluid as source of gravitation is examined. Since the matter motion is geodetic, the absence of heat flow implies that the temperature is a function of time alone. For a subclass approaching homogeneity and isotropy at large cosmological times an expression for the temperature is derived. It does not coincide with the law of temperature satisfied by the FRW universes, even asymptotically. However, by assuming an equation of state explicitly dependent of the space coordinates, it is shown that the FRW thermodynamics may be recovered. In all cases the Euler and Gibbs-Duhem relations are no longer valid.

Key-words: Cosmology; Inhomogeneous models; Thermodynamics.

It has been conjectured that the present symmetric phase of our universe is a final result of the evolution of an initial highly inhomogeneous and anisotropic state[1-3]. This belief has had an important role in the search of more general cosmological models evolving to the Friedmann-Robertson-Walker (FRW) universes. A remarkable set of solutions of the Einstein Field Equations (EFE) fulfilling such a condition is the class of inhomogeneous dust-filled universes found by Szekeres[4-6]. Generalizations of these models with pressure are available in the literature[7-11]. These extended solutions present at least one subclass for which an homogeneous and isotropic phase is established at large cosmological times.

In the present note we examine the thermodynamic behavior of the Szekeres type models. It is shown that the thermodynamic limit does not follow necessarily, however, from the dynamic one. More clearly, if a class of perfect fluid Szekeres' type cosmologies approaching the FRW ones is given its thermodynamics do not evolve asymptotically, in general, to the standard thermodynamics of the FRW universes. For the sake of brevity, we shall consider only the parabolic subclass of Szekeres type models of class II recently derived by us[12].

The metric of such models is (in our units $8\pi G=c=k_B=1$)

$$ds^2 = dt^2 - Q^2dx^2 - R^2\ (dy^2+y^2dz^2) \quad , \tag{1}$$

where the function Q (t,x,y,z) takes the form

$$Q = AR + R_0S \quad , \tag{2}$$

with the functions $A(x,y,z)$, $S(x,t)$ and $R(t)$ given by

$$A = (\lambda \cos z + \nu \sin z) y + \omega \quad , \tag{3}$$

$$S = \mu (R/R_0)^{(3\gamma-4)/2} \quad , \tag{4}$$

$$R = R_0 \left[1 + \frac{3\gamma}{2} (t-t_0) \right]^{3\gamma/2} \quad , \tag{5}$$

where λ, ν, ω and μ are arbitrary functions of x, R_0 and t_0 are constants and γ is a parameter which may be identified, for large cosmological times, with the "adiabatic index" of the asymptotic equation of state $p = (\gamma-1)\rho$. This space-time contains a perfect fluid whose isotropic pressure in the comoving frame $(u^\alpha = \delta^\alpha{}_0)$ takes the form

$$p = \frac{3(\gamma-1)}{R_0^2} (R_0/R)^{3\gamma} \quad , \tag{6}$$

and its net energy density, heuristically considered in the Ref. (12) as originated from a two fluid mixture, is

$$\rho = \frac{p}{\gamma-1} \left[\gamma-1 + (2-\gamma) \frac{AR}{Q} \right] \quad , \tag{7}$$

which may also, for further reference, be rewritten in the following forms:

$$\rho = \frac{p}{\gamma-1} \left[1 + (\gamma-2)(1 - \frac{AR}{Q}) \right] \quad , \tag{7a}$$

and

$$\rho = \frac{p}{\gamma-1} \left[\frac{AR}{Q} + (\gamma-1)(1 - \frac{AR}{Q}) \right] \quad . \tag{7b}$$

Here, we assume the fluid to be simple. If $\gamma = 1$ $(p = 0)$, these spacetimes reduce to a particular case of the dust-filled Szekeres parabolic models[4,5]. Note also that the functions R and p are the same ones present in the FRW flat models[13] with a "gamma-law". Actually, both if S vanishes ($Q = AR$) or if the limit of large cosmological times ($R >> R_0$) is reached, the FRW flat models in Szekeres' type coordinates are recovered[12].

Next we compute the temperature distribution for the spacetimes described by eqs. (1)-(7). First we remark that since there is no heat flow ($q^\alpha = 0$), the matter motion being geodetic ($a_\beta = 0$), the heat conducting equation of Eckart[14-15] (semi--colon denotes covariant derivative)

$$q^\alpha = \chi h^{\alpha\beta} (T_{;\beta} - Ta_\beta) \quad , \tag{8}$$

implies that the temperature of any Szekeres' type model with perfect fluid is a function of time t alone. Particularly, for the models of class II, since $R = R(t)$ it follows that $T = T(R)$ as in the FRW case. We now recall that combining the motion equations of the fluid, contained in the conservation laws

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad , \tag{9}$$

$$(nu^\alpha)_{;\alpha} = 0 \quad , \tag{10}$$

where n is the particle number density, with the Gibbs law[15]

$$nTd\sigma = d\rho - \left(\frac{\rho+p}{n}\right) dn \quad , \tag{11}$$

where σ is the specific entropy (per particle), one finds that the temperature of a perfect fluid model satisfies at a point $\vec{x} \equiv (x,y,z)$

$$\frac{\dot{T}}{T} = \left(\frac{\partial p}{\partial \rho}\right)_n \frac{\dot{n}}{n} \quad , \tag{12}$$

where an overdot means comoving time derivative and the quantities ρ and p were taken as functions of the thermodynamics variables T and n.

In the standard FRW models with $p = (\gamma-1)\rho$, since $\dot{n}/n = -3\dot{R}/R$, eq. (12) gives

$$T_{FRW} = T_0 \left(R_0/R\right)^{3(\gamma-1)} \quad , \tag{13}$$

where T_0 is the temperature at $R = R_0$. In the case $\gamma = 1$, this solution yields the expected limiting result for dust

$$T_{FRW} = T_0 = \text{const} \quad . \tag{13a}$$

For the Szekeres type models it is necessary to establish the equation of state defined by the expressions (6) and (7) for ρ and p which are, in principle, functions of T and n. First one needs to find $n(\vec{x},t)$. In the background (1), from the conservation law (10), one may write

$$n = \frac{f(x,y,z)}{QR^2} \quad , \tag{14}$$

where the arbitrary function f is found by considering that if $Q \to AR$, $n \to n_{FRW} = n_0(R_0/R)^3$. Hence,

$$n = \frac{n_0 A R_0{}^3}{QR^2} \quad . \tag{15}$$

It should be noticed that, as the energy density, the term $\dot{n}/n$ in eq. (10) may be rewritten using (2), (4) and (15) in the following forms:

$$\dot{n}/n = -3\dot{R}/R \left[\frac{\gamma}{2} + \left(\frac{2-\gamma}{2}\right) \frac{AR}{Q}\right] \quad , \tag{16}$$

$$= -3\dot{R}/R \left[1 + \left(\frac{\gamma-2}{2}\right)\left(1 - \frac{AR}{Q}\right)\right] , \tag{16a}$$

$$= -3\dot{R}/R \left[\frac{AR}{Q} + \frac{\gamma}{2}\left(1 - \frac{AR}{Q}\right)\right] \quad . \tag{16b}$$

As will be shown, the two sets of equations (7-7b) and (16-16b) play corresponding roles with respect to the possible thermodynamic behaviors presented by the models.

First, let us consider the conventional procedure. Note that since p and T are functions only of the time, instead of $\rho = \rho(n,T)$ one may take $\rho = \rho(n,p)$. Now, using eqs. (2), (4), (6) and (15), the terms $\frac{AR}{Q}$ and $1 - \frac{AR}{Q}$ present in eqs. (7-7b) may be computed. Moreover, one can see that such an equation of state not explicitly dependent on $\vec{x}$, will be obtained only if we use (7) and not (7a) or (7b). In this case eq.(7) gives

$$\rho = p + \frac{\alpha(2-\gamma)}{\gamma-1}\, np^{1-\frac{1}{\gamma}} \quad , \tag{17}$$

where

$$\alpha = \frac{1}{n_0}\left(\frac{3(\gamma-1)}{R_0^2}\right)^{1/\gamma} \quad , \tag{18}$$

is a constant. A similar procedure for (16) gives

$$\dot{n}/n = -\frac{3\gamma}{2}\frac{\dot{R}}{R}\left[1 + \alpha\left(\frac{2-\gamma}{\gamma}\right) np^{-1/\gamma}\right] \quad , \tag{19}$$

or using (17)

$$\dot{n}/n = -\frac{3\gamma}{2}\left(\frac{\partial\rho}{\partial p}\right)_n \frac{\dot{R}}{R} \quad . \tag{20}$$

Substituting the above result into (12), a straightforward integration yields

$$T = T_0\left(R_0/R\right)^{3\gamma/2} \quad . \tag{21}$$

This decay temperature law furnishes the thermal behavior of the considered Szekeres' type models during all of its evolution. Of course, it does not evolve to the standard FRW result (cf. eq. (13)). In particular, it gives a rather unexpected result, namely: the temperature of the dust-filled Szekeres parabolic model approaching the respective FRW one scale with $R^{-3/2}$ instead of being constant.

As an attempt to circumvent this paradoxical result i.e., to try to recover the standard law of temperature obeyed by the FRW models, we assume next that the equation of state in an inhomogeneous background may also depend explicitly of the space coordinates(16) in the form $\rho = \rho(\vec{x},n,p)$. Now, using eqs. (2), (4), (6) and (15), the couple of eqs. (7b) and (16b) can be rewritten as

$$\rho = \frac{p}{\gamma-1}\left[1 + (\gamma-2)\ \beta n p^{-1/2}\right] \tag{22}$$

and

$$\dot{n}/n = -3\,\frac{\dot{R}}{R}\left[1 + \left(\frac{\gamma-2}{2}\right)\ \beta n p^{-1/2}\right] \quad , \tag{23}$$

where the function $\beta(\vec{x})$ is

$$\beta = \frac{\mu\sqrt{3(\gamma-1)}}{An_0R_0} \tag{24}$$

It thus follows that $\dot{n}/n = -3(\gamma-1)(\frac{\partial\rho}{\partial p})_n \frac{\dot{R}}{R}$ and, from eq. (12), the FRW temperature law (13) is readily recovered.

Seemingly, we solved the paradox, since in this case the dynamic and thermodynamic limits are made consistent with each other. However, the particular cases studied suggest that we may consider other $\vec{x}$ dependent ρ's. Thus, we introduce a γ-dependent parameter b such that the sets (7-7b) and (16-16b) may be rewritten in the following unified forms:

$$\rho = \frac{p}{\gamma-b}\left[b + \gamma(\frac{1-b}{\gamma-1})\frac{AR}{Q} + (\gamma-2b)(1-\frac{AR}{Q})\right] \tag{25}$$

and

$$\dot{n}/n = -3\frac{\dot{R}}{R}\left[b + (1-b)\frac{AR}{Q} + (\frac{\gamma}{2}-b)(1-\frac{AR}{Q})\right] ; \tag{26}$$

from which the particular decomposition (7-7b) and (16-16b) are recovered by taking $b = \frac{\gamma}{2}$, 1 and 0 respectively. Of course, (25) and (26) is only a more general decomposition of ρ and $\dot{n}/n$ in terms of $\frac{AR}{Q}$ and $1-\frac{AR}{Q}$, which do not alter their values, but lead to distinct equations of state when such terms are expressed as functions of p, n and of the space coordinates. In fact, in this case the equation of state is

$$\rho = \frac{p}{\gamma-b}\left[b+\gamma(\frac{1-b}{\gamma-1})\,\alpha n p^{-1/\gamma} + (\gamma-2b)\beta n p^{-1/2}\right] , \tag{27}$$

with α and β defined respectively by (18) and (24). Also from (26) and (27)

$$\frac{\dot{n}}{n} = -\ 3(\gamma-b)\left(\frac{\partial\rho}{\partial p}\right)_n \frac{\dot{R}}{R} \quad . \tag{28}$$

Thus, combining (12), (27) and (28) one obtains

$$T = T_0 \left[R_0/R\right]^{3(\gamma-b)} \quad , \tag{29}$$

which generalizes eqs. (13) and (21) for b equal 1 and $\gamma/2$ respectively. In addition, if $b = 0$ eq. (29) yields

$$T = T_0 \left[R_0/R\right]^{3\gamma} \quad , \tag{30}$$

which, as one can easily see, corresponds exactly to temperature law generated by the decompositions (7b) and (16b).

We see now that the equation of state (27) does not depend explicitly on the space coordinates only if $b = \gamma/2$, the first case considered. If this condition is relaxed there is now an infinite (one-parametric) class of possible temperature distributions corresponding to the possible choice of b. For $b = 1$, the FRW temperature law is recovered. It seems to us that this indetermination of the temperature law has its origin in two facts: (i) In the Szekeres type one fluid models no simple equation of state conecting ρ and p may, without loss of generality, be imposed a priori in order to integrate the EFE. In fact, the equation of state (27) has been derived using the particle number density after the integration and, as in the standard FRW solutions, n does not play any dynamical role. (ii) The energy conservation law (contained in the EFE) is always satisfied regardless the value of b. This condition implies that the specific entropy remains constant along each fluid line for any value of the b parameter and time dependence of T.

We remark that the usual local form of the Euler relation(17) (zero chemical potential), $\sigma = \frac{\rho+p}{nT}$, is not valid in any of these models. This is more easily seen looking to its counterpart, the so-called Gibbs-Duhem relation

$$\frac{dp}{dT} = -\frac{\rho+p}{T} \quad . \tag{31}$$

The above equation cannot be satisfied because ρ depends of the space coordinates whereas p and T do not. In fact, if $b = 1$, for instance, by integrating the Gibbs law (11) one finds, except for a possible additive constant, that

$$\sigma = \frac{\rho-p}{(2-\gamma)nT} \quad , \tag{32}$$

whose constant value, $\sigma = \frac{3\gamma}{n_0 T_0 R_0^2}$, may be easily checked by substituting the expressions of ρ, p, n and T into (32). It coincides with the value of σ for the FRW models with a "gamma-law".

Finally, it should be noticed that we consider here only the class of one-fluid Szekeres like parabolic models which smoothly approach the FRW limit regardless of the value of γ. Of course in the framework of a two-fluid interpretation as suggested in Ref. (12), it is possible that the thermodynamics of these models may be free of the anomalies here found.

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