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NON EQUILIBRIUM RELATIVISTIC COSMOLOGY(*)

by

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ABSTRACT

The purpose of the present work is to give a certain systematization through the discussion of results already known and the presentation of new ones. In section 2 we give a brief review of the necessary mathematical background. The theory of perturbation of Friedmann-like Universes is presented in section 3. The reduction of Einstein's equations for homogeneous Universes to an autonomous planar system of differential equations is done in section 4. Finally in section 5 the alternative gravitational non-minimal coupling and its consequences to cosmology are discussed.

1. INTRODUCTION

Classical Relativistic Cosmology has been, in recent years, a very active subject. Many authors have revised some of its main topics. However, the range of this subject is so large that it is far from being exhausted. In this vein,we will present here some topics which were not fully discussed previously. We limit our analysis to three subjects: the theory of perturbation of expanding universes (the quasi-Maxwellian version); the qualitative analysis of irreversible cosmos and the properties of universes filled with scalar and vector fields coupled non-minimally to gravity.

The theory of perturbation attracted the attention of many cosmologists which have tried to related the observed inhomogenities to small disturbances present in our universe. Here we limit our analysis to the discussion of the quasi-Maxwellian equations of motion. The reason for this is twofold. Firstly, as we shall see, such method presents many advantages over the conventional one, at least for those background geometries which are conformally flat. Secondly, it permits a straightforward generalization for alternative gauge-like theories of gravitation. Indeed, the quasi-Maxwellian approach of gravity is based on the fact that we can interpret Bianchi identities as true equations of motion for a given set of metric initial conditions. Using the curvature tensor $R^{\alpha\beta\mu\nu}$ we write

$$R^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu} \quad . \qquad (1)$$

Different choices for the current $J^{\alpha\beta\mu}$ will give origin to dis tinct theories of gravitation. In this work we limit our discussion to the choice

$$J_{\alpha\beta\mu} = T_{\mu\alpha;\beta} - T_{\mu\beta;\alpha} - \frac{1}{2}\left[g_{\mu\alpha}T_{,\beta} - g_{\mu\beta}T_{,\alpha}\right] \qquad (2)$$

which, as we shall see in section 2, is a necessary condition for the equivalence of system (1) to Eintein's equation. However, distinct models for the current $J_{\alpha\beta\mu}$ have been proposed either suggested as a consequence of quantum fluctuations of the geometry [Novello, 1978] or dictated by other reasons [Camenzind,1977]. We then turn our analysis to the influence of viscosity on the behavior of the Universe at large. The whole system of Eintein's equations can be reduced (in the case of homogeneous and isotropic universes) to a planar autonomous dynamical system. We discuss the qualitative features of this system, the generic properties of the solutions and their stability. In section 5 we deal with some consequences of non-minimal coupling of both scalar and vector fields to gravity. We present the mechanism of spontaneous symmetry breaking and, as a consequence, the generation of repulsive gravitational forces induced by a scalar field. As a consequence of the non-minimal coupling of electromagnetic fields to gravitation the equation of motion obeyed by photons becomes non-linear. We then examine an Universe filled with such non-linear photons and find out that this allows cosmological solutions devoid of singularities to exist.

2. SOME MATHEMATICAL TOOLS OF GENERAL RELATIVITY

The purpose of the present section is to give a general overview of some of the most important techniques which will be used in the investigation of cosmological questions in subsequent sections of this work. Although we do not intend to extend this presentation very further, we have tried to make it as self-contained as possible. Here and there we have pointed out in the literature those papers/books in which further material can be found.

2.1 - NOTATIONS AND DEFINITIONS

We list here some of the symbols which are used in the text.

$g_{\mu\nu}$ metric tensor

greek indices 0, 1, 2, 3

latin indices 1, 2, 3

$\eta_{\mu\nu}$ Minkowskii metric tensor with standard form

$\eta_{\mu\nu} = (+1, -1, -1, -1)$

n_μ four-velocity (normalized $n_\mu n_\nu g^{\mu\nu} = 1$)

$W_{\alpha\beta\mu\nu}$ Conformal Weyl tensor

$R_{\alpha\beta\mu\nu}$ Riemann curvature tensor

$\Gamma^\alpha_{\mu\nu}$ Christofell symbols

$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$

, simple derivative ($\psi_{,\mu} = \frac{\partial\psi}{\partial x^\mu}$)

; co-variant derivative ($\psi^\alpha{}_{;\mu} = \psi^\alpha{}_{,\mu} + \Gamma^\alpha_{\varepsilon\mu}\,\psi^\varepsilon$)

∇_α alternative index for the co-variant derivative

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\beta\nu,\mu} + \Gamma^{\alpha}_{\sigma\nu}\,\Gamma^{\sigma}_{\mu\beta} - \Gamma^{\alpha}_{\sigma\mu}\,\Gamma^{\sigma}_{\nu\beta}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

$\eta^{\alpha\beta\mu\nu}$ completely anti-symmetric tensor

$\varepsilon^{\alpha\beta\mu\nu}$ completely anti-symmetric pseudo tensor

g determinant of $g_{\mu\nu}$

$F^{*}_{\mu\nu}$ dual of the anti-symmetric tensor $F_{\mu\nu}$

$E_{\mu\nu}$ electric part of Weyl tensor

$H_{\mu\nu}$ magnetic part of Weyl tensor

$T^{\alpha\beta\mu\nu}$ Super energy-momentum tensor of gravity

$T_{\mu\nu}$ energy-momentum tensor of matter

$h_{\mu\nu}$ projected tensor

θ expansion factor

$\sigma_{\mu\nu}$ shear

$\omega_{\mu\nu}$ vorticity

$\dot{V}_{\mu}$ acceleration

2.2 - THE WEYL TENSOR. PROPERTIES -

The description of that part of the geometry, the evolution of which is not coupled directly to matter in Einstein's theory is represented by the conformal Weyl tensor $W_{\alpha\beta\mu\nu}$. We define $W_{\alpha\beta\mu\nu}$ in terms of the curvature tensor $R_{\alpha\beta\mu\nu}$ and its contraction $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$ and, $R \equiv R^{\mu}{}_{\mu}$ by means of the expression

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - H_{\alpha\beta\mu\nu} + \frac{1}{6} R \, g_{\alpha\beta\mu\nu} , \qquad (1)$$

in which

$$g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu} \, g_{\beta\nu} - g_{\alpha\nu} \, g_{\beta\mu} \qquad (2)$$

$$H_{\alpha\beta\mu\nu} = \frac{1}{2} \{ R_{\alpha\mu} \, g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} \, g_{\beta\mu} - R_{\beta\mu} \, g_{\alpha\nu} \} . \qquad (3)$$

The conformal tensor has all the symmetries of the curvature tensor, namely

$$W_{\alpha\beta\mu\nu} = -W_{\alpha\beta\mu\nu} = -W_{\alpha\beta\mu\nu} = W_{\mu\nu\alpha\beta} .$$

Besides, it is completely trace-free. Thus, it has only ten (10) independent components. We can decompose the Weyl tensor with respect to an arbitrary observer which moves with velocity n^{μ} (normalized $n^{\mu} n^{\mu} g_{\mu\nu} = 1$) in electric ($E_{\mu\nu}$) and magnetic ($H_{\mu\nu}$) parts defined by analogy to the electromagnetic field:

$$E_{\alpha\beta} = -W_{\alpha\mu\beta\nu}\, n^{\mu} n^{\nu} \tag{4a}$$

$$H_{\alpha\beta} = -\overset{*}{W}_{\alpha\mu\beta\nu}\, n^{\mu} n^{\nu}. \tag{4b}$$

The star operator represents the <u>dual</u>, defined by means of the Levi-civita completelly anti-symmetric pseudo tensor $\varepsilon_{\mu\nu\rho\sigma}$ by the expression

$$W^{*}_{\alpha\mu\rho\sigma} = \frac{1}{2}\eta_{\alpha\mu}{}^{\varepsilon\tau}\, W_{\varepsilon\tau\rho\sigma}\ , \tag{5}$$

in which

$$\eta_{\alpha\mu\rho\sigma} \equiv \sqrt{-g}\ \varepsilon_{\alpha\mu\rho\sigma} \tag{6}$$

and $g \equiv \det g_{\mu\nu}$.

We then have for $\eta^{\alpha\beta\mu\nu}$ the value

$$\eta^{\alpha\beta\mu\nu} = -\frac{1}{\sqrt{-g}}\ \varepsilon^{\alpha\beta\mu\nu}\ .$$

We remark that although the Levi-Civita symbol is a pseudo-tensor, the dual object $\eta_{\mu\nu\rho\sigma}$ is a <u>true</u> tensor. This is due to the fact that $\sqrt{-g}$ is also a pseudo quantity which transforms with the inverse power of the Jacobian, making up for the Jacobian dependence of $\varepsilon_{\mu\nu\rho\sigma}$.

For later use and for completeness we recall here some properties of $\eta_{\alpha\beta\mu\nu}$ and $g_{\alpha\beta\mu\nu}$.

We have

$$\eta^{\sigma\nu\rho\varepsilon}\ \eta_{\lambda\alpha\beta\varepsilon} = -\delta^{\sigma\nu\rho}_{\lambda\alpha\beta} \tag{7}$$

in which the symbol $\delta^{\sigma\nu\rho}_{\lambda\alpha\beta}$ represents the determinant construc ted with $\delta^{\mu}{}_{\nu}$,

$$\delta^{\sigma\nu\rho}_{\lambda\alpha\beta} = \begin{vmatrix} \delta^{\sigma}_{\lambda} & \delta^{\sigma}_{\alpha} & \delta^{\sigma}_{\beta} \\ \delta^{\nu}_{\lambda} & \delta^{\nu}_{\alpha} & \delta^{\nu}_{\beta} \\ \delta^{\rho}_{\lambda} & \delta^{\rho}_{\alpha} & \delta^{\rho}_{\beta} \end{vmatrix} \tag{8}$$

$$= \delta^{\rho}_{\beta}\, \delta^{\sigma\nu}_{\lambda\alpha} - \delta^{\nu}_{\beta}\, \delta^{\sigma\rho}_{\lambda\alpha} + \delta^{\sigma}_{\beta}\, \delta^{\rho\nu}_{\lambda\alpha} \; ,$$

in which

$$\delta^{\sigma\nu}_{\lambda\alpha} \equiv \begin{vmatrix} \delta^{\sigma}_{\lambda} & \delta^{\sigma}_{\alpha} \\ \delta^{\nu}_{\lambda} & \delta^{\nu}_{\alpha} \end{vmatrix} = \delta^{\sigma}_{\lambda}\, \delta^{\nu}_{\alpha} - \delta^{\sigma}_{\alpha}\, \delta^{\nu}_{\lambda} \; .$$

By contraction we obtain

$$\eta_{\alpha\beta\varepsilon\lambda}\, \eta^{\sigma\nu\varepsilon\lambda} = -2\delta^{\sigma\nu}_{\alpha\beta} \tag{9}$$

and contracting once more

$$\eta^{\sigma\nu\varepsilon\lambda}\, \eta_{\beta\nu\varepsilon\lambda} = -6\delta^{\sigma}_{\beta} \; . \tag{10}$$

Finally ,

$$\eta^{\sigma\nu\varepsilon\lambda}\, \eta_{\sigma\nu\varepsilon\lambda} = -24. \tag{11}$$

Let us come back to the Weyl tensor. From the properties of symmetry of $W_{\alpha\beta\mu\nu}$ we show that the dual operation is independent of the pair on which it is applied. Indeed, we have

$$W^{*}_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu}^{\;*} \equiv \overset{*}{W}_{\alpha\beta\mu\nu}. \tag{12}$$

It is worth to remark that this property is not, in general, valid for the Riemann curvature tensor.

Indeed, taking the dual of $R_{\alpha\beta\mu\nu}$ on the first and on the second pair of indices we obtain

$$R^{*}_{\alpha\beta\mu\nu} - R_{\alpha\beta\mu\nu}^{\;*} = -2H_{\alpha\beta\mu\nu}^{\;*} + \frac{1}{2} R\eta_{\alpha\beta\mu\nu} , \tag{13}$$

in which we have used the fact that

$$g^{*}_{\alpha\beta\mu\nu} = g_{\alpha\beta\mu\nu}^{\;*} = \eta_{\alpha\beta\mu\nu} . \tag{14}$$

Consequently, $\eta^{*}_{\alpha\beta\mu\nu} = \eta_{\alpha\beta\mu\nu}^{\;*} = -g_{\alpha\beta\mu\nu}$, which could be used as an alternative equivalent way to define the metric bi-tensor $g_{\alpha\beta\mu\nu}$.

Then the necessary and sufficient condition to have independence of the order of the application of the dual, that is,

$$R^{*}_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}^{\;*} ,$$

is contained in the relation

$$H_{\alpha\beta\mu\nu}^{\;*} = \frac{1}{4} R\eta_{\alpha\beta\mu\nu} ,$$

or, taking the dual (in $\mu\nu$) of this expression and noting that taking twice the dual on the same anti-symmetric pair of indices is equivalent to a mere change of sign, we obtain

$$H_{\alpha\beta\mu\nu} = \frac{1}{4} R g_{\alpha\beta\mu\nu} ,$$

which is satisfied if $R_{\mu\nu}$ is proportional to the metric tensor, that is, for the class of geometry which is called <u>Einstein spaces</u>:

$$R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu} .$$

Incidentally, we remark that from (13) we can write

$$R^{*}_{\alpha\beta}{}^{*}_{\mu\nu} = R_{\alpha\beta\mu\nu} - 2W_{\alpha\beta\mu\nu} - \frac{1}{6} R\, g_{\alpha\beta\mu\nu} , \qquad (15)$$

which is a useful property to convert double dual of Riemann tensor in terms of the invariant decomposition tensors of the curvature.

We remark that although the trace of the dual curvature tensor vanishes, the trace of the double dual does not. Indeed, taking the trace of (13) and (15) gives

$$R_{\alpha\beta}{}^{*}_{\mu\nu}\, g^{\alpha\mu} = 0 \qquad (16a)$$

$$R^{*}_{\alpha\beta}{}^{*}_{\mu\nu}\, g^{\alpha\mu} = R_{\beta\nu} - \frac{1}{2} R g_{\beta\nu} . \qquad (16b)$$

So much for the basic properties of the curvature tensor. Let us go back to the Weyl decomposition in terms of the electric and magnetic tensor $E_{\mu\nu}$ and $H_{\mu\nu}$.

From definitions (4) it follows:

$$E_{\mu\nu} = E_{\nu\mu}$$

$$E_{\mu\nu}\, n^{\nu} = 0 \qquad (17)$$

$$E_{\mu\nu}\, g^{\mu\nu} = 0$$

and analogous relations for $H_{\mu\nu}$:

$$H_{\mu\nu} = H_{\nu\mu}$$

$$H_{\mu\nu}\, n^{\nu} = 0 \qquad (18)$$

$$H_{\mu\nu}\, g^{\mu\nu} = 0 .$$

A simple inspection on equations (4a) and (4b) shows that the dual operation is equivalent to the map

$$E_{\mu\nu} \longrightarrow H_{\mu\nu} \qquad (19a)$$

$$H_{\mu\nu} \longrightarrow -E_{\mu\nu} . \qquad (19b)$$

Such transformation, which is very similar to the rotation of the plane of polarization of an electromagnetic wave, is a special case of the so-called dual rotation.

2.3 - DUAL ROTATION AND INVARIANTS OF THE METRIC

We define the operator of dual rotation as an abstract map of the electric and magnetic parts of Weyl tensor generated by a rotation in the plane ($E_{\mu\nu}$, $H_{\xi\nu}$). Thus defined, this map constitutes a one-parameter continuous group. We have:

$$\begin{pmatrix} E'_{\mu\nu} \\ H'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_{\mu\nu} \\ H_{\mu\nu} \end{pmatrix} . \qquad (20)$$

The dual map (19) is the special case $\theta = \pi/2$.

Let us define the complex quantity $\Sigma_{\mu\nu\rho\sigma}$ as:

$$\Sigma_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + i\overset{*}{W}_{\mu\nu\rho\sigma} \quad . \qquad (21)$$

Multiply this expression by $\Sigma^{\mu\nu\rho\sigma}$ to obtain

$$\Sigma_{\mu\nu\rho\sigma}\ \Sigma^{\mu\nu\rho\sigma} = 2(I_1 + iI_2),$$

in which I_1 and I_2 are the second order invariants

$$I_1 \equiv W^{\alpha\beta}{}_{\mu\nu}\ W^{\mu\nu}{}_{\alpha\beta} \qquad (22)$$

$$I_2 \equiv W^{\alpha\beta}{}_{\mu\nu}\ \overset{*}{W}{}^{\mu\nu}{}_{\alpha\beta} \ . \qquad (23)$$

The norm of $\Sigma_{\mu\nu\alpha\beta}$ vanishes, that is,

$$\Sigma^{\alpha\beta}{}_{\mu\nu}\ \overline{\Sigma}{}^{\mu\nu}{}_{\alpha\beta} = 0$$

(a bar means complex conjugate)

Besides the two invariants I_1 and I_2 of second order, with the Weyl tensor one can construct two more invariants I_3 and I_4, of third order:

$$I_3 \equiv W^{\alpha\beta}{}_{\mu\nu} \; W^{\mu\nu}{}_{\rho\sigma} \; W^{\rho\sigma}{}_{\alpha\beta} \tag{24}$$

$$I_4 \equiv \overset{*}{W}{}^{\alpha\beta}{}_{\mu\nu} \; W^{\mu\nu}{}_{\rho\sigma} \; W^{\rho\sigma}{}_{\alpha\beta} \; . \tag{25}$$

The third order relation takes the form:

$$\Sigma^{\alpha\beta}{}_{\mu\nu} \; \Sigma^{\mu\nu}{}_{\rho\sigma} \; \bar{\Sigma}^{\rho\sigma}{}_{\alpha\beta} = 0$$

and for the triple product one obtains the identity:

$$\Sigma_{\alpha\beta}{}^{\mu\nu} \; \Sigma_{\mu\nu}{}^{\rho\sigma} \; \Sigma_{\rho\sigma}{}^{\alpha\beta} = 4(I_3 + iI_4) \, .$$

The dual rotation (20) can be written directly in terms of $W_{\alpha\beta\mu\nu}$:

$$W'_{\alpha\beta\mu\nu} = \cos\theta \; W_{\alpha\beta\mu\nu} + \sin\theta \; \overset{*}{W}_{\alpha\beta\mu\nu} \tag{26}$$

or more concisely in terms of $\Sigma_{\alpha\beta\mu\nu}$:

$$\Sigma'_{\alpha\beta\mu\nu} = e^{-i\theta} \Sigma_{\alpha\beta\mu\nu} \, .$$

The main interest on dual rotation is related to the fact that it constitutes an exact symmetry of Einstein's theory of gravity in the absence of sources for the geometry. This situation is very similar to the case of Electrodynamics in the absence of charge. We will discuss this result later on, after the presentation of Einstein's equation in quasi-Maxwellian formulation.

The four scalars I_1, I_2, I_3, I_4 are invariants constructed uniquely with the Weyl tensor. They can be associated to pure gravitational fields (that is, to those regions of spacetime free of any other kind of energy except of gravitational origin).

However, the total number of functional independent invariants which can be constructed as functions of the metric are in number of 14. Besides the above four ones, the remaining ten are constructed with products containing contractions of curvature tensor, that is, with the irreducible objects $C_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} Rg_{\mu\nu}$ and R. We ennumerate them here, just for completeness.

Let us define the quantities $D_{\mu\nu}$ and $\overset{*}{D}_{\mu\nu}$ by the expressions

$$D_{\mu\nu} \equiv W_{\mu\alpha\nu\beta}\ C^{\alpha\beta} \tag{27a}$$

$$\overset{*}{D}_{\mu\nu} = \overset{*}{W}_{\mu\alpha\nu\beta}\ C^{\alpha\beta}\ . \tag{27b}$$

We write below the remaining 10 invarians

$$I_5 = C_{\mu\nu}\ C^{\mu\nu}$$

$$I_6 = C_{\mu}{}^{\alpha}\ C_{\alpha}{}^{\nu}\ C_{\nu}{}^{\mu}$$

$$I_7 = C_\mu{}^\alpha \, C_\alpha{}^\beta \, C_\beta{}^\nu \, C_\nu{}^\mu$$

$$I_8 = R$$

$$I_9 = D_{\mu\nu} \, C^{\mu\nu}$$

$$I_{10} = D_{\mu\nu} \, D^{\nu\mu}$$

$$I_{11} = D^\alpha{}_\beta \, D^\beta{}_\lambda \, C^\lambda{}_\alpha$$

$$I_{12} = \overset{*}{D}_{\mu\nu} \, C^{\mu\nu}$$

$$I_{13} = \overset{*}{D}_{\mu\nu} \, D^{\mu\nu}$$

$$I_{14} = \overset{*}{D}{}^\alpha{}_\beta \, \overset{*}{D}{}^\beta{}_\mu \, C^\mu{}_\alpha \, .$$

2.4 - DUAL IDENTITIES

In the study of the anti-symmetric electromagnetic tensor $f_{\mu\nu}$ we generally use two important identities involving $f_{\mu\nu}$ and its dual $f^*_{\mu\nu}$, in order to simplify certains expressions. These identities are:

$$\overset{*}{f}_{\mu\lambda}\ \overset{*}{f}{}^{\lambda}{}_{\nu} - f_{\mu\lambda}\ f^{\lambda}{}_{\nu} = A\ g_{\mu\nu} \tag{28a}$$

$$\overset{*}{f}_{\mu\nu}\ f^{\nu\lambda} = f_{\mu\nu}\ \overset{*}{f}{}^{\nu\lambda} = -\frac{1}{2}\ B\ g_{\mu\nu}\ , \tag{28b}$$

in which the invariants A and B are given by

$$A \equiv \frac{1}{2}\ f_{\mu\lambda}\ f^{\mu\lambda}$$

$$B \equiv \frac{1}{2}\ f_{\mu\lambda}\ \overset{*}{f}{}^{\mu\lambda}\ .$$

As a simple example of the use of these invariants we can consider the stress-energy tensor of the electromagnetic field, given by

$$T_{\mu\nu} = f_{\mu}{}^{\alpha}\ f_{\alpha\nu} + \frac{1}{4}\ g_{\mu\nu}\ f_{\alpha\beta}\ f^{\alpha\beta}\ ,$$

which can be re-written using (28) in a manifestly dual symmetric form

$$2T_{\mu\nu} = f_{\mu\alpha}\ f^{\alpha}{}_{\nu} + \overset{*}{f}_{\mu\alpha}\ \overset{*}{f}{}^{\alpha}{}_{\nu}\ .$$

The similarities of the properties of Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ and Maxwell tensor $f_{\mu\nu}$ induce us to suspect that similar relations should exist relating $W^{\alpha\beta\mu\nu}$ and its dual. Two of these relations were known from long date due to Lanczos( ) and Debever( ) and are the following:

$$W_{\alpha\rho\mu\sigma}\, W_{\beta\ \lambda}^{\ \rho\ \sigma} - \overset{*}{W}_{\alpha\rho\lambda\sigma}\, \overset{*}{W}_{\alpha\ \mu}^{\ \rho\ \sigma}\, \overset{*}{W}_{\alpha\ \mu}^{\ \rho\ \sigma} = \frac{1}{8}\, I_1\, g_{\mu\lambda}\, g_{\alpha\beta} \quad (29a)$$

$$W_{\alpha\rho\lambda\sigma}\, W_{\beta}^{\ \rho\lambda\sigma} = \frac{1}{4}\, I_1\, g_{\alpha\beta} \quad (29b)$$

in which $I_1 \equiv W_{\alpha\beta\mu\nu}\, W^{\alpha\beta\mu\nu}$.

Recently, Novello and Duarte have exhibited a set of four identities constructed not only with quadratic products of $W_{\alpha\beta\mu\nu}$ but also with cubic terms. These last identities have not analog in the case of electrodynamics, once triple products of $f_{\mu\nu}$ can be simplified by means of the above identities and reduced to zero.

Indeed, we have for instance:

$$f_{\mu\nu}\, f^{\nu}_{\ \lambda}\, f^{\lambda\mu} = f_{\mu\nu}\{\overset{*}{f}{}^{\nu}_{\ \lambda}\, f^{\lambda\mu} - I_1\, g^{\nu\mu}\} = (f_{\mu\nu}\, \overset{*}{f}{}^{\nu}_{\ \lambda})f^{\lambda\mu} =$$

$$= -\frac{1}{2}\, I_2\, g_{\mu\lambda}\, f^{\lambda\mu} = 0$$

$$f_{\mu\nu}\, f^{\nu}_{\ \lambda}\, \overset{*}{f}{}^{\lambda\mu} = -\frac{1}{2}\, f_{\mu\nu}\, g^{\mu\nu}\, I_2 = 0$$

and so on.

The complete set of identities of double and triple products of Weyl tensor are (for the proofs of these identities we quote Novello and Duarte (1980))

$$W_{\alpha\rho\mu\sigma}\, W_{\beta\ \lambda}^{\ \rho\ \sigma} - \overset{*}{W}_{\alpha\rho\lambda\sigma}\, \overset{*}{W}_{\beta\ \mu}^{\ \rho\ \sigma} = I_1\, g_{\lambda\mu}\, g_{\alpha\beta} \qquad (30)$$

$$W^{(\alpha}{}_{\rho\mu\sigma}\, W^{\beta)\rho}{}_{\lambda}{}^{\sigma} - \overset{*}{W}{}^{(\alpha}{}_{\rho\mu\sigma} \overset{*}{W}{}^{\beta)\rho}{}_{\lambda}{}^{\sigma} = 2 I_1 g_{\lambda\mu}\, g^{\alpha\beta}$$

$$\overset{*}{W}{}^{\alpha(\mu}{}_{\beta\nu}\, W_{\alpha}{}^{\varepsilon)\beta}{}_{\tau} = I_2\, g^{\mu\varepsilon}\, g_{\nu\tau}$$

$$W_{\alpha\mu\beta\nu}\, \overset{*}{W}{}^{\alpha\varepsilon\beta\rho} = \overset{*}{W}_{\alpha\mu\beta\nu}\, W^{\alpha\varepsilon\beta\rho}$$

$$W_{\alpha\beta\mu}{}^{\nu}\, W^{\alpha\beta\mu}{}_{\tau} = 2 I_1\, \delta^{\nu}_{\tau} \qquad (31)$$

$$W_{\alpha\beta\mu}{}^{\nu} \overset{*}{W}{}^{\alpha\beta\mu} = 2 I_2\, \delta^{\nu}_{\tau} \qquad (32)$$

$$\overset{*}{W}{}^{\alpha\mu}{}_{\beta\nu} W_{\alpha}{}^{\varepsilon\beta}{}_{\tau} + W^{\alpha\varepsilon}{}_{\beta\nu} \overset{*}{W}_{\alpha}{}^{\mu\beta}{}_{\tau} = I_2\, g^{\mu\varepsilon}\, g_{\nu\tau} \qquad (33)$$

$$W_{\mu\alpha\varepsilon\beta}\, W^{\varepsilon}{}_{\tau}{}^{\lambda\gamma}\, W_{\lambda\gamma}{}^{\mu\alpha} - \overset{*}{W}_{\mu\alpha\varepsilon\tau}\, \overset{*}{W}{}^{\varepsilon}{}_{\beta}{}^{\lambda\gamma} W_{\lambda\gamma}{}^{\mu\alpha} = 8\, I_3 g_{\beta\tau} \qquad (34)$$

$$W_{\mu\alpha\varepsilon\beta}\, W^{\mu\alpha\lambda\gamma}\, W_{\lambda\gamma}{}^{\varepsilon}{}_{\tau} = 4\, I_3\, g_{\beta\tau} \qquad (35)$$

$$W_{\mu\alpha\varepsilon\beta}\, \overset{*}{W}{}^{\mu\alpha\lambda\gamma}\, W_{\lambda\gamma}{}^{\varepsilon}{}_{\tau} = 4\, I_4\, g_{\beta\tau} \qquad (36)$$

$$W_{\mu\alpha\varepsilon\beta}\, W^{\varepsilon}{}_{\tau}{}^{\lambda\gamma}\, \overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} + W_{\mu\alpha\varepsilon\tau}\, W^{\varepsilon}{}_{\beta}{}^{\lambda\gamma}\, \overset{*}{W}_{\lambda\gamma}{}^{\mu\alpha} = 8\, I_4\, g_{\beta\tau}\ . \qquad (37)$$

If we consider the definition (Bel) of the super tensor $T^{\mu\nu\rho\sigma}$ we can use these identities in order to re-write $T^{\mu\nu\rho\sigma}$ in a form which is analogous to the expression of the energy-momentum tensor of the electromagnetic case.

This gives

$$T^{\alpha\beta\lambda\mu} = W^{\alpha\rho\lambda\sigma} W^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} + \overset{*}{W}{}^{\alpha\rho\lambda\sigma} \overset{*}{W}{}^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} .$$

2.5 - FUNDAMENTAL KINEMATICAL QUANTITIES

A curve $\Gamma(s)$ defined on a 4-dimensional Riemannian manifold M_4 is the result of a linear mapping carrying elements of a sub-set S of R^1 into a region of M_4. Let $\vec{n}(s)$ be the tangent vector to the curve Γ, and s be a parameter on the curve. A congruence is defined in M_4 if instead of an isolated curve $\Gamma(s)$ there exists a whole set of distinct trajetories characterized by a new parameter p which selects a curve of congruence. Thus the congruence will be represented by two parameters s and p and will be noted $\Gamma(s,p)$.

The congruence is time-like if the vectors $\vec{n}(s)$ of the curves which belong to $\Gamma(s,p)$ are time-like. We consider here only time-like congruences, the vector $\vec{n}(s)$ being normalized:

$$n^{\mu} n^{\nu} g_{\mu\nu} = +1. \qquad (40)$$

The metric $g_{\mu\nu}$ and the vector $\vec{n}$ of M_4 induce a projector tensor $h_{\mu\nu}$ which separates any geometric (tensorial) object of M_4 in terms of quantities defined along Γ plus quantities defined on the 3-dimensional space H orthogonal to $\vec{n}$. We write

$$h_{\mu\nu} \equiv g_{\mu\nu} - n_{\mu} n_{\nu}. \qquad (41)$$

This tensor $h_{\mu\nu}$ satisfies the properties of symmetry

$$h_{\mu\nu} = h_{\nu\mu}, \qquad (42a)$$

it is orthogonal to $\vec{n}$,

$$h_{\mu\nu}\, n^{\nu} = 0 \tag{42b}$$

and is a true projector

$$h_{\mu}{}^{\nu}\, h_{\nu}{}^{\lambda} = h_{\mu}{}^{\lambda}. \tag{42c}$$

Such quantity $h_{\mu\nu}$ can thus be identified with the metric induced by $g_{\mu\nu}$ on the hypersurface H. Indeed, we can write for the fundamental distance between two arbitrary points of M_4:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = h_{\mu\nu}\, dx^{\mu}\, dx^{\nu} + (n_{\mu}\, dx^{\mu})^2 \tag{43}$$

which separates the length ds into a pure spatial part $(h_{\mu\nu}\, dx^{\mu}\, dx^{\nu})^{1/2}$ plus a time interval $dt = n_{\mu}\, dx^{\mu}$ by the identification of the time-coordinate t with the parameter defined on $\Gamma(s)$.

Given any vector A^{μ} we can define its restriction on H, noted $\hat{A}^{\mu}$, by setting

$$\hat{A}_{\mu} = h^{\mu}{}_{\alpha}\, A^{\alpha}. \tag{44}$$

The covariant derivative in H operates on objects thus defined by (44) through the expression

$$\hat{\nabla}_{\alpha}\, \hat{A}_{\mu} = h_{\alpha}{}^{\varepsilon}\, h_{\mu}{}^{\lambda}\, \nabla_{\varepsilon}\, \hat{A}_{\lambda}. \tag{45}$$

The quantity $\hat{A}_{\alpha\mu} \equiv \hat{\nabla}_{\alpha}\, \hat{A}_{\mu}$ is indeed a quantity completely contained in H, in the sense that its contraction with the vec- tor $\vec{n}$ vanishes identically:

$$n^{\alpha} \hat{A}_{\alpha\mu} = 0$$

$$n^{\mu} \hat{A}_{\alpha\mu} = 0 .$$

Thus, we have

$$\hat{\nabla}_{\alpha} \hat{A}_{\beta} = h_{\alpha}{}^{\varepsilon} h_{\beta}{}^{\lambda} (A_{\rho} h^{\rho}{}_{\varepsilon})_{;\lambda} = h_{\alpha}{}^{\varepsilon} h_{\beta}{}^{\lambda} h^{\rho}{}_{\varepsilon} A_{\rho;\lambda} +$$

$$+ h_{\alpha}{}^{\varepsilon} h_{\beta}{}^{\lambda} A_{\rho} h_{\varepsilon}{}^{\rho}{}_{;\lambda} = h_{\alpha}{}^{\rho} h_{\beta}{}^{\lambda} A_{\rho;\lambda} -$$

$$- h_{\alpha}{}^{\varepsilon} h_{\beta}{}^{\lambda} A_{\rho} n^{\rho} n_{\varepsilon;\lambda} ,$$

using property (3b).

We can then write

$$\hat{\nabla}_{\alpha} \hat{A}_{\beta} = h_{\alpha}{}^{\varepsilon} h_{\beta}{}^{\lambda} (A_{\rho} h^{\rho}{}_{\varepsilon})_{,\lambda} - \Gamma^{\mu}_{\varepsilon\lambda} h_{\alpha}{}^{\varepsilon} h_{\beta}{}^{\lambda} h^{\rho}{}_{\mu} A_{\rho} . \qquad (45')$$

Thus we are led to define the restricted connection on H by the expression

$$\hat{\Gamma}^{\rho}_{\alpha\beta} \equiv \Gamma^{\mu}_{\varepsilon\lambda} h_{\mu}{}^{\rho} h^{\varepsilon}{}_{\alpha} h^{\lambda}{}_{\beta} . \qquad (46)$$

Using (46) into (45'):

$$\hat{\nabla}_{\alpha} \hat{A}_{\beta} = h_{\alpha}{}^{\rho} h_{\beta}{}^{\lambda} \hat{A}_{\rho,\lambda} - \hat{\Gamma}^{\rho}_{\alpha\beta} \hat{A}_{\rho} .$$

Let us exemplify the properties of such derivative in the geometry of an expanding homogeneous and isotropic universe. The 4-dimensional Friedmann geometry written in a gaussian system of coordinates takes the form (Robertson and Nooman, (1968))

$$ds^2 = dt^2 - A^2(t)\, \gamma ij(\vec{x})\, dx^i\, dx^j . \quad (47)$$

Remember that latin indices run in the domain {1,2,3}.

The 3-dimensional geometry has constant curvature and thus the Riemannian tensor ${}^{(3)}R_{ijkl}$ can be written in this case in the compact form

$$^{(3)}R_{ijkl} = \varepsilon \gamma_{ijkl} , \quad (48)$$

where ε is a constant and we have used the definition $\gamma_{ijkl} \equiv \gamma_{ik}\, \gamma_{jl} - \gamma_{il}\, \gamma_{jk}$

Contracting with the metric γ_{ij} of H we obtain

$$^{(3)}R_{ik} = 2\varepsilon\, \gamma_{ik} . \quad (49)$$

Contracting once more we have

$$^{(3)}R = 6\varepsilon . \quad (50)$$

Remark that (47) is already in the form (43). Making the choice

$$n^\mu = \delta^\mu_0 ,$$

we have

$$h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \delta^{\mu}{}_{0}\, \delta^{0}{}_{\nu} \ ,$$

that is,

$$h^{0}{}_{i} = 0$$

$$h^{i}{}_{j} = \delta^{i}{}_{j} \ .$$

Then we have for an arbitrary vector A^{μ}:

$$\hat{A}_{0} = h_{0}{}^{\mu}\, A_{\mu} = 0$$

$$\hat{\nabla}_{0}\, \hat{A}_{\mu} = 0$$

$$\hat{\nabla}_{k}\, \hat{A}_{l} = h_{k}{}^{m}\, h_{l}{}^{n}\, \hat{A}_{m,n} - \hat{\Gamma}^{m}_{kl}\, \hat{A}_{m} = \hat{A}_{k,l} - \hat{\Gamma}^{m}_{kl}\, \hat{A}_{m} \ .$$

However, from (18)

$$\hat{\Gamma}^{m}_{kl} = h^{m}{}_{\alpha}\, h^{\lambda}{}_{k}\, h_{l}{}^{\varepsilon}\, \Gamma^{\alpha}_{\lambda\varepsilon} = \Gamma^{m}_{kl} \ .$$

Thus in this case the restriction of the connection on H coincides with the values of the connection of M_4.

Using the metric of Friedmann (12) we obtain:

$$\hat{\Gamma}^{m}_{kl} = \frac{1}{2}\, g^{mn}\, \{g_{kn,l} + g_{ln,k} - g_{kl,n}\}$$

$$= \frac{1}{2}\, \gamma^{mn}\, \{\gamma_{kn,l} + \gamma_{ln,k} - \gamma_{kl,n}\} \ .$$

Thus,

$$\hat{\Gamma}^m_{kl} = {}^{(3)}\Gamma^m_{kl} \ ,$$

in which ${}^{(3)}\Gamma^m_{kl}$ is constructed with the 3-dimensional geometry γ_{kl}. We remark that the projector $h_{\mu\nu}$ is indeed the (Riemannian) geometry of H, that is, its co-variant derivative restricted on H vanishes. The proof of this is straightforward:

$$\hat{\nabla}_\alpha \ h_{\mu\nu} = h_\alpha{}^\varepsilon \ h_\mu{}^\lambda \ h_\nu{}^\rho \ \nabla_\varepsilon \ h_{\lambda\rho} = -h_\alpha{}^\varepsilon \ h_\mu{}^\lambda \ h_\nu{}^\rho \ \nabla_\varepsilon (n_\lambda \ n_\rho) =$$

$$= -h_\alpha{}^\varepsilon \ h_\mu{}^\lambda \ h_\nu{}^\rho \ n_\lambda \ \nabla_\varepsilon \ n_\rho - h_\alpha{}^\varepsilon \ h_\mu{}^\lambda \ h_\nu{}^\rho \ n_\rho \ \nabla_\delta \ n_\lambda$$

$$= 0 \ .$$

The Riemann tensor $\hat{R}_{\mu\varepsilon\beta\alpha}$ in H is given by the definition:

$$\hat{\nabla}_\alpha \ \hat{\nabla}_\beta \ \hat{A}_\mu - \hat{\nabla}_\beta \ \hat{\nabla}_\alpha \ \hat{A}_\mu = \hat{R}_{\mu\varepsilon\beta\alpha} \ \hat{A}^\varepsilon \ .$$

A good example of the uses of the above expression is given by the search of the Gauss-Codazzi formula which relates the 4-dimensional scalar of curvature R to the 3-dimensional scalar $\hat{R}$ and to the extrinsic curvature $K_{\mu\nu}$. This quantity $K_{\mu\nu}$ measures the bending of the 3-surface H in the embedding space M_4. Such relation is of great help in the examination of continuity conditions of the gravitational field, for instance.

Let us then evaluate the expression $\hat{\nabla}_\beta \ \hat{\nabla}_\alpha \ \hat{A}_\mu$:

$$\hat{\nabla}_\beta \hat{\nabla}_\alpha \hat{A}_\mu = h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda \nabla_\varepsilon (\hat{\nabla}_\sigma \hat{A}_\lambda)$$

$$= h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda \nabla_\varepsilon \{h_\sigma{}^\rho h_\lambda{}^\tau \nabla_\rho (h_{\tau\phi} A^\phi)\}$$

$$= h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda \nabla_\varepsilon \{h_\sigma{}^\rho h_\lambda{}^\tau \nabla_\rho A_\tau + h_\sigma{}^\rho h_\lambda{}^\tau A^\phi \nabla_\rho h_{\tau\phi}\} =$$

$$= h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda \nabla_\varepsilon \{h_\lambda{}^\tau \nabla_\sigma A_\tau - n_\sigma n^\rho h_\lambda{}^\tau \nabla_\rho A_\tau +$$

$$+ h_\lambda{}^\tau A^\phi \nabla_\sigma h_{\tau\phi} - n_\sigma n^\rho h_\lambda{}^\tau A^\phi \nabla_\rho h_{\tau\phi}\}$$

$$= h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau (\nabla_\varepsilon h_\lambda{}^\tau)(\nabla_\sigma A_\tau) + h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau \nabla_\varepsilon \nabla_\sigma A_\tau -$$

$$- h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau (\nabla_\varepsilon n_\sigma)(\nabla_\rho A_\tau) n^\rho +$$

$$+ h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau \nabla_\varepsilon \{A^\phi \nabla_\sigma h_{\tau\phi}\} +$$

$$+ h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda (\nabla_\varepsilon h_\lambda{}^\tau) A^\phi \nabla_\sigma h_{\tau\phi} -$$

$$- h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau (\nabla_\varepsilon n_\sigma) A^\phi n^\rho \nabla_\rho h_{\tau\phi} =$$

$$= -h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda n^\tau (\nabla_\varepsilon n_\lambda)(\nabla_\sigma A_\tau) + h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau \nabla_\varepsilon \nabla_\sigma A_\tau -$$

$$- h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau (\nabla_\varepsilon n_\sigma)(\nabla_\rho A_\tau) - h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau (\nabla_\varepsilon A^\phi)(\nabla_\sigma n_\tau) n_\phi$$

$$+ h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau A^\phi \nabla_\varepsilon \nabla_\sigma h_{\tau\phi} + h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\lambda (\nabla_\varepsilon n_\lambda) n^\tau \{(\nabla_\sigma n_\tau) n_\phi +$$

$$+ n_\tau \nabla_\sigma n_\phi\} A^\phi - h_\beta{}^\varepsilon h_\alpha{}^\sigma h_\mu{}^\tau (\nabla_\varepsilon n_\sigma) A^\phi n^\rho \nabla_\rho h_{\tau\phi} .$$

Anti-symmetrization in α, β yields:

$$\hat{R}_{\mu\epsilon\alpha\beta}\,\hat{A}^{\epsilon} = h_{\beta}{}^{\epsilon}\, h_{\alpha}{}^{\sigma}\, h_{\mu}{}^{\tau}\, R_{\tau\lambda\sigma\epsilon}\, A^{\lambda} + h_{\beta}{}^{\epsilon}\, h_{\alpha}{}^{\sigma}\, h_{\mu}{}^{\tau}\, A^{\phi}\{R_{\tau\nu\sigma\epsilon}\, h^{\nu}{}_{\phi} +$$

$$+ R_{\phi\nu\sigma\epsilon}\, h_{\tau}{}^{\nu}\} + h_{[\beta}{}^{\epsilon}\, h_{\alpha]}{}^{\sigma}\, h_{\mu}{}^{\lambda}\, K_{\lambda\epsilon}\, A^{\phi} K_{\phi\sigma} \qquad (51)$$

where we used the fact that $\vec{n}$ is orthogonal to H, that is, $n_{\nu,\mu} - n_{\nu,\mu} = 0$ and defined the extrinsic curvature $K_{\mu\nu} \equiv \nabla_{\mu}\, n_{\nu}$. Now,

$$h_{\beta}{}^{\rho}\, h_{\alpha}{}^{\sigma}\, h_{\mu}{}^{\tau}\{R_{\tau\epsilon\sigma\rho} + R_{\tau\nu\sigma\rho}\, h^{\nu}{}_{\epsilon} + R_{\epsilon\nu\sigma\rho}\, h_{\tau}{}^{\nu}\} =$$

$$= h_{\beta}{}^{\rho}\, h_{\alpha}{}^{\sigma}\, h_{\mu}{}^{\tau}\{R_{\tau\epsilon\sigma\rho} + R_{\tau\epsilon\sigma\rho} - R_{\tau\nu\sigma\rho}\, n^{\tau}\, n^{\nu}\} +$$

$$+ R_{\epsilon\tau\sigma\rho} - R_{\epsilon\nu\sigma\rho}\, n^{\tau}\, n^{\nu}\}$$

$$= h_{\beta}{}^{\rho}\, h_{\alpha}{}^{\sigma}\, h_{\mu}{}^{\tau}\, h_{\epsilon}{}^{\nu}\, R_{\tau\nu\sigma\rho} \;.$$

Using this result in (51) we obtain

$$\hat{R}_{\mu\tau\alpha\beta} = h_{\mu}{}^{\lambda}\, h_{\tau}{}^{\rho}\, h_{\alpha}{}^{\sigma}\, h_{\beta}{}^{\epsilon}\, R_{\lambda\rho\sigma\epsilon} + h_{[\beta}{}^{\epsilon}\, h_{\alpha]}{}^{\sigma}\, h_{\mu}{}^{\lambda}\, K_{\lambda\epsilon}\, K_{\tau\sigma} \;. \qquad (52)$$

Contracting with the metric in H:

$$\hat{R}_{\mu\alpha} \equiv h^{\tau\beta}\, \hat{R}_{\mu\tau\alpha\beta} = h_{\mu}{}^{\lambda}\, h^{\rho\epsilon}\, h_{\alpha}{}^{\sigma}\, R_{\lambda\rho\sigma\epsilon} +$$

$$+ h^{\epsilon\tau}\, h_{\alpha}^{\sigma}\, h_{\mu}^{\lambda}\, K_{\lambda\epsilon}\, K_{\tau\sigma} - h_{\alpha}{}^{\epsilon}\, h^{\sigma\tau}\, h_{\mu}{}^{\lambda}\, K_{\lambda\epsilon}\, K_{\tau\sigma}$$

or

$$\hat{R}_{\mu\alpha} = h_{\mu}{}^{\lambda} \, h_{\alpha}{}^{\sigma} \, h^{\rho\varepsilon} \, R_{\lambda\rho\sigma\varepsilon} + k_{\mu\tau} \, k^{\tau}{}_{\alpha} - k_{\mu\alpha} \, k \, , \tag{53}$$

in which

$$k \equiv k_{\mu\nu} \, h^{\mu\nu} .$$

Contracting once again with the metric on H yields the desired result,

$$\hat{R} = h^{\lambda\sigma} \, h^{\rho\varepsilon} \, R_{\lambda\rho\sigma\varepsilon} + k_{\mu\nu} \, k^{\mu\nu} - k^{2} \tag{54}$$

$$\hat{R} = R - 2R_{\mu\nu} \, n^{\mu} \, n^{\nu} + k_{\mu\nu} \, k^{\mu\nu} - k^{2} , \tag{54'}$$

which relates R, the Riemann scalar of curvature of M_4, to the intrinsic curvature $\hat{R}$ and the embedding property of H into M_4, specified by tensor $k_{\mu\nu}$.

Let us now consider the decomposition of the extrinsic curvature ($n_{\mu;\nu}$) into irreducible components.
We set

$$n_{\mu;\nu} = \sigma_{\mu\nu} + \frac{\theta}{3} \, h_{\mu\nu} + \omega_{\mu\nu} + a_{\mu} \, n_{\nu} , \tag{55}$$

in which $\omega_{\mu\nu}$ is the vorticity tensor:

$$\omega_{\mu\nu} = \frac{1}{2} \, h_{[\mu}{}^{\alpha} h_{\nu]}{}^{\beta} \, n_{\alpha;\beta} , \tag{56}$$

$\sigma_{\mu\nu}$ is the trace free dilatation tensor (shear):

$$\sigma_{\mu\nu} = \frac{1}{2} h_{(\mu}{}^{\alpha} h_{\nu)}{}^{\beta} n_{\alpha;\beta} - \frac{1}{3} \theta h_{\mu\nu}. \qquad (57)$$

The expansion θ, is defined by

$$\theta = n^{\mu}{}_{;\mu} \qquad (58)$$

and

$$a_{\mu} = n_{\mu;\nu} n^{\nu} \qquad (59)$$

is the acceleration of the field of velocities.

From the definition of the curvature tensor we have

$$n_{\alpha;\beta;\gamma} - n_{\alpha;\gamma;\beta} = R_{\alpha\epsilon\beta\gamma} n^{\epsilon} .$$

Multiplying by n^{γ} and projecting twice on H we obtain

$$h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} (n_{\mu;\nu})^{\bullet} - h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} a_{\mu;\nu} + h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} n_{\mu;\gamma} n^{\gamma}{}_{;\nu} = R_{\mu\epsilon\nu\gamma} n^{\epsilon} n^{\gamma} h^{\mu}{}_{\alpha} h^{\nu}{}_{\beta} , \qquad (60)$$

where the point means derivative in the direction of the field $\vec{n}$.

Let us define the quantity $Q_{\mu\nu}$ by the expression.

$$Q_{\mu\nu} = n_{\mu;\nu} - a_{\mu} n_{\nu} .$$

We can then re-write equation (60) in the form

$$h_{\alpha}^{\ \mu} h_{\beta}^{\ \nu} \dot{Q}_{\mu\nu} + a_{\alpha} a_{\beta} - h_{\alpha}^{\ \mu} h_{\beta}^{\ \nu} a_{\mu;\nu} + Q_{\alpha\gamma} Q^{\gamma}_{\ \beta} = R_{\alpha\varepsilon\beta\gamma} n^{\varepsilon} n^{\gamma} . \qquad (60)'$$

This is the fundamental equation which enables us to obtain all the required expressions for the evolution of the kinematical quantities. Contracting eq.(60)' in α,β we easily obtain the equation of evolution of the expansion factor θ (known as Raychaudhuri's equation):

$$\dot{\theta} + \frac{\theta^2}{3} + 2\sigma^2 - 2\omega^2 - a^{\alpha}_{\ ;\alpha} + \dot{a}_{\alpha} v^{\alpha} + a_{\alpha} a^{\alpha} = R_{\mu\nu} n^{\mu} n^{\nu}, \qquad (61)$$

in which σ^2 and ω^2 are defined by:

$$\sigma^2 \equiv \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} \qquad (62a)$$

$$\omega^2 \equiv \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} . \qquad (62b)$$

Symmetrising equation (60)' we obtain:

$$h_{\alpha}^{\ \mu} h_{\beta}^{\ \nu} \dot{\sigma}_{\mu\nu} + \frac{\dot{\theta}}{3} h_{\alpha\beta} - \frac{1}{2} h_{\alpha}^{\ \mu} h_{\beta}^{\ \nu} \left[a_{\mu;\nu} + a_{\nu;\mu}\right] +$$

$$+ a_{\alpha} a_{\beta} + \frac{1}{2} Q_{(\alpha}^{\ \ \mu} Q_{\mu\beta)} = R_{\alpha\varepsilon\beta\nu} n^{\varepsilon} n^{\nu} - \frac{1}{3} R_{\mu\nu} n^{\mu} n^{\nu} h_{\alpha\beta} .$$

But,

$$Q_{(\alpha}^{\ \ \mu} Q_{\mu\beta)} = \frac{2}{9} \theta^2 h_{\alpha\beta} + \frac{4}{3} \theta \sigma_{\alpha\beta} + 2\sigma_{\alpha\mu} \sigma^{\mu}_{\ \beta} + 2\omega_{\alpha\mu} \omega^{\mu}_{\ \beta} .$$

Let us introduce the spin vector $\vec{\omega}$ associated to the vorticity $\omega_{\mu\nu}$ through the definition

$$\omega^{\tau} = \frac{1}{2} \eta^{\alpha\beta\rho\tau} \omega_{\alpha\beta} n_{\rho} . \tag{63}$$

Thus defined, the spin vector is orthogonal to the vector field:

$$\omega^{\mu} n_{\mu} = 0 .$$

We have:

$$\omega_{\alpha\mu} \omega^{\mu}{}_{\beta} = \omega_{\mu} \omega^{\mu} h_{\alpha\beta} - \omega_{\alpha} \omega_{\beta}$$

and thus

$$\omega_{\alpha\mu} \omega^{\alpha\mu} = -2\omega_{\alpha} \omega^{\alpha} .$$

Using these results into the symmetric form of equation (60) we obtain the equation of evolution of shear:

$$h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \dot{\sigma}_{\mu\nu} + \frac{1}{3} h_{\alpha\beta} \left[-\omega^2 - 2\sigma^2 + a^{\lambda}{}_{;\lambda}\right] +$$

$$+ a_{\alpha} a_{\beta} - \frac{1}{2} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} (a_{\mu;\nu} + a_{\nu;\mu}) + \frac{2}{3} \theta \sigma_{\alpha\beta} +$$

$$+ \sigma_{\alpha\mu} \sigma^{\mu}{}_{\beta} - \omega_{\alpha} \omega_{\beta} = R_{\alpha\varepsilon\beta\nu} v^{\varepsilon} v^{\nu} - \frac{1}{3} R_{\mu\nu} v^{\mu} v^{\nu} h_{\alpha\beta} .$$

(64)

The equation of evolution of the rotation tensor ω is obtained by anti-symmetrisation of equation (60)'. A straightforward calculation gives the desired expression:

$$h_\alpha{}^\mu \, h_\beta{}^\nu \, \dot{\omega}_{\mu\nu} - \frac{1}{2} h_\alpha{}^\mu \, h_\beta{}^\nu \, (a_{\mu;\nu} - a_{\nu;\mu}) +$$

$$+ \frac{2}{3} \theta \, \omega_{\alpha\beta} + \sigma_{\alpha\mu} \, \omega^\mu{}_\beta - \sigma_{\beta\mu} \, \omega^\mu{}_\alpha = 0. \qquad (65)$$

2.6 - QUASI-MAXWELLIAN EQUATIONS OF GRAVITY

In this section we will present the so called quasi-Maxwellian formulation of the dynamics of gravity. This procedure explores the fact that we can use Bianchi identities as true dynamical equations of propagation of gravitational disturbances. In this way, the equation of gravity assumes a form which has a great resemblance with Maxwell's equations of eletrodynamics, as we shall see.

In any Riemannian space V_4 the curvature tensor $R_{\alpha\beta\mu\nu}$ satisfies identically the equation

$$R^{\alpha\beta}{}_{\mu\nu;\lambda} + R^{\alpha\beta}{}_{\nu\lambda;\mu} + R^{\alpha\beta}{}_{\lambda\mu;\nu} = 0, \qquad (66)$$

which is known as Bianchi identity (Eisenhart, 1949).

Multiplying equation (66) by the Levi-Civita tensor $\eta^{\mu\nu\rho\sigma}$, we have

$$\overset{*}{R}{}^{\alpha\beta\rho\sigma}{}_{;\lambda} + \frac{1}{2} R^{\alpha\beta}{}_{\nu\lambda;\mu} \eta^{\mu\nu\rho\sigma} + \frac{1}{2} \eta^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\lambda\mu;\nu} = 0$$

or

$$\overset{*}{R}{}^{\alpha\beta\rho\sigma}{}_{;\lambda} + R^{\alpha\beta}{}_{\lambda\mu;\nu} \eta^{\mu\nu\rho\sigma} = 0 . \qquad (67)$$

Contracting σ and λ:

$$\overset{*}{R}{}^{\alpha\beta\rho\sigma}{}_{;\sigma} + R^{\alpha\beta}{}_{\sigma\mu;\nu} \eta^{\mu\nu\rho\sigma} = 0$$

or

$$R^{\alpha\beta\overset{*}{\rho\sigma}}{}_{;\sigma} = 0 \ . \qquad (68)$$

Contracting indices α and μ in equation (67) we obtain:

$$R^{\alpha\beta\mu\nu}{}_{;\nu} = R^{\mu}[\alpha;\beta], \qquad (69)$$

which is equivalent to (66).

The set of equations (68) and (69) constitutes the point of departure of many theories of gravitation. Here we will restrict our analysis only to Einstein's model and postpone discussion of alternative theories of gravity to next sections.

We consider, primarily, the vacuum case. The dynamical equations for $g_{\mu\nu}$ are given by

$$R^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \ . \qquad (70)$$

Let Σ be a space-like hypersurface endowed with a given orientation specified by the vector n^{μ} normal to Σ. We impose on Σ the validity of vacuum Einstein's equations:

$$R_{\mu\nu}(\Sigma) = 0 \ . \qquad (71)$$

Thus our problem consists in showing that equation $\left[(R^{\alpha\beta\mu\nu}{}_{;\nu}=0)\right]$ plus the constraint condition (71) is equivalent to vacum Einstein's equations.

From equation (70) and (71) we obtain the tenporal derivative $\frac{\partial R^{\nu}{}_{i}}{\partial \eta^{\alpha}} \equiv R^{\nu}{}_{i,\alpha}\eta^{\alpha}$, and $\frac{\partial R^{0}{}_{0}}{\partial \eta^{\alpha}} \equiv R^{0}{}_{0,\alpha}\eta^{\alpha}$ characterized by the variation of the parameter along the normal $\vec{n}$. These quantities, once known, can be used to propagate the condition [equation (71)] on Σ, to the future of Σ, in terms of known quantities.

Indeed, we can write

$$R^{\nu}{}_{i,o} = \Gamma_{io}^{\alpha} R^{\nu}{}_{\alpha} - \Gamma_{\alpha o}^{\nu} R^{\alpha}{}_{i} + R^{\nu}{}_{o,i} + \Gamma^{\nu}_{\alpha i} R^{\alpha}{}_{o} - \Gamma^{\alpha}_{oi} R^{\nu}{}_{\alpha}$$

and

$$R^{0}{}_{o,o} = -R^{i}{}_{o,i} + \left[\Gamma^{\lambda}{}_{o\nu} - \Gamma_{\nu\rho}{}^{\rho}\right] R^{\nu}{}_{\lambda} \, .$$

Consequently, if Einstein's equations for the vacuum (71) are valid on the hypersurface Σ,then Bianchi identities will propagate these equations throughout the space-time, beyond Σ and we obtain the validity of Einstein's equations in V_4. In the case there is matter present a straightfoward generalization of the above procedure shows that,indeed, we can use Bianchi identities to propagate the dynamics of gravity. We recognize then that Einstein's equations act like a set of constraint conditions for the identities (69).

After this result, one should look for a more formal deduction of the mixed group of Einstein's equations and Bianchi identities from a variational principle. We postpone the discussion of this question. Here we will explore the consequence of this new way of looking at these equations.

In terms of the conformal Weyl tensor $W^{\alpha\beta}{}_{\mu\nu}$, Bianchi identities (69) take the form:

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha} R^{,\beta]} . \qquad (72)^*$$

Using Einstein's equations, we can re-write this expression in the form

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2} T^{\mu[\alpha;\beta]} + \frac{1}{6} g^{\mu[\alpha} T^{,\beta]} . \qquad (73)$$

From now on, equations (72) and (73) will be called the W-representation of General Relativity(Lichnerowicz, 1960; Jordan (1960)).

In absence of matter we have

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = 0 . \qquad (74a)$$

$$\overset{*}{W}{}^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \qquad (74b)$$

These equations are invariant under a dual rotation of an arbitrary constant angle θ characterized by the transformation

$$W_{\alpha\beta\mu\nu} \longrightarrow W'_{\alpha\beta\mu\nu} = \cos\theta \, W_{\alpha\beta\mu\nu} + \sin\theta \, \overset{*}{W}_{\alpha\beta\mu\nu} , \qquad (75)$$

as one can easily verify from equations (74a) and (74b). This symmetry still needs a clearer understanding*.

The W-representation of Einstein's equations is very

*See Apendix I

useful in many respects. Here we will explore the similarities between gravity and electrodynamics which are almost explicitly contained in this representation.

The source of the curvature characterized by the energy-momentum tensor $T_{\mu\nu}$ will be taken as a perfect fluid with density of energy ρ and pressure p:

$$T_{\mu\nu} = \rho\, n^{\mu}\, n^{\nu} - p\, h^{\mu}{}_{\nu} \,. \tag{76}$$

The fundamental observer, co-moving with matter, has velocity n^{μ}, which we normalize: $n^{\mu}\, n^{\nu}\, g_{\mu\nu} = +1$.

We will then project equations (72) and (73) multiplying respectively by products of n^{μ} and $h^{\mu\nu}$. We have four independent projections for the divergence of the Weyl tensor. They are:

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, n_{\beta}\, n_{\mu}\, h_{\alpha}{}^{\sigma} \tag{77a}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, \eta_{\alpha\lambda\alpha\beta}\, n^{\lambda}\, n_{\mu} \tag{77b}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, h_{\mu(\sigma}\, \eta_{\nu)\lambda\alpha\beta}\, n^{\lambda} \tag{77c}$$

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\, n_{\beta}\, h_{\mu(\rho}\, h_{\sigma)\alpha}$$

and the corresponding projection on the right hand side of equation (73).

Let us develop each one of these projections separately.

First projection: $W_{\alpha\beta\mu}{}^{\nu}{}_{;\nu}\, n^{\beta}\, n^{\mu}\, h^{\alpha\sigma}$

Using the decomposition (4) of the Weyl tensor in terms of the electric and magnetic parts we can write:

$$W_{\alpha\beta\rho}{}^{\tau}{}_{;\tau} = (\eta_{\alpha\beta\mu\nu}\,\eta_{\rho}{}^{\sigma}{}_{\lambda\tau} - g_{\alpha\beta\mu\nu}\,g_{\rho}{}^{\sigma}{}_{\lambda\tau})\cdot(n^{\mu}\,n^{\lambda}\,E^{\nu\tau})_{;\tau} \; +$$

$$(\eta_{\alpha\beta\mu\nu}\,g_{\rho}{}^{\sigma}{}_{\lambda\tau} + g_{\alpha\beta\mu\nu}\,\eta_{\rho}{}^{\sigma}{}_{\lambda\tau})(n^{\mu}\,n^{\lambda}\,H^{\nu\tau})_{;\sigma} \,.$$

Projecting in case (a) gives:

$$g_{\alpha\beta\mu\nu}\,g_{\rho}{}^{\sigma}{}_{\lambda\tau}\,h^{\alpha\varepsilon}\,n^{\beta}\,n^{\rho}(n^{\mu}\,n^{\lambda}\,E^{\nu\tau})_{;\sigma} = -h_{\nu}{}^{\varepsilon}\,E^{\nu\sigma}{}_{;\sigma} + h_{\nu}{}^{\varepsilon}\,n_{\tau}\,n^{\sigma}\,E^{\nu\tau}{}_{;\sigma} =$$

$$= -h_{\nu}{}^{\varepsilon}\,h^{\lambda\mu}\,E^{\nu}{}_{\lambda;\mu} \,.$$

The second term can be transformed in the following way:

$$\eta_{\alpha\beta\mu\nu}\,(g_{\rho\lambda}\delta^{\sigma}_{\tau} - g_{\rho\tau}\delta^{\sigma}_{\lambda})\,n^{\mu}{}_{;\sigma}\,n^{\lambda}\,H^{\nu\tau}\,n^{\rho}\,h^{\alpha\varepsilon}\,n^{\beta} = \eta_{\alpha\beta\mu\nu}\,n^{\mu}{}_{;\tau}\,h^{\alpha\varepsilon}n^{\beta}H^{\nu\tau} =$$

$$= \eta_{\alpha\beta\mu\nu}\,(\sigma^{\mu}{}_{\tau} + \frac{\theta}{3}\,h^{\mu}{}_{\tau} + \omega^{\mu}{}_{\tau} - a^{\mu}n_{\tau})\,h^{\alpha\varepsilon}\,\eta^{\beta}\,H^{\nu\tau} =$$

$$= \eta^{\varepsilon}{}_{\beta\mu\nu}\,\sigma^{\mu}{}_{\tau}\,\eta_{\beta}\,H^{\nu\tau} + \eta^{\varepsilon}{}_{\beta\mu\nu}\,\omega^{\mu}{}_{\tau}\,\eta^{\beta}\,H^{\nu\tau} \,.$$

The third term gives

$$(g_{\alpha\mu}\,g_{\beta\nu} - g_{\alpha\nu}\,g_{\beta\mu})\,\eta_{\rho}{}^{\sigma}{}_{\lambda\tau}\,n^{\mu}\,n^{\lambda}{}_{;\sigma}\,H^{\nu\tau}\,n^{\beta}\,n^{\rho}\,h^{\alpha\varepsilon} = -\eta_{\rho}{}^{\sigma}{}_{\lambda\tau}\,n^{\lambda}{}_{;\sigma}\,H^{\varepsilon\tau}\,n^{\rho} =$$

$$= -\eta_{\rho}{}^{\sigma\lambda\tau}\,\omega_{\lambda\delta}\,H^{\varepsilon}{}_{\tau}\,\eta^{\rho} \,.$$

Thus, collecting these terms we obtain

$$\omega_{\alpha\beta\rho}{}^{\sigma}{}_{;\sigma}\, n^{\beta}\, n^{\rho}\, h^{\alpha\varepsilon} = h_{\nu}{}^{\varepsilon}\, h^{\lambda\mu}\, E^{\nu}{}_{\lambda;\mu} + \eta^{\varepsilon}{}_{\beta\mu\nu}\, \sigma^{\mu}{}_{\tau}\, n^{\beta}\, H^{\nu\tau} +$$

$$+ \eta^{\varepsilon}{}_{\beta\mu\nu}\, \omega^{\mu}{}_{\tau}\, n_{\beta}\, H^{\nu\tau} - \eta^{\rho\sigma\lambda\tau}\, \omega_{\lambda\sigma}\, H^{\varepsilon}{}_{\tau}\, n_{\rho} \,.$$

Remembering the definition of the spin vector ω^{τ}:

$$\omega^{\tau} = \frac{1}{2}\, \eta^{\alpha\beta\rho\tau}\, \omega_{\alpha\beta}\, n_{\rho}\,,$$

we have:

$$\eta^{\varepsilon}{}_{\beta\mu\nu}\, \omega^{\mu}{}_{\tau}\, n^{\beta}\, H^{\nu\tau} = -\eta^{\varepsilon\beta\nu}{}_{\mu}\, \eta^{\mu}{}_{\tau\rho\sigma}\, \omega^{\rho}\, n^{\sigma}\, n_{\beta}\, H_{\nu}{}^{\tau} = -\delta^{\varepsilon\beta\nu}_{\tau\rho\sigma}\, \omega^{\rho}\, n^{\sigma}\, n_{\beta}\, H_{\nu}{}^{\tau} =$$

$$= (\delta^{\varepsilon}_{\tau}\delta^{\beta}_{\sigma}\delta^{\nu}_{\rho} - \delta^{\varepsilon}_{\tau}\delta^{\beta}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\varepsilon}_{\rho}\delta^{\beta}_{\sigma}\delta^{\nu}_{\tau} + \delta^{\varepsilon}_{\sigma}\delta^{\beta}_{\rho}\delta^{\nu}_{\tau} - \delta^{\varepsilon}_{\sigma}\delta^{\beta}_{\tau}\delta^{\nu}_{\rho} + \delta^{\varepsilon}_{\rho}\delta^{\beta}_{\tau}\delta^{\nu}_{\sigma})\, \omega^{\rho}\, n^{\sigma}\, n_{\beta}\, H_{\nu}{}^{\tau} =$$

$$= \omega^{\nu}\, H_{\nu}{}^{\varepsilon}$$

and

$$\eta^{\rho\sigma\lambda\tau}\, \omega_{\lambda\sigma}\, n_{\rho}\, H^{\varepsilon}{}_{\tau} = -2\omega_{\tau}\, H^{\varepsilon\tau}.$$

Finally, we obtain

$$\omega_{\alpha\beta\rho}{}^{\sigma}{}_{;\sigma}\, n^{\beta}\, n^{\rho}\, h^{\alpha\varepsilon} = h_{\nu}{}^{\varepsilon}\, h^{\lambda\mu}\, E^{\nu}{}_{\lambda;\mu} + \eta^{\varepsilon}{}_{\beta\mu\nu}\, \sigma^{\mu}{}_{\tau}\, n^{\beta}\, H^{\tau\nu} + 3\omega_{\tau}\, H^{\tau\varepsilon} \quad (79).$$

Let us now evaluate the right-hand side of equations (73) projected by (77a).

We have

$$T^{\mu[\alpha;\beta]} n_\beta n_\mu h_\alpha{}^\varepsilon = (\rho n^\mu n^{[\alpha})^{;\beta]} n_\beta n_\mu h_\alpha{}^\varepsilon - (p h^{\mu[\alpha})^{;\beta]} n_\beta n_\mu h_\alpha{}^\varepsilon =$$

$$= \rho n^\mu n^{\alpha;\beta} n_\beta n_\mu h_\alpha{}^\varepsilon - (\rho n^\mu n^\beta)^{;\alpha} n_\beta n_\mu h_\alpha{}^\varepsilon - p h^{\mu[\alpha;\beta]} n_\beta n_\mu h_\alpha{}^\varepsilon =$$

$$= \rho a^\varepsilon - \rho_{,\alpha} h^{\alpha\varepsilon} + p a^\varepsilon .$$

Thus:

$$\{-\frac{1}{2} T^{\mu[\alpha;\beta]} + \frac{1}{6} g^{\mu[\alpha} T^{,\beta]}\} n_\beta n_\mu h_\alpha{}^\varepsilon = -\frac{1}{2}\left[(\rho+p)a^\varepsilon - \rho_{,\alpha} h^{\alpha\varepsilon}\right] -$$

$$-\frac{1}{6}(\rho-3p)_{,\alpha} h^{\alpha\varepsilon} = \frac{1}{3}\rho_{,\alpha} h^\alpha{}_\varepsilon + \frac{1}{2} p_{,\alpha} h^\alpha{}_\varepsilon - \frac{1}{2}(\rho+p) a_\varepsilon .$$

Now, from the "conservation" law $T^{\mu\nu}{}_{;\nu} = 0$ projecting on the rest space of n^μ we obtain

$$T^{\mu\nu}{}_{;\nu} h_{\mu\alpha} = 0 ,$$

which implies,

$$(\rho+p) a_\varepsilon - p_{,\alpha} h^\alpha{}_\varepsilon = 0 .$$

Finally, collecting all those results we obtain:

$$h_\nu{}^\varepsilon h^{\lambda\mu} E^\nu{}_{\lambda;\mu} + \eta^\varepsilon{}_{\beta\mu\nu} \sigma^\mu{}_\tau n_\beta H^{\nu\tau} + 3\omega_\tau H^{\tau\varepsilon} = \frac{1}{3}\rho_{,\alpha} h^{\alpha\varepsilon} . \quad (80)$$

The remaining projection can be evaluated in a very similar way as we did for the previous case. We do not reproduce here the rather long and tedious calculation but we give only the final result:

<u>Second Projection</u>:

$$W_{\alpha\beta\mu}{}^{\nu}{}_{;\nu}\ \eta^{\sigma\lambda\alpha\beta}\ n_\lambda\ n^\mu\ .$$

We obtain,

$$h_\nu{}^\varepsilon\ h^{\lambda\mu}\ H^\nu{}_{\lambda;\mu} - \eta^\varepsilon{}_{\beta\mu\nu}\ \sigma^\mu{}_\tau\ n^\beta\ E^{\nu\tau} - 3\omega_\tau\ E^{\tau\varepsilon} = 0. \tag{81}$$

<u>Third Projection</u>:

$$W_{\alpha\beta\mu}{}^{\nu}{}_{;\nu}\ h^{\mu(\sigma}{}_\eta{}^{\nu)\lambda\alpha\beta}\ n_\lambda\ .$$

We obtain:

$$\dot{E}^{\mu\nu}\ h_\mu{}^\rho\ h_\nu{}^\tau + \theta\ E^{\rho\tau} - \frac{1}{2}\ E_\nu{}^{(\rho}{}_\sigma{}^{\tau)\nu} - \frac{1}{2}\ E_\nu{}^{(\rho}{}_\omega{}^{\tau)} +$$
$$+ \eta^{\tau\nu\mu\varepsilon}\ \eta^{\rho\lambda\alpha\beta}\ n_\mu\ n_\lambda\ E_{\varepsilon\alpha}\ \sigma_{\beta\nu} + a_\alpha\ H_\beta{}^{(\rho}{}_\eta{}^{\tau)\lambda\alpha\beta}\eta_\lambda -$$
$$- \frac{1}{2}\ H_\beta{}^\mu{}_{;\alpha}\ h_\mu{}^{(\tau}{}_\eta{}^{\rho)\lambda\alpha\beta}\eta_\lambda = -\frac{1}{2}\ (\rho+p)\sigma^{\rho\tau}\ . \tag{82}$$

<u>Fourth Projection</u>:

$$W^{\alpha\beta\mu\nu}{}_{;\nu}\ n_\beta\ h_{\mu(\rho}h_{\sigma)\alpha}\ .$$

We obtain:

$$\dot{H}^{\mu\nu}\ h_\mu{}^\rho\ h_\nu{}^\sigma + \theta\ H^{\rho\sigma} - \frac{1}{2}\ H_\nu{}^{(\rho}{}_\sigma{}^{\sigma)\nu} - \frac{1}{2}\ H_\nu{}^{(\rho}{}_\omega{}^{\sigma)} +$$
$$+ \eta^{\sigma\nu\xi\varepsilon}\ \eta^{\rho\lambda\alpha\beta}\ n_\mu n_\lambda H_{\varepsilon\alpha}\ \sigma_{\beta\nu} - a_\alpha\ E_\beta^{(\rho}\eta^{\sigma)\lambda\alpha\beta}\ n_\lambda + \frac{1}{2}\ E^\mu_{\beta;\alpha}\ h_\mu{}^{(\sigma}{}_\nu{}^{\rho)\lambda\alpha\beta}\ n_\lambda = 0. \tag{83}$$

Equations (80), (81), (82) and (83) constitute the set of quasi-Maxwellian system obtained in the W-representation. These equations, complemented by the kinematical evolution equations and the conservation of the energy-momentum tensor, constitute the whole set which describes the dynamics of matter and geometry, in Einstein's General Relativity.

<u>Two Examples of the Use of Quasi-Maxwellian Equations of Gravity Complemented by the Evolution of Kinematical Quantities</u>.

In order to gain some insight on the behavior of these equations, let us apply them in two special and simple distinct cases: i) a conformally flat geometry having a perfect fluid as its source(Friedmann, Cosmos) and ii) a vacuum solution with anisotropic spatially homogeneous three-dimensional flat section (Kasner Vacuum Universe).

Let us examine each of these solutions separately:

(i) <u>Friedmann Geometry</u> - The fundamental lenght has the form

$$ds^2 = dt^2 - A^2(t)\ g_{ij}(\chi^{\kappa}) d\chi^i\ dxj.$$

The source of the geometry is a perfect fluid with density energy ρ and pressure p. It has a non-vanishing expansion θ, no shear , no rotation, no acceleration. In the co-moving coordinate system the velocity of the fluid takes the form $v^{\alpha} = \delta^{\alpha}_{0}$.

From equation (80) we obtain

$$\rho_{,\alpha}\ h^{\alpha}{}_{\lambda} = 0\ . \tag{84}$$

This means that ρ is spatially homogeneous, that is, $\rho = \rho(t)$.

From the conservation equations we obtain

$$\dot{\rho} + (\rho + p)\theta = 0 \tag{85a}$$

and

$$p_{,i} = 0 \, , \tag{85b}$$

that is, p= p(t).

From eq. (61),

$$\dot{\theta} + \frac{1}{3}\,\theta^2 = -\frac{1}{2}\,(\rho + 3p) + \Lambda \, . \tag{86}$$

All these are consequence of the quasi-Maxwellian equations supplemented by the evolution of the kinematical quantities. To these equations, we have to impose the initial conditions that Einstein's equations of motion are satisfied in a given hypersurface Σ.

This gives a constraint relating the expansion θ to the density of energy ρ:

$$\frac{1}{3}\,\theta^2 + \frac{3}{A^2}\,\varepsilon = \rho - \Lambda \, , \tag{87}$$

in which ε is proportional to the three-dimensional scalar ($\varepsilon = \frac{{}^{(3)}R}{6}$).

We recognize that eq. (85) and (86) are precisely the dynamical set of Einstein's equations for the Friedmann Universe.

(ii) Kasner Geometry

This is a vacuum solution of Einstein's equations without cosmological term, which has a 3-dimensional hypersurface of homogeneity which constitutes the section t= constant, for the global time t.

In the gaussian system of coordinates we write the fundamental element of the lenght in the form

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \quad (88)$$

The velocity of the fundamental observer in this system of coordinates is given by

$$V^\mu = \delta^\mu_0 . \quad (89)$$

The expansion factor θ and the shear σ^i_j are, respectively,

$$\theta = \frac{\dot{V}}{V} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} , \quad (90)$$

in which V is the volume of the 3-dimensional section ,

$$\sigma^1_1 = \frac{\dot{a}}{a} - \frac{1}{3}\theta \quad (91a)$$

$$\sigma^2_2 = \frac{\dot{b}}{b} - \frac{1}{3}\theta \quad (91b)$$

$$\sigma^3_3 = \frac{\dot{c}}{c} - \frac{1}{3}\theta . \quad (91c)$$

The other kinematical quantities vanish.

We have chosen this model to exemplify the power of the quasi-Maxwellian equations of gravity, since Kassner geometry is not conformally flat, and thus, in this case the equations are not trivial.

A straightforward calculation shows that the magnetic part $H_{\mu\nu}$ of the Weyl tensor vanishes identically, and that the electric part $E_{\mu\nu}$ is diagonal. We have:

$$E^1_1 = -\frac{1}{3}\frac{\ddot{a}}{a} + \frac{1}{6}\left(\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right) - \frac{1}{3}\frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{1}{6}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)\frac{\dot{a}}{a} \tag{92a}$$

$$E^2_2 = -\frac{1}{3}\frac{\ddot{b}}{b} + \frac{1}{6}\left(\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c}\right) - \frac{1}{3}\frac{\dot{a}}{a}\frac{\dot{c}}{c} + \frac{1}{6}\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right)\frac{\dot{b}}{b} \tag{92b}$$

$$E^3_3 = -\frac{1}{3}\frac{\ddot{c}}{c} + \frac{1}{6}\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b}\right) - \frac{1}{3}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{1}{6}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\frac{\dot{c}}{c} \,. \tag{92c}$$

Let us denote, for convenience ,

$$E^1_1 = \alpha \;;\; E^2_2 = \beta \;;\; E^3_3 = \gamma \,.$$

The traceless condition imposes

$$\alpha + \beta + \gamma = 0 \,. \tag{93}$$

After a rather long but direct calculation, we can transform equations (80, 81, 82, 83) in the set:

$$\dot{\alpha} + 2\alpha\theta - 2\alpha\frac{\dot{a}}{a} + \beta\frac{\dot{b}}{b} + \gamma\frac{\dot{c}}{c} = 0 \tag{94}$$

$$\dot{\beta} + 2\beta\theta - 2\beta\frac{\dot{b}}{b} + \alpha\frac{\dot{a}}{a} + \gamma\frac{\dot{c}}{c} = 0 \tag{95}$$

$$\dot{\gamma} + 2\gamma\theta - 2\gamma\frac{\dot{c}}{c} + \alpha\frac{\dot{a}}{a} + \beta\frac{\dot{b}}{b} = 0 . \tag{96}$$

These are the quasi-Maxwellian equations which must be suplemented by the initial condition

$$R_{\mu\nu}(\Sigma) = 0 . \qquad (97)$$

We choose Σ to be the surface $t = t_0 =$ constant.
We set the ansatz:

$$a(t) = a_0 \; t^{p_1} \qquad (98a)$$

$$b(t) = b_0 \; t^{p_2} \qquad (98b)$$

$$c(t) = c_0 \; t^{p_3} . \qquad (98c)$$

Solving the Cauchy condition (97) gives two constraints:

$$p_1 + p_2 + p_3 = 1 \qquad (99)$$

$$p_1^2 + p_2^2 + p_3^2 = 1 . \qquad (100)$$

For the ansatz (98) the equations of evolution reduce to the following algebraic relations

$$p_1^2 \; (p_1 - 1) = p_2^2 \; (p_2 - 1) \qquad (101)$$

$$p_1^2 \; (p_1 - 1) + p_1 \; p_3 (p_1 - 1) + p_2 \; p_3 (p_2 - 1) = 0 . \qquad (102)$$

It needs a simple algebraic manipulation to show that equations (101, 102) are consequence of equations (99,100) and thus the system merely propagates the initial conditions throughout sucessive epochs, in the future of Σ.

2.7 - CONSERVATION LAWS

Let $T^{\mu}{}_{\nu}$ represent the most general fluid, as seen by an arbitrary observer n^{μ} which co-moves with the fluid. We write

$$T_{\mu\nu} = \rho\, n_{\mu}\, n_{\nu} - p\, h_{\mu\nu} + q_{(\mu}\, n_{\nu)} + \pi_{\mu\nu} \,, \tag{103}$$

in which besides the density of energy ρ, the isotropic pressure p, we have introduced the four heat conduction q_{μ} and the anisotropic (traceless and symmetric) pressure $\pi_{\mu\nu}$. These two quantities satisfy the properties

$$q^{\mu}\, n_{\mu} = 0, \tag{104}$$

wich states that the heat flux rests on the surface orthogonal to the direction of the velocity, and

$$\begin{aligned} \pi_{\mu\nu} &= \pi_{\nu\mu} \\ \pi_{\mu\nu}\, g^{\mu\nu} &= 0 \\ \pi_{\mu\nu}\, n^{\nu} &= 0 \end{aligned} \tag{105}$$

which say that n^{ν} is an eigen-vector of $\pi^{\mu}{}_{\nu}$ with null eigenvalue.

From the divergenceless of $T_{\mu\nu}$ projecting parallel and orthogonal to n^{μ} we obtain:

$$T^{\mu\nu}{}_{;\nu}\, n_{\mu} = 0 \,,$$

which, using section (2.5), can be written in the form:

$$\dot{\rho} + (\rho + p)\theta + \dot{q}^{\mu}\, n_{\mu} + q^{\mu}{}_{;\mu} - \pi^{\mu\nu}\, \sigma_{\mu\nu} = 0. \qquad (106)$$

Projecting $T^{\mu\nu}{}_{;\nu}=0$ by $h_{\mu\nu}$,

$$T^{\mu\nu}{}_{;\nu}\, h_{\mu\alpha} = 0,$$

which yields

$$(\rho + p)a_{\alpha} - p_{,\mu}\, h^{\mu}{}_{\alpha} + \dot{q}_{\mu} h^{\mu}{}_{\alpha} + \theta q_{\alpha} + q^{\nu}\sigma_{\alpha\nu} + q^{\nu}\, \omega_{\alpha\nu} +$$

$$+ \pi_{\alpha}{}^{\nu}{}_{;\nu} + \pi^{\mu\nu}\, \sigma_{\mu\nu}\, n_{\alpha} = 0\,. \qquad (107)$$

For the special case of a perfect fluid, which will be the case most extensively treated in these notes, we have

$$q_{\mu} = 0$$

$$\pi_{\mu\nu} = 0$$

and thus the equations of conservation reduce to:

$$\dot{\rho} + (\rho + p)\, \theta = 0 \qquad (108a)$$

$$(\rho + p)\, a_{\mu} - p_{,\nu}\, h^{\nu}{}_{\mu} = 0. \qquad (109)$$

Remember that a dot means derivative in the direction of $\vec{n}$.

2.8 - CONSTRAINT RELATIONS FOR THE KINEMATICAL QUANTITIES

Besides the equations of evolution for the kinematical parameters (θ, $\tau_{\mu\nu}$, $\omega_{\mu\nu}$), there are three more equations envolving these parameters and the metric quantities which do not contain any derivative with respect to the time (that is, a derivative projected in the direction of the fluid velocity). Let us write explicitly these equations, in order to obtain a complete and self-consistent set of equations.

From the definition of the curvature tensor we write

$$\eta_{\alpha;\beta;\gamma} - \eta_{\alpha;\gamma;\beta} = R_{\alpha\varepsilon\beta\gamma}\, n^{\varepsilon} \, . \qquad (110)$$

Contracting α and β and projecting into H we obtain

$$h^{\gamma}{}_{\lambda}(n^{\alpha}{}_{;\alpha})_{;\gamma} - h^{\gamma}{}_{\lambda}\, n^{\alpha}{}_{;\gamma;\alpha} = R_{\varepsilon\gamma}\, n^{\varepsilon}\, h^{\gamma}{}_{\lambda} \, . \qquad (111)$$

However, using (55) we can write

$$h^{\gamma}{}_{\lambda}\, n^{\alpha}{}_{;\gamma;\alpha} = (\sigma^{\alpha}{}_{\gamma} + \omega^{\alpha}{}_{\gamma})_{;\alpha} h^{\gamma}{}_{\lambda} + \frac{1}{3}\,\theta_{,\alpha}\, h^{\alpha}{}_{\lambda} + a^{\alpha}(\sigma_{\gamma\alpha} + \omega_{\gamma\alpha}) \, .$$

Using this result into equation (111) we obtain

$$\frac{2}{3}\theta_{,\mu}\, h^{\mu}{}_{\lambda} - (\sigma^{\alpha}{}_{\gamma} + \omega^{\alpha}{}_{\gamma})_{;\alpha}\, h^{\gamma}{}_{\lambda} - a^{\alpha}(\sigma_{\gamma\alpha} + \omega_{\gamma\alpha}) = R_{\mu\alpha}\, n^{\mu}\, h^{\alpha}{}_{\lambda} , \qquad (112)$$

which is the first constraint equation. Let us go into the second relation.

Using (110) three times and changing indices we obtain

$$(n_{\alpha;\beta} - n_{\beta;\alpha})_{;\gamma} + (n_{\gamma;\alpha} - n_{\alpha;\gamma})_{;\beta} + (n_{\beta;\gamma} - n_{\gamma;\beta})_{;\alpha} =$$

$$= (R^{\mu}{}_{\alpha\gamma\beta} + R^{\mu}{}_{\beta\alpha\gamma} + R^{\mu}{}_{\gamma\beta\alpha})n_{\mu}.$$

The right hand side of this expression vanishes due to the symmetries of the Riemann tensor.

Using equation (50) we have

$$n_{\alpha;\beta} - n_{\beta;\alpha} = 2\omega_{\alpha\beta} + a_{\alpha}\, n_{\beta} - a_{\beta}\, n_{\alpha} \;.$$

Thus,

$$(2\omega_{\alpha\beta} + a_{\alpha}n_{\beta} - a_{\beta}n_{\alpha})_{;\gamma}\, \eta^{\alpha\beta\gamma\lambda} = 0,$$

or, using the fact that $g_{\mu\nu;\lambda}=0$, this equation reduces to

$$(\omega_{\alpha\beta}\, \eta^{\alpha\mu\gamma\lambda})_{;\gamma} + (a_{\alpha}\, n_{\beta})_{;\gamma}\, \eta^{\alpha\beta\gamma\lambda} = 0 \;.$$

Multiplying by n^{λ}:

$$(\omega_{\alpha\beta}\, \eta^{\alpha\beta\gamma\lambda})_{;\gamma}\, n_{\lambda} + a_{\alpha}\, n_{\beta;\gamma}\, n_{\lambda}\, \eta^{\alpha\beta\gamma\lambda} = 0.$$

Remembering the definition of the spin vector ω_{ϕ} we can transform this equation to its final form

$$\omega^{\alpha}{}_{;\alpha} + 2\omega^{\alpha}\, a_{\alpha} = 0 \;, \tag{113}$$

which constitutes the second equation of constraint. Finally , let us deduce the third and last constraint.

Multiplying equations (110) by the tensor $\eta_{\rho}{}^{\gamma\beta\varepsilon} n_{\varepsilon}$ we have

$$n_{\alpha;\beta;\gamma} \eta_{\rho}{}^{\gamma\beta\varepsilon} n_{\varepsilon} = \frac{1}{2} R_{\alpha\mu\beta\gamma} n^{\mu} \eta_{\rho}{}^{\gamma\beta\varepsilon} n_{\varepsilon}$$

Using equations (50) and a little of algebraic manipulations yields the third equation of constraint

$$(\sigma_{\beta(\tau} - \omega_{\beta(\tau)}{}_{;\gamma} \eta_{\sigma)}{}^{\gamma\beta\varepsilon} n_{\varepsilon} + 2a_{(\tau}\omega_{\sigma)} = \frac{1}{2} R^{\mu\beta\gamma}_{(\tau} n^{\mu} n_{\varepsilon} \eta_{\sigma)\gamma\beta}{}^{\varepsilon} \tag{114}$$

We can simplify the right-hand side of this expresssion by noting the definition of dual given by (5)

$$\frac{1}{2} R_{\tau}{}^{\mu\beta\gamma} \eta_{\sigma\gamma\beta\varepsilon} n_{\mu} n^{\varepsilon} = -R_{\tau\mu\overset{*}{\sigma}\varepsilon} n^{\mu} n^{\varepsilon}.$$

Using expression (1)

$$\frac{1}{2} R_{\tau}{}^{\mu\beta\gamma} \eta_{\sigma\gamma\beta\varepsilon} n_{\mu} n^{\varepsilon} = -\overset{*}{W}_{\tau\lambda\sigma\varepsilon} n^{\mu} n^{\varepsilon} - H_{\tau\mu\overset{*}{\sigma}\varepsilon} n^{\mu} n^{\varepsilon} +$$

$$+ \frac{1}{6} R \eta_{\tau\mu\sigma\varepsilon} n^{\mu} n^{\varepsilon}$$

$$= -\overset{*}{W}_{\tau\mu\sigma\varepsilon} n^{\mu} n^{\varepsilon} - \frac{1}{2} \eta_{\sigma\varepsilon}{}^{\rho\lambda} H_{\tau\mu\rho\lambda} n^{\mu} n^{\varepsilon}$$

$$= -\overset{*}{W}_{\tau\mu\sigma\varepsilon} n^{\mu} n^{\varepsilon} - \frac{1}{2} \eta_{\sigma\varepsilon}{}^{\rho\lambda} n^{\mu} n^{\varepsilon} (R_{\mu\beta} g_{\tau\alpha} - g_{\tau\beta} R_{\mu\alpha})$$

symmetrization of this expression gives:

$$\frac{1}{2} R_{(\tau}{}^{\mu\beta\gamma} \eta_{\sigma)\gamma\beta\varepsilon} n_{\mu} n^{\varepsilon} = -2\overset{*}{W}_{\tau\mu\sigma\varepsilon} n^{\mu} n^{\varepsilon}$$

By the definition (4) this is equivalent to $H_{\tau\sigma}$.

Thus equation (114) simplifies to

$$\frac{1}{2} (\sigma_{\beta(\tau} - \omega_{\beta(\tau})_{;\gamma} \eta_{\sigma)}{}^{\gamma\beta\varepsilon} n_{\varepsilon} + a_{(\tau}\omega_{\sigma)} = H_{\tau\sigma} \qquad (115)$$

This accomplishes the task of obtaining the complete system of equations (dynamics plus constraints) for the kinematical parameters of the fluid.

2.9 - THE EQUATIONS OF STRUCTURE OF CARTAN'S FORMALISM

Let us define a set of four linearly independent 4-vectors, $e^{\alpha}_{A}(x)$, throughout the manifold V_4, in which a metric has been assigned and defined by its components $g_{\mu\nu}(x)$ in a given coordinate system $\{x^{\alpha}\}$.

The metric g in the tetrad frame takes the form

$$g_{AB} \equiv \vec{e}_A \cdot \vec{e}_B = g_{\mu\nu}(x)\; e^{\mu}_{A}(x)\; e^{\nu}_{B}(x) \,. \tag{116}$$

We can invert this expression, since we can obtain e^{B}_{α}(*) from the vectors $e^{\alpha}_{A}(x)$, through:

$$e^{A}_{\alpha}(x)\; e^{\alpha}_{B}(x) = \delta^{A}{}_{B} \tag{117a}$$

$$e^{A}_{\alpha}(x)\; e^{\beta}_{A}(x) = \delta^{\beta}_{\alpha} \,. \tag{117b}$$

Thus we can write

$$g_{\mu\nu}(x) = e^{A}_{\mu}(x)\; e^{B}_{\nu}(x)\; g_{AB} \,. \tag{118}$$

Although a given set of tetrads fixes univocally the metric $g_{\mu\nu}(x)$ (once g_{AB} is given), the converse is not true.

Let us choose the tetrads as an inertial frame and set $g_{AB} \equiv \eta_{AB} = \mathrm{diag}(+1,-1,-1,-1)$.

Any local rotation of the tetrads, characterized by a Lorentz matrix $\Lambda^{A}{}_{B}$, transforms the e^{α}_{A} frame into another inertial frame e'^{α}_{A}:

$$e'^{A}_{\alpha}(x) = \Lambda^{A}_{B}(x)\; e^{B}_{\alpha}(x). \tag{119}$$

Since Λ^{A}_{B} is a Lorentz matrix the tetrad metric g_{AB} is left unchanged:

$$g'_{AB} = \Lambda_{A}^{\;C}\; g_{CD}\; \Lambda^{D}_{\;B} = g_{AB}$$

and consequently the metric $g_{\mu\nu}(x)$ is left unchanged too:

$$g'_{\mu\nu}(x) = e'^{A}_{\mu}(x)\; e'^{B}_{\nu}(x)\; g'_{AB} = \Lambda^{A}_{\;C}\; \Lambda^{B}_{\;D}\; e^{C}_{\;\mu}\; e^{D}_{\;\nu}\; g'_{AB} =$$

$$= \Lambda^{A}_{\;C}\; \Lambda^{B}_{\;D}\; e^{C}_{\;\mu}\; e^{D}_{\;\nu}\; \Lambda_{A}^{\;M}\; \Lambda_{B}^{\;N}\; g_{MN}.$$

Now,

$$\Lambda^{A}_{\;B}\; \Lambda_{A}^{\;C} = \delta^{C}_{\;B},$$

then,

$$g'_{\mu\nu}(x) = \delta^{M}_{\;C}\; \delta^{N}_{\;D}\; e^{C}_{\;\mu}\; e^{D}_{\;\nu}\; g_{MN} = e^{C}_{\;\mu}\; e^{D}_{\;\nu}\; g_{CD} = g_{\mu\nu}(x).$$

We remark that the rotation (119) is space-time dependent, since Lorentz matrix Λ^{A}_{B} may be different from one point to another. This independence of the metric on a space-time point dependent Lorentz rotation is usually called the gauge invariance of the gravitational field. In case the geometry is invar iant for a global Lorentz rotation then the metric of the space-time reduces to the flat Monkowskian structure.

We define 1-forms θ^A in terms of the tetrads by the expression:

$$\theta^A = e^A{}_\alpha \, dx^\alpha . \tag{120}$$

Taking the exterior derivative of this 1-form we obtain the connection 1-form $\omega^A{}_B$:

$$d\theta^A = -\omega^A{}_B \wedge \theta^B \tag{121}$$

in which the symbol $\wedge$ means the wedge product of the Grassmann algebra [see Flanders (1963) for a concise and very simple introduction to the calculus with differential forms].

Since θ^A constitutes a basis for 1-forms we can develop the connection $\omega^A{}_B$ in this basis introducing the so-called Ricci coefficients $\gamma^A{}_{BC}$:

$$\omega^A{}_B \equiv \gamma^A{}_{BC} \, \theta^C . \tag{122}$$

Then, from (121) and (122) we obtain directly the expression for $\gamma^A{}_{BC}$ in terms of the tetrads:

$$\gamma^A{}_{MN} = -e^A{}_{\alpha;\mu} \, e^\alpha{}_M \, e^\mu{}_N . \tag{123}$$

Or, taking the inverse of this expression, using (117):

$$e^{A}{}_{\alpha;\beta} = -\gamma^{A}{}_{BC}\, e^{B}{}_{\alpha}\, e^{C}{}_{\beta}. \qquad (124)$$

From the definition of the curvature tensor we have:

$$e^{A}{}_{\alpha;\mu;\nu} - e^{A}{}_{\alpha;\nu;\mu} = R_{\alpha}{}^{\lambda}{}_{\mu\nu}\, e^{A}{}_{\lambda}\, . \qquad (125)$$

Taking covariant derivative of (124) and anti-symmetrising we obtain, using (125)

$$-\frac{1}{2} R^{A}{}_{BCD}\, \theta^{C} \wedge \theta^{D} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B}\ ;$$

or, denoting the left hand side of this expression as the 2-form curvature $\Omega^{A}{}_{B}$ we can write

$$\Omega^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B}\ . \qquad (126)$$

Expressions (121) and (126) are the basic equations of structure of Cartan for the formalism of exterior differentials. They describe in a very concise form the content of Riemannian metric geometry.

2.10 - THE COMPLEX VECTORIAL FORMALISM

The essential point of the complex vectorial formalism introduced by Debever (1964), Cahen, Debever and Defrise (1969) depends on the observation of the existence of an isomorphism between the group SO_3C of the rotations in the complex three dimensional space and the Lorentz group L.

Thus, one can map bi-vectors of the 4-dimensional manifold V_4 into vectors of a 3-dimensional complex space C_3 and locally construct a formalism by means of which all relevant quantities of the geometry of V_4 are mapped into objects of C_3. Then, using Cartan method of moving tetrad frame (Cartan) this map can be extended to cover the whole manifold. Here we will briefly sketch the main lines of this procedure.

Let us choose the tetrad to be generated by a set of complex null vectors $e^{\alpha}{}_{A}$ such that θ^0, θ^1 are real and θ^2, θ^3 are complex conjugated, that is

$$\begin{aligned} \overline{\theta}^0 &= \theta^0 \\ \overline{\theta}^1 &= \theta^1 \\ \overline{\theta}^2 &= \theta^3. \end{aligned} \qquad (127)$$

The fundamental length takes the form

$$ds^2 = 2(\theta^0\theta^1 - \theta^2\theta^3). \qquad (128)$$

The tetrad metric g_{AB} is, in this case, generated by Pauli matrix σ_1:

$$g_{AB} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} = g^{AB} \tag{129}$$

and the 2x2 matrix σ_1 has the standard value

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Let us consider now a set of complex anty-symmetric matrices $Z^m{}_{AB}$, $\bar{Z}^m{}_{AB}$ satisfying the following algebra [see Israel ( )]:

$$Z^m{}_{AB}\, Z^{nB}{}_C + Z^m{}_{CB}\, Z^{nB}{}_A = -\gamma^{mn}\, g_{AC} \tag{130a}$$

$$Z^m{}_{AB}\, Z^{nB}{}_C - Z^m{}_{CB}\, Z^{nB}{}_A = -\varepsilon^{mnq}\, Z_{qAC} \tag{130b}$$

$$Z^m{}_{AB}\, \bar{Z}^{nB}{}_C - Z^m{}_{CB}\, \bar{Z}^{nB}{}_A = 0 , \tag{130c}$$

in which ε^{mnq} is the Levi-Civita completelly anti-symmetric tensor, g_{AB} is the metric (129) and γ^{mn} is the metric in C_3 defined by the internal product

$$\vec{Z}^m \cdot \vec{Z}^n = 2\gamma^{mn}, \tag{131}$$

which means

$$\frac{1}{4}\, Z^m{}_{AB}\, Z^n{}_{CD}\, g^{ABCD} = \gamma^{mn} . \tag{132}$$

In this formula g^{ABCD} behaves like a metric for the space of bi-vectors of V_4 and is defined in terms of the metric g_{AB} by the formula:

$$g_{ABCD} = g_{AC}\, g_{BD} - g_{AD}\, g_{BC} \cdot \qquad (133)$$

From equations (130a,b) we obtain

$$Z^m{}_{AB}\, Z^{nB}{}_C = -\frac{1}{2}\left[\gamma^{mn}\, g_{AC} + \varepsilon^{mnq}\, Z_{qAC}\right] \cdot \qquad (134)$$

A specific realization of such set of Z's can be constructed and will be used later on for calculations. We set the 4x4 matrices $Z^m{}_{AB}$ in the form:

$$Z^1{}_{AB} = \begin{pmatrix} 0 & -\frac{1}{2}(\sigma_1 + i\sigma_2) \\ \frac{1}{2}(\sigma_1 - i_{\sigma_2}) & 0 \end{pmatrix}$$

$$Z^2{}_{AB} = \begin{pmatrix} 0 & \frac{1}{2}(\sigma_1 - i\sigma_2) \\ -\frac{1}{2}(\sigma_1 + i\sigma_2) & 0 \end{pmatrix}$$

$$Z^3{}_{AB} = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$\overline{Z}^{1}_{AB} = \begin{pmatrix} 0 & -\frac{1}{2}(\sigma_3 + I_2) \\ \frac{1}{2}(\sigma_3 + I_2) & 0 \end{pmatrix}$$

$$\overline{Z}^{2}_{AB} = \begin{pmatrix} 0 & -\frac{1}{2}(\sigma_3 - I_2) \\ \frac{1}{2}(\sigma_3 - I_2) & 0 \end{pmatrix}$$

$$\overline{Z}^{3}_{AB} = -\frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \tag{135}$$

in which $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ form with σ_1 the set of Pauli matrices, and I_2 is the identity in two dimensions.

These definitions enable us to introduce a basis for 2-forms in V_4 represented by vectors in C_3. We define the 2-forms vectors in C_3, Z^m, by means of the above matrices $Z^m{}_{AB}$:

$$Z^m = \frac{1}{2} Z^m{}_{AB}\, \theta^A \wedge \theta^B \tag{136a}$$

$$\overline{Z}^m = \frac{1}{2} \overline{Z}^m{}_{AB}\ \theta^A \wedge \theta^B, \tag{136b}$$

which gives

$$Z^1 = -\theta^0 \wedge \theta^3$$

$$Z^2 = \theta^1 \wedge \theta^2$$

$$Z^3 = -\frac{1}{2}\left[\theta^0 \wedge \theta^1 + \theta^2 \wedge \theta^3\right]$$

$$\overline{Z}^1 = -\theta^0 \wedge \theta^2$$

$$\overline{Z}^2 = \theta^1 \wedge \theta^3$$

$$\overline{Z}^3 = -\frac{1}{2}\left[\theta^0 \wedge \theta^1 - \theta^2 \wedge \theta^3\right].$$

Defined in this way the set $(Z^m{}_{AB}, \overline{Z}^n{}_{AB})$ constitutes a basis for bivectors in V_4 and $(Z^m, \overline{Z}^m)$ constitutes a basis for the corresponding two-forms.

We can use such basis to describe the bi-vectors of the Riemannian geometry V_4, that is we develop the connection 1-form ω_{AB} and the curvature 2-form $\Omega^A{}_B$ in this basis:

$$\omega_{AB} = \omega_m Z^m{}_{AB} + \overline{\omega}_m \overline{Z}^m{}_{AB} \tag{137}$$

$$\Omega_{AB} = \Omega_m Z^m{}_{AB} + \overline{\Omega}_m \overline{Z}^m{}_{AB}\ . \tag{138}$$

The first equation of structure of Cartan can be written in C_3 by a direct derivation of the 1-form Z^A. We have

$$Z^m = \frac{1}{2} Z^m{}_{AB}\, \theta^A \wedge \theta^B.$$

Taking the exterior derivative of this expression and noting that $Z^m{}_{AB}$ are constant numbers, we obtain:

$$dZ^m = Z^m{}_{AB}\, d\theta^A \wedge \theta^B = -Z^m{}_{AB}\, \omega^A{}_C \wedge \theta^C \wedge \theta^B =$$

$$= -Z^m{}_{AB}\, (\omega_m Z^{mA}{}_C + \bar{\omega}_m \bar{Z}^{mA}{}_C) \wedge \theta^C \wedge \theta^B ;$$

using identities (130) we obtain

$$dZ^m = \frac{1}{2} \varepsilon^{mpq}\, \omega_p\, Z_{qAB} \wedge \theta^A \wedge \theta^B. \tag{139}$$

This equation represents in C_3 the first equation of structure of Cartan.

In a similar manner we obtain the second Cartan structural equation in C_3.

We have:

$$\Omega_{AB} = \Omega_m Z^m{}_{AB} + \bar{\Omega}_m \bar{Z}^m{}_{AB}$$

and

$$\Omega_{AB} = d\omega_{AB} + \omega_{AC} \wedge \omega^C{}_B .$$

Thus,

$$\Omega_m Z^m{}_{AB} + \bar{\Omega}_m \bar{Z}^m{}_{AB} = d(\omega_m Z^m{}_{AB} + \bar{\omega}_m \bar{Z}^m{}_{AB}) + (\omega_m Z^m{}_A{}^C +$$

$$+ \bar{\omega}_m \bar{Z}^m{}_A{}^C) \Lambda (\omega_n Z^n{}_{CB} + \bar{\omega}_n \bar{Z}^n{}_{CB}) =$$

$$= d\omega_m Z^m{}_{AB} + d\bar{\omega}_m \bar{Z}^m{}_{AB} + \frac{1}{2} \omega_m \Lambda\omega_n (Z^m{}_A{}^C Z_{nCB} -$$

$$- Z^m{}_B{}^C Z_{nCA}) + \omega_m \Lambda\bar{\omega}_n Z^m{}_A{}^C \bar{Z}^n{}_{CB} +$$

$$+ \bar{\omega}_m \Lambda\omega_n \bar{Z}^m{}_A{}^C Z^n{}_{CB} + \bar{\omega}_m \Lambda\bar{\omega}_n \bar{Z}^m{}_A{}^C \bar{Z}^n{}_{CB}.$$

Using identities (130) we obtain

$$\Omega_m = d\omega_m - \frac{1}{2} \varepsilon_{mnq} \omega^n \Lambda\omega^q, \qquad (140)$$

which is the analog of equation (126) in C_3.

This form of writing the curvature tensor, Ω_m, has an interesting and practical advantage: one can exhibit explicity the irreducible components of the curvature tensor. Indeed, we write:

$$\Omega_m = (C_{mn} + \frac{R}{12} \gamma_{mn}) Z^n + \varepsilon_{mn} \bar{Z}^n . \qquad (141)$$

In this formula, C_{mn} (which is trace- free, $C_{mn} \gamma^{mn} = 0$) represents the Weyl conformal tensor, R is the scalar of curvature and ε_{mn} represents the Ricci tensor without trace $R^A_B - \frac{1}{4} R\delta^A{}_B$.

We arrive at this interpretation using the previous identities (130) and by noting the additional properties:

$$Z^m \Lambda \bar{Z}^n = 0 \qquad (142a)$$

$$Z^m \wedge Z^n = -\gamma^{mn} V, \qquad (142b)$$

in which V represents the 4-form $\theta^0 \wedge\theta^1 \wedge\theta^2 \wedge\theta^3$.

Using these properties (142) into the expansion (141) we obtain the set

$$\Omega_m \wedge\bar{Z}_n = -\varepsilon_{mn} V \qquad (143)$$

$$\Omega_m \wedge Z^m = -\frac{1}{2} R V. \qquad (144)$$

These relations enable us to write Einstein's equation using objects of C_3 in the form:

$$\Omega_m \wedge\bar{Z}_n = M^{AB}{}_{mn} T_{AB} V \qquad (145)$$

$$\Omega_m \wedge Z^m = -\frac{1}{2} T V . \qquad (146)$$

The mixed quantity $M^{AB}{}_{mn}$ is defined in terms of the Z's:

$$M_{AB}{}^{mn} = g^{CD} Z^m{}_{AC} Z^n{}_{DB} \qquad (147)$$

and satisfies the property of completeness

$$M^{mn}{}_{AB} M^{AB}{}_{pq} = \delta^m{}_p \delta^n{}_q . \qquad (148)$$

2.11 - MODES OF VIBRATION OF FRIEDMANN UNIVERSES

Perturbations of Friedmann Universes are particularly easy to be analysed by using a decomposition of scalar, vector and tensor perturbations in terms of eigen-functions of the generalized Laplacian operator. In this section we present some properties of these eigen-functions (Lifshitz (1980), Harrison (1967)).

(i) SCALAR EIGEN-FUNCTIONS: $Q_{(n)}$

We have

$$h^{\mu\nu}\,\hat{\nabla}_{\mu}\,\hat{\nabla}_{\nu}\;Q_{(n)} = \frac{K^2_{(n)}}{A^2}\;Q_{(n)}\;, \tag{149}$$

where we used the notation of section 2.5 and A(t) is the radius of the Universe defined by expression (47).

In order to express conditions only on spatial perturbations we impose on $Q_{(n)}$ a restriction of time invariance

$$\dot{Q}_{(n)} \equiv Q_{(n),\alpha}\;n^{\alpha} = 0 \tag{150}$$

Vector and tensor perturbations can be obtained from $Q_{(n)}$ by operating with co-variant derivative:

$$\Pi_{(n)\alpha} = h_{\alpha}{}^{\beta}\,\hat{\nabla}_{\beta}\,Q_{(n)}\,\frac{A^2}{K^2_{(n)}} = \frac{A^2}{K^2_{(n)}}\,\hat{\nabla}_{\alpha}\,Q_{(n)}\,. \tag{151}$$

This vector satisfies the properties:

$$h^{\mu\nu}\,\hat{\nabla}_{\nu}\,\Pi_{\mu} = Q \tag{152}$$

$$\hat{\nabla}^2 \Pi_\mu = h^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \Pi_\mu = \frac{2\varepsilon + K^2}{A^2} \Pi_\mu \tag{153}$$

(For sake of simplicity, from now on we will write Q instead of $Q_{(n)}$) where ε measures the 3-curvature, cf. equation (50).

A direct calculation shows that

$$\hat{\nabla}_\mu \Pi_\nu - \hat{\nabla}_\nu \Pi_\nu = 0 \ . \tag{154}$$

This enables us to define the symmetric eigen-tensor $P_{\mu\nu}$:

$$P_{\mu\nu} \equiv \hat{\nabla}_\nu \Pi_\mu - \frac{1}{3} Q h_{\mu\nu} \tag{155}$$

This definition implies immediately the following properties:

$$P^\mu{}_\mu = 0 \tag{156}$$

$$h^{\alpha\nu} \hat{\nabla}_\alpha P_{\mu\nu} = \frac{2}{3} \frac{3\varepsilon + K^2}{A^2} \Pi_\mu \tag{157}$$

$$h^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta P_{\mu\nu} = \frac{(6\varepsilon + K^2)}{A^2} P_{\mu\nu} \ . \tag{158}$$

(ii) <u>VECTOR EIGEN-FUNCTIONS</u>: $\hat{S}_{\alpha(n)}$

They are defined by the equations

$$h^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \hat{S}_\alpha = \frac{K^2}{A^2} \hat{S}_\alpha \tag{159}$$

$$\hat{S}_\alpha \, n^\alpha = 0 \tag{160}$$

$$\hat{S}^\alpha{}_{;\alpha} = 0 \, . \tag{161}$$

This vector $\hat{S}_\alpha$ allows us to define the tensorial quantity $\hat{\Sigma}_{\mu\nu}$ by the expression

$$\hat{\Sigma}_{\alpha\beta} = \hat{\nabla}_\alpha \, \hat{S}_\beta + \hat{\nabla}_\beta \, \hat{S}_\alpha . \tag{162}$$

A direct calculation shows that we have

$$h^{\mu\nu} \, \hat{\nabla}_\nu \, \hat{\Sigma}_{\rho\mu} = \frac{1}{A^2} \, (2\varepsilon + K^2) \, \hat{S}_\rho \tag{163}$$

$$\hat{\Sigma}_{\mu\nu;\lambda} \, n^\lambda = - \frac{\theta}{3} \, \hat{\Sigma}_{\mu\nu} \, . \tag{164}$$

We define two more derived quantities from this vector $\hat{S}^\alpha$ which will be of great help in the analysis of the perturbation quantities in Friedmann universes. They are

$$\hat{F}_{\alpha\beta} \equiv \hat{\nabla}_\beta \, \hat{S}_\alpha - \hat{\nabla}_\alpha \, \hat{S}_\beta \tag{165}$$

$$*\hat{S}^\mu \equiv \eta^{\mu\varepsilon\beta\lambda} \, n_\varepsilon \, \hat{S}_{\beta;\lambda} \, . \tag{166}$$

Corresponding to (162) we have

$$*\hat{\Sigma}_{\mu\nu} \equiv \hat{\nabla}_\mu \, *\hat{S}_\nu + \hat{\nabla}_\nu \, *\hat{S}_\mu . \tag{167}$$

We list below some useful properties of these quantities, which are not difficult to be obtained:

$$h^{\mu}{}_{(\varepsilon}\ h^{\nu}{}_{\alpha)}\ \eta_{\mu}{}^{\beta\gamma\lambda}\ n_{\lambda}\ \hat{\Sigma}_{\nu\beta;\gamma} = h^{\mu}{}_{(\varepsilon}\ h^{\nu}{}_{\alpha)}\ \hat{\nabla}_{\nu}\ {}^{*}\hat{S}_{\mu} \tag{168}$$

$$h^{\mu}{}_{(\varepsilon}\ h^{\nu}{}_{\alpha)}\ \eta_{\mu}{}^{\beta\gamma\lambda}\ n_{\lambda}\ \hat{\nabla}_{\gamma}\ \hat{F}_{\nu\beta} = -h^{\nu}{}_{(\varepsilon}\ h_{\alpha)}{}^{\mu}\ \hat{\nabla}_{\nu}\ {}^{*}\hat{S}_{\mu} \tag{169}$$

$$h^{\alpha}{}_{\varepsilon}\ h^{\gamma\nu}\ \hat{\nabla}_{\nu}\ \hat{\nabla}_{\gamma}{}^{*}\hat{S}_{\alpha} = \frac{K^2}{A^2}\ {}^{*}\hat{S}_{\varepsilon} \tag{170}$$

$$h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}(\hat{\nabla}_{\mu}\ {}^{*}\hat{S}_{\nu})_{;\lambda}\ n^{\lambda} = -\frac{2}{3}\theta\ h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}\ \hat{\nabla}_{\mu}\ {}^{*}\hat{S}_{\nu} \tag{171}$$

$$h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}(\hat{\nabla}_{\mu}\ \hat{S}_{\nu})_{;\lambda}\ V^{\lambda} = -\frac{\theta}{3}\ h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}\ \hat{\nabla}_{\mu}\ \hat{S}_{\nu} \tag{172}$$

$$h^{\mu(\rho}\ \eta^{\varepsilon)\lambda\nu\alpha}\ n_{\lambda}\ {}^{*}\hat{\Sigma}_{\alpha\mu;\nu} = (2\varepsilon - K^2)\ h^{\mu(\varepsilon}\ h^{\rho)\nu}\ \hat{S}_{\nu;\mu}\ . \tag{173}$$

(iii) TENSOR EIGEN-FUNCTIONS

We have

$$h^{\beta\alpha}\ \hat{\nabla}_{\alpha}\ \hat{\nabla}_{\beta}\ \hat{U}_{\mu\nu} = \frac{K^2}{A^2}\ \hat{U}_{\mu\nu}\ , \tag{174}$$

in which the constant K has the following spectrum (for distinct Bianchi-types):

Type I: $0 < |k| < \infty$

Type V: $K^2 = q^2 + 3 \qquad 0 < q < \infty$

Type IX: $K^2 = n^2 - 3 \qquad n = 3,4...$

Besides, the tensor $\hat{U}_{\mu\nu}$ satisfy the properties:

$$\hat{U}_{\mu\nu;\alpha}\ V^{\alpha} = 0 \tag{175}$$

$$h^{\mu\alpha} \hat{\nabla}_{\alpha} \hat{U}_{\mu\nu} = 0 \qquad (176)$$

$$h^{\mu\nu} \hat{U}_{\mu\nu} = 0 \, . \qquad (177)$$

2.12 - THE METHOD OF QUALITATIVE INVESTIGATION OF NON-LINEAR DIFFERENTIAL EQUATIONS

Let a system of two non-linear equations be written in the form

$$\dot{x} = F(x,y) \tag{178a}$$

$$\dot{y} = G(x,y) \tag{178b}$$

for the variables x, y. Functions F and G are regular (say of class C^n) which depend only on x and y but not on t. The dot means $\frac{d}{dt}$. The range of variable t is the real axis $(-\infty,+\infty)$. In general, system (178) is not easy to integrate and consequently we have to look for alternative methods of investigation of the properties of the integral curves $x= x(t)$ and $y= y(t)$.

We call a point on the phase space (x,y) a singular point if it annihilates simultaneously functions F and G, making the derivative $\frac{dx}{dy}$ indeterminate. We usually say that (x_0,y_0) for which $F(x_0,y_0) = G(x_0,y_0) = 0$ is also a point of equilibrium of the system. The main important property of this system is related to a theorem which states (Andronov, 1973) that in the neighborhood of an equilibrium point (x_0,y_0) the general behavior of the integral curves of the system can be deduced by a slight investigation of the linearization of the functions F and G in the neighborhood of (x_0,y_0). This allows us to limit our resumé to the characterization of linear systems.

The most general form in this case is:

$$\dot{y} = ax + by + \alpha \tag{179a}$$

$$\dot{x} = cx + dy + \beta \qquad (179b)$$

in which a, b, c, d, α and β are real constants.

The singular points are obtained by the commom solutions of $\dot{y} = \dot{x} = 0$. Making a translation $x \to x - x_o$ and $y \to y - y_o$ the system (179) can be reduced to

$$\dot{y} = ax + by \qquad (180a)$$

$$\dot{x} = cx + dy. \qquad (180b)$$

Let us call $\Omega = \det \hat{\Omega} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $\Omega \neq 0$ the origin $(x_o, y_o) = (0,0)$ is a singular (isolated) point.

Let us examine some special cases (see Sansone-Conti (19)).

<u>Case i</u>: a = d = 0 (<u>SADDLE</u>)

The system reduces to

$$\dot{x} = cx \qquad \dot{y} = by. \qquad \text{(with } cb < 0\text{)}$$

The solutions are given by:

$$x = \alpha e^{ct}$$

$$y = \beta e^{bt}$$

(α and β are arbitrary constants).

We can easily draw in the phase plane the behavior of the integral curves (see figure 1).

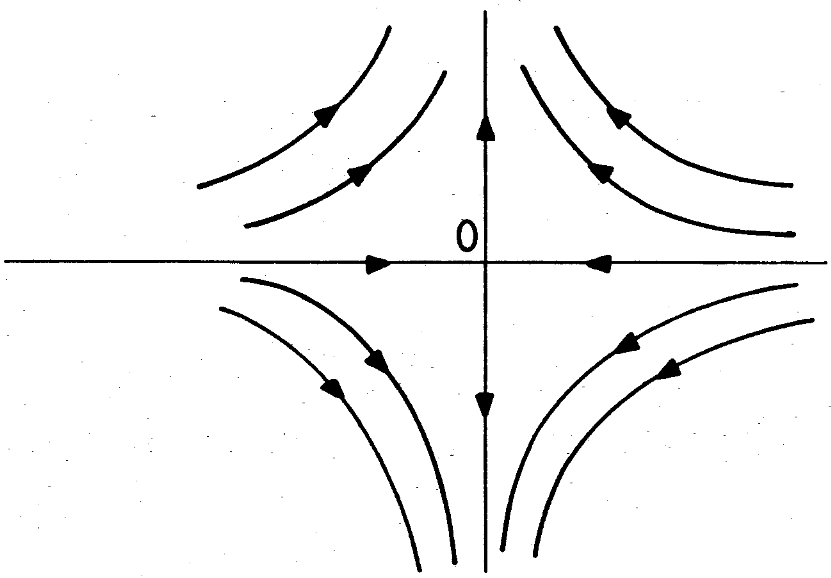

Figure 1 - SADDLE POINT

We call the singular origin a <u>saddle</u> point. (The arrows point in the direction of increasing values of parameter t).

<u>Case ii</u>: (<u>TWO-TANGENT NODE</u>)

We have

$$\dot{x} = cx$$

$$y = by$$

with $cb > 0$.

Only the configuration in the phase plane changes (see Fig. 2).

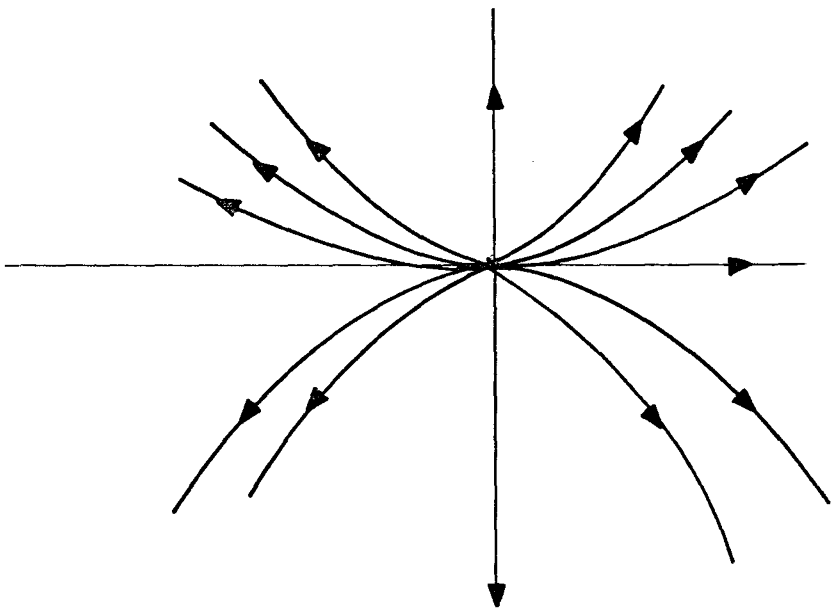

Figure 2a - TWO-TANGENT NODE
(unstable case)

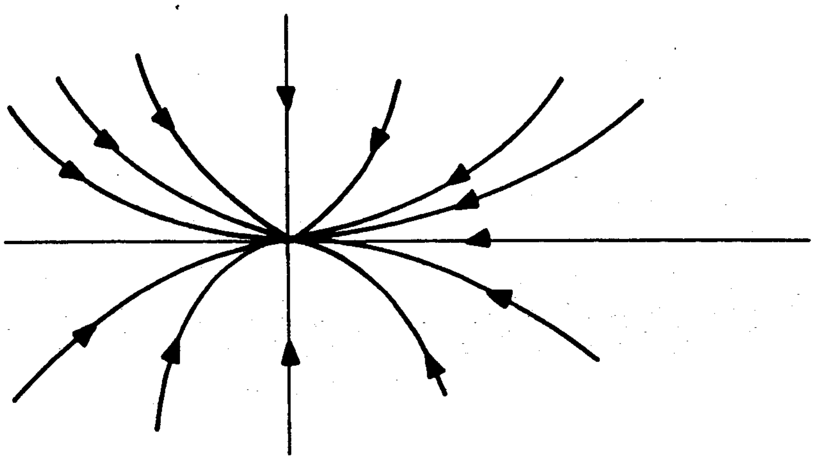

Figure 2b - TWO TANGENT NODE
(Stable case)

Case iii: (ONE-TANGENT NODE)

$$\dot{x} = ax + ay \qquad (a \neq 0)$$

$$\dot{y} = ay.$$

The solutions are easily explicity found:

$$x = \alpha e^{at} + a\beta t e^{at}$$

$$y = \beta e^{at}$$

(see figure 3).

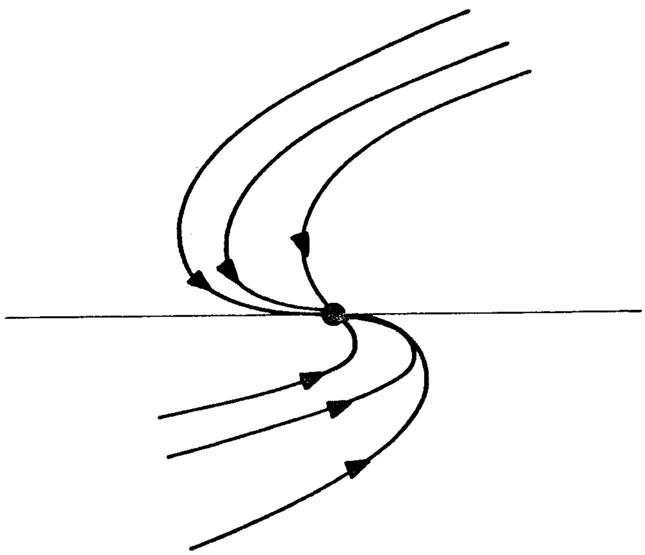

Figure 3a - ONE-TANGENT NODE
(Stable case)

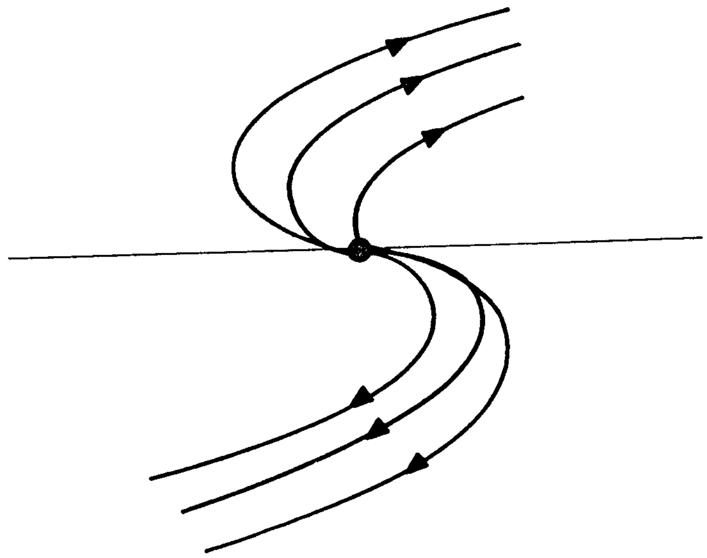

Figure 3b - ONE-TANGENT NODE
(unstable case)

Case IV: (STELLAR NODE)

The system reduces to

$$\dot{x} = ax$$

$$\dot{y} = ay.$$

The phase space is easily found to be given by figures 4a, b.

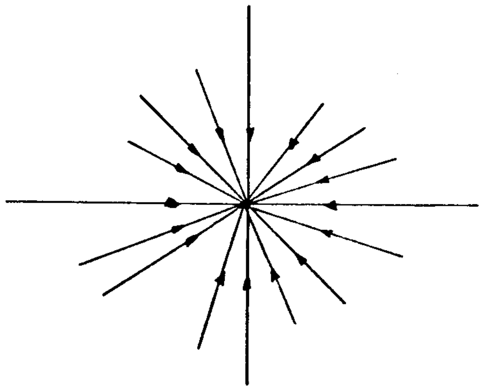

Figure 4a - STELLAR NODE
(Stable case)

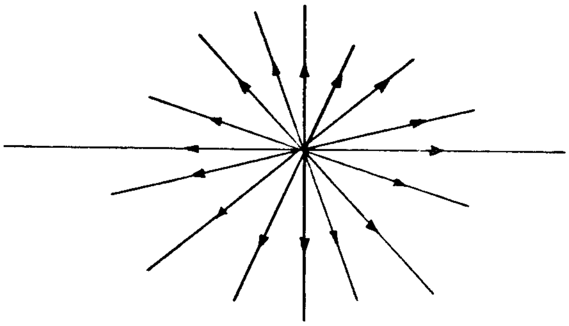

Figure 4a - STELLAR NODE
(Unstable case)

<u>Case v</u>: (FOCUS)

The system is given by

$$\dot{x} = ax - by$$
$$\dot{y} = bx + ax . \qquad \text{(with } ab \neq 0\text{)}$$

The phase plane is easily drawn as focus (stable or unstable).

<u>Case vi</u>: (CENTER)

$$\dot{x} = -by$$
$$\dot{y} = bx . \qquad (b \neq 0)$$

Solutions are : $x = \alpha \cos (bt + \gamma)$

$y = \alpha \sin (bt + \gamma)$.

(a) (b)

Figure 5 - FOCUS
(Stable (a) and unstable (b) cases)

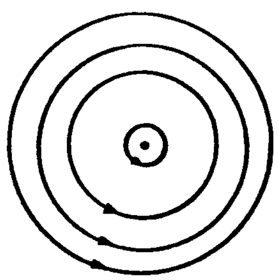

FIGURE 6 - CENTER

Let us now pass to the general case in which the system takes the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy\ ,$$

with det $\Omega \neq 0$.

It is possible to show (see Sansone-Conti pg. 44) that for different values of a, b, c and d we can always reduce the system to one of the previous simple cases by linear transformations.

We give here the theorem without proof. The reader can work it out by himself or accompany Sansone-Conti demonstra-tion.

<u>Theorem 1.</u>

Given the system

$$\dot{x} = \alpha x + \beta y$$
$$\dot{y} = \gamma x + \delta y$$

with $\Omega \equiv \alpha\delta - \beta\gamma \neq 0$

$I \equiv \alpha + \delta.$

Then, there exist real non-degenerate linear transforma-tions

$$x' = ax + by$$
$$y' = cx + dy$$

by which the system is changed into:

(i) the system

$$\dot{x}' = \lambda x' \qquad (\lambda\mu < 0) \qquad \text{if } \Omega < 0;$$
$$\dot{y}' = \mu y'$$

(ii) the system

$$\dot{x}' = \lambda x'$$
$$\dot{y}' = \mu y'$$

with $(\lambda\mu > 0,\ \lambda \neq \mu)$

if $0 < 4\Omega < I^2$;

(iii) the system

$$\dot{x}' = \lambda x' + \lambda y'$$
$$\dot{y}' = \lambda y'$$

(with $\lambda \neq 0$)

if $0 < I^2 = 4\Omega, \quad \beta^2 + \gamma^2 > 0$;

(iv) the system

$$\dot{x}' = \lambda x'$$
$$\dot{y}' = \lambda y'$$

(with $\lambda \neq 0$)

if $0 < I^2 = 4\Omega, \quad \beta^2 + \gamma^1 = 0$;

(v) the system

$$\dot{x}' = \lambda x' - \mu y'$$
$$\dot{y}' = \mu x' + \lambda y'$$

(with $\lambda\mu \neq 0$)

if $0 < I^2 < 4\Omega$;

(vi) the system

$$\dot{x}' = -\mu y'$$
$$\dot{y}' = \mu x'$$

(with $\mu \neq 0$)

if $0 = I^2 < 4\Omega$.

This theorem allows us to obtain a very simple characterization of the behavior in phase space of distinct linear planar autonomous system. We obtain the table below

Center	$0 = I^2 < 4\Omega$
Focus	$0 < I^2 < 4\Omega$
Stellar Node	$0 < I^2 = 4\Omega,\ \beta^2 + \gamma^2 = 0$
One-Tangent Node	$0 < I^2 = 4\Omega,\ \beta^2 + \gamma^2 > 0$
Two-Tangent Node	$0 < 4\Omega < I^2$
Saddle Point	$4\Omega < 0\ (\leq I^2)$

Table of Classification of distinct linear planar autonomous systems - (See Sansone and Conti (1964))

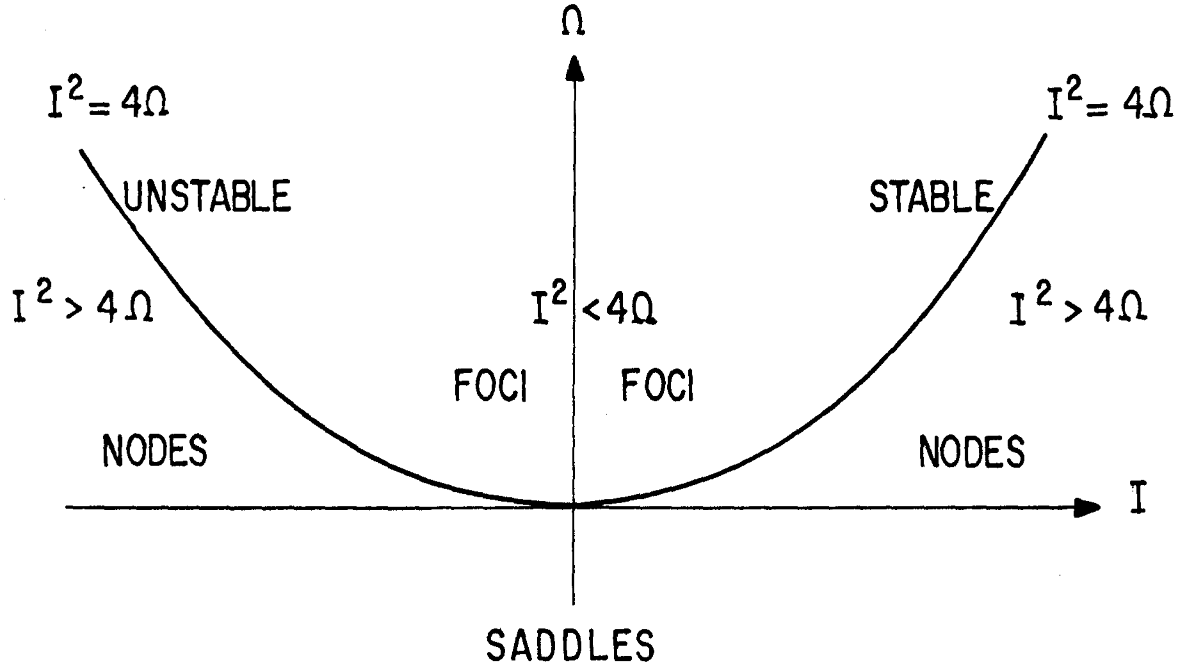

Figure of Classification of distinct linear planar autonomous system

3. THE GENERAL THEORY OF PERTURBATION OF EXPANDING UNIVERSES

In this chapter we present the methods to investigate the evolution of small perturbation of Einstein's equation of General Relativity. Although we concentrate in one specific method (the quasi-Maxwellian approach) we briefly review two other possible alternative procedures. We apply this method to the exam of the evolution of homogeneous and isotropic evolutionary Universes (Friedmann Universes).

3.1 - INTRODUCTION

The investigation of the stability of a large gravitational system was the subject of two memorable works, one by J. Jeans (1902), for the Newtonian treatment, and other by E. Lifshitz (1946), for the relativistic case. Since then, many authors have dedicated to review the subject (see, for instance, E.R. Harrison (1967), G.B. Field (1875) and P.J. Peebles (1980)).

In all these reviews the standard technique discussed in Lifshitz's paper on perturbation of Einstein's equation of motion is used. This is certainly the most direct and a very general procedure. However, in certain special cases of interest in Cosmology, like, for instance, in Friedmann homogeneous and isotropic expanding Universes, there is a competitive procedure used for the first time by Hawking (1966), some years ago. Although there have been a few works using the conformal technique employed by Hawking, as far as we know this procedure has not been largely used and a complete analysis of the whole set of the perturbation equations using the conformal technique has not been worked out in the literature. We intend to remedy this situation by providing a complete investigation of this method here.

Three methods have been used in the investigation of small perturbations of Einstein's equation:

(i) The perturbation of the metric tensor in the standard form of Einstein's equations.

(ii) The perturbation of Weyl tensor in the schema of the quasi-Maxwellian equations.

(iii) The perturbation of the complex vectorial version of

Einstein's equations.

Let us briefly comment each of these methods and sketch their main lines of procedure.

3.1.1 - PERTURBATION OF THE METRIC TENSOR $g_{\mu\nu}$ AND EINSTEIN'S EQUATIONS IN STANDARD FORM

The perturbation of the geometry is represented by small variations of the components of the metric tensor, namely

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x). \qquad (1)$$

Such variation induces on the derived quantities (the curvature tensor $R_{\alpha\beta\mu\nu}$ and its contractions) corresponding transformations:

$$R^{\mu\nu}{}_{\rho\sigma}(x) \longrightarrow \tilde{R}^{\mu\nu}{}_{\rho\sigma}(x) = R^{\mu\nu}{}_{\rho\sigma}(x) + \delta R^{\mu\nu}{}_{\rho\sigma}(x) \qquad (2a)$$

$$R^{\mu}{}_{\nu}(x) \longrightarrow \tilde{R}^{\mu}{}_{\nu}(x) = R^{\mu}{}_{\nu}(x) + \delta R^{\mu}{}_{\nu}(x) \qquad (2b)$$

$$R(x) \longrightarrow \tilde{R}(x) = R(x) + \delta R(x) \ . \qquad (2c)$$

The dynamics of the perturbing quantities constitutes equations of motion for a tensorial field in a Riemannian curved geometry determined by the background gravitational field. In general, one imposes that the perturbed quantities obey equations of motion which are analogous to the equations for the unperturbed geometry, that is,

$$\delta R^{\mu}{}_{\nu} - \frac{1}{2}(\delta R)\ \delta^{\mu}{}_{\nu} = -k\ \delta T^{\mu}{}_{\nu} \ . \qquad (3)$$

It seems worthwhile to remark that, due to the non-linear character of the theory, the perturbed quantities could obey a more general non-einstenian equation:

$$\delta R^{\mu}{}_{\nu} - \frac{1}{2} (\delta R)\ \delta^{\mu}{}_{\nu} = -K\ \delta T^{\mu}{}_{\nu} + \Phi^{\mu}{}_{\nu} \qquad (4)$$

in which $\Phi^{\mu}{}_{\nu}$ is a functional of the perturbed metric. The choice $\Phi^{\mu}{}_{\nu} \equiv 0$ neglects all deviations from Einstein's equations which could be due to the fluctuation of the geometry (see Novello (1978) for a different point of view). This is the standard procedure. It is straighforward and by far the simplest one. It is no surprise the fact that it was the first perturbation method used and the most largely employed until nowadays [see Weinberg (1972) for a review].

3.1.2 - PERTURBATION OF THE WEYL TENSOR IN THE SCHEMA OF THE QUASI-MAXWELLIAN EQUATIONS

The perturbation of the geometry in this method is represented by the variation of the Weyl conformal tensor $W^{\alpha\beta}{}_{\mu\nu}$

$$W^{\alpha\beta}{}_{\mu\nu}(x) \longrightarrow \tilde{W}^{\alpha\beta}{}_{\mu\nu}(x) = W^{\alpha\beta}{}_{\mu\nu}(x) + \delta W^{\alpha\beta}{}_{\mu\nu}(x). \qquad (5)$$

However, instead of using Einstein's equations in the standard form the dynamics of the perturbed tensor $\delta W^{\alpha\beta}{}_{\mu\nu}$ is described by the quasi-Maxwellian equations of motion (see section 2) (Hawking (1966); Olson (1976)).

The main interest in this approach is for the case of the perturbations of those geometries which are conformally flat. The reason for this is related to the simplicity of this method in the characterization of the perturbation as a true modification of the geometry and not merely a transformation of coordinates. Indeed, as the quantity $W^{\alpha\beta}{}_{\mu\nu}$ is a tensor, if it vanishes in the background geometry then we can be sure that all quantities $\delta W^{\alpha\beta}{}_{\mu\nu}$ are true perturbations and not merely a consequence of a change of coordinates.

This is precisely the weakest point of the previous method; if we consider a modification of the metric tensor,

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x),$$

how can we be sure that such a modification is not a consequence of a change of coordinate mapping?
Here, we will concentrate our main efforts on the quasi-Maxwellian form.

3.1.3 PERTURBATION OF EINSTEIN'S EQUATIONS USING THE COMPLEX VECTORIAL FORMALISM

Although this method will not be pursued here, it seems worth to give an overview of its general features since as far as we know, it has not been previously used for the treatment of perturbations.

We start by choosing a set of null tetrads $\{e^A_\alpha\}$ and corresponding 1-forms θ^A such that the fundamental length is given by $ds^2 = 2(\theta^0\theta^1 - \theta^2\theta^3)$. The internal geometry g_{AB} reduces to the form

$$g_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \qquad (6)$$

A perturbation of the above set of tetrads can be generated by a matrix $M^A{}_B$ (which is not a Lorentz rotation)by the map

$$e^A_\alpha(x) \longrightarrow \tilde{e}^A{}_\alpha(x) = e^A{}_\alpha(x) + M^A{}_B(x)\ e^B{}_\alpha(x) . \qquad (7)$$

In what follows we will consider the quantities $M^A{}_B$ as infinitesimals and we will neglect any non-linear term on M.

The independence of the metric quantities on a local Lorentz rotation of the tetrads allows us to choose the perturbation in such way to leave the tetrad metric g_{AB} unchanged.

Thus

$$\tilde{g}_{AB} = \tilde{e}_A{}^\alpha \tilde{e}_B{}^\beta \tilde{g}_{\alpha\beta} = g_{AB} .$$

The 1-forms θ^A change correspondingly to the modification

$$\theta^A \longrightarrow \tilde{\theta}^A = \theta^A + M^A{}_B\ \theta^B \qquad (8)$$

and the metric $g_{\mu\nu}$

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) \simeq g_{\mu\nu} + M^A{}_B(e^B{}_\mu e_{A\nu} + e^B{}_\nu e_{A\mu}) .$$

We can choose a basis Z^m in the auxiliary complex 3-dimensional space C_3, in such a way that under the above perturbation neither the quantities $Z^m{}_{AB}$ nor the internal metric γ_{mn} of C_3 change.

That is, we set:

$$\delta Z^m{}_{AB} = 0 \tag{10a}$$

$$\delta\gamma_{mn} = 0 . \tag{10b}$$

We thus have for the perturbation of the 2-form Z^m:

$$\delta Z^m \simeq Z^m{}_{AM} M^B{}_C \theta^A \Lambda\theta^C . \tag{11}$$

The first equation of structure of Cartan, that is ,

$$d\theta^A + \omega^A{}_B \Lambda\theta^B = 0 ,$$

defines the 1-form $\omega^A{}_B$. In the C_3 basis we write the decomposition (see section 2)

$$\omega_{AB} = \omega_m Z^m{}_{AB} + \bar{\omega}_m \bar{Z}^m{}_{AB} . \tag{12}$$

Using (11) it is straightforward to obtain

$$\delta\omega_m \simeq \left[\frac{1}{2} M_A{}^C \omega_{CB} - \omega_A{}^C M_{CB} - M_{AB,C}\theta^C\right] Z^{AB}_m . \tag{13}$$

In a completely analogous way, for the 2-form of curvature $\Omega^A{}_B$ we have:

$$\delta\Omega_m \simeq \frac{1}{2}\left[\phi_{AB} - \varepsilon_{mnq}\ \omega^n \wedge M_A{}^C\ \omega_{CB} - \omega_A{}^C\ M_{CB} - M_{AB,C}\theta^C\ \right] Z^{AB}{}_m\ , \tag{14}$$

in which we defined the quantity ϕ_{AB} as:

$$\phi_{AB} \equiv M_{AB,CD}\ \theta^C \wedge \theta^D + M_{AB,C}\ \omega^C{}_E\ \wedge\theta^E + M_{CB,E}\ \omega_A{}^C\ \wedge\theta^E -$$

$$- M_{AC,E}\ \omega^C{}_B\ \wedge\theta^E + M_A{}^C(\Omega_C{}^B - \omega_C{}^E\ \wedge\omega_E{}^B) - M^C{}_B(\Omega_{AC} - \omega_{AE}\wedge\omega^E{}_C). \tag{15}$$

These formula allow us to proceed in the investigation of the perturbation of the gravitational field in a straightforward way. Although such formulation may appear rather involved, it has certain advantages in some special cases depending on the symmetries of the background gravitational field. Besides, it produces a direct mechanism to eliminate ab initio those modifications in the geometry which are simple gauge transformations, by specifying the properties of the matrix of perturbation $M^A{}_B$.

3.2 - FRIEDMANN UNIVERSES

The purpose of the present section 3.2 is to use the method of the quasi-Maxwellian equations of motion in the study of perturbation of expanding homogeneous and isotropic Universes. These constitute the class of Friedmann cosmos. Let us start by a compact review of the basic properties of these universes.

The fundamental length of Friedmann cosmos in (t, χ, θ, ϕ) system of coordinates takes the form

$$ds^2 = dt^2 - A^2(t)\left[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta)d\phi^2\right] , \qquad (16)$$

in which $\sigma(\chi)$ may assume the values χ, $\sin\chi$ or $\sinh\chi$.Accordingly,the 3-dimensional space section is euclidean, closed or open.

The source of this geometry is a perfect fluid with density of energy ρ, pressure p and velocity $\vec{v}$.

In co-moving coordinates we set

$$v^\mu = \delta^\mu_0 \ .$$

For future references,we list some properties of this model in Table I.

ρ	λ	ε	θ	$A(\eta)$	$t(\eta)$
$\frac{4}{3}\ t^{-2}$	0	0	$2t^{-1}$	$A_0\eta^{2/3}$	η
$\frac{3}{4}\ t^{-2}$	1/3	0	$\frac{3}{2}\ t^{-1}$	$A_0\eta^{1/2}$	η
$\frac{6}{a_0^2}\ (1\text{-}\cos\eta)^{-3}$	0	1	$\frac{3}{a_0}\ \frac{\sin\eta}{(1-\cos\eta)^2}$	$A_0(1\text{-}\cos\eta)$	$A_0(\eta\text{-}\sin\eta)$
$\frac{3}{a_0^2}\ \frac{1}{\sin^4\eta}$	1/3	1	$\frac{3}{a^2}\ \frac{\cos\eta}{\sin^2\eta}$	$A_0\sin\eta$	$A_0(1\text{-}\cos\eta)$
$\frac{6}{a_0^2}\ \frac{1}{(\cosh\eta-1)^3}$	0	-1	$\frac{3}{a_0}\ \frac{\sinh\eta}{(\cosh\eta-1)^2}$	$A_0(\text{consh}\eta\text{-}1)$	$A_0(\sinh\eta\text{-}\eta)$
$\frac{3}{a_0^2}\ \frac{1}{\sinh^4\eta}$	1/3	-1	$\frac{3}{a_0}\ \frac{\cosh\eta}{\sinh^2\eta}$	$A_0\ \sinh\eta$	$A_0(\cosh\eta\text{-}1)$

TABLE I - Fundamental quantities of Friedmann Universe. λ is specified by the equation of state $p= \lambda\rho$, relating the pressure to the density of energy. The radius of the Universe is given by the function $A(\eta)$. The expansion factor θ measures the time variation of the volume V per unit of volume. We take units K= 1, c= 1.

In this geometry only the expansion θ does not vanish. All remaining kinematical quantities vanish (that is for the fluid velocity $\sigma^{\mu}{}_{\nu}= 0$, $\omega^{\mu}{}_{\nu}= 0$, $a^{\mu}= 0$). This implies that we can write $v_{\mu;\nu} = \frac{\theta}{3}\ h_{\mu\nu}$, a result which will be of great help in the simplification of some calculus of the perturbation.

Let us make one remark here. In the standart Big-Bang model of the Universe the initial state is represented by a radiation gas (mainly photons), the equation of state of which is given by $p = \frac{1}{3}\rho$. The Universe starts with a big explosion (Big-Bang Phase), expands in equilibrium and after the decoupling of matter and radiation we enter a period in which the evolution of each one of the constituents of the cosmos evolves separately. From the conservation of energy, we conclude that ρ_γ, the density of radiation, depends on the radius of the universe through the simple expression

$$\rho_\gamma \sim A^{-4},$$

and for the density of matter ρ_M we find $\rho_M \sim A^{-3}$.

This fact leads to the separation of the characteristics of the geometry of the Universe into two, non-mixing eras: the radiation era (in which $\rho_\gamma >> \rho_M$) and the matter dominated era (for $\rho_\gamma << \rho_M$).

The simple dependence of ρ on the radius $A(t)$ shows that the Universe starts in a radiation era and turns into a matter dominated era for later periods, since $A(t)$ is assumed to be a well-behaved monotonic function.

However it is a dificult task to find a smooth function which connects continuosly both regions.

In general this causes no problem, once we are interested only on asymptotic situations in which one of these regimes dominates. However, if we insist in finding a continuous transition from one era to the other, we are led to consider very intricate expressions for the equation of state $p = p(\rho)$, assuming that a perfect fluid description could still be a faithful representation of the cosmical fluid. Just to exemplify this procedure, we can refer to a case in which the equation of state is represented by a time-dependent relation $p = p_0 \,\mathrm{sech}(mt)\rho$.

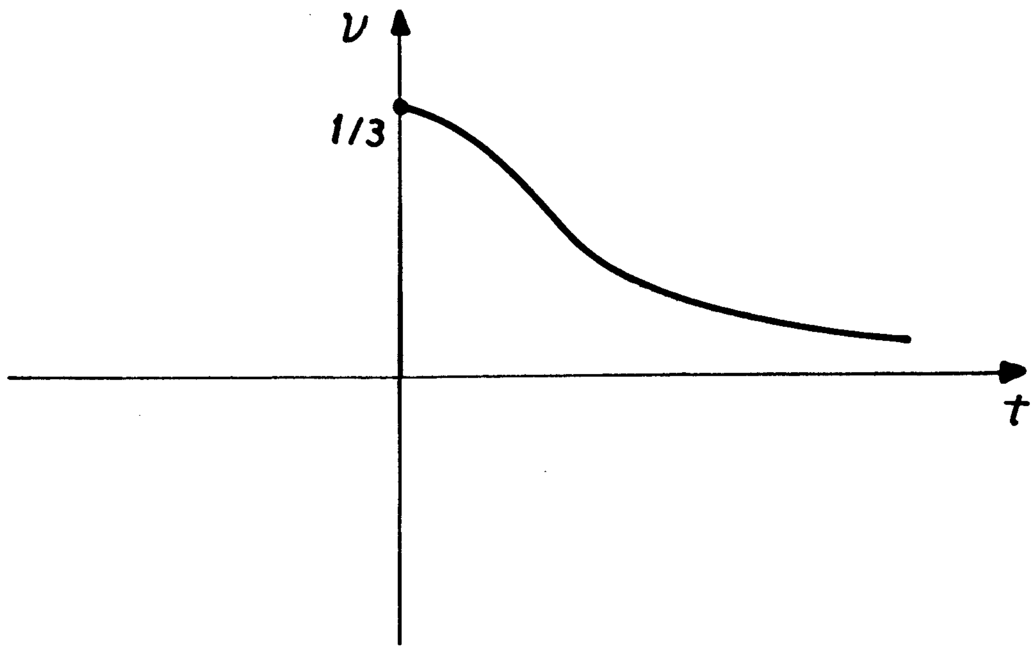

Fig. 1 - A model for the dependence of the equation of state with the cosmological time. The function ν measures the ratio p/ρ.

Such function has good asymptotic limits: for $t \simeq 0$ the equation of state is approximated by a radiation gas ($p = \rho/3$) and for large values of time it tends to the matter dominated era ($p=0$).

So much for these preliminaries, let us go directly into the perturbation schema.

3.3 - PERTURBATION OF THE GEOMETRY

II. THE QUASI-MAXWELLIAN PROCEDURE

The basic quantities of perturbation, are the following elements:

(i) The Metric: $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$

(ii) The Density of Energy: $\rho \rightarrow \tilde{\rho} \simeq \rho + \delta\rho$

(iii) The pressure: $p \rightarrow \tilde{p} = p + \delta p$

(iv) The fluid velocity: $v^{\mu} \rightarrow \tilde{v}^{\mu} = v^{\mu} + \delta v^{\mu}$

in which $\delta g_{\mu\nu}$, $\delta\rho$, δp and δv^{μ} are "small" quantities, that is, $(\delta\rho)^2 << \delta\rho$, etc. Besides these quantitites,we will analyse the possibility that the fluid becomes non-perfect during the perturbing era.

One can argue that such perturbations of the metric $\delta g_{\mu\nu}$ may be nothing but a consequence of transformation of coordinates and that a real perturbation theory should deal with super-space S (which is a collection of geometries) in which contiguity of metric should be defined in a very precise way by the specification of the topology of S. This is a well founded critics. Although it is not a general result, in some special cases of interest we can characterize in a precise and definite way real perturbations and examine its evolution on the unperturbed background manifold.

The Friedmann universe, for instance, constitutes an example in which the method of investigation of the evolution of the perturbation in the quasi-Maxwellian formalism can overcome such difficulty, as we shall see later on. In general this method is unambiguous if the background is conformally flat.

Let us turn our attention now to the perturbation of derived quantities which are a consequence of the above modifications of the velocity field.

The Expansion θ:

From the definition of θ ,

$$\theta = \nabla_\alpha V^\alpha ,$$

we have for the perturbed quantity:

$$\tilde{\theta} = \tilde{V}^\alpha{}_{;\tilde{\alpha}}$$

$$\tilde{\theta} = (v^\alpha + \delta v^\alpha)_{,\alpha} + (\Gamma^\alpha_{\varepsilon\alpha} + \delta\Gamma^\alpha_{\varepsilon\alpha})(V^\alpha + \delta V^\alpha)$$

$$\simeq \nabla_\alpha v^\alpha + \nabla_\alpha(\delta v^\alpha) + V^\varepsilon \delta\Gamma^\alpha_{\varepsilon\alpha};$$

thus ,

$$\delta\theta \equiv \tilde{\theta} - \theta \simeq \nabla_\alpha \delta v^\alpha + v^\varepsilon \delta\Gamma^\alpha_{\varepsilon\alpha} . \qquad (16)$$

The Rotation $\omega_{\mu\nu}$:

We have, by definition ,

$$\tilde{\omega}_{\mu\nu} = \frac{1}{2} \tilde{h}^\lambda_\mu \tilde{h}^\varepsilon_\nu \tilde{\nabla}_{[\varepsilon} \tilde{v}_{\lambda]}$$

$$= \frac{1}{2} \tilde{h}^\lambda_\mu \partial_{[\varepsilon} \tilde{v}_{\lambda]} ,$$

in which

$$\tilde{h}^{\lambda}_{\mu} = \delta^{\lambda}_{\mu} - \tilde{v}_{\mu}\tilde{v}^{\lambda} . \qquad (17)$$

From the definition of the perturbation of the contra-variant vector field v^{μ} we can set

$$\tilde{v}_{\mu} = \tilde{g}_{\mu\lambda}\tilde{v}^{\lambda}$$

$$\tilde{v}_{\mu} = \{g_{\mu\lambda} + \delta g_{\mu\lambda}\}\{v^{\lambda} + \delta v^{\lambda}\}$$

$$\tilde{v}_{\mu} \simeq v_{\mu} + g_{\mu\nu}\delta v^{\lambda} + v^{\lambda}\ \delta g_{\mu\nu} ;$$

thus

$$\delta v_{\mu} \equiv \tilde{v}_{\mu} - v_{\mu} \simeq g_{\mu\lambda}\ \delta v^{\lambda} + v^{\lambda}\ \delta g_{\mu\lambda} . \qquad (18)$$

Using this result in definition of $\delta\omega_{\mu\nu}$,

$$\delta\omega_{\mu\nu} \equiv \omega_{\mu\nu} - \omega_{\mu\nu} , \qquad (19)$$

we have

$$\delta\omega_{\mu\nu} \simeq \frac{1}{2}\ h^{\lambda}_{\mu}\ h^{\varepsilon}_{\nu}\ \partial_{[\varepsilon}\ \delta v_{\lambda]} - \frac{1}{2}\ h^{\lambda}_{\mu}\ \partial_{[\varepsilon}v_{\lambda]}\,\delta(v_{\nu}v^{\varepsilon}) - \frac{1}{2}\ h^{\varepsilon}_{\nu}\ \partial_{[\varepsilon}v_{\lambda]}\,\delta(v_{\mu}v^{\lambda}) ,$$

which is the general expression for the perturbation of the vortex tensor.

<u>The shear</u> $\sigma_{\mu\nu}$:

$$\tilde{\sigma}_{\mu\nu} = \frac{1}{2}\, \tilde{h}^{\lambda}_{\mu}\, \tilde{h}^{\varepsilon}_{\nu}\, \tilde{\nabla}_{(\varepsilon}\tilde{v}_{\lambda)} - \frac{1}{3}\, \tilde{\theta}\, \tilde{h}_{\mu\nu}.$$

Defining the perturbed shear by the expression

$$\delta\sigma_{\mu\nu} \equiv \tilde{\sigma}_{\mu\nu} - \sigma_{\mu\nu},$$

we obtain:

$$\delta\sigma_{\mu\nu} \simeq -\frac{1}{2}\{h^{\lambda}_{\mu}\ \delta(v_{\nu}v^{\lambda}) + h^{\varepsilon}_{\nu}\ \delta(v_{\mu}v^{\lambda})\}\ \nabla_{(\varepsilon}v_{\lambda)}$$
$$- h^{\lambda}_{\mu}\, h^{\varepsilon}_{\nu}\, v_{\alpha}\delta\Gamma^{\alpha}_{\varepsilon\lambda} + \frac{1}{2}\, h^{\lambda}_{\mu}\, h^{\varepsilon}_{\nu}\, \nabla_{(\varepsilon}\delta v_{\lambda)} - \qquad (20)$$
$$-\frac{1}{3}\,\delta\theta\ h_{\mu\nu} + \frac{\theta}{3}\,\delta(v_{\mu}v_{\nu}).$$

<u>The acceleration $a^{\mu} \equiv \dot{v}^{\mu}$</u>:

We have

$$\tilde{a}^{\mu} = \tilde{v}^{\lambda}\ \nabla_{\lambda}\tilde{v}^{\mu}$$

$$\tilde{a}^{\mu} = a^{\mu} + (\delta v^{\mu})^{\bullet} + \delta\Gamma^{\mu}_{\lambda\varepsilon}\ v^{\varepsilon}v^{\lambda} + (\nabla_{\lambda}v^{\mu})\delta v^{\lambda}$$

or

$$\delta a^{\mu} \equiv \tilde{a}^{\mu} - a^{\mu} \simeq (\delta v^{\mu})^{\bullet} + v^{\varepsilon}v^{\lambda}\ \delta\Gamma^{\mu}_{\lambda\varepsilon} + (\nabla_{\lambda}v^{\mu})\delta v^{\lambda}. \qquad (21)$$

Formulae (16), (19), (20) and (21) give the general rule for obtaining the perturbation of the kinematical quantities as functions of the perturbed velocity δV_{μ}, the metric $\delta g_{\mu\nu}$

and quantities defined in the background geometry.

3.4 - PERTURBED QUANTITIES IN FRIEDMANN UNIVERSES

The freedom we have in the choice of coordinate systems allows us to impose a gauge condition. We make the choice

$$\delta g_{0\alpha} = 0 , \tag{22}$$

which has the useful property of not changing the gaussian character of the system.

The normalization condition of the fluid velocity gives

$$\delta(g_{\mu\nu} v^{\mu} v^{\nu}) = 0$$

$$(\delta g_{\mu\nu}) v^{\mu} v^{\nu} + 2 g_{\mu\nu} (\delta v^{\mu}) v^{\nu} = 0 ,$$

which implies

$$\delta v^{0} = 0 . \tag{23}$$

Remark that this system of coordinates is no more comoving, although it still rests a gaussian system.

Using these simplifications, due to the gauge choice, in the formulas which we deduced previously for the Kinematical quantities we obtain:

$$\delta\theta = \delta v^{k}{}_{,k} + \Gamma^{k}_{ik}\, \delta v^{i} + \delta\Gamma^{\alpha}_{0\alpha} \tag{24}$$

$$\delta\omega_{ij} = \frac{1}{2}\, \delta v_{[i,j]} \tag{25}$$

$$\delta\sigma_{ij} = -\frac{\theta}{3}\,\delta g_{ij} + \frac{1}{2}\delta v_{(i;j)} - \frac{1}{3}\,\delta\theta g_{ij} + \frac{1}{2}(\delta g_{ij})_{,\,o} \qquad (26)$$

$$\delta a^{k} = (\delta v^{k})_{,0} + 2\,\frac{\dot{a}}{2}\,\delta v^{k}. \qquad (27)$$

Let us now turn to the perturbation of the quasi-Maxwellian equations of gravity for the Friedmann background.

The Friedmann geometry has a null Weyl tensor. Thus we can, without ambiguity, denote the perturbation quantity by $E_{\mu\nu} \equiv \delta E_{\mu\nu}$ and $H_{\alpha\mu} \equiv \delta H_{\alpha\mu}$.

Equation (80) section 2, gives:

$$E_{\alpha\mu;\nu} h^{\alpha\varepsilon} h^{\mu\nu} = \frac{1}{3}(\delta\rho)_{,\alpha} h^{\alpha\varepsilon} - \frac{1}{3}\rho_{,\alpha}\,\delta(V^{\alpha}V^{\varepsilon}). \qquad (28)$$

Now, in the background $\rho = \rho(t)$, which implies

$$\rho_{,\alpha}\,\delta(V^{\alpha}V^{\beta}) = \dot{\rho}\;\delta V^{\beta}\;.$$

For $\beta = 0$, eq. (28) reduces to an identity.

For $\beta = k$, it gives:

$$E^{ik}{}_{;k} = \frac{1}{3}\,\delta\rho_{,\ell}\, g^{\ell i} - \frac{1}{3}\,\dot{\rho}\,\delta V^{i}, \qquad (29)$$

in which we have used condition (22) for the motion of the fluid and the consequent relations

$$H_{\mu 0} = 0 \qquad (30a)$$

$$E_{\mu 0} = 0, \qquad (30b)$$

Eq. (81) section 2 reduces to

$$H_{\alpha\mu;\nu}h^{\alpha\varepsilon}h^{\mu\nu} = (\rho + p)\,\delta\omega^{\varepsilon} \ , \tag{31}$$

or, equivalently in this case, to

$$H^{ik}{}_{;k} = (\rho+p)\,\delta\omega^{i} \, . \tag{32}$$

In the same vein, the equation of evolution involving time derivatives of $E_{\mu\nu}$ and $H_{\mu\nu}$ is

$$\dot{E}^{\mu\nu}h_{\mu}{}^{\rho}h_{\nu}{}^{\sigma} + \theta E^{\rho\sigma} - \frac{1}{2}\, E_{\nu}{}^{(\rho}h^{\sigma)}{}_{\mu}\, V^{\mu;\nu} - \frac{\Theta}{3}\eta^{\sigma\nu\mu\varepsilon}\,\eta^{\rho\lambda\alpha\beta}\, V_{\mu}V_{\lambda}E_{\varepsilon\alpha}h_{\beta\nu} -$$

$$- \frac{1}{2}\, H_{\beta}{}^{\mu}{}_{;\alpha}\, h_{\mu}^{(\sigma}\eta^{\rho)\lambda\alpha\beta}\, V_{\lambda} = -\frac{1}{2}(\rho+p)\,\delta\sigma^{\rho\sigma} \ , \tag{33}$$

or, equivalently ,

$$\dot{E}^{ij} + \theta E^{ij} - \frac{1}{2}\{H^{i}{}_{m;n}\eta^{0jmn} + H^{j}{}_{m;n}\eta^{0imn}\} = \frac{1}{2}(\rho+p)\,\delta\sigma^{ij} \, . \tag{34}$$

Finally, the fourth equation

$$\dot{H}^{\mu\nu}\, h_{\mu}{}^{\rho}h_{\nu}{}^{\sigma} + \theta H^{\rho\sigma} - \frac{1}{2}\, H_{\nu}{}^{(\rho}h^{\sigma)}{}_{\mu}\, V^{\mu;\nu} - \frac{\theta}{3}\eta^{\sigma\nu\mu\varepsilon}\,\eta^{\rho\lambda\alpha\beta}\, V_{\mu}V_{\lambda}H_{\varepsilon\alpha}h_{\beta\nu} +$$

$$+ \frac{1}{2}\, E_{\beta}{}^{\mu}{}_{;\alpha}\, h_{\mu}{}^{(\sigma}\eta^{\rho)\lambda\alpha\beta}\, V_{\lambda} = 0 \, , \tag{35}$$

or, equivalently,

$$\dot{H}^{ij} + \theta H^{ij} + \frac{1}{2}\, \{\varepsilon^{i}{}_{m;n}\, \eta^{0jmn} + \varepsilon^{j}{}_{m;n}\eta^{0imn}\} = 0 \, . \tag{36}$$

Equations (28), (31), (33) and (35) constitute the set of quasi-Maxwellian equations of the gravitational field. Besides them, we have to take into account the evolution of the perturbation of the variables which characterize the state of motion of the galactic fluid, the perturbation of the equations of conservation of energy and the constraint equation. Let us now turn to them.

3.5 - PERTURBATION OF THE EQUATIONS OF EVOLUTION OF THE KINEMATICAL QUANTITIES

(a) The Equation of the Expansion θ:

Raychaudhuri's equation for the expansion factor θ has the form

$$\dot{\theta} + \frac{\theta^2}{3} + 2\sigma^2 - 2\omega^2 - a^{\alpha}{}_{;\alpha} = R_{\mu\nu}v^{\mu}v^{\nu} \ .$$

In first order approximation this equation reduces to:

$$(\delta\theta)^{\bullet} + \frac{2}{3}\,\theta\delta\theta - a^{\alpha}{}_{;\alpha} = \delta(R_{\mu\nu}v^{\mu}v^{\nu}) \ . \qquad (37)$$

In the gauge in which $\delta v^0 = 0 = \delta g^{0\alpha}$ we have $\delta\Gamma^0_{00} = 0$.

From the definition of the acceleration a^{α}, the component for $\alpha = 0$ is given by

$$a^0 = (\delta v^0)^{\bullet} + \delta\Gamma^0_{00} + \frac{\theta}{3}\,h^0{}_{\lambda}\ \delta V^{\lambda} \ ;$$

we obtain

$$a^0 = 0 \ . \qquad (38)$$

From Einstein's equations,

$$\delta(R_{\mu\nu}v^{\mu}v^{\nu}) = \delta(-T_{\mu\nu}v^{\mu}v^{\nu} + \frac{T}{2}\,g_{\mu\nu}v^{\mu}v^{\nu})$$

$$= -(\delta T_{\mu\nu})v^{\mu}v^{\nu} - 2T_{\mu\nu}(\delta v^{\mu})v^{\mu} + \frac{1}{2}\,\delta T$$

$$= -\delta T_{00} - 2T_{0i}\delta v^i + \frac{\delta T}{2}$$

$$= -\delta\rho + \frac{1}{2}\,\delta(\rho - 3p)$$

$$= -\frac{1}{2}\,(\delta\rho + 3\delta p)$$

$$= -\frac{(1+3\lambda)}{2}\,\delta\rho \; . \qquad (39)$$

Thus equation (37) reduces to

$$(\delta\theta)^{\bullet} + \frac{2}{3}\,\theta\,\delta\theta - a^k{}_{;k} = -\frac{(1+3\lambda)}{2}\,\delta\rho \; . \qquad (40)$$

(b) <u>The Equation of Shear $\sigma^{\mu\nu}$</u>:

Just as in the case of the Weyl tensor, we write $\sigma_{\mu\nu} \equiv \delta\sigma_{\mu\nu}$, since the fluid motion in the background is shear free (and also $\delta a^{\alpha} \equiv a^{\alpha}$, $\delta\omega^{\alpha\beta} \equiv \omega^{\alpha\beta}$, for the same reason).

From section (2) the equation of evolution (64) is:

$$h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \dot{\sigma}_{\mu\nu} + \frac{1}{3} h_{\alpha\beta}\, a^k{}_{;k} - \frac{1}{2}\, h_{\alpha}{}^{\mu} h_{\beta}^{\nu}(a_{\mu;\nu} + a_{\nu;\mu}) + \frac{2}{3}\,\theta\sigma_{\alpha\beta} =$$

$$= \delta(R_{\alpha\varepsilon\beta\nu} v^{\varepsilon} v^{\nu} - \frac{1}{3}\, R_{\mu\nu} v^{\mu} v^{\nu} h_{\alpha\beta}) \; . \qquad (41)$$

Let us evaluate the right hand side of this equation in the case of a perfect fluid.

We have

$$R_{\alpha\varepsilon\beta\nu} v^{\varepsilon} v^{\nu} = W_{\alpha\varepsilon\beta\nu} v^{\varepsilon} v^{\nu} + H_{\alpha\varepsilon\beta\nu} v^{\varepsilon} v^{\nu} - \frac{R}{6}\, g_{\alpha\varepsilon\beta\nu} v^{\varepsilon} v^{\nu} . \qquad (42)$$

Now, from the definitions of $E_{\mu\nu}$, $H_{\alpha\beta}$ and $g_{\alpha\varepsilon\beta\nu}$ we have:

$$W_{\alpha\varepsilon\beta\nu}v^{\varepsilon}v^{\nu} = -E_{\alpha\beta} \tag{43a}$$

$$g_{\alpha\varepsilon\beta\nu}\, v^{\varepsilon}v^{\nu} = (g_{\alpha\beta}g_{\varepsilon\nu} - g_{\alpha\nu}g_{\varepsilon\beta})v^{\varepsilon}v^{\nu} = g_{\alpha\beta} - v_{\alpha}v_{\beta} = h_{\alpha\beta} \tag{43b}$$

$$H_{\alpha\varepsilon\beta\nu}v^{\varepsilon}v^{\nu} = \frac{1}{2}\{R_{\alpha\beta}g_{\varepsilon\nu} + R_{\varepsilon\nu}g_{\alpha\beta} - g_{\alpha\nu}R_{\varepsilon\beta} - g_{\varepsilon\beta}R_{\alpha\nu}\}v^{\varepsilon}v^{\nu}$$

$$= \frac{1}{2}\{R_{\alpha\beta} + R_{00}\, g_{\alpha\beta} - v_{\alpha}R_{0\beta} - v_{\beta}R_{0\alpha}\} . \tag{43c}$$

For a perfect fluid with $T_{\mu\nu} = \rho v_{\mu}v_{\nu} - ph_{\mu\nu}$:

$$R_{oi} = 0 .$$

Thus,

$$R_{\alpha\varepsilon\beta\nu}v^{\varepsilon}v^{\nu} = -E_{\alpha\beta} + \frac{1}{2} R_{\alpha\beta} + \frac{1}{2} R_{00}g_{\alpha\beta} - \frac{1}{2} \delta^{0}_{\alpha} R_{0\beta} -$$

$$- \frac{1}{2} \delta^{0}_{\beta} R_{0\alpha} - \frac{1}{6} Rh_{\alpha\beta} \tag{44}$$

$$R_{\alpha\varepsilon\beta\nu}v^{\varepsilon}v^{\nu} - \frac{1}{3} R_{\mu\nu}v^{\mu}v^{\nu}h_{\alpha\beta} = -E_{\alpha\beta} + \frac{1}{2} R_{\alpha\beta} + \frac{1}{2} R_{00}\, g_{\alpha\beta}$$

$$- \frac{1}{2} \delta^{0}_{\alpha}R_{0\beta} - \frac{1}{2} \delta^{0}_{\beta}R_{0\alpha} - \frac{1}{6} Rh_{\alpha\beta} - \frac{1}{3} R_{00}\, h_{\alpha\beta} . \tag{45}$$

For $\alpha = 0$, $\beta = k$,

$$R_{0\varepsilon\beta\nu}\, v^{\varepsilon}v^{\nu} - \frac{1}{3} R_{\mu\nu}\, v^{\mu}v^{\nu}h_{0\beta} = 0.$$

For $\alpha = \ell$ and $\beta = k$,

$$R_{\ell\varepsilon k\nu}\, v^{\varepsilon} v^{\nu} - \frac{1}{3} R_{\mu\nu}\, v^{\mu} v^{\nu} h_{\ell k} = -E_{\ell k} \,.$$

Using these results into equation (41) we obtain for the spatial components (the o-ν components gives no information):

$$\dot{\sigma}_{ik} + \frac{1}{3} g_{ik}\, a^{\ell}{}_{;\ell} - \frac{1}{2} a_{(i;k)} + \frac{2}{3} \theta \sigma_{ik} = -E_{ik}, \qquad (46)$$

in which $\dot{\sigma}_{ik} \equiv \sigma_{ik;0}$.

(c) <u>The Equation for the Vorticity $\omega^{\mu\nu}$</u>:

From section 2 equation (65) we have:

$$h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \dot{\omega}_{\mu\nu} - \frac{1}{2} h_{\alpha}{}^{[\mu} h_{\beta}{}^{\nu]} a_{\mu;\nu} + \frac{2}{3} \theta \omega_{\mu\nu} = 0. \qquad (47)$$

The only surviving terms are the spatial components:

$$\dot{\omega}_{k\ell} - \frac{1}{2} a_{k;\ell} + \frac{1}{2} a_{\ell;k} + \frac{2}{3} \theta\, \omega_{k\ell} = 0.$$

Multiplying this equation by $\frac{1}{2} \eta^{okij}$ and using the definition of the vector $\vec{\omega}$ this equations reduces to

$$\dot{\omega}^{k} + \frac{2}{3} \theta\, \omega^{k} = \frac{1}{2} \eta^{okij}\, a_{i;j} \,. \qquad (48)$$

3.6 - PERTURBATION OF THE CONSTRAINT EQUATIONS

Let us now perturb the equations for the congruence which do not contain time-derivatives.

First constraint equation:

$$\frac{2}{3}\theta_{,\alpha} h^{\alpha}{}_{\mu} - (\sigma^{\alpha}{}_{\rho} + \omega^{\alpha}{}_{\rho})_{;\alpha} h^{\rho}{}_{\mu} - a_{\alpha}(\sigma_{\mu}{}^{\alpha} + \omega_{\mu}{}^{\alpha}) = R_{\varepsilon\alpha} v^{\varepsilon} h^{\alpha}{}_{\mu}, \quad (49)$$

For a perfect fluid we can re-write the right hand side using Einstein's equation, to obtain:

$$R_{\varepsilon\alpha} v^{\varepsilon} h^{\alpha}{}_{\mu} = (-T_{\varepsilon\alpha} + \frac{T}{2} g_{\varepsilon\alpha}) v^{\varepsilon} h^{\alpha}{}_{\mu}$$

$$= \{-\rho v_{\varepsilon} v_{\alpha} + p h_{\varepsilon\alpha} + \frac{(\rho - 3p)}{2} g_{\varepsilon\alpha}\} v^{\varepsilon} h^{\alpha}{}_{\mu} \quad (50)$$

$$= 0 .$$

Thus, if the perturbation does not change the properties of the fluid that $q^{\alpha} = 0$ and $\Pi^{\alpha\beta} = 0$, we then have

$$\delta(R_{\varepsilon\alpha} v^{\varepsilon} h^{\alpha}{}_{\mu}) = 0 . \quad (51)$$

Thus, equation (49) gives:

$$\frac{2}{3}(\delta\theta)_{,\alpha} h^{\alpha}{}_{\mu} - \frac{2}{3}\theta_{,\alpha}\delta(v^{\alpha} v_{\mu}) - (\sigma^{\alpha}{}_{\rho} + \omega^{\alpha}{}_{\rho})_{;\alpha} h^{\rho}{}_{\mu} = 0$$

Or

$$\frac{2}{3}\ \delta\theta_{,\mu} - \frac{2}{3}\ \dot{\theta}\delta v_{\mu} - (\sigma^{\alpha}{}_{\rho} + \omega^{\alpha}{}_{\rho})_{;\alpha}\ h^{\rho}{}_{\mu} = 0 .$$

If $\mu = 0$ this equation gives no information, and for $\mu = k$ we obtain

$$\frac{2}{3}(\delta\theta)_{,k} - \frac{2}{3}\ \dot{\theta}\delta v_{k} - (\sigma^{\ell}{}_{k} + \omega^{\ell}{}_{k})_{;\ell} = 0 . \qquad (52)$$

For the second constraint equation:

$$\omega^{\alpha}{}_{;\alpha} + 2\omega^{\alpha}\ a_{\alpha} = 0 .$$

In the first order we have

$$\omega^{\alpha}{}_{;\alpha} = 0 . \qquad (53)$$

For the third constraint equation:

$$\{\sigma_{\beta(\alpha} - \omega_{\beta(\alpha}\}_{;\mu}\ \eta_{\rho)}{}^{\mu\beta\varepsilon}\ v_{\varepsilon} - \frac{2}{3}\ \theta\ v_{(\alpha}\ \omega_{\rho)} + 2a_{(\alpha}\ \omega_{\rho)} =$$

$$= -\frac{1}{2}\ R_{\beta\gamma\mu(\alpha}\ \eta_{\rho)}{}^{\gamma\beta\varepsilon}\ v^{\mu} v_{\varepsilon} .$$

We obtain, by a straighforward calculation :

$$H_{\mu\nu} = -\frac{1}{2}\ h_{(\mu}^{\alpha}\ h_{\nu)}{}^{\beta}(\sigma_{\alpha\rho;\lambda} + \omega_{\alpha\rho;\lambda})\ \eta_{\beta}{}^{\varepsilon\rho\lambda} v_{\varepsilon} \qquad (54)$$

Thus the whole system of perturbed equations is given by equations (29)(31)(34)(36)(40)(46)(48)(52)(53) and (54). Let us now turn to a systematic discussion of these equations in the three possible cases: scalar, vector and tensor perturbations, respectively, the density of energy. The vortex rotations of the fluid and the disturbances of metric, as gravitational waves.

3.7 - BASIC EQUATIONS OF PERTURBATION OF THE DENSITY OF ENERGY IN FRIEDMANN UNIVERSE (SCALAR PERTURBATION)

In this section we will examine the following question: given, at certain time t_0, a small perturbation on the Friedmann Universe, how this perturbation will evolve with time? In other words: is Friedmann cosmos stable or unstable for gentle modification of its basic features?

This problem has been examined prior in the literature, with the principal motivation to give a simple explanation of the observed inhomogeneities which are seen in small scales of length in the cosmos. Here we will present the quasi-Maxwellian version of it. We give a complete analysis of the full system of equations (quasi-Maxwellian equations of gravity, plus the equations of evolution of kinematical quantities associated to the cosmic fluid) and solve all of them in some special cases of interest showing the compatibility of the whole system. In this case we will use the explicit 3+1 decomposition of space-time and consider the quantities on the space-like surface Σ as a 3-dimensional space (characterized by coordinates x^i (i=1,2,3)). That is, we take $h^0{}_\mu = 0$, $h^i{}_j = \delta^i{}_j$. This choice somehow simplifies the calculations and is a point of contact with standard procedures in the calculus of perturbation (see Lifshitz (1946), for instance).

The important point we want to stress from the very beginning is that, in order to obtain the correct equation of evolution of the variation of the density of energy, we only need to consider the equation of conservation of the energy-momentum tensor plus Raychaudhuri's equation for the expansion factor.

Let us expand the perturbation of energy $\delta\rho$ in a convenient basis $Q_{(k)}$. In the Euclidean section ($\varepsilon=0$) we can take, for instance, $Q_{(k)}(x)= e^{i\vec{k}\cdot\vec{x}}$ (cf. section 2.11).

We set

$$\delta\rho = N(t)Q \tag{55}$$

$$\delta V_k = R(t) \frac{\partial Q}{\partial x^\alpha} \equiv R(t)Q,_k \tag{56}$$

[it is understood that this expression contains a complete series and that we should write, in a more precise way, for instance

$$\delta\rho = \sum_k N(t)\ C_k\ Q_{(k)}(\vec{x})] . \tag{57}$$

Then, from equation (27) the acceleration vector is given by:

$$a_k = \dot{R}Q,_k \ . \tag{58}$$

From the conservation equations we have:

$$(\delta\rho)^{\bullet} + (\rho+p)\ \delta\theta + (\delta\rho + \delta p)\theta = 0 \tag{59}$$

$$(\rho+p)\ a_k - \delta p,_k + \dot{p}\ \delta v_k = 0\ . \tag{60}$$

In this section we assume that the equation of state $p= \lambda\rho$ does not change during the perturbation.

Thus:

$$\delta p = \lambda \delta\rho \, . \tag{61}$$

Using (61) into (60) we have

$$a_k = \frac{\lambda}{(1+\lambda)} \frac{\delta\rho,k}{\rho} - \frac{\lambda}{1+\lambda} \frac{\dot{\rho}}{\rho} \delta v_k$$

or

$$a_k = \frac{\lambda}{(1+\lambda)} \frac{N}{\rho} a_{;k} + \lambda\theta \; R \; Q_{,k}$$

$$a_k = \lambda \left[\frac{N}{(1+\lambda)} + \theta R \right] Q_{,k} \, . \tag{62}$$

Thus, from (60) and (62) we have:

$$\dot{R} = \lambda \left[\frac{N}{\rho(1+\lambda)} + R\theta \right] . \tag{63}$$

Now, from equation (59) we have

$$\delta\theta = - \frac{1}{(1+\lambda)\rho} \left[\dot{N} + (1+\lambda) N\theta \right] Q$$

or

$$\delta\theta = - \frac{1}{1+\lambda} \left(\frac{N}{\rho} \right)^{\bullet} Q \, . \tag{64}$$

Using (64) and (62) into Raychaudhuri's equation,we have:

$$\left[-\frac{1}{(1+\lambda)}\left(\frac{N}{\rho}\right)^{\cdot\cdot}-\frac{2}{3}\,\theta\,\frac{1}{1+\lambda}\left(\frac{N}{\rho}\right)^{\cdot}\right]Q+\frac{\dot{R}}{A^2}\,(\nabla^2 Q)=-\frac{1}{2}\,(1+3\lambda)NQ\,.$$

Remark that in this expression we have used the fact that

$$a^k=-\frac{\dot{R}}{A^2}\,Q^{,k}$$

in which

$$Q^{,k}=Q_{,\ell}\,\delta^{\ell k}.$$

Re-arranging the above expression, we obtain:

$$\frac{1}{1+\lambda}\left[\left(\frac{N}{\rho}\right)^{\cdot\cdot}+\frac{2}{3}\,\theta\left(\frac{N}{\rho}\right)^{\cdot}\right]+\frac{\dot{R}}{A^2}\,k^2=\frac{1}{2}(1+3\lambda)\,N\,. \qquad (65)$$

The quantity $\frac{N}{\rho}\equiv\mu$ is known as the density of contrast (of energy). Using equation (63) into (65) we obtain:

$$\frac{1}{1+\lambda}\left[\ddot{\mu}+\frac{2}{3}\,\theta\dot{\mu}\right]+\frac{\lambda}{A^2}\,k^2\left(\frac{N}{(1+\lambda)\rho}+R\theta\right)-\frac{1}{2}\,(1+3\lambda)\,\rho\mu=0$$

or equivalently

$$\frac{1}{1+\lambda}\,(A^2\dot{\mu})^{\cdot}-\frac{(1+3\lambda)}{2}\,A^2\rho\mu+\frac{k^2\lambda}{1+\lambda}\,\mu+\lambda k^2\,\theta R=0. \qquad (66)$$

Equations (66) and (63) enable us to know the time dependence of the perturbation of the density of energy $\delta\rho$ and of the fluid velocity v_k. Indeed, derivating (66) and using once more equation (63), we obtain

$$\frac{1}{(1+\lambda)}(A^2\dot{\mu})^{\cdot\cdot} - \frac{1+3\lambda}{2}(A^2\rho\mu)^{\cdot} + \lambda\theta\left[-\frac{1}{1+\lambda}(A^2\dot{\mu})^{\cdot} + \frac{(1+3\lambda)}{2}A^2\rho\mu\right] +$$

$$+\frac{\dot{\theta}}{\theta}\left[\frac{1+3\lambda}{2}A^2\rho\mu - \frac{(A^2\dot{\mu})^{\cdot}}{1+\lambda}\right] + \frac{\lambda}{1+\lambda}k^2\left[\dot{\mu} - \frac{\dot{\theta}}{\theta}\mu\right] = 0 \,. \qquad (67)$$

This is the general expression which allows us to know the density of contrast μ in terms of the known quantities of the background geometry. However, instead of dealing directly with this equation let us simplify it a little further.

A direct inspection on equation (67) shows that a particular solution of it is given by $\mu = -(1+\lambda)R_0\theta$. This is the special case in which $\dot{R} = 0$, that is, there is no perturbation of the acceleration (see eq. 58).

We can use this fact to reduce the degree of equation (92). Accordingly, we define

$$\mu = \mu_0 F \,. \qquad (68)$$

By setting

$$J \equiv \dot{F} \,, \qquad (69)$$

a direct calculation gives from (67) the equation of evolution of J:

$$\theta\ddot{J} + \left[\left(\frac{2}{3} - \lambda\right)\theta^2 - (1+3\lambda)\rho\right]\dot{J} + (1+3\lambda)\left[2\lambda\theta^2 - \frac{\rho}{2}(1+3\lambda)\right]\frac{\rho}{\theta}J +$$

$$+ \frac{\lambda k^2}{A^2}\theta J = 0 \,. \qquad (70)$$

Before proceeding to the analysis of equation (70) it seems wothwhile to make some comments on the special solution μ_0, about which there have been misleading remarks in the literature. This solution represents a perturbation that is damped as the Universe expands, typically $\mu_0 \sim t^{-1}$. The fact that this solution appears as a trick to reduce the degree of the differential equation of the density of contrast may rise a suspiction that it is nothing but a consequence of a coordinate transformation and not a true perturbation. Some authors have conjectured that this should be the situation for a fluid endowed with non-null pressure $p=\lambda\rho$, but that a singular situation could occur in case $p=0$, thus making this special solution μ_0 to be a real perturbation only in such case.

Let us prove now that this conjecture is false, and that μ_0 may be eliminated for all values of $0 \leq \lambda \leq 1$ by a simple coordinate transformation.

In order to prove this we will use the complete set of Kinematical equations plus Einstein's equations in the quasi-Maxwellian form.

We have

$$\delta\rho = N(t)\ Q(\vec{x}) \tag{71a}$$

$$\delta V_k = R(t)\ Q_{,k}(\vec{x}) \tag{71b}$$

$$E_{ij} = E(t)\ P_{ij} \tag{71c}$$

$$H_{ij} = H(t)\ P_{ij} \tag{71d}$$

$$\delta\sigma_{ij} = \Sigma(t)\ P_{ij}. \tag{71e}$$

As a consequence of (71b) there are no rotational perturbations. From equation (29) we have

$$E_{ik;j}\ g^{ij} = \frac{1}{3}\ (\delta\rho)_{,k} - \frac{1}{3}\ \dot{\rho}\ \delta V_k$$

or

$$-\ 2E\ \frac{k^2+3}{k^2} = A^2\ \rho\left[\mu + (1+\lambda)\,\theta R\right]. \tag{72}$$

From equation (32) we obtain

$$g^{ij}\ H_{ik\|j} = 0$$

or

$$H(t) = 0\ . \tag{73}$$

The equation of evolution of E_{ij} is obtained from (34)

$$\dot{E} + \frac{\dot{A}}{A}\ E = -\ \frac{1}{2}\ (1+\lambda)\,\rho\ \Sigma \tag{74}$$

From the equation of evolution of the shear, we have

$$\dot{\Sigma} + E - \frac{k^2}{A^2}\ \dot{R} = 0\ . \tag{75}$$

Besides these equations, we have to consider the constraints. The only equation which is not trivially satisfied reduces to

$$\theta R + \frac{1}{1+\lambda}\,\dot{\mu} + \frac{k^2+3}{k^2}\;\frac{\Sigma}{A^2} = 0 . \tag{76}$$

When the solution $\mu = \mu_0 = -(1+\lambda)\theta R_0$ is inserted in these equations we get $E = \Sigma = 0$.

Thus, this solution μ_0 makes all tensorial quantities to vanish identically. Only the perturbation of the scalar quantities does not vanish identically, but this can be eliminated by a simple re-scaling $t \to \tilde{t} = t + f(\vec{x}, t)$.

The case $p = 0$

In this case we can deal directly with equation (66). We have

$$\ddot{\mu} + \frac{2}{3}\theta\,\dot{\mu} - \frac{1}{2}\,\rho\mu = 0 . \tag{77}$$

Writing

$$\mu = \theta F \tag{78}$$

and substituting into eq. (77) we obtain

$$\theta \dot{f} - \rho f = 0 , \tag{79}$$

in which

$$\dot{f} \equiv F .$$

Thus,

$$f = f_0 \exp \int \frac{\rho}{\theta} \, dt \ . \qquad (80)$$

Let us make two comments here:

Firstly, we notice that if the perturbation is such that the initial condition is taken as $(\delta V_k)_0 = 0$, then at any time, the perturbation of the velocity vanishes.

Secondly, we remark that the equation (77) for μ does not contain the wavelength of the perturbation k. The term which couples the wavelength with the density of contrast disappears as a consequence of the vanishing of the acceleration of the fluid.

This has the very important consequence that the time dependence of the contrast factor is the same for all wavelengths of perturbation, even for those which can stay beyond the horizon. This is due to the fact that the only interaction between distinct parts of the fluid has a gravitational origin.

Furthermore, in the linear approximation, the gravitational field which interacts with the perturbed density $\delta\rho$ has its origin in the uniform distribution $\rho_0(t)$ and, consequently, the wavelength of the perturbation is not relevant for the dynamics of $\delta\rho$.

This is not the case when the fluid has a non-null pressure p, as we shall see later.

For the sake of completeness, let us present the solutions of the perturbation for distinct topologies of the 3-dimensional section.

(i) Euclidean Section

In this case (see Table I) we have:

$$A(t) = A_0 t^{2/3}$$

$$\theta(t) = 2t^{-1}$$

$$\rho(t) = \frac{4}{3} t^{-2} .$$

Equation (79) becomes

$$\dot{f} = \frac{2}{3} t^{-1} f ,$$

and thus we obtain

$$f(t) = f_0 t^{2/3} . \tag{81}$$

The density of contrast takes the value

$$\mu = \mu_{(1)} t^{2/3} + \mu_{(2)} t^{-1} , \tag{82}$$

in which $\mu_{(1)}$ and $\mu_{(2)}$ are constants. The evanescent mode t^{-1} may be eliminated by a simple coordinate transformation (a re scaling of time).

(ii) Closed Section

We have

$$A(\eta) = A_0(1-\cos\eta)$$

$$t(\eta) = A_0(\eta-\sin\eta)$$

$$\theta = \frac{3}{A_0}\ \frac{\sin\eta}{(1-\cos\eta)^2}$$

$$\rho = \frac{6}{A_0^2}\ \frac{1}{(1-\cos\eta)^3}\ .$$

In this case we have:

$$\sin\eta\,\frac{df}{d\eta} - 2f = 0\ .$$

The solution is

$$f = f_0\ tg^2(\eta/2) \qquad (83)$$

and thus,for the density of the constrast,

$$\mu = \frac{3\ \sin\eta}{A_0(1-\cos\eta)^2}\left[(\sin\eta - 3\eta)f_0 + f_1\right] + \frac{3f_0}{A_0(1-\cos\eta)}\ . \qquad (84)$$

Eliminating the fictitious solution, the physical pertur bation is given by:

$$\mu = 3f_a\left[\frac{\sin\eta}{(1-\cos\eta)^2}(\sin\eta-3\eta) + \frac{1}{1-\cos\eta}\right]\ . \qquad (85)$$

(iii) Open Section

We have:

$$A(\eta) = A_0(\cosh \eta - 1)$$

$$t(\eta) = A_0(\sinh \eta - \eta)$$

$$\theta = \frac{3}{A_0} \frac{\sinh \eta}{(\cosh\eta - 1)^2}$$

$$\rho = \frac{6}{A_0} (\cosh\eta - 1)^3 .$$

The physical perturbation, in this case, is :

$$\mu = f_0 \sinh^{-2} (\frac{\eta}{2})(1 - \frac{\eta}{2} \coth \frac{\eta}{2}) + 3 f_b \cdot \quad (86)$$

Let us now examine the case λ= 1/3 (pure radiation)

(i) Euclidean Section

From Table I we obtain the fundamental values for the unperturbed quantities:

$$A(t) = A_0 t^{1/2}$$

$$\rho(t) = 3/4 t^{-2}$$

$$\theta(t) = 3/2 t^{-1} .$$

Equation (70) becomes:

$$t^2 \ddot{J} - \frac{1}{2} t \dot{J} + \left[\frac{1}{2} + \frac{k^2}{3A_0^2} t\right] J = 0 \cdot \qquad (87)$$

The solutions for J(t) are

$$J(t) = J_a \phi \sin\phi + J_b \phi \cos\phi , \qquad (88)$$

in which J_a and J_b are arbitrary (small) constants and

$$\phi = \frac{2}{\sqrt{3A_0}} K t^{1/2}.$$

The density of contrast, after eliminating the fictitious solu tion proportional to θ,is:

$$\mu = 2\alpha\left[\frac{\sin\phi}{\phi} + \left(\frac{1}{\phi^2} - \frac{1}{2}\right)\cos\phi\right] + 2\beta\left[\frac{\cos\phi}{\phi} - \left(\frac{1}{\phi^2} - \frac{1}{2}\right)\sin\phi\right] \cdot \qquad (89)$$

In this case the dynamics of the perturbation depends on the associated wavelength. In the case of very long wavelength, that is, $\frac{1}{K} >> \frac{t^{1/2}}{A_0}$, we can approximate this expression by the form.

$$\mu \simeq \frac{\alpha}{4} \phi^2 + \frac{\beta}{3} \phi ,$$

in which the term proportional to t^{-1} has been eliminated, as before. This result was obtained previously by Olson (1976). See also Lifshitz (1963).

(ii) Open Section

From Table I we extract the values of the unperturbed background:

$$A(\eta) = A_0 \sinh \eta$$

$$t(\eta) = A_0 (\cosh \eta - 1)$$

$$\rho(\eta) = \frac{3}{A_0{}^2} \sinh^{-4} \eta$$

$$\theta(\eta) = \frac{3}{A_0} \frac{\cosh \eta}{\sinh^2 \eta} .$$

Defining

$$J' \equiv \frac{dJ}{d\eta} ,$$

we have

$$\dot{J} = \frac{J'}{A_0(\cosh\eta - 1)} .$$

Then, eq. (70) becomes

$$\sinh(2\eta)\, J'' - 4\, J' + \left[8 \coth 2\eta + \frac{k^2}{3} \sinh 2\eta \right] J = 0 \;. \quad (90)$$

For very early times we can approximate this equation (for $\eta << 1$) by the equation

$$\eta^2 J'' - 2\eta J' + \left[2 + \frac{k^2}{3}\eta^2\right] J = 0 \; ; \qquad (91)$$

writing

$$J = \eta^{3/2} H$$

we obtain for H, a Bessel differential equation:

$$\eta^2 H'' + \eta H' + \left(\frac{k^2}{3}\eta^2 - \frac{1}{4}\right) H = 0 \qquad (92)$$

the solution of which is given by

$$J(\eta) = \eta^{3/2} \{C_1 J_{1/2}(q\eta) + C_2 Y_{1/2}(q\eta)\} \quad (93)$$

in which $q^2 \equiv \frac{k^2}{3}$.

For small values of the parameter, i.e., $q\eta \ll 1$, we approximate such expression by the form

$$J_<(\eta) \sim C_1 q^{1/2} \eta^2 - \frac{C_2}{2(\pi q)^{1/2}} \eta \, , \qquad (94)$$

and in the other extreme case ($q\eta \gg 1$),

$$J_>(\eta) \sim \eta \left(\frac{2}{\pi q}\right)^{1/2} \{C_1 \sin q\eta - C_2 \cos q\eta\}. \quad (95)$$

For the density of contrast we have

$$\mu = \theta \int J \, A(\eta) \, d\eta \, ,$$

that is,

$$\mu_{<} \sim \frac{3}{4} C_1 q^{1/2} \eta^2 - \frac{C_2}{2}(\pi q)^{-1/2} \eta , \qquad (96)$$

and for very large wavelengths

$$\mu_{>} \sim q^{-3/2} . \qquad (97)$$

Closed Section

The result of the perturbation of $\delta\rho$ in the case $\varepsilon= +1$ (closed section) can be obtained from the previous case by the formal mapping

$$A \rightarrow iA$$

$$\eta \rightarrow i\eta \qquad (98)$$

$$K \rightarrow iK .$$

Let us make an additional remark. In the standard procedüre (Lifshitz 1946) one deals directly with perturbations of the metric. One should ask if it is possible to find $\delta g_{\mu\nu}$ from our solution. The answer for this is yes. Indeed, from the definition of shear we have

$$\delta\sigma_{ij} = -\frac{\theta}{3}\,\delta g_{ij} + \frac{1}{2}\,\delta V_{(i;j)} - \frac{1}{3}\,\delta\theta g_{ij} + \frac{1}{2}(\delta g_{ij})_{,o};$$

setting

$$\delta g_{ij} = B(t)\hat{\sum}_{ij}$$

we obtain, in the case of perfect fluid,

$$\dot{B} - 2\,\frac{\dot{A}}{A}\,B = \alpha A^{-2} \quad ,$$

in which $\alpha = \dfrac{2(1+\lambda)\rho_0}{2\epsilon + k^2} =$ constant.

We can easily integrate this expression to obtain

$$M = \text{constant}.\; A^2 \int \frac{d\eta}{A^3(\eta)} \quad .$$

3.8 - VECTOR PERTURBATIONS

In the previous section we have worked out the full system of perturbation equations for a general fluctuation of the geometry and the matter. Let us restrict our study in this section to those perturbations in which only the state of motion of the fluid changes, leaving unperturbed the density of energy. This corresponds to the case in which a vortex is introduced and we want to examine the future evolution of such small vorticity.

It enormously simplifies our analysis if we follow Lifshitz et al. (1946) and use a complete basis of the hyperspherical harmonics to develop the general perturbation. For the case of pure vortex fluctuations ($\delta\rho=0$), this is accomplished by the vector $\hat{S}_\alpha$ which has the following properties:

(i) $\hat{S}_\alpha$ is defined in the 3-dimensional rest space M orthogonal to V^μ, obtained using the projector operator $h_{\nu\mu}$, that is

$$V^\mu \hat{S}_\mu = 0 \qquad (99a)$$

$$h_\mu{}^\nu \hat{S}_\nu = \hat{S}_\mu \ . \qquad (99b)$$

(ii) $\hat{S}_\alpha$ is an eigen-vector of the 3-dimensional Laplacian operator.

$$h^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu \hat{S}_\alpha = \frac{K^2}{A^2} \hat{S}_\alpha \qquad (100)$$

(K is an integer)

in which the co-variant derivative $\hat{\nabla}_\mu$ is defined as the restriction on M of the covariant derivative ∇_μ. Thus,

$$\hat{\nabla}_\mu \hat{A}_\lambda \equiv h_\mu{}^\varepsilon \nabla_\varepsilon (h_\lambda{}^\rho A_\rho) \tag{101}$$

(iii) The eigen-vector $\hat{S}_\alpha$ is stationary.

This means

$$\hat{S}_{\alpha;\lambda} V^\lambda = 0. \tag{102}$$

(iv) The eigen-vector $\hat{S}_\alpha$ is divergence-free

$$\hat{\nabla}_\alpha \hat{S}^\alpha = 0 . \tag{103}$$

From this vector $\hat{S}^\alpha$, we can construct tensors $\hat{\Sigma}_{\alpha\beta}$ and $\hat{F}_{\alpha\beta}$ by taking the symmetric and anti-symmetric derivative, respectivelly:

$$\hat{\Sigma}_{\alpha\beta} = \hat{\nabla}_{(\alpha} \hat{S}_{\beta)} \tag{104a}$$

$$\hat{F}_{\beta\alpha} = \hat{\nabla}_{[\alpha} \hat{S}_{\beta]}. \tag{104b}$$

Besides we can construct the dual vector $*\hat{S}^\mu$:

$$*\hat{S}^\mu \equiv \eta^{\mu\varepsilon\beta\lambda} V_\varepsilon \hat{S}_{\beta;\lambda} \quad . \tag{104c}$$

Using the properties of $\hat{S}_\alpha$ it is not difficult to show that $\hat{\Sigma}_{\alpha\beta}$ and $\hat{F}_{\alpha\beta}$ satisfy the following properties:

$$h^{\mu\nu}\ \hat{\nabla}_{\nu}\ \hat{\Sigma}_{\mu\rho} = \frac{2\varepsilon+K^2}{A^2}\ \hat{S}_{\rho} \tag{105a}$$

$$\hat{\Sigma}_{\mu\nu;\lambda}\ V^{\lambda} + \frac{\Theta}{3}\ \hat{\Sigma}_{\mu\nu} = 0 \tag{106}$$

$$h^{\mu}{}_{(\rho}\ h^{\nu}{}_{\alpha)}\ \eta_{\mu}{}^{\beta\gamma\lambda}\ V_{\lambda}\ \hat{\Sigma}_{\nu\beta;\gamma} = h^{\mu}{}_{(\rho}\ h^{\nu}_{\alpha)}\ \hat{\nabla}_{\nu}\ \hat{S}^{*}{}_{\mu} \tag{107}$$

$$h^{\mu}{}_{(\rho}\ h^{\nu}{}_{\alpha)}\ \eta_{\mu}{}^{\beta\gamma\lambda}\ V_{\lambda}\ \hat{\nabla}_{\gamma}\ \hat{\mathbf{F}}_{\nu\beta} = -h^{\nu}_{(\rho}\ h^{\mu}_{\alpha)}\ \hat{\nabla}_{\nu}\ \hat{S}^{*}_{\mu} \tag{108}$$

$$h^{\alpha}{}_{\rho}\ h^{\gamma\nu}\ \hat{\nabla}_{\nu}\hat{\nabla}_{\gamma}\ \hat{S}^{*}_{\alpha} = \frac{K^2}{A^2}\ \hat{S}^{*}_{\rho} \tag{109}$$

$$h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}\ (\hat{\nabla}_{\mu}\hat{S}^{*}_{\nu})^{\bullet} + \frac{2}{3}\ \theta\ h_{(\alpha}{}^{\mu}h_{\beta)}{}^{\nu}\ \hat{\nabla}_{\mu}\hat{S}^{*}_{\nu} = 0 \tag{110}$$

$$h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}\ (\hat{\nabla}_{\mu}\hat{S}_{\nu})^{\bullet} + \frac{\theta}{3}\ h_{(\alpha}{}^{\mu}\ h_{\beta)}{}^{\nu}\ \hat{\nabla}_{\mu}\hat{S}_{\nu} = 0 \tag{111}$$

$$h^{\mu(\rho}\ \eta^{\sigma)\lambda\nu\alpha}\ V_{\lambda}\ \hat{\Sigma}^{*}{}_{\alpha\mu;\nu} = (2\varepsilon-K^2)\ h^{\mu(\sigma}\ h^{\rho)\nu}\ \hat{S}_{\nu;\mu}\ , \tag{112}$$

in which ε is defined in terms of the scalar of the 3-curvature by means of the formula

$$\hat{R}_{\alpha\beta\mu\nu} = -\frac{\varepsilon}{A^2}\ h_{\alpha\beta\mu\nu}$$

and $h_{\alpha\beta\mu\nu} \equiv h_{\alpha\mu}\ h_{\beta\nu} - h_{\alpha\nu}\ h_{\beta\mu}$.

Thus, we can develop any vortex perturbation in the basis

$\hat{S}_\mu$. We set*

$$\delta V_\mu = V(\tau)\hat{S}_\mu \tag{113a}$$

$$q_\mu = q(\tau)\hat{S}_\mu \tag{113b}$$

$$\Pi_{\mu\nu} = \Pi(\tau)\hat{\Sigma}_{\mu\nu} , \tag{113c}$$

in which τ means the proper time of the fundamental co-moving observer V^μ.

Using (24) we obtain for the vorticity

$$\omega_{\mu\nu} = \frac{1}{2} V(\tau)\hat{F}_{\mu\nu}. \tag{114}$$

For the shear and the electric tensor $E_{\mu\nu}$ we set:

$$\sigma_{\mu\nu} = L(\tau)\hat{\Sigma}_{\mu\nu} \tag{115}$$

$$E_{\mu\nu} = E(\tau)\hat{\Sigma}_{\mu\nu}. \tag{116}$$

Equation (26) for the definition of the acceleration gives

$$a_\mu = (\dot{V} + \frac{1}{3}\theta V)\ \hat{S}_\mu . \tag{117}$$

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*Remark that this decomposition is to be understood as a series $\delta V_\mu = \sum_m c_{(m)} V(\tau)\ \hat{S}_\mu^{(m)}$, for instance. We will write only a generic term of the series in the text, just to simplicity of writing.

Using this value into equation (38) we have:

$$\dot{V} + \frac{(1-3\lambda)}{3}\,\theta V + \frac{1}{(1+\lambda)}\,\frac{1}{\rho}\left[\dot{q} + \frac{4}{3}\theta q + \frac{2\varepsilon+k^2}{A^2}\,\Pi\right] = 0 \qquad (118)$$

The constraint relation (30) gives

$$h_\mu{}^\beta(\sigma^{\mu\nu} - \omega^{\beta\nu})_{;\nu} + \frac{2}{3}\,\dot{\theta}\,\delta\,V_\mu = q_\mu. \qquad (119)$$

From Raychaudhuri equation (5) we know $\dot{\theta}$:

$$\dot{\theta} = -3\left[\frac{\varepsilon}{A^2} + \frac{1+\lambda}{2}\,\rho\right]. \qquad (120)$$

Using this value of $\dot{\theta}$ into eq.(119) we obtain the value of the shear L:

$$L = \frac{A^2}{2\varepsilon+K^2}\left[q + (1+\lambda)\rho V\right] + \frac{V}{2}. \qquad (121)$$

The expansion (56) for $E_{\mu\nu}$ and the equation for the divergence of $E_{\mu\nu}$ (eq. 33) gives the value of E:

$$E = \frac{1}{3(2\varepsilon+K^2)}\,A^2\,\theta\left[(1+\lambda)\rho V + q\right] + \frac{\pi}{2}. \qquad (122)$$

Let us now turn to the calculus of $H_{\mu\nu}$. Equation (32) gives:

$$H_{\alpha\beta} = -\frac{1}{2}(L - \frac{V}{2})\,h_{(\alpha}{}^\mu\,h_{\beta)}{}^\nu\,\hat{\nabla}_\nu\,\hat{S}^*_\mu. \qquad (123)$$

This induces us to define the expansion

$$H_{\alpha\beta} = H(\tau)\,\hat{\Sigma}^*_{\alpha\beta} \equiv H(\tau)\,\hat{\nabla}_{(\alpha}\,\hat{S}^*_{\beta)} \qquad (124)$$

and consequently,

$$H(\tau) = - \frac{A^2}{2(2\varepsilon+K^2)} \left[(1+\lambda)\rho V + q\right]. \qquad (125)$$

It seems worth to remark that although we used only four equations (Raychaydhuri equation, the conservation of energy-momentum tensor, the constraint relation which connects $H_{\mu\nu}$ with spatial derovatives of the shear and the vorticity and the equation for the divergence of $E_{\mu\nu}$) we have solved our problem of finding the evolution of the vortex perturbation. Indeed, we know $E_{\mu\nu}$, $H_{\mu\nu}$ and $\sigma_{\mu\nu}$. Equation (58) is a condition envolving V, q and Π. There is no more non-trivial condition left for the whole set of equations of the complete quasi-Maxwellian system. All other equations are identically satisfied. This can be shown by a rather long(*) but straightforward substitution of the values of H, E, L and condition (58) into the remaining equations.

Let us examine two particular cases of these perturbations which are of great interest.

<u>case i</u>: <u>Perfect Fluid</u>

We set $q = 0 = \Pi$

In this case equation (58) can be immediately integrated to give the value of the perturbation V . We find

$$V = V_o \, A^{3\lambda-1} \qquad (126c)$$

(*) J.M. Salim - PhD Thesis - CBPF (1982) unpublished.

$$E = \frac{(1+\lambda)}{3(2\varepsilon+K^2)} \rho_o V_o \theta A^{-2} \tag{126b}$$

$$H = -\frac{2}{3} \frac{E}{\theta} \tag{126c}$$

$$L = \frac{1+\lambda}{2\varepsilon+K^2} \rho_o V_o + \frac{1}{2} V_o A^{3\lambda-1} . \tag{126d}$$

In the case of radiation ($\lambda=1/3$), the velocity perturbation and the corresponding shear and vorticity are constant, a result which was known since Lifshitz et al paper (1963). Remark, however, that for stiff matter, in case $\frac{1}{3}<\lambda<1$, the vorticity (and the shear) increases as time goes on, once we are considering standard Friedmann expanding background in which $A(\tau)$ is a monotonic function. Note that (66b,c) shows that for those perturbations which preserve the condition of the source as a perfect fluid, short wavelenghts ($\lambda_K = \frac{A}{K} \ll 1$) of the gravitational disturbance are inhibited.

<u>case ii</u>: <u>Stokes Fluid</u>

Let us consider that the perturbation is characterized by $q=0$ but, has anisotropic pressure $\Pi_{\mu\nu}$, linearly related to the shear,

$$\Pi_{\mu\nu} = \alpha \sigma_{\mu\nu} , \tag{127}$$

with α=constant.

In this case we can integrate equation (58) to obtain the value of $V(\tau)$:

$$V = V_o \, e^{-\alpha t} \, e^{-\frac{3}{8}\frac{\alpha}{\rho_o}(2\varepsilon+v^2)\int A^2 dt} \quad (128)$$

in which V_o is a constant.

In general, for a Stokes fluid (67) the only restriction we set on constant α is that it must be positive. This garantees, through the use ad hoc of the second principle of thermodynamics, that entropy only increases in the direction of the arrow of time. However, for the present case in which $\Pi_{\mu\nu}$ is to be considered as a first order perturbation term, equation (37) states that this condition on α may be relaxed, once the contribution of anisotropy to the variation of entropy is a second order effect (the additional term is $\Pi_{\mu\nu}\,\sigma^{\mu\nu} = \alpha\sigma^2$).

Thus,the instability of this kind of perturbation which may occur for $\alpha<0$, is not forbidden.

Let us make a final remark on the general features of the vortex perturbation.

Suppose we want to consider a pure electric perturbation by setting $H=0$. Equations (58)(61)(62) and (51) imply that then $\Pi=0=E$. That is, the perturbed geometric is conformally flat too. This is possible only if there is a heat flux such that $q= -(1+\lambda)\rho V$, in which case the shear is given by $L= \frac{V}{2}$. This shows that we can prescribe arbitrarily the function $V(\tau)$ for the perturbation of the universe. We have just to proceed as follows: consider $L(\tau)$ as a given function (either by theory or by any kind of observation) of the distortion of the cosmic fluid. Then we obtain $V(\tau)$. From this we evaluate the heat flux which is necessary to satisfy the whole system of perturbed equations. Note that this result is completely in dependent of the wavelength of the perturbation. It remains only to consider a physical model to generate heat. This has to be examined for each case individually.

3.9 - TENSOR PERTURBATIONS

In this section we consider perturbations of the geometry which are not linked to perturbations of the matter. This represents gravitational waves propagating in the Friedmann background.

The basic equations (of the quasi-Maxwellian system) reduce in this case to the set:

$$h^{\alpha}{}_{\varepsilon}\, E_{\alpha\mu;\nu}\, h^{\mu\nu} = 0 \qquad (129)$$

$$h^{\alpha}{}_{\varepsilon}\, H_{\alpha\mu;\nu}\, h^{\mu\nu} = 0 \qquad (130)$$

$$h_{\varepsilon}{}^{\alpha}\, h_{\rho}{}^{\beta}\, \frac{DE_{\alpha\beta}}{D\tau} - \frac{1}{2}\, h^{\mu}{}_{(\rho}\, h_{\varepsilon)}{}^{\lambda}\, \eta_{\lambda}{}^{\tau\nu\alpha}\, H_{\alpha\mu;\nu}\, V_{\tau} + \theta E_{\varepsilon\rho} = M_{\varepsilon\rho} \qquad (131)$$

$$h_{\varepsilon}{}^{\alpha}\, h_{\rho}{}^{\beta}\, \frac{DH_{\alpha\beta}}{D\tau} + \frac{1}{2}\, h^{\mu}{}_{(\rho}\, h_{\varepsilon)}{}^{\lambda}\, \eta_{\lambda}{}^{\tau\nu\alpha}\, E_{\alpha\mu;\nu}\, V_{\sigma} + \Theta H_{\varepsilon\rho} = N_{\varepsilon\rho} \qquad (132)$$

$$\frac{D\sigma_{\alpha\beta}}{D\tau} + \frac{2}{3}\,\theta\,\sigma_{\alpha\beta} = -E_{\alpha\beta} - \frac{1}{2}\,\Pi_{\alpha\beta}\ , \qquad (133)$$

in which $M_{\mu\nu}$ and $N_{\mu\nu}$ are defined by the expressions:

$$M_{\mu\nu} = -\frac{1}{2}(\rho+p)\,\sigma_{\mu\nu} + \frac{1}{2}\, h^{\varepsilon}_{\mu}\, h^{\lambda}_{\nu}\, \frac{D\Pi_{\varepsilon\lambda}}{D\tau} + \frac{1}{6}\,\theta\Pi_{\mu\nu} \qquad (134)$$

$$N_{\mu\nu} = -\frac{1}{4}\, h^{\alpha}_{(\mu}\, h^{\varepsilon}_{\nu)}\, \eta_{\varepsilon}{}^{\lambda\beta\rho}\, V_{\lambda}\, \Pi_{\alpha\rho;\beta}\ . \qquad (135)$$

Using the fact that in the unperturbed geometry we have

$\frac{D}{D\tau} h_{\mu\nu} \equiv \dot{h}_{\mu\nu} = 0$ and defining the operator $P_{\alpha\beta}$ given by

$$P_{\alpha\beta}[H] \equiv \frac{1}{2} h^{\mu}_{(\alpha} h_{\beta)}^{\nu} \eta_{\nu}^{\ \lambda\varepsilon\rho} V_{\lambda} H_{\rho\mu;\varepsilon} \qquad (136)$$

we can re-write the above equations in the compact form:

$$\dot{E}_{\alpha\beta} + \theta E_{\alpha\beta} - P_{\alpha\beta}[H] = M_{\alpha\beta} \qquad (137a)$$

$$\dot{H}_{\alpha\beta} + \theta H_{\alpha\beta} + P_{\alpha\beta}[E] = N_{\alpha\beta} \qquad (137b)$$

$$\dot{\sigma}_{\alpha\beta} + \frac{2}{3} \theta \sigma_{\alpha\beta} = -E_{\alpha\beta} - \frac{1}{2} \Pi_{\alpha\beta} \qquad (137c)$$

$$H_{\mu\nu} = P_{\mu\nu}[\sigma] . \qquad (137d)$$

Applying the operator $\frac{D}{D\tau}$ to equation (137a) we obtain:

$$\ddot{E}_{\alpha\beta} + \dot{\theta} E_{\alpha\beta} + \theta \dot{E}_{\alpha\beta} - \dot{P}_{\alpha\beta}[H] = \dot{M}_{\alpha\beta} .$$

In order to evaluate the term $\dot{P}_{\alpha\beta}[H]$ we can proceed as follows: We apply the operator $P_{\alpha\beta}$ to equation (137b) and obtain

$$P_{\alpha\beta}[\dot{H}] + P_{\alpha\beta}[\theta H] + P_{\alpha\beta}[P[E]] = P_{\alpha\beta}[N] . \qquad (139)$$

Let us evaluate each term of this equation separately. The first term gives

$$P_{\alpha\beta}[\dot{H}] = \frac{1}{2} h_{(\alpha}^{\ \gamma} h_{\beta)}^{\rho} \eta_{\rho}^{\ \lambda\nu\mu} V_{\lambda} (H_{\mu\gamma;\varepsilon} V^{\varepsilon})_{;\nu} . \qquad (140)$$

The double derivative on $H_{\mu\nu}$ can be written:

$$H_{\mu\gamma;\varepsilon;\nu} V^{\varepsilon} = H_{\mu\gamma;\nu;\varepsilon} V^{\varepsilon} - \frac{1}{6} (\rho + 3p) H_{\nu(\mu} V_{\gamma)} . \tag{141}$$

The proof of this is as follows. Using the definition of Riemann curvature tensor we can invert the order of the derivative to obtain

$$H_{\mu\gamma;\varepsilon;\nu} V^{\varepsilon} = H_{\mu\gamma;\nu;\varepsilon} V^{\varepsilon} + R_{\mu}{}^{\lambda}{}_{\varepsilon\nu} H_{\lambda\gamma} V^{\varepsilon} + R_{\gamma}{}^{\lambda}{}_{\varepsilon\nu} H_{\mu\lambda} V^{\varepsilon} \tag{142}$$

In the Friedmann geometry we have used equation (1) section (2.2):

$$R_{\mu\lambda\varepsilon\nu} = \frac{1}{3} T(g_{\mu\varepsilon} g_{\lambda\nu} - g_{\mu\nu} g_{\lambda\varepsilon}) - \frac{1}{2} (g_{\mu\varepsilon} T_{\lambda\nu} + g_{\lambda\nu} T_{\mu\varepsilon} -$$

$$- g_{\mu\nu} T_{\lambda\varepsilon} - g_{\lambda\varepsilon} T_{\mu\nu}).$$

Contracting this tensor with $H_{\mu\nu}$ and V^{λ} to obtain the right hand side of (142) we find:

$$R_{\mu\lambda\varepsilon\nu} H^{\lambda}{}_{\gamma} V^{\varepsilon} = - \frac{1}{6}(\rho + 3p) H_{\nu\gamma} V_{\mu}$$

and

$$R_{\gamma\lambda\varepsilon\nu} H_{\mu}{}^{\lambda} V^{\varepsilon} = - \frac{1}{6}(\rho + 3p) H_{\nu\gamma} V_{\gamma} .$$

Thus,

$$H_{\mu\gamma;\varepsilon;\nu} V^{\varepsilon} = H_{\mu\gamma;\nu;\varepsilon} V^{\varepsilon} - \frac{1}{6} (\rho+3p) H_{\nu(\mu} V_{\gamma)} .$$

Substituting this result into equation (142), expression (140) gives:

$$P_{\alpha\beta}[\dot{H}] = \dot{P}_{\alpha\beta}[H] + \frac{1}{3}\theta\, P_{\alpha\beta}[H] \,. \tag{143}$$

Now, using the fact that $h_{\mu}{}^{\nu}\,\theta_{,\nu} = 0$,

$$P_{\alpha\beta}[\theta H] = \theta P_{\alpha\beta}[H] \,, \tag{144}$$

which implies that equation (140) reduces to:

$$\dot{P}_{\alpha\beta}[H] + \frac{4}{3}\,\theta P_{\alpha\beta}[H] + P_{\alpha\beta}[P[E]] = P_{\alpha\beta}[N] \,. \tag{145}$$

Using equation (137a) to isolate $P_{\alpha\beta}[H]$ as a function of the electric tensor $E_{\mu\nu}$ and the quantity $M_{\mu\nu}$, equation (145) above takes the form:

$$\dot{P}_{\alpha\beta}[H] = \frac{4}{3}\theta\,[M_{\alpha\beta} - \theta E_{\alpha\beta} - \dot{E}_{\alpha\beta}] - P_{\alpha\beta}[P[E]] + P_{\alpha\beta}[N] \,. \tag{146}$$

Finally, equation (138) reduces to:

$$\ddot{E}_{\alpha\beta} + \frac{7}{3}\,\theta\dot{E}_{\alpha\beta} + (\dot{\theta} + \frac{4}{3}\,\theta^{2})E_{\alpha\beta} + P_{\alpha\beta}[P[E]] = \dot{M}_{\alpha\beta} + \frac{4}{3}\theta M_{\alpha\beta} + P_{\alpha\beta}[N] \,. \tag{147}$$

In an analogous way, one finds that the magnetic tensor $H_{\alpha\beta}$ satisfies a similar equation:

$$\ddot{H}_{\alpha\beta} + \frac{7}{3}\theta\,\dot{H}_{\alpha\beta} + (\dot{\theta} + \frac{4}{3}\theta^{2})H_{\alpha\beta} + P_{\alpha\beta}[P[H]] = \dot{N}_{\alpha\beta} + \frac{4}{3}\,\theta N_{\alpha\beta} - P_{\alpha\beta}[M] \,. \tag{148}$$

Equations (147) and (148) are generalizations of the equation of wave propagation in Friedmann background. In order to make such similarity explicit let us find the relation between the operator $P[P[X]]$ and the generalized Laplacian operator ∇^2 which is defined by

$$\nabla^2 H_{\varepsilon\rho} \equiv -h^{\alpha}_{\varepsilon}\, h^{\delta}_{\rho}\, h^{\gamma\lambda}(h^{\mu}_{\alpha}\, h^{\nu}_{\delta}\, h^{\beta}_{\gamma}\, H_{\mu\nu;\beta})_{;\lambda}. \tag{149}$$

We have, using definition (136):

$$P_{\varepsilon\rho}[P[H]] = \frac{1}{4}\, h_{(\varepsilon}{}^{\gamma}\, h_{\rho)}{}^{\delta} \Big[\frac{\theta}{3}\, h_{\phi\lambda}\, V_{\nu}\, \eta_{\delta}{}^{\nu\lambda\alpha}\, \eta_{(\gamma}{}^{\phi\sigma\tau}\, h^{\mu}_{\alpha)} h^{\psi}_{\tau}\, h^{\beta}_{\sigma} H_{\mu\psi;\beta} +$$

$$+ V_{\nu}\, V_{\phi}\, \eta_{\delta}^{\nu\lambda\alpha}\, \eta_{(\gamma}{}^{\phi\sigma\tau} \big[h_{\alpha)}{}^{\mu}\, h^{\psi}_{\tau}\, h^{\beta}_{\sigma}\, H_{\mu\psi;\beta}\big]_{;\lambda}\Big].$$

A simple manipulation shows that

$$h^{\gamma}{}_{(\varepsilon}\, h_{\rho)}{}^{\delta}\, V_{\nu}\, h_{\phi\lambda}\, \eta_{\delta}{}^{\nu\lambda\alpha}\, \eta_{(\gamma}{}^{\phi\sigma\tau}\, h^{\mu}_{\alpha)}\, h^{\psi}_{\tau}\, h^{\beta}_{\sigma}\, H_{\mu\psi;\beta} = 0 .$$

Thus, we can write

$$P_{\varepsilon\rho}[P[H]] = h^{\alpha}_{\varepsilon}\, h^{\tau}_{\rho}\, (h^{\mu}_{\alpha}\, h^{\nu}_{\tau}\, h^{\beta\lambda}\, H_{\mu\nu;\beta})_{;\lambda} -$$

$$- \frac{1}{2}\, h_{(\varepsilon}{}^{\alpha}\, h_{\rho}{}^{\tau}{}_{)}\, (h^{\mu}_{\alpha}\, h^{\beta}_{\tau}\, h^{\nu\lambda}\, H_{\mu\nu;\beta})_{;\lambda},$$

or, in the equivalent form

$$P_{\varepsilon\rho}[P[H]] = h^{\alpha}_{\varepsilon}\, h^{\tau}_{\rho}\, h^{\gamma\lambda}(h^{\mu}_{\alpha}\, h^{\nu}_{\tau}\, h^{\beta}_{\gamma}\, H_{\mu\nu;\beta})_{;\lambda} - \frac{1}{3}\, \theta^2\, H_{\varepsilon\rho} + \rho H_{\varepsilon\rho}. \tag{150}$$

The term $P_{\alpha\beta}[M]$ can be written in a more convenient way as:

$$P_{\varepsilon\rho}[M] = - \frac{1}{4} (\rho+p) h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\nu\alpha}_{\tau} V_{\lambda} \tau_{\alpha\mu;\nu} +$$

$$+ \frac{1}{4} h_{(\varepsilon}^{\delta} h^{\tau}_{\rho)} \eta^{\lambda\mu\alpha}_{\tau} V_{\lambda} \dot{\Pi}_{\gamma\delta;\nu} +$$

$$+ \frac{1}{12} \theta h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\nu\alpha}_{\tau} V_{\lambda} \Pi_{\alpha\mu;\nu} .$$

Or, using equation (137d), we find

$$P_{\varepsilon\rho}[M] = - \frac{1}{2} (\rho+p) H_{\varepsilon\rho} + \frac{1}{12} \theta h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\nu\alpha}_{\tau} V_{\lambda} \Pi_{\alpha\mu;\nu} +$$

$$+ \frac{1}{4} h^{\delta}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\nu\gamma}_{\tau} V_{\lambda} \dot{\Pi}_{\gamma\delta;\nu} . \qquad (151)$$

We have also

$$\dot{M}_{\varepsilon\rho} + \frac{4}{3} \theta M_{\varepsilon\rho} = - \frac{1}{4} h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\beta\alpha}_{\tau} V_{\lambda} \dot{\Pi}_{\mu\alpha;\beta} -$$

$$- \frac{1}{4} \theta h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\beta\alpha}_{\tau} V_{\lambda} \Pi_{\mu\alpha;\beta} . \qquad (152)$$

These results allow us to re-write the equation of evolution of $H_{\mu\nu}$ (eq. 148) under the form:

$$\Box H_{\varepsilon\rho} + \frac{7}{3} \theta H_{\varepsilon\rho} + \{\dot{\theta}+\theta^{2}+ \frac{1}{2}(\rho-p)\} H_{\varepsilon\rho} = - \frac{1}{2} h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\beta\alpha}_{\tau} V_{\lambda} \dot{\Pi}_{\mu\alpha;\beta} -$$

$$- \frac{1}{3} \theta h^{\mu}_{(\varepsilon} h^{\tau}_{\rho)} \eta^{\lambda\nu\alpha}_{\tau} V_{\lambda} \Pi_{\alpha\mu;\nu} , \qquad (153)$$

in which the generalized D'Allembert operator $\Box$ is given by

$$\Box H_{\varepsilon\rho} \equiv \ddot{H}_{\varepsilon\rho} - \nabla^2 H_{\varepsilon\rho}.$$

In a similar way one finds:

$$\Box E_{\varepsilon\rho} + \frac{7}{3}\,\theta E_{\varepsilon\rho} + \{\dot{\theta} + \theta^2 + \frac{1}{2}(\rho - p)\}E_{\varepsilon\rho} = -\left[\frac{1}{2}(\dot{\rho} + \dot{p}) + \frac{1}{3}(\rho + p)\theta\right]\sigma_{\varepsilon\rho} +$$

$$+ \frac{1}{2}\,\ddot{\Pi}_{\varepsilon\rho} - \frac{1}{2}\, h^{\alpha}_{\varepsilon} h^{\tau}_{\rho} h^{\gamma\lambda}\left[h^{\mu}_{\alpha} h^{\nu}_{\tau} h^{\beta}_{\gamma}\, \Pi_{\mu\nu;\beta}\right]_{;\lambda} + \left[\frac{\dot{\theta}}{6} + \frac{1}{4}\,(p-q) + \frac{7}{18}\,\theta^2\right]\Pi_{\varepsilon\rho} \tag{154}$$

in which we assumed the restriction on $\Pi_{\mu\nu}$,

$$\Pi_{\alpha\mu;\nu}\, h^{\alpha}{}_{\varepsilon}\, h^{\mu\nu} = 0,$$

a condition which is fulfilled in case, for instance, of a linear Stokesian fluid in which the anisotropic pressure is linearly related to the shear

$$\Pi_{\mu\nu} = \xi\, \sigma_{\mu\nu} \tag{155}$$

(with ξ=constant or at most a function of the energy density). We can now proceed by examination of the perturbed equations by taking a decomposition(as in section 2.11, equations (174) and the following ones).

We set

$$E_{\mu\nu} = \sum_{n} \underset{(n)}{E}\,(\tau)\; U^{(n)}_{\mu\nu} \tag{156a}$$

$$H_{\mu\nu} = \sum_{n} \underset{(n)}{H}\,(\tau)\; U^{(n)}_{\mu\nu} \tag{156b}$$

$$\sigma_{\mu\nu} = \sum_n L_{(n)}(\tau)\; U^{(n)}_{\mu\nu} \qquad (156c)$$

$$\Pi_{\mu\nu} = \sum_n \Pi_{(n)}(\tau)\; U^{(n)}_{\mu\nu} \; . \qquad (156d)$$

Using this decomposition into expression (153), we obtain

$$\ddot{H} + \frac{7}{3}\theta\dot{H} + \{\dot{\theta} + \theta^2 + \frac{1}{2}(\rho - p) + \frac{K^2}{A^2}\}H = -\frac{H}{L}(\dot{\Pi} + \frac{2}{3}\theta\Pi) \; . \qquad (157)$$

Now, using the constitutive relation (155) the right hand side takes the form

$$\text{rhs} = -\frac{H}{L}\xi(\dot{L} + \frac{2}{3}\theta L) \; ,$$

which can be simplified, using eq.(137c),

$$\text{rhs} = -\frac{H}{L}\xi\,(-E - \frac{1}{2}\xi L) \; ,$$

obtaining, thus:

$$\ddot{H} + \frac{7}{3}\theta\dot{H} + \{\dot{\theta} + \theta^2 + \frac{1}{2}(\rho - p) + \frac{K^2}{A^2} + \xi\frac{E}{L} + \frac{1}{2}\xi^2\}H = 0 \; .$$

In an analogous way, we obtain the equation for $E_{\mu\nu}$:

$$\ddot{E} + \frac{7}{3}\theta\dot{E} + \{\dot{\theta} + \theta^2 + \frac{1}{2}(\rho - p) + \frac{K^2}{A^2}\}E = -\{\frac{1}{2}(\dot{\rho} + \dot{p}) + \frac{1}{3}(\rho + p)\theta\}L + \frac{1}{2}\ddot{\Pi} +$$

$$+ \frac{5}{6}\theta\dot{\Pi} + \{\frac{\dot{\theta}}{6} + \frac{1}{4}(\rho - p) + \frac{7}{18}\theta^2 - \frac{K^2}{2A^2}\}\Pi$$

or

$$\ddot{E}+(\frac{7}{3}\theta + \frac{1}{2}\xi)\dot{E} + (\dot{\theta}+\theta^2+ \frac{\theta}{2}\xi + \frac{1}{2}(\rho-p) + \frac{K^2}{A^2})E = -\{\frac{1}{2}(\dot{\rho}+\dot{p})+ \frac{1}{3}(\rho+p)\theta +$$

$$+ \frac{1}{9}\xi\theta^2 + \frac{1}{6}\xi\rho - \frac{1}{2}\frac{K^2}{A^2}\xi\}L \tag{159}$$

and the equation of evolution of shear

$$\dot{L} + \frac{2}{3}\theta L = -E - \frac{1}{2}\xi L$$

or

$$\dot{L} + (\frac{2}{3}\theta + \frac{1}{2}\xi) L = -E . \tag{160}$$

Equations (158), (159) and (160) constitute the basic system of equations which govern the evolution of the perturbed gravitational wave-like field. In order to gain some insight into this system let us examine it in some special and simple circumstance.

Gravitational Waves in De Sitter Universe

De Sitter Universe can be characterized by setting:

$$\theta = \theta_o = \text{constant}$$
$$p = -\rho = \Lambda ,$$

in which Λ is the cosmological constant. So, it represents an evolutionary cosmos with a constant value of its expansion $(\theta_o^2=-3\Lambda)$ in a steady state situation.

We consider only two extreme cases in which either the electric tensor $E_{\mu\nu}$ or the magnetic tensor $H_{\mu\nu}$ vanishes.

<u>Case i</u>: $E_{\mu\nu} = 0$

From the equation of evolution of the shear (160) we obtain immediately

$$L = L_o\, e^{-2/3\theta_o t} \qquad (161)$$

and for H the equation

$$\ddot{H} + \frac{7}{3}\theta_o \dot{H} + (\theta_o^2 + \Lambda + \frac{K^2}{A^2})H = 0 \ .$$

For very short wavelenghts (e.g., $\theta_o^2 >> \frac{K^2}{A^2}$) equation (162) can be easily integrated to give

$$H(t) = H_o\, e^{-|q|^2 t},$$

in which q may assume two values: $q_1 = -\frac{1}{3}\theta_o$ or $q_2 = -2\theta_o$. If we make a two dimensional picture, with coordinates x, y, related to H by the autonomous system

$$\dot{x} = -\frac{7}{3}\theta_o x - \frac{2}{3}\theta_o^2 y$$

$$\dot{y} = x ,$$

(in which x = H), we can use the methods of section (2.12) to depict the qualitative picture of the system showing the estability of the perturbation.

<u>Case ii</u>: $H_{\mu\nu} = 0$

We leave the analysis of this case to the reader, for we can proceed in a very similar way as in the previous case.

4. QUALITATIVE ANALYSIS

4.1 INTRODUCTION

The difficulty of obtaining exact solutions of the nonlinear system of Einstein's equations of gravity has led to investigations of alternative ways to extract information from this system in certain special situations. Among these,we can quote the spinorial formalism, null tetrad frame, complexification of known exact solutions, the inverse scattering technique,etc. One of these methods, the qualitative analysis of investigation of an autonomous system of non-linear differential equations, has been of great help in recent years. We will present such method in this section (see, however, section 2.12 for the non-initiated reader).

It is almost astonishing that the highly non-linear and complicated system of Einstein's equations admits its reduction,in some special cases, to an autonomous planar system of equations. This, of course, is possible only in some restricted situations and certainly it is not applicable in general. However, the important point is that those cases in which such reduction is possible are of great interest and they include precisely the most typical and traditional cosmological models like,for instance,Friedmann, Kasner and Gödel.

This method of analysis has been applied by many authors, e.g. Collins and Stewart (1971), Shikrin (1973),Belinski and Khalatnikov (1975, 1977), Novello et al (1979, 1980), Bogoyavlenskii and S.P. Novikov(1973) and Ellis ( ).

It seems worth to remark that the term qualitative in this context is not a synonimous of imprecise, vague: it only means that the method does not look for exact solutions in an explicit form, but it intends to investigate the behavior of classes of solutions and the topological properties of these classes in an abstract (phase) space of collections of integral curves-each one representing an exact solution. The method is important for, at least, two reasons:

(i) It permits an overview of a collection of exact solutions and its topological behavior in the space of the solutions.

(ii) It allows the set up of a program of search of new exact solutions giving some general properties of a large set of solutions.

Besides this, in a more deep context, this method can be used as a tool to analyse the stability of a given geometry in a collection of geometries. Indeed, it is possible to analyse the variation of the topology of a set of integral solutions of Einstein's equations by an examination of a family of solutions which depend on one parameter. For instance, Friedmann's universes, which depend on the equation of state $p= \lambda\rho$, is a one-parameter family characterized by the constant λ. One could ask about the modification of the topological properties of the Friedmann solutions under a change of the parameter λ. This can be applied to more general situations, and even to discuss the generic stability of Einstein's system of equations itself.

4.2 SPATIALLY HOMOGENEOUS AND ISOTROPIC UNIVERSES

Friedmann Universe

The fundamental length of the spatially homogeneous and isotropic world is given in a gaussian system of coordinates under the form

$$ds^2 = dt^2 - A^2(t)\left[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right] . \qquad (1)$$

The standard (Friedmann) Cosmology assumes that the main responsible for the curvature of the space-time can be represented by a perfect fluid, the stress-energy of which is given by

$$T^\mu{}_\nu = \rho V^\mu V_\nu - p\, h^\mu{}_\nu , \qquad (2)$$

where

$$h^\mu{}_\nu = \delta^\mu{}_\nu - V^\mu V_\nu \qquad (3)$$

is the projector in the 3-dimensional space orthogonal to the normalized vector V^μ (see section 3.2).
Einstein's equation of motion reduces to the set:

$$3\left(\frac{\dot{A}}{A}\right)^2 + \frac{3\varepsilon}{A^2} = \rho - \Lambda \qquad (4)$$

$$2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 - \frac{1}{A^2}\frac{\sigma''}{\sigma} = -p - \Lambda \qquad (5)$$

$$2\,\frac{\sigma''}{\sigma} - \frac{\sigma'^2}{\sigma^2} + \frac{1}{\sigma^2} = 0\ . \qquad (6)$$

We remark that due to the spatial homogeneity of Friedmann model the quantities ρ and p depend only on the global time t.

From equation (6) we conclude that the model admits three possibilities for the 3-dimensional geometry;accordingly, the three dimensional scalar of curvature[(3)] $R \equiv 6\varepsilon$ is positive, negative or null.

We have:

case i: Euclidean section, $\varepsilon=0$

$\sigma = \chi$

case ii:open Universe, $\varepsilon= -1$

$\sigma = \sinh \chi$

case iii: closed Universe, $\varepsilon= +1$

$\sigma= \sin \chi$

Thus, it remains to solve only equations (4) and (5) for the three unknown A(t), $\rho(t)$ and p(t). In order to solve these equations we must provide an equation of state $p= p(\rho)$. Let us limit our analysis, in this section, to the case in which this relation is simply given by a linear expression:

$$p= \nu\rho \qquad (7)$$

and limit ν to the range $[0,1]$. The lower limit is given by the requirement of positiveness of the pressure, the upper limit is a restriction imposed by the relativistic barrier on the sound velocity, $v_s \leq$ light velocity = 1.

From the conservation of the stress-energy tensor we write

$$T^{\mu\nu}{}_{;\nu} = 0.$$

Multiplying this expression by $V^\mu = \delta^\mu{}_0$ and using the form (2) for the energy tensor we obtain

$$\dot{\rho} + (\rho+p)\theta = 0 . \qquad (8)$$

Using (7) it takes the form

$$\dot{\rho} + (1+\nu)\ \rho\theta = 0 . \qquad (9)$$

The evolution of the expansion parameter θ is given by Raychaudhuri equation, (cf. section 2.5)

$$\dot{\theta} + \frac{\theta^2}{3} + 2\sigma^2 - 2\omega^2 - a^\alpha{}_{;\alpha} = R_{\mu\nu}V^\mu V^\nu . \qquad (10)$$

Using Einstein's equation we obtain

$$\dot{\theta} + \frac{\theta^2}{3} + 2\sigma^2 - 2\omega^2 - a^\alpha{}_{;\alpha} = -T_{\mu\nu}V^\mu V^\nu + \frac{T}{2} . \qquad (11)$$

This equation has been of great help in the study of the behavior of the gravitational field in regions of high value of the curvature(*).

(*) see R. Penrose in Batelle Rencontres (1967) and references therein.

Using $V^{\mu} = \delta^{\mu}_{0}$ for the velocity vector field and expression (2) for the energy-momentum tensor, Raychaudhuri's equation takes the form

$$\dot{\theta} = -\frac{\theta^2}{3} - \frac{(1+3\nu)}{2}\rho - \Lambda . \tag{12}$$

Let us re-write Einstein's equations (4) and (5) in terms of the expansion factor. We have

$$\frac{\theta^2}{3} + \frac{3\varepsilon}{A^2} = \rho - \Lambda \tag{13}$$

$$\frac{2}{3}\dot{\theta} + \frac{\theta^2}{3} + \frac{\varepsilon}{A^2} = -p - \Lambda . \tag{14}$$

Equation (14) is a consequence of taking the derivation of (13) with respect to time and using the equation of conservation of the energy as given by (8).

Thus, the whole system of Einstein's equations can be reduced to equations (8), (12) and (13).

Equation (13) may be regarded as a constraint, since it does not involve any time derivative. The dynamic system in the (ρ, θ) variables are given by eq. (8) and (12) which can be written in the generic form:

$$\dot{\rho} = F(\rho, \theta) \tag{15}$$

$$\dot{\theta} = L(\rho, \theta) , \tag{16}$$

with

$$F(\rho,\theta) \equiv -(1+\nu)\rho\theta \qquad (17)$$

$$L(\rho,\theta) \equiv -\frac{\theta^2}{3} - \frac{(1+3\nu)}{2}\rho - \Lambda \, . \qquad (18)$$

It is a very fortunate consequence of the double choice of the geometry (1) and the perfect fluid configuration of the sources that the equations which govern the evolution of the density of energy and the expansion factor reduce to a planar autonomous system. Let us undertake now the qualitative exam of this system.

Qualitative Analysis of Friedmann's Universes

In the examination of the planar system (15,16), we follow the methods and theorems quoted in section 2.12, which gives a pedestrian outline of the main rules of the qualitative analysis of a non-linear system of differential equations. The singular points of the system are the values of ρ and θ which annihilate simultaneously the right-hand side of equations (15,16), that is

$$F(\theta_0,\rho_0) = 0$$

$$L(\theta_0,\rho_0) = 0 .$$

They are given by:

$$P_1:\ (\rho_0,\theta_0) = \left(-\frac{2\Lambda}{1+3\nu}\ ,0\right)$$

$$P_2:\ (0,+(-3\Lambda)^{1/2})$$

$$P_3:\ (0,-(-3\Lambda)^{1/2})\ .$$

If Λ is positive, there is no singular point at all (for ρ positive); if Λ vanishes, the unique singular point is the origin 0. We examine the case $\Lambda<0$.

The fundamental matrix of the coefficients of functions F and L in the neighborhood of the singular points is $\hat{\Omega}$:

$$\hat{\Omega} = \begin{pmatrix} \frac{\partial F}{\partial \rho} & \frac{\partial F}{\partial \theta} \\ \frac{\partial L}{\partial \rho} & \frac{\partial L}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -(1+\nu)\theta & -(1+\nu)\rho \\ -\frac{1}{2}(1+3\nu) & -\frac{2}{3}\theta \end{pmatrix}$$

The trace I and the determinant Ω, at the different singular points, take the values:

$$I(P_1) = 0$$

$$I(P_2) = -\left(\frac{5}{3}+\nu\right)(-3\Lambda)^{1/2}$$

$$I(P_3) = \left(\frac{5}{3}+\nu\right)(-3\Lambda)^{1/2}$$

$$\Omega(P_1) = (1+\nu)\Lambda$$

$$\Omega(P_2) = -2(1+\nu)\Lambda$$

$$\Omega(P_3) = -2(1+\nu)\Lambda$$

Thus, we obtain the results of Table I (see section 2.12).

In order to simplify the drawing of the integral paths in the phase plane, it is useful to know the isoclines J_0 and J_∞, the curves of constant value of the derivative $\frac{d\theta}{d\rho} \equiv J$. A simple inspection of equations (15,16) gives for J_0 the expression

$$\theta^2 = -\frac{3}{2}(1+3\nu)\rho - 3\Lambda .$$

The curves J_∞ coincide with the coordinate axis ($\theta=0$ and $\rho=0$). These results allow us to depict figure 1.

Singular Point	Ω	$4\Omega - I^2$	Nature of the singular point
P_1	<0	<0	saddle
P_2	>0	<0	two-tangent node
P_3	>0	<0	two-tangent node

Table I - Nature of the singular points for the system (15,16) in the case of a negative cosmological constant.

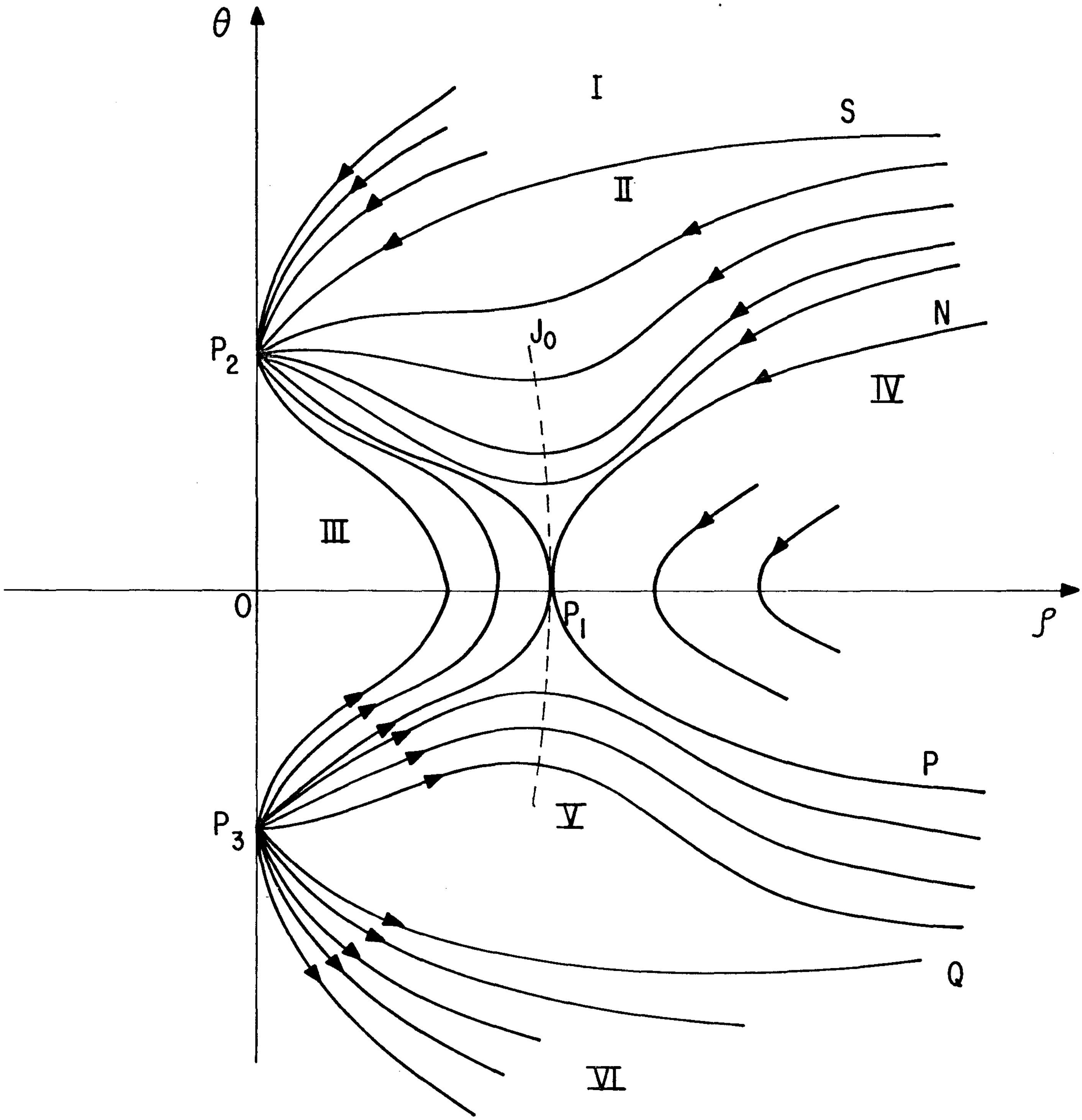

Figure 1 - Behavior of integral curves of Einstein's equations for homogeneous isotropic Universes filled with perfect fluid (the figure is drawn for the case of negative cosmological constant). The arrows on the curves point in the direction of increasing time.

A useful limiting situation of the above investigation is the case in which the cosmological constant vanishes. In this case, points P_1, P_2 and P_3 coalesce to the origin. Figure 2 illustrates this case.

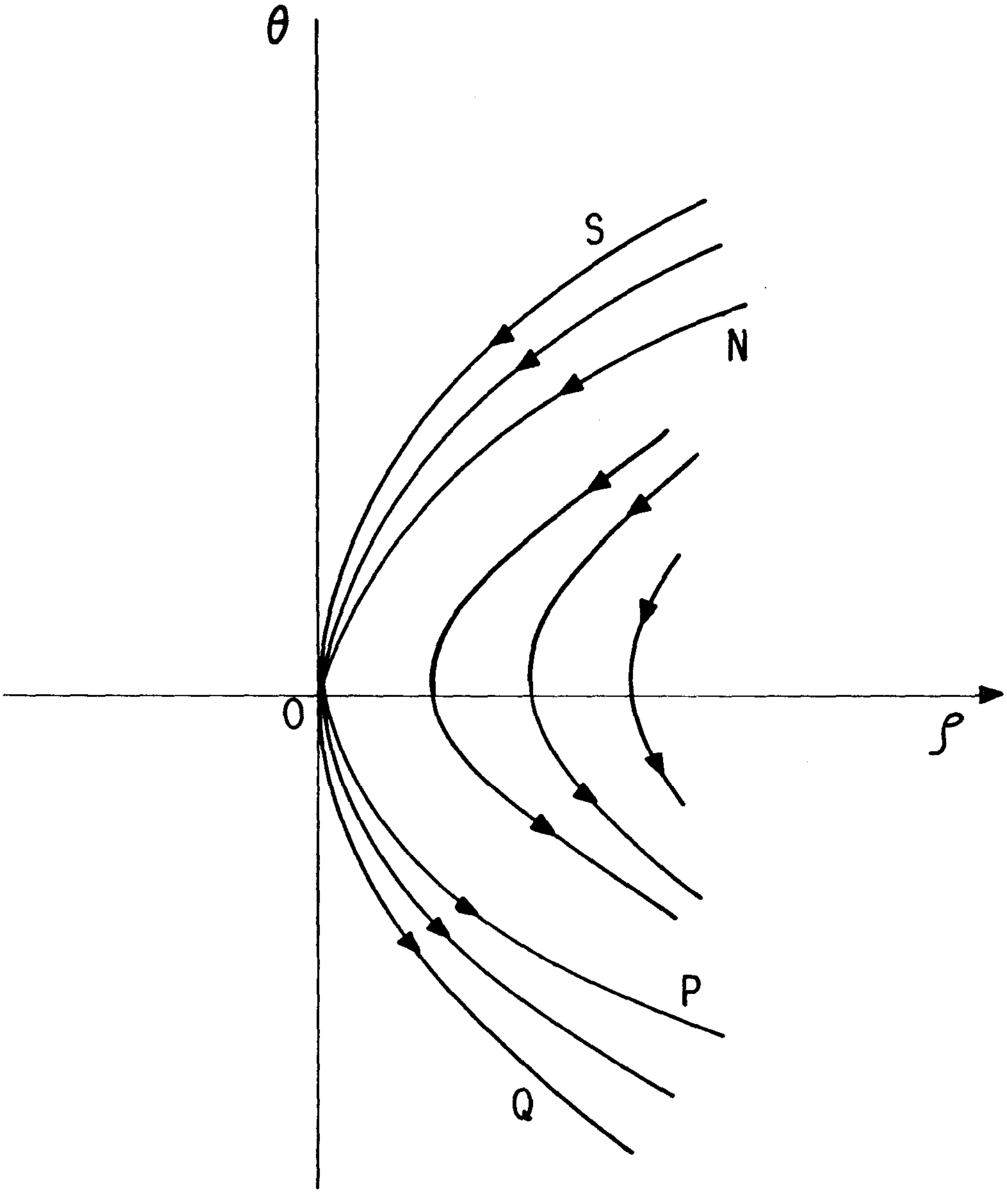

Figure 2 - Behavior of integral curves of Einstein's equations for homogeneous isotropic Universes filled with perfect fluid (case in which $\Lambda=0$)

Before commenting these figures, let us make a remark on equation (13). A simple inspection shows that points P_2 and P_3 are true singularities: the radius of the Universe diverge at these points. This does not happens at point P_1. For the case $\Lambda = 0$ the radius diverge at the origin showing real singularity.

Comments on the figures

We separate figure 1 in six distinct regions:

Region I : Limited by the curve SP_2 and axis $\rho = 0$ (for $\theta > \theta_{P_2}$)
Region II: Limited by the curve NP_2S
Region III: Limited by P_2 P_1 P_3 (including 0)
Region IV: Inside the parabola NP_1P
Region V: Inside PP_1P_3Q
Region VI: Outside QP_3 (for $\theta < \theta_{P_3}$)

Let us briefly sketch the general behavior of a typical solution in each region.

Region I:

The model starts with an infinite density and infinite expansion. Then both, the expansion and the density, decrease until the point P_2 is attained. At this point the density vanishes, the expansion arrives at its minimum value and the radius of the Universe is infinite.

Region II:

The model starts at $(\rho, \theta) = (+\infty, -\infty)$. The density and the expansion dimish with the increasing of the parameter t and tend asymptotically to the singularity P_2 with a similar behavior as in region I. Although the behavior of the energy a long the whole trajectory mimics that of region I, the expansion θ passes by a minimum which is a consequence of a non-catastrophic attraction of the singular point P_1. After the minimum (for $\rho \sim P_1$) the expansion increases until the point P_2. We realize that the cosmical repulsion, due to the presence of Λ is responsible for such very strange behavior expressed in these models.

Region III:

This is the only section of the phase plane in which both the density of energy and the expansion remain bounded.

This region represents models of the Universe which are dominated by the cosmological constant. Indeed, a simple inspection of figure 2 shows that in the limit $\Lambda \to 0$, this whole subregion shrinks to the singular point 0.

A typical model starts at P_3 with an infinite radius and $(\rho, \theta) = (0, \theta_{P_3} < 0)$. Then as the expansion becomes less negative, the density of energy increases, passes through a maximum at the moment in which the expansion factor changes sign (from contraction to expansion) and as time goes on the density of energy decreases towards the value zero at the maximum of the expansion θ_{P_2}. At this point the radius of the Universe diverges.

These solutions represent bouncing models, in which the radius of the Universe has a minimum value A_0, corresponding to the point in which θ changes sign, the Universe stops collapsing and an expanding era starts. For such situation to become possible, the cosmological constant must be high enough to prevent collapse, that is, the maximum value of the total energy and Λ must satisfy the inequality.

$$\frac{1+3\nu}{2}\,\rho_{max} < -\Lambda \;.$$

Region IV:

It represents a series of closed Universes with analogous behaviour as in the absence of cosmological constant.

The models start at $(\rho,\theta)_i = (+\infty,+\infty)$ and ends at $(\rho,\theta)_f = (+\infty,-\infty)$.

The density of energy has a different minimum for each integral curve.

The curve NP_1P represents the Euclidean case.

Region V:

Similar to region II.

Region VI:

It represents a time inversion situation of the models of region I.

Figure 2 presents similar features as regions I, IV and VI of figure 1. The other regions of figure 1 are absent here since the singular points P_1, P_2 and P_3 coalesce to the unique singular point 0.

4.3 - THE INFLUENCE OF VISCOSITY

Although a perfect fluid description of the matter content of the Universe in a homogeneous and isotropic geometry may seem a very idealized picture, it gives a model of the Cosmos which, in its general features, seems to be in good agreement with the main observations.

However, even without a deeper analysis of its observational properties, there are reasons to believe that such a model cannot represent the history of the Universe for all times. To get convinced of this it is enough to recall the existence in such a model of a global singularity, e.g., the vanishing of the spatial volume V and, as a consequence the formidable value for the density of energy (which in the very limit of zero volume goes to infinity), packing all existing matter in a very compact region (in the limit, into a point).

To escape from this naïve model one may either modify the properties of the geometry, or change the character of the fluid or both.

It seems reasonable to take the point of view that the transition from a perfect fluid behavior (latter era) to some less restringent regime (primordial era) is not so drastic, but that it is smooth in such a way that the whole energy content of the Universe should admit a hydrodynamical description. In this vein, we maintain the fluid description but introduce dissipative terms on it. It is clear that even such description could not be mantained forever. However, it seems a fashionable hypothesis to assume that a viscous era was present during a certain period of time, the duration of which should be matter for future concern.

Stokes Fluidity Principle

In this section we will treat the energy-momentum tensor as a fluid with density of energy ρ, isotropic pressure p and anisotropic pressure $\Pi^i{}_j$.

The trace of $\Pi^i{}_j$ vanishes, in order that p can represent the total isotropic pressure. The stokes fluidity condition is stated in terms of the response of the fluid to a given perturbation which is characterized solely by the dilatation tensor $\theta^i{}_j$ (sometimes called the deformation tensor).

If the response of the fluid is such that it can be described by a functional dependence of the anisotropic stress tensor $\Pi^i{}_j$ in terms of the dilation $\theta^i{}_j$, then we say that such fluid satisfies the Stokesian fluidity condition. We write

$$\Pi^i{}_j = F^i{}_j\left[\theta^\ell{}_m\right] . \tag{17}$$

Choosing coordinates in such way that the value $V^\mu = \delta^\mu{}_0$, we can write, using a well-known theorem of Cayley for matrix

$$\Pi^i{}_j = \alpha\ \delta^i{}_j + \beta\ \theta^i{}_j + \gamma\ \theta^i{}_\ell\ \theta^\ell{}_j\ , \tag{18}$$

in which α, β and γ are polynomials in the principal invariants of the matrix $\theta^i{}_j$. These invariants are:

$$I \equiv \theta^i{}_j = \mathrm{Tr}\ \theta^i{}_j \equiv \theta \tag{19a}$$

$$II \equiv \frac{1}{2}(\theta^2 - \theta^i{}_j\ \theta^j{}_i) \tag{19b}$$

$$III = \det \theta^i_j . \tag{19c}$$

Thus, we can write for the most general Π^i_j, the expansion

$$\Pi^i_j = [\alpha_0 + \alpha_1 I + (\alpha_2 I^2 + \alpha'_2 \Pi) + (\alpha_3 I^3 + \alpha'_3 III + \alpha''_3 I\ II) + \ldots]\delta^i_j +$$
$$+ [\beta_0 + \beta_1 I + (\beta_2 I^2 + \beta'_2 II) + \ldots]\theta^i_j + [\gamma_0 + \gamma_1 I + (\gamma_2 I^2 + \gamma'_2 II) + \ldots]\theta^i_k \theta^k_j , \tag{20}$$

in which the coefficients α_ℓ, β_ℓ, γ_ℓ may still depend on the density of energy ρ.

Although such Stokes fluid represents a generalization of the perfect fluid, it still does not have the highest degree of generality.

We will discuss later on how to generalize further the Stokes principle in order to describe other more elaborated situations.

Viscous Isotropic Models

Let us come back to Friedmann geometry. In this case, as the shear vanishes, the most general expression of the Stokesian fluid takes the form of a polynomial of the pressure in terms of the expansion. We limit our analysis here to the quadratic order and set

$$p' = p - \alpha\theta - \beta\theta^2 , \tag{21}$$

in which $\alpha = \alpha(\rho)$ and $\beta = \beta(\rho)$.

Einstein's equation (15,16) gives:

$$\dot{\theta} = -\frac{\rho}{2}(1+3\nu) - \frac{\theta^2}{3} - \Lambda + \frac{3}{2}\alpha\theta + \frac{3}{2}\beta\theta^2 \equiv f(\theta,\rho) \quad (22)$$

$$\dot{\rho} = -(1+\nu)\rho\theta + \alpha\theta^2 + \beta\theta^3 \equiv h(\theta,\rho) \; . \quad (23)$$

Let us examine this set of equations in the case of vanishing cosmological constant.

We distinguish two special situation:

case i: α and β are constants

case ii: $\beta = 0$
$\alpha = \alpha_0 \rho^{\mu}$,

in which μ is an arbitrary parameter (for the time being) and α_0 is a positive constant. We analyse both cases.

Case i: α and β are constants.

In this case the singular points are

$$P_1(\rho,\theta) = (0,0)$$
$$P_2(\rho,\theta) = \left(\frac{\theta_0^2}{3}, \theta_0\right),$$

with
$$\theta_0 = \frac{3\alpha}{1+\nu-3\beta} \; .$$

The fundamental matrix of the singularities $\hat{\Omega}$ takes the form

$$\hat{\Omega} = \begin{pmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \theta} \\ \frac{\partial h}{\partial \rho} & \frac{\partial h}{\partial \theta} \end{pmatrix}$$

$$\hat{\Omega} = \begin{pmatrix} -(1+\nu)\theta + \frac{d\alpha}{d\rho}\theta^2 + \frac{d\beta}{d\rho}\theta^3 & -(1+\nu)\rho + 2\alpha\theta + 3\beta\theta^2 \\ -\frac{(1+3\nu)}{2} + \frac{3}{2}\frac{d\alpha}{d\rho}\theta + \frac{3}{2}\frac{d\beta}{d\rho}\theta^2 & -\frac{2}{3}\theta + \frac{3}{2}\alpha + 3\beta\theta \end{pmatrix}. \tag{24}$$

Substituting the values of P_1 and P_2 we obtain for the trace I and the determinant Ω the following values at the singular points:

$$\Omega(P_1) = 0$$

$$\Omega(P_2) = \frac{3a^2}{1+\nu-3\beta}$$

$$I(P_1) = \frac{3}{2}\alpha$$

$$I(p_2) = -\frac{\alpha}{2}\frac{9\beta-3\nu-7}{3\beta-\nu-1}$$

The second law of Thermodynamics imposes that α and β must be positive, by identification of the direction of evolution (for increasing values of parameter <u>t</u>) with the direction of the expansion of the Universe.

Let us analyse the behavior of the integral curves in the neighborhood of point P_2. Using the techniques displayed in section 2.12 we obtain the results:

If $1+\nu<3\beta$ then $\Omega<0$ and thus, $4\Omega<I^2$.

We conclude that point P_2 is a saddle point. On the other hand if $1+\nu > 3\beta$, point P_2 is a two-tangent node.

Finally, if $\beta=\frac{3\nu-1}{9}$ (in which case, $4\Omega=I^2$), point P_2 is a one-tangent node.

The result of this analysis is contained in figure 3.

Comments on figure 3

We divide the phase plane in five regions:

Region I : Limited by the curves QOS.

Region II: The interior of SON

Region III: The interior of NOR

Region IV: Limited by the curve RO and axis $\rho = 0$

Region V : Limited by the curve QO and axis $\rho = 0$.

We will sketch the general behavior of the solutions in each of these regions.

Before doing this, let us make two remarks:

(i) In a similar manner as in the previous case of a perfect fluid, equation (13) implies that the exterior region of the parabola NOQ consists in a class of open models, and the interior of it on closed models.

(ii) Contrary to the previous case, the axis $\rho = 0$ is not an integral curve of the planar system. Consequently,it can be crossed by the solutions,as we see in figure 3. We have drawn the lines for negative values of the density in a discontinuous way in order to recall the reader that it represents a classically forbidden region.

Let us now turn to the analysis of the five distinct regions.

RegionI.

It contains solutions which represent closed Uni-

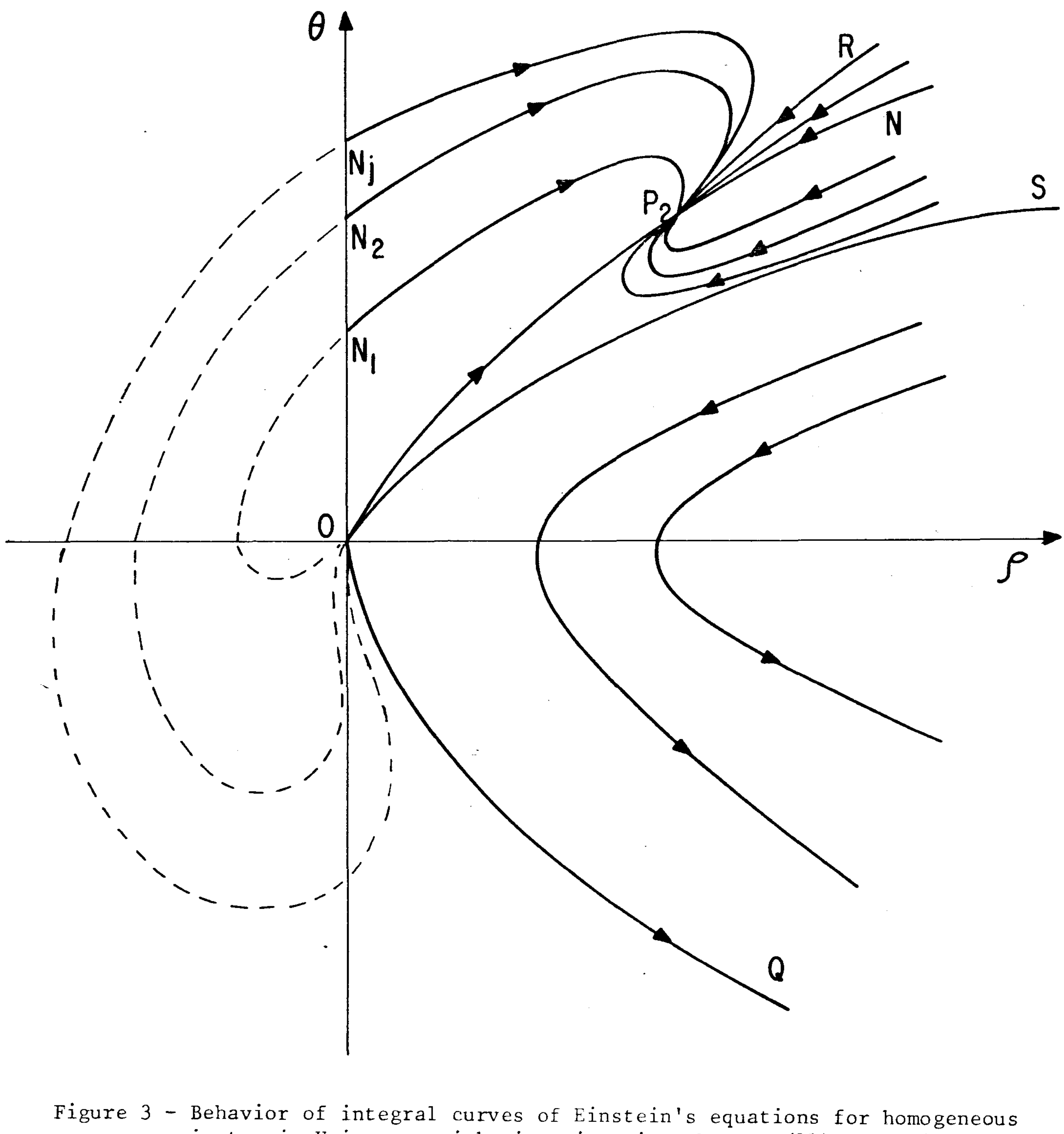

Figure 3 - Behavior of integral curves of Einstein's equations for homogeneous isotropic Universes with viscosity given by eq. (21). The figure is drawn for the case $\frac{1+\nu}{3} - \frac{4}{9} < \beta < \frac{1+\nu}{3}$

verses, starting at $(\rho,\theta) = (+\infty,+\infty)$ with a big explosion and ending at $(\rho,\theta) = (+\infty,-\infty)$ in collapse. Such behavior is the same as in the non-viscous case.

Region II.

In this region constant ε (which measures the 3-dimensional curvature) is still positive, once $\rho - \frac{\theta^2}{3} > 0$. These Universes start like in the previous case with a big explosion ($\rho = \theta = +\infty$), but their future fate are, however, very distinct. They end at P_2 with a finite value of both the energy and the expansion. No singularity at the end.

From the big explosion the density of energy decreases until a minimum (different for each curve) is attained. After this point the energy increases until its final value $\rho(P_2) = \frac{\theta_0^2}{3} = \frac{3\alpha^2}{(1+\nu-3\beta)^2}$.

At this point the radius of the Universe becomes infinite.

The responsible for the increase of the energy in the neighborhood of P_2 is the viscous term, and we can talk of this region as a viscous dominated era.

Region III.

A typical Universe of this region has an explosive beginning in which $(\rho,\theta) = (+\infty,+\infty)$. Then the energy and the expansion decrease until arriving at the singular point P_2, in which the radius of the Universe is infinite. The final configuration is the same as in the preceding case.

Region IV.

Here we find models which are created at a finite time with zero energy and finite expansion N_i. As the Universe expands matter is being created until the value $\rho(P_2)$ is attained at the singularity, after an infinite time.

These curves may be continued untill classically unphysical regions, for negative values of the energy. If we proceed to the left of the θ-axis then we see the singular origin of those solutions: they all start at $\theta = \rho = 0$ with an infinite radius, pass a contracting era (in which the energy is negative) and enters an expanding era in which the energy is still negative. Beyond this region they enter into the classically permitted situation, for positive values of ρ.

Region V.

If we do not continue the integral curves for negative values of the energy this region V admits very strange models.

Tipically, a Universe in this region has a finite duration without nothing exceptional occuring during its whole life time. A Universe starts with a finite contraction and $\rho = 0$. As time goes on matter is created, the energy passes a maximum and then diminishes, while the contraction increases. After a finite time, the model disappears (that is, enters a classically forbiden region). If we follow this solution to the left of the θ axis, then we come into the situation previously examined (Regeion IV).

Finally, let us comment the three possible cases of models with $\varepsilon=0$. They are all contained in the parabola $\frac{\theta^2}{3} = \rho$. There is a model which starts at the point 0 with null energy and null expansion, Then as the expansion increases the density increases until the second singular point is attained. The final configuration of this model is an Universe with density ρ_0 and expansion θ_0 at P_2.

The second solution on the parabola $\rho = \frac{\theta^2}{3}$ represents an Universe with a singular behavior: $(\rho,\theta) = (+\infty,+\infty)$.As the Universe expands the density of energy becomes lower until the minimum at P_2 is attained with the same end as in the previous case.

Finally, the third integral curve is the branch in the negative region (contraction) of θ, with a similar behavior as the one found in the Friedmann cosmos in which the matter behaves like a perfect fluid.

We present at figures 4, 5 and 6 the configurations for some special values of the coefficients of viscosity (*).

Let us now turn to the analysis of the case in which the coefficients of viscosity are not constant but depend on the energy.

The case $\beta = 0$, $\alpha = \alpha_0 \rho^{\mu}$ (in which α_0 and μ are constants) was examined by Belinski and Khalatnikov (1977) and the case $\beta = \beta_0 \rho^{m}$, $\alpha = 0$ was examined by Novello and Araujo (1980).

The features of these models depend deeply on the constants μ and m. We will present a few configurations for

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(*) These figures have been examined by Ruben Araujo (Master Thesis, Rio de Janeiro, 1979) (unpublished)

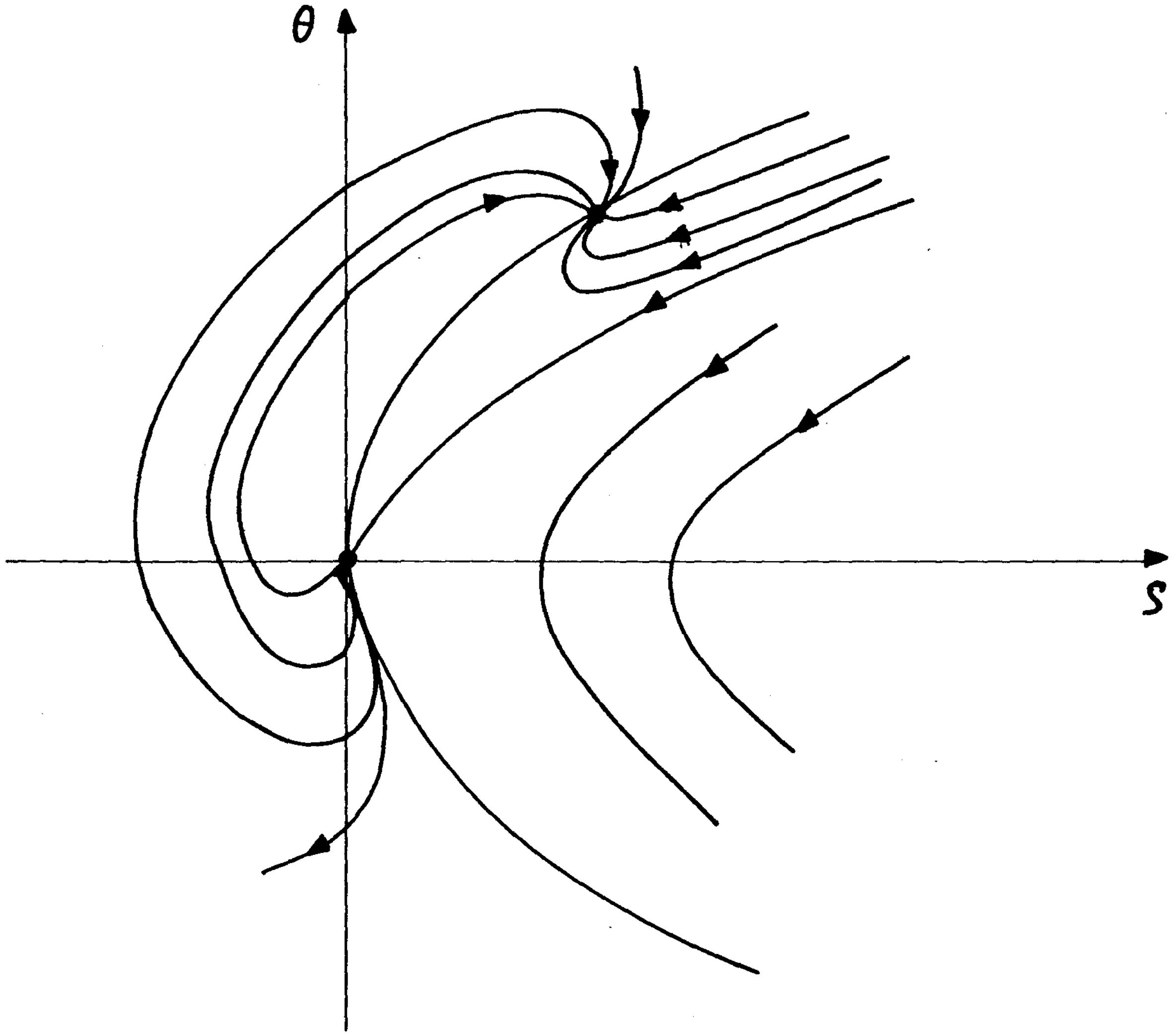

Figure 4 - Behavior of integral curves of Einstein's equations for homogeneous isotropic universes with viscosity given by eq.(21). The figure is drawn in case $\beta = \frac{1+\nu}{3} - \frac{4}{9}$.

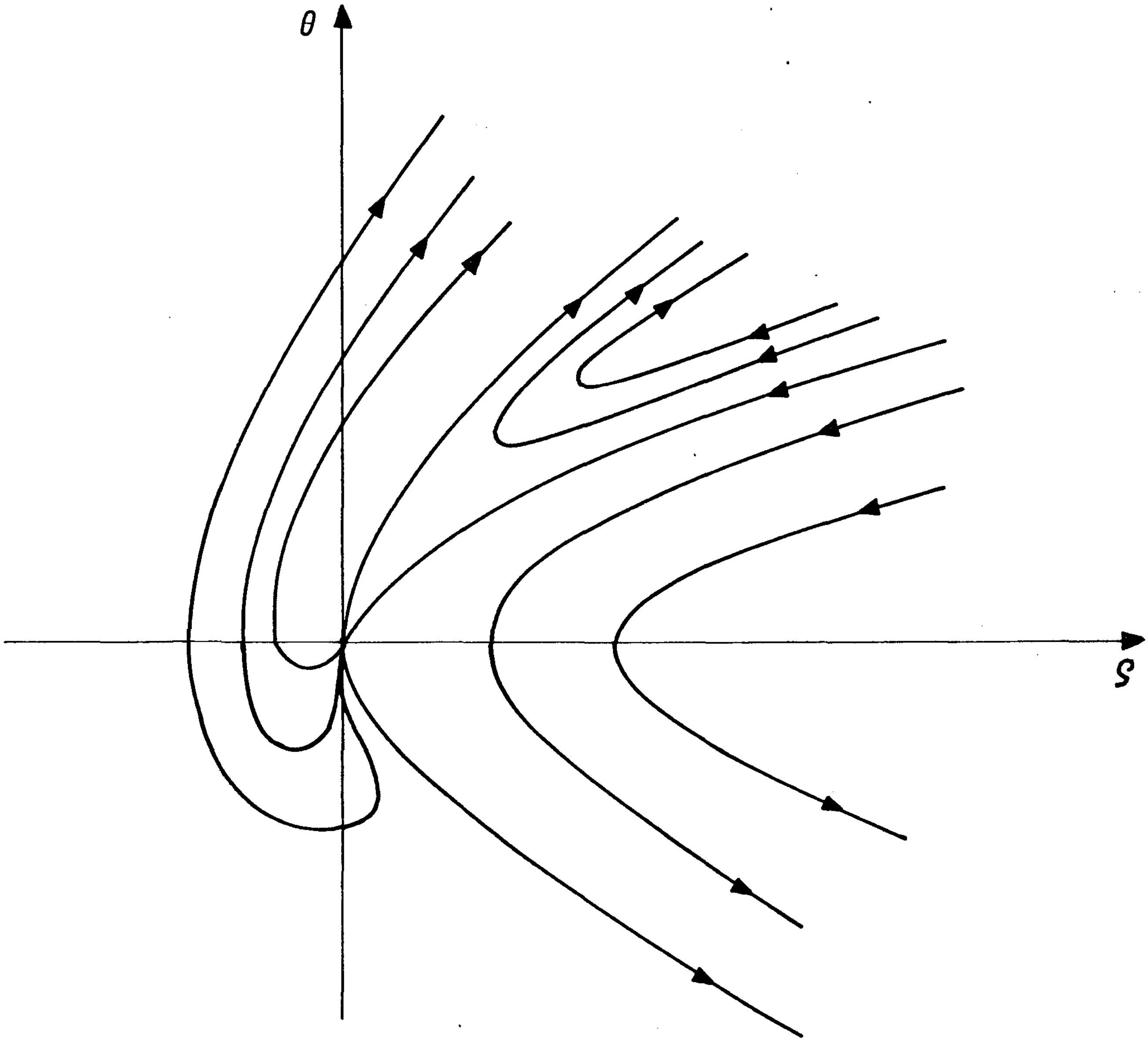

Figure 5 - Behavior of integral curves of Einstein's equations for homogeneous isotropic Universes with viscosity given by eq.(21). The figure is drawn for the case $\beta = \frac{1+\nu}{3}$.

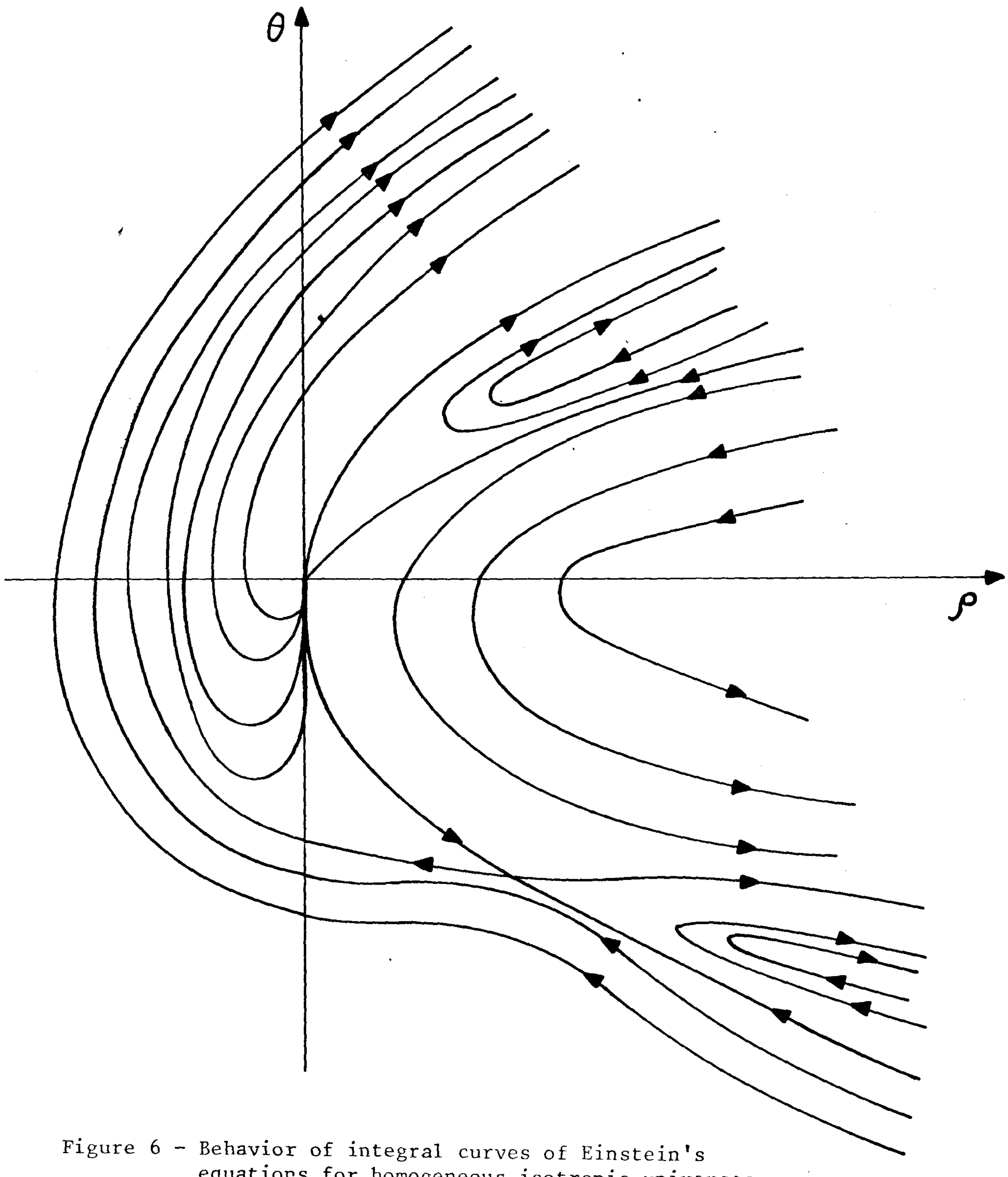

Figure 6 - Behavior of integral curves of Einstein's equations for homogeneous isotropic universes with viscosity given by eq.(21). The figure is drawn in the case $\beta > \frac{1+\nu}{3}$.

some cases of interest.

The linear regime

$$\alpha = \alpha_0 \rho^{\mu}$$

$$\beta = 0$$

In this case the singular points are:

$$P_1:\ (\rho,\theta)_1 = (0,0)$$

$$P_2:\ (\rho,\theta)_2 \approx \left(\frac{\theta_0^{\ 2}}{3}, \theta_0 \right) ,$$

with $$\theta_0 = \left[\frac{1+\nu}{\alpha_0} \quad 3^{\mu-1} \right]^{1-2\mu}$$

The behavior in the phase plane is represented by figures 7 and 8.

The quadratic regime

We set

$$\alpha = 0$$

$$\beta = \beta_0 \rho^{m} .$$

In this case we have, from equation (22) and (23):

$$h(\theta,\phi) = \rho - \frac{1}{3} \theta^2 + \frac{3}{2} \beta_0 \rho^{m} \theta^2 - \frac{3}{2} \gamma\rho$$

$$f(\theta,\phi) = \beta_0 \theta^3 \rho^{m} - \gamma\rho^{\theta} ,$$

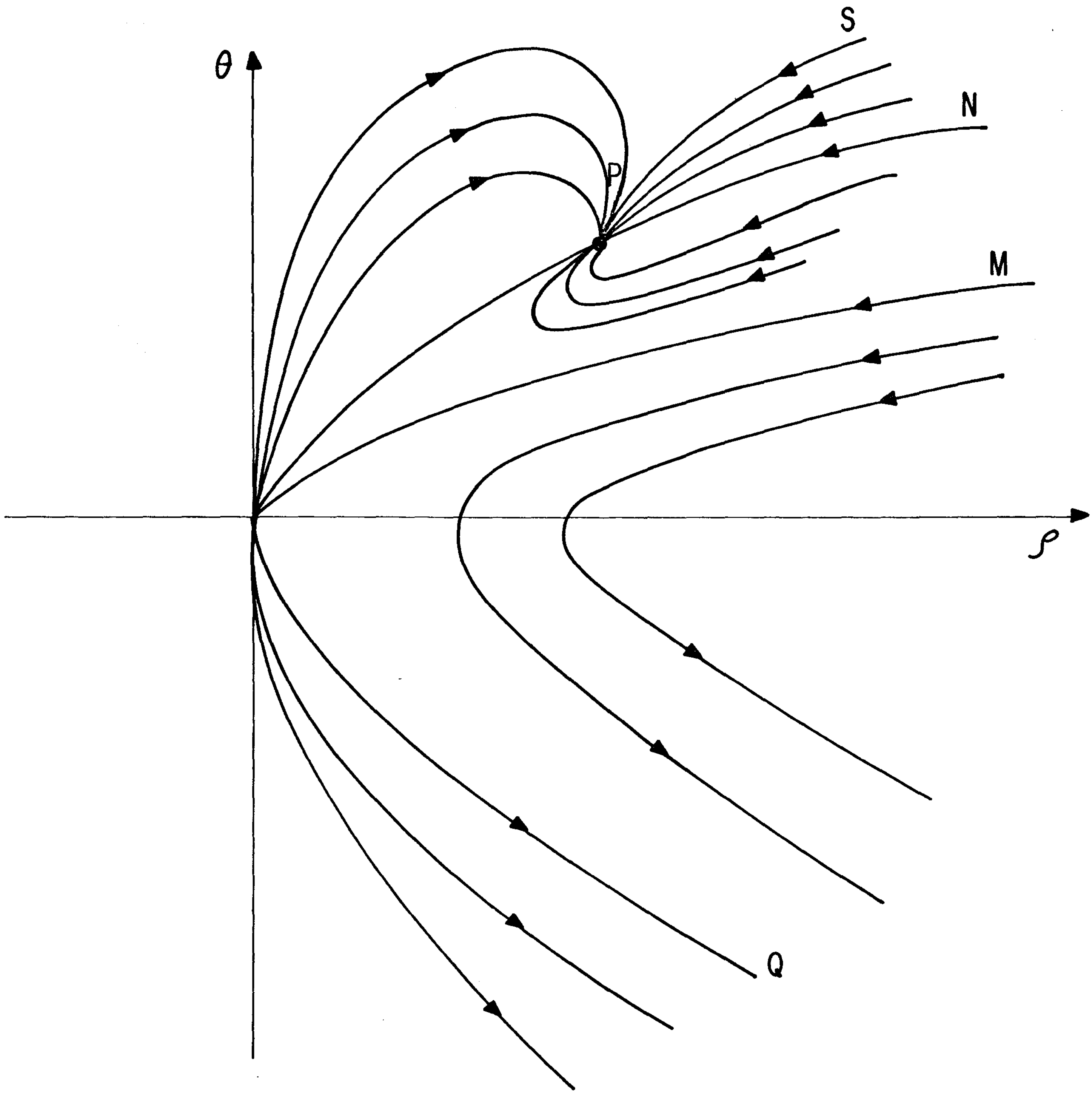

Figure 7 - Behavior of integral curves of Einstein's equations for homogeneous isotropic Universes with linear viscosity. The figure is drawn in the case $\beta=0$, $\alpha=\alpha_0\rho^\mu$ for $\frac{2}{3}\frac{1}{1+\nu}<\mu<\frac{1}{2}$ (see the text).

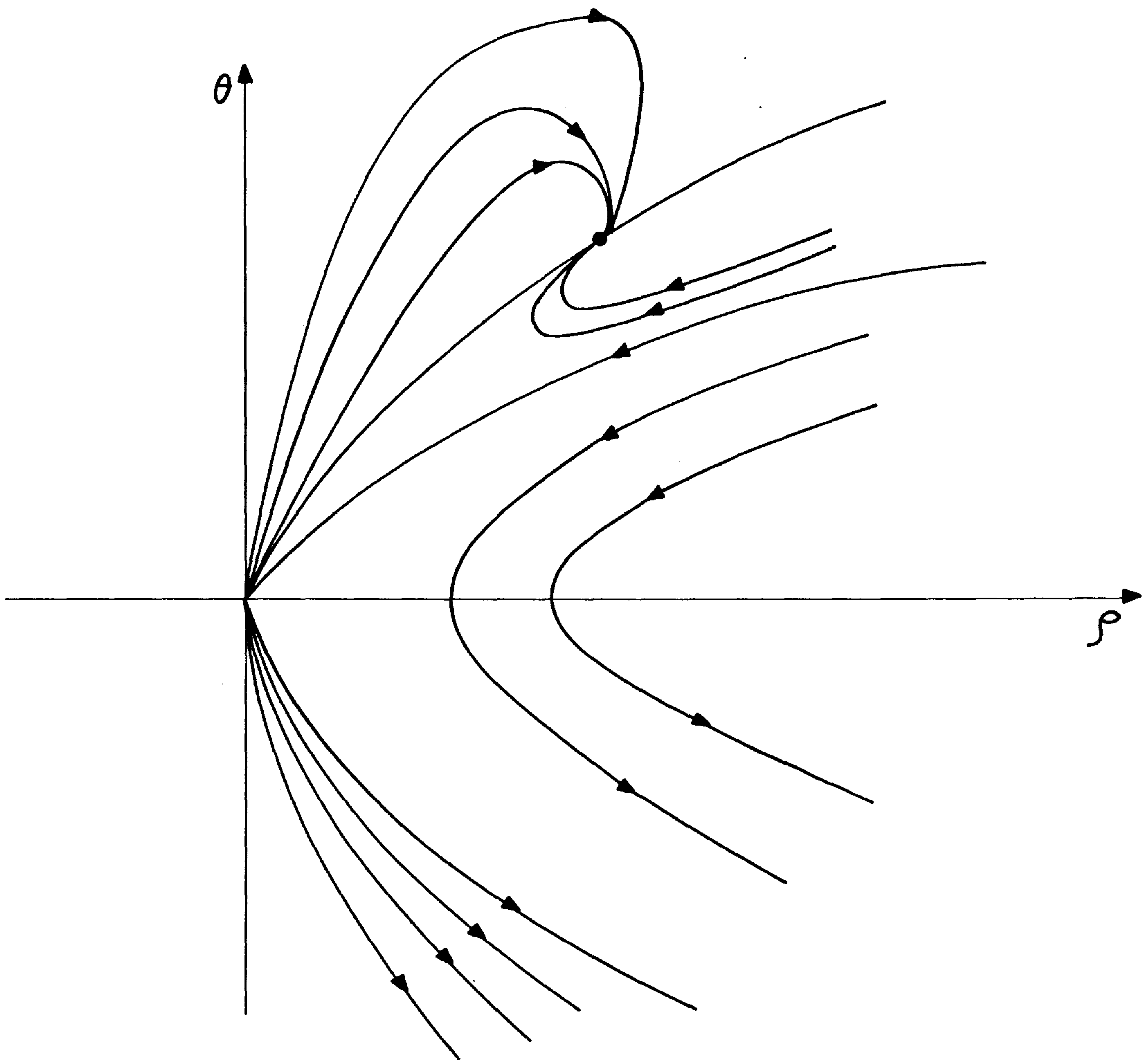

Figure 8 - Behavior of integral curves of Einstein's equatios for homogeneous and isotropic Universes with linear viscosity. The figure is drawn in tge case $\beta = 0$, $\alpha = \alpha_0 \rho^{\mu}$ for $\mu = \frac{1}{2} - \frac{2}{3(1+\nu)}$

in which $p = (\gamma-1)\rho$.

The singular points are:

$$P_0 = (\rho,\theta) = (0,0)$$

$$P_1 = (\rho_0,\theta_0)$$

$$P_2 = (\rho_0,-\theta_0),$$

with

$$\rho_0 = \frac{1}{3}\theta_0^2$$

$$3\beta_0\rho_0^m = \gamma .$$

Developping $h(\theta,\rho)$ and $f(\theta,\rho)$ in a power series in the neighborhood of the singular points we obtain

$$\hat{\Omega} = \begin{pmatrix} -\frac{2}{3}\theta_0 + \gamma\theta_0 & 1 + \frac{3}{2}\gamma(m-1) \\ \frac{2}{3}\gamma\theta_0^2 & \gamma\theta_0(m-1) \end{pmatrix}.$$

The trace I and the determinant Ω are given by:

$$\Omega = -\frac{2}{3}\gamma m \theta_0^2$$

$$I = \theta_0(\gamma m - \frac{2}{3}).$$

Using the theorems of section 2.12 we easily arrive to the following results:

If $m>0$ P_1, P_2 are saddle points.

If $m<0$ and $4\Omega - I^2<0$ points P_1, P_2 are two- tangent nodes;

If $m = \frac{2}{3} \gamma$, points P_1, P_2 are one-tangent nodes.

In the expansion era the node is stable; in contracting era it is unstable.

See figures 9, 10 and 11.

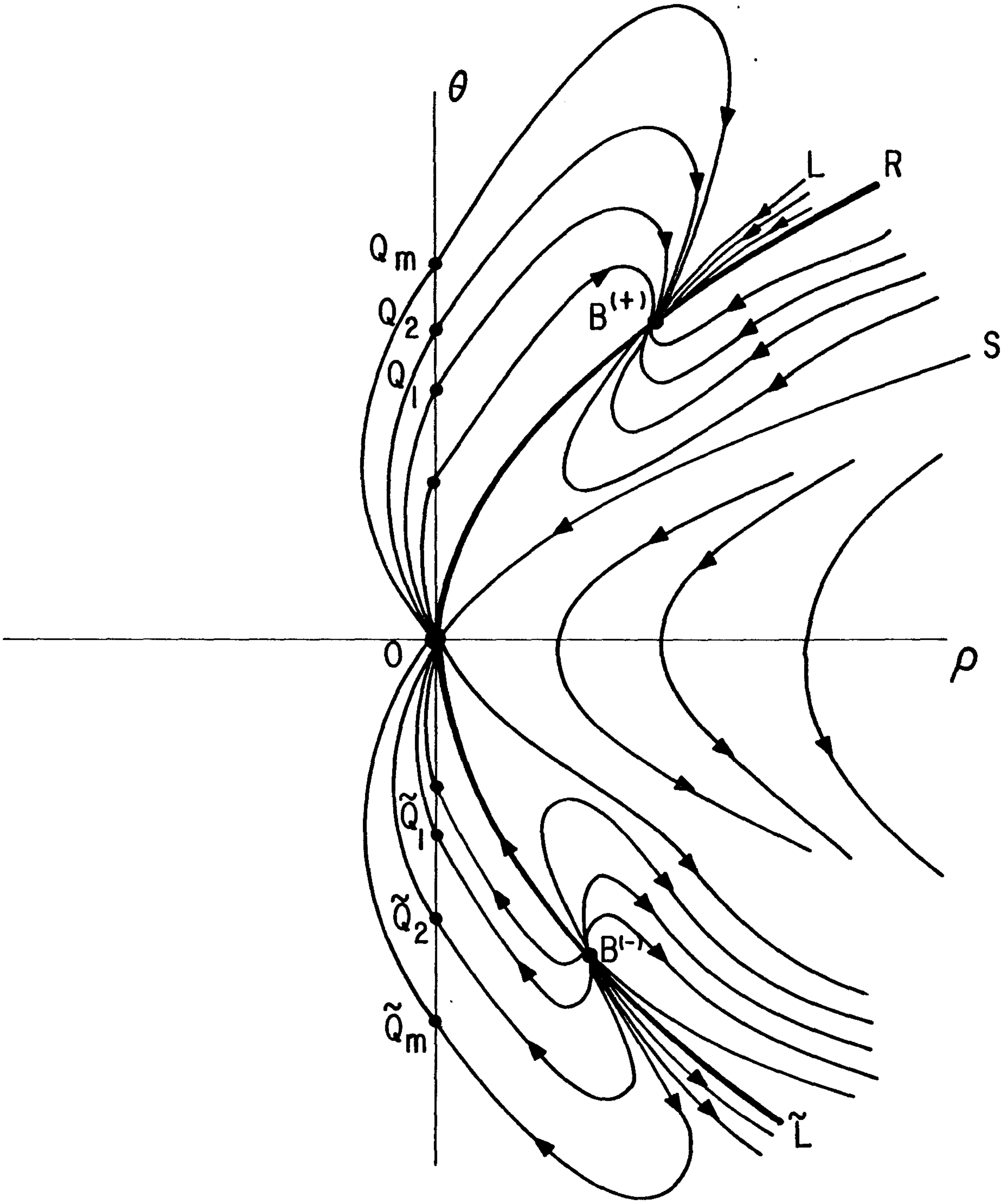

Figure 9 - The fluid has viscosity represented by $\tilde{p} = p - M\,\rho^{\mu}\theta^2$, M and μ are constants. The figure is drawn for the case in which $\frac{-2}{3\gamma} < \mu < 0$. Points B are two-tangent nodes.

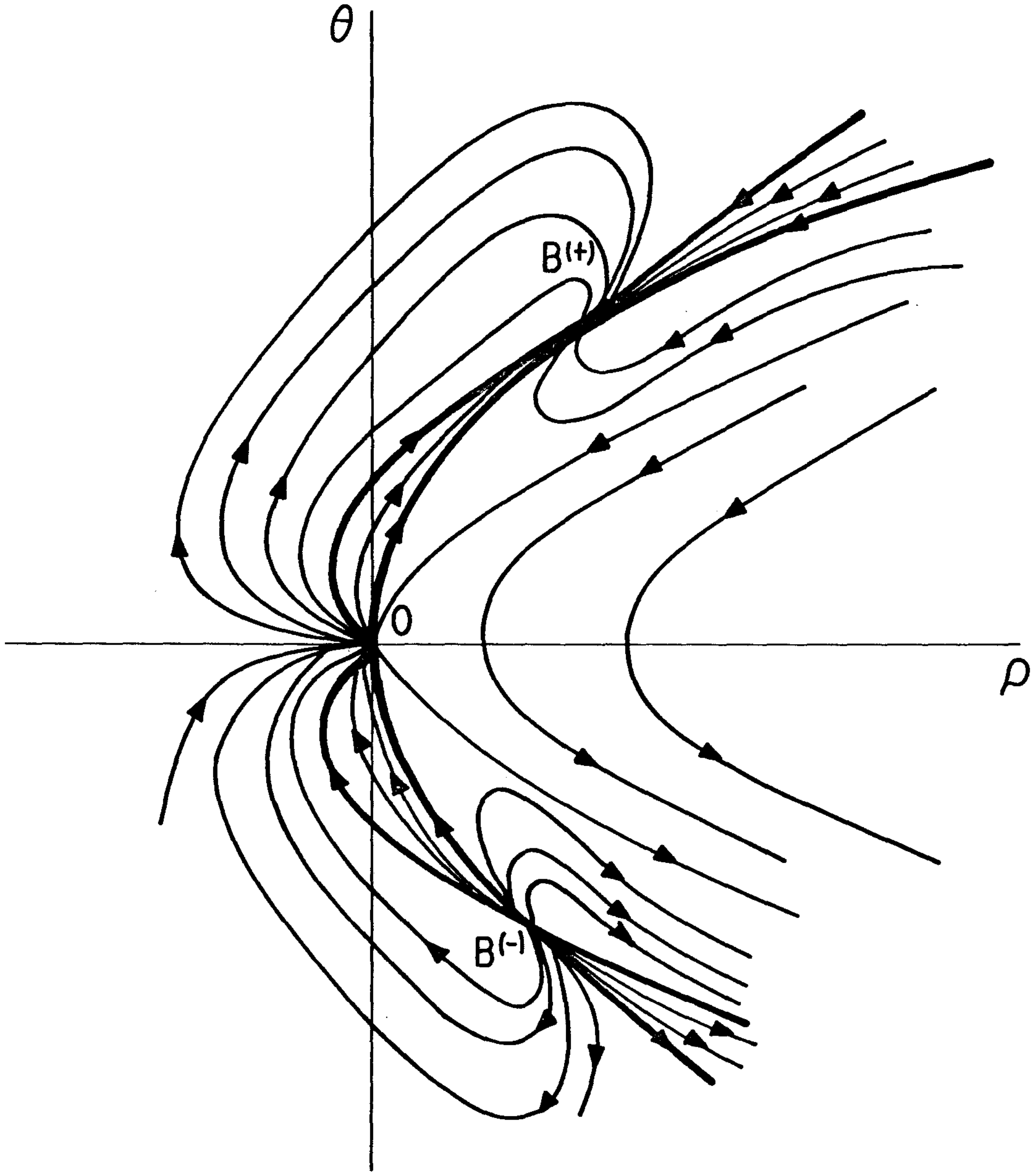

Figure 10 - Case $\tilde{p} = p - M\rho^{\mu}\theta^2$; M and μ are constant. The figure is drawn for $\mu < -\frac{2}{3}\gamma$. Points B are two-tangent nodes.

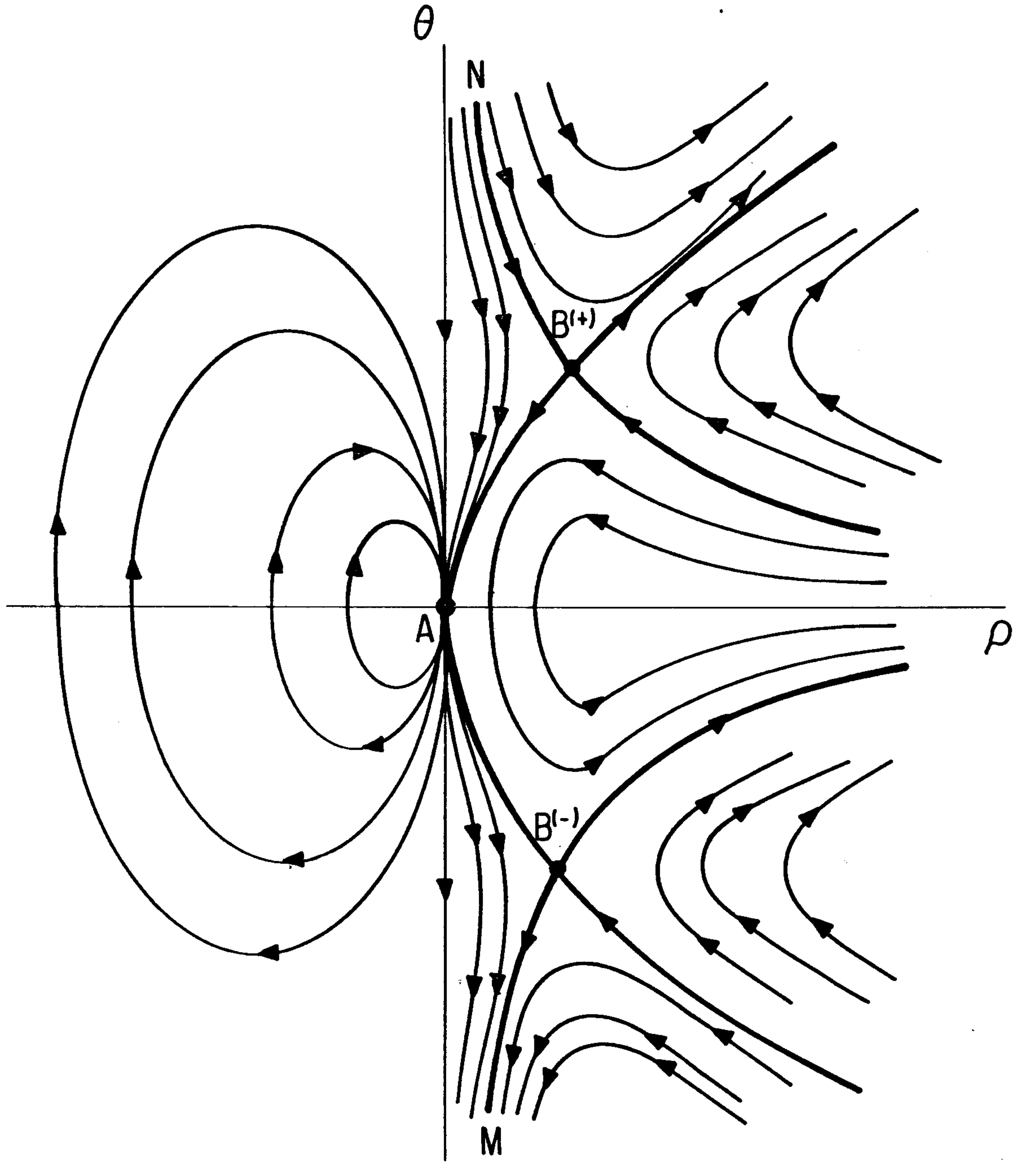

Figure 11 - Case $\tilde{p} = p - M\rho^{\mu}\theta^2$; M and μ are constants with $\mu > 1$. Points B are saddle points.

4.4 HOMOGENEOUS ROTATING UNIVERSES

Some years ago K. Gödel presented a cosmological model in which the cosmic fluid had an intrinsic rotation. Although this model could not represent our real Universe (for instance, it is a static configuration of rotating galaxies, which does not fit well with the observed portion of space-time we can see), it has some intrinsic curious properties which have stimulated deeper investigation of certain types of Riemannian geometries. Our interest here in presenting this model is linked with the possibility of examining some more general rotating models which allow the qualitative analysis.

We start by writing the fundamental length of our geometry as being given by

$$ds^2 = dt^2 + 2A(x,t)dy\, dt - B(x,t)dy^2 - H^2(t)dz^2 - F^2(t)dx^2. \tag{25}$$

The fluid velocity v^α takes the values $V^0 = 1$, $V^i = 0$, $(i=1,2,3)$.

The rotation and the acceleration vectors are given respectivelly by

$$\omega^\alpha = (0,0,0,\Omega), \tag{26a}$$

with

$$\Omega \equiv \frac{A'}{2FH} \quad (A^2+B)^{-1/2}$$

$$a^{\alpha} = \left(\frac{A\dot{A}}{A^2+B}, 0, - \frac{\dot{A}}{A^2+B}, 0 \right), \qquad (26b)$$

in which $A' \equiv \frac{\partial A}{\partial X}$ and $\dot{A} \equiv \frac{\partial A}{\partial t}$.

The source of this geometry is taken a non-perfect fluid, given by

$$T_{\mu\nu} = (\rho+p)\, V_\mu V_\nu - p g_{\mu\nu} + q_\mu V_\nu + q_\nu V_\mu , \qquad (27)$$

in which q_μ is the four-vector heat flux.

The vanishing of T_{12} imposes

$$\frac{d}{dt}\left[\frac{H}{F} A'(A^2+B)^{1/2} \right] - \frac{(B'+2AA')}{(A^2+B)^{1/2}} \frac{1}{F} \frac{d}{dt}(AH) = 0 \qquad (28)$$

If we set

$$B = (m-1)A^2 , \qquad (29)$$

where m is a constant, we obtain

$$A(x,t) = A_0\, e^{qx} A_2(t) , \qquad (30)$$

in which A_0 and q are arbitrary constants.

The absence of anisotropic pressure imposes $R_{11} = R_{22} = R_{33}$, that is,

$$2(n-1)\left[\frac{\ddot{F}}{F} + \frac{\dot{F}}{F}\frac{\dot{A}}{A} - \frac{\dot{F}^2}{F^2} \right] - \frac{(2m-1)}{2} \frac{q^2}{F^2} = 0 \qquad (31)$$

$$(m-1)\left[\frac{\ddot{A}}{A} + \frac{\dot{F}^2}{F^2} - \frac{\dot{A}}{A}\frac{\dot{F}}{F} - \frac{\ddot{F}}{F} \right] + (2m-1)\frac{q^2}{F^2} - 2\frac{\dot{F}^2}{F^2} = 0 . \qquad (32)$$

Novello and Rebouças (1978) presented a solution of these equations which contains Gödel's model as a special case, in which the value of the constant m is 1/2. Let us follow them and set this value for m. We then reduce (31) and (32) to an autonomous homogeneous planar system:

$$\frac{\ddot{F}}{F} + \frac{\dot{F}}{F}\frac{\dot{A}}{A} - \frac{\dot{F}^2}{F^2} = 0 \qquad (33)$$

$$\frac{\ddot{A}}{A} - \frac{\ddot{F}}{F} + 5\frac{\dot{F}^2}{F^2} - \frac{\dot{A}}{A}\frac{\dot{F}}{F} = 0\ . \qquad (34)$$

Indeed, let us define the new variables x(t) and y(t) as

$$x(t) \equiv \frac{\dot{F}}{F}$$

$$y(t) \equiv \frac{\dot{A}}{A}\ .$$

Then equations (33) and (34) reduce to the set:

$$\dot{x} = -xy \qquad (35)$$

$$\dot{y} = -y^2-4x^2. \qquad (36)$$

Figure (12) represents the integral curves of this system. Let us make somme comments on it. The unique singular point is the origin (0,0). This is precisely Gödel's metric.We then conclude from figure (12) that there is only one solution which tends to Gödel's geometry for asymptotic values of time

($t\to\pm\infty$). This solution, which corresponds to $\dot{F}=0$, was presented in Novello-Rebouças' paper (Novello and Rebouças, 1978)

For $\frac{\dot{A}}{A}>0$ the solution tends to Gödel in the infini- future ($t\to+\infty$); and if $\frac{\dot{A}}{A}<0$ it tends to Gödel in the infinite past ($t\to-\infty$). A small perturbation of Gödel's model (by a small quantity of heat, for instance) makes the model stay far away from Gödel's geometry forever.

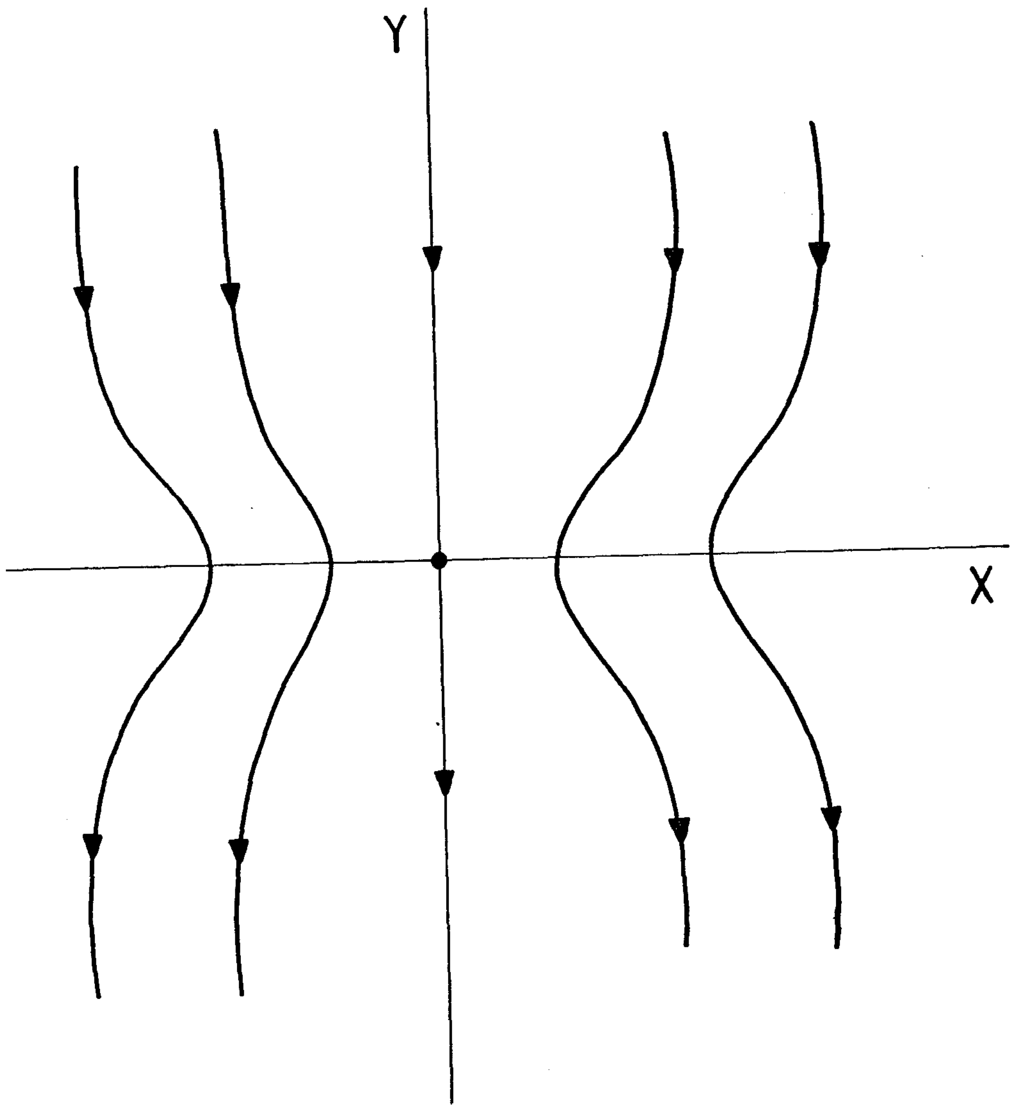

Fig. 12 - Integral curves of the system (35,36) representing rotating cosmological models (non-perfect fluid). The origin represents Gödel's solution.

5.1 - INTRODUCTION

In the preceding section we have examined the properties of the standard cosmological model, that is, Friedmann Universe. We have reviewed its stability property against any type of perturbation. Although the great majority of the cosmologists believe that we live in a Friedmann-like Universe, theoretical reasons allow alternative expanding models to be thought as candidates of the description of the structure of space-time at large, at least for ancient eras. To be precise, even for later eras structures more complex than Friedmann's can be considered, like, for instance, the possible existence of White Holes and/or the sudden irruption of acausal structures in the cosmos at large, with their unpredictable behavior.

The purpose of the present section is to give alternatives models for the global properties induced by the presence of cosmic fields coupled non-minimally to gravity. It has been argued in the literature that non singular cosmologies are consequence of quantum effects (Melnikov and Orlov (1979), Starobinsky (1980).

Here we show that this is possible within classical theory just by taking non minimal coupling with gravity, as we show in section 5.7 for non-linear photons. We examine here also the mechanism of spontaneous breaking of symmetry in cosmology. In section 5.4 we present the consequences of this non-invariance in the creation of a repulsive gravity induced by radiation.

5.2 - THE SCALAR FIELD: ALTERNATIVE EQUATIONS OF MOTION IN A CURVED SPACE TIME

In flat Minkowskii space-time the equation of motion of a scalar field $\phi(x^\mu)$ is obtained, using a variational principle, by the Lagrangian

$$L_{(1)} = \partial_\mu \phi^* \partial_\nu \phi \ \eta^{\mu\nu} - m^2 \ \phi^* \phi \ , \tag{1}$$

in which ϕ^* is the complex conjugate of ϕ.

The corresponding equation of motion is

$$\overset{(o)}{\Box} \phi + m^2 \phi = 0 \ , \tag{2}$$

in which $\overset{(o)}{\Box}\phi \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \phi$ is the Laplacian operator.

The correspondent of this equation in arbitrary curved space-time, endowed with a metric $g_{\mu\nu}(x)$, introduces an arbitrariness in the theory once there is not a unique candidate to such generalization, Two are the main lines of procedure which have been adopted to deal with such situation. Let us review both of them.

(i) Minimal Coupling Principle

In the Minimal Coupling Principle the equation of motion of ϕ is assumed to be free from any functional dependence on the curvature tensor.

This led unequivocally to the Lagrangian

$$L_{(2)} = \sqrt{-g} \ \{\partial_\mu \phi^* \ \partial_\nu \phi \ g^{\mu\nu} - m^2 \ \phi^* \phi\} \tag{3}$$

and the corresponding equation to ϕ).

$$\Box\phi + m^2\phi = 0 , \tag{4}$$

in which

$$\Box\phi \equiv \frac{1}{\sqrt{-g}} (\sqrt{-g}\ \phi_{,\alpha}\ g^{\alpha\beta})_{,\beta} \tag{5}$$

is the generalization of the Laplacian operator $\overset{(o)}{\Box}$.

(ii) Conformal Invariance

In this procedure one states that the equation of ϕ in curved space-time must be such that, in case the mass of the ϕ field vanishes, the resulting equation becomes invariant under a conformal mapping. This map is due to a space-time point dependent scale transformation characterized by

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x)\ g_{\mu\nu}(x). \tag{6}$$

It induces in the curvature tensor the corresponding modification

$$\tilde{R}^{\alpha\beta}{}_{\mu\nu} = \Omega^{-2} R^{\alpha\beta}{}_{\mu\nu} - \frac{1}{4}\ \delta^{[\alpha}_{[\mu}\ Q^{\beta]}_{\nu]} , \tag{7}$$

in which

$$Q^{\alpha}{}_{\beta} \equiv 4\Omega^{-1}(\Omega^{-1})_{;\beta;\lambda}\ g^{\alpha\lambda} - 2(\Omega^{-1})_{,\mu}(\Omega^{-1})_{,\nu}\ g^{\mu\nu}\ \delta^{\alpha}{}_{\beta} .$$

Contracting indices ,

$$\tilde{R}^{\alpha}{}_{\mu} = \Omega^{-2} R^{\alpha}{}_{\mu} - \frac{1}{2}(Q^{\alpha}{}_{\mu} + \frac{1}{2}\ Q\gamma^{\alpha}{}_{\mu}). \tag{8}$$

Finally, for the scalar of curvature R:

$$R \to \tilde{R} = \Omega^{-2}\{R + 6\ \frac{\Box\Omega}{\Omega}\} . \tag{9}$$

The Weyl (conformal) tensor changes only by a multiplicative factor

$$\tilde{W}^{\alpha\beta}{}_{\mu\nu} = \Omega^{-2} \, W^{\alpha\beta}{}_{\mu\nu} \, . \tag{10}$$

Now, if we adopt the principle that the theory must be invariant under a conformal mapping and note that the associated change of the ϕ field is

$$\phi(x) \rightarrow \tilde{\phi}(x) = \Omega^{-1}(x) \quad \phi(x) \, , \tag{11}$$

then we are led to the Lagrangian

$$L_{(3)} = \sqrt{-g} \, \{\partial_\mu \phi^* \partial_\nu \phi g^{\mu\nu} - m^2 \phi^* \phi - \frac{1}{6} R \phi^* \phi\} \tag{12}$$

and the corresponding equation of motion:

$$\Box \phi + (\frac{1}{6} R + m^2) \phi = 0 \, . \tag{13}$$

It seems worth to remark here that this equation is not univocally defined. Indeed, let us add to $L_{(3)}$ the term

$$\sqrt{-g} \quad I_{(1)}^{1/2} \, \phi^2 \, , \tag{14}$$

in which the invariant $I_{(1)}$ is defined as the second order Weyl contraction (see section 2.3)

$$I_{(1)} \equiv W_{\alpha\beta\mu\nu} \, W^{\alpha\beta\mu\nu} \, .$$

It is easy to show that expression (14)is invariant under the conformal map (6)-(9)-(11). There are other terms which can be added to the Lagrangian of the ϕ-field and which do not break the conformal invariance of the theory. These terms are generated by higher powers of the Weyl tensor multiplied by higher powers of the ϕ-field. Terms of the form $\phi^2 I^{1/2}_{(1)}$, ϕ^4, $\phi^6 I^{-1/2}_{(1)}$ and so on, conserve the conformal invariance of the theory. How to decide among them? This is still an open question. Let us turn now to a more intimate study of the non-minimal (conformal invariant) equation (13) of motion of the scalar field.

5.3 - CONFORMAL COUPLING AND THE FUNDAMENTAL SOLUTION

Let us assume that the Lagrangian of the metric $g_{\mu\nu}(x)$ and of the scalar field $\phi(x)$ is given by

$$L = \sqrt{-g}\left[R + \partial_\mu\phi^*\partial_\nu\phi\ g^{\mu\nu} - m^2\phi^*\phi - \frac{1}{6}R\ \phi^*\phi + 2\Lambda\right] \tag{15}$$

in which we have introduced an additional constant term (Λ is the so-called cosmological constant).

The equations of motion which result from this Lagrangian are

$$\Box\phi + (m^2 + \frac{1}{6}R) = 0 \tag{16}$$

$$(1 - \frac{1}{6}\ \phi^2)\quad (R_{\mu\nu} - \frac{1}{2}R\ g_{\mu\nu}) = -\ \frac{1}{2}(\phi^*_{,\mu}\ \phi_{,\nu} + \phi^*_{,\nu}\ \phi_{,\mu}) +$$
$$+\ \frac{1}{2}\ g_{\mu\nu}\{\phi^*_{,\lambda}\ \phi_{,\varepsilon}\ g^{\lambda\varepsilon} - m^2\phi^*\phi\} -\ \frac{1}{6}\ \Box\ (\phi^*\phi)g_{\mu\nu} +$$
$$+\ \frac{1}{6}(\phi^*\phi)_{,\mu;\nu} + \Lambda g_{\mu\nu}\ . \tag{17}$$

Taking the trace of equation (17) gives

$$R = m^2\phi^*\phi - 4\Lambda\ . \tag{18}$$

Substituting this value into equations (16) we find:

$$\Box\phi + (m^2 - \frac{2}{3}\ \Lambda)\phi + \frac{m^2}{6}\ \phi^2\phi = 0\ . \tag{19}$$

The form (19) of the equation of ϕ shows explicitly that the final consequence of the introduction of the conformal term proportional to $R\phi^2$ in Lagrangian (15) is equivalent to a quartic self-coupling of ϕ. This fact suggests a further generalization of the equation of ϕ by introducing ab initio the self-interacting term

$$\sigma(\phi^*\phi)^2$$

in the Lagrangian. Thus we set for L the form:

$$L = \sqrt{-g}\left[R + \partial_\mu\phi^*\partial_\nu\phi\ g^{\mu\nu} - m^2\phi^*\phi - \frac{1}{6}R\ \phi^*\phi + \sigma(\phi^*\phi)^2 + 2\Lambda\right]. \tag{20}$$

The corresponding equation for ϕ:

$$\Box\phi + (m^2 - \frac{2}{3}\Lambda)\ \phi + (\frac{m^2}{6} - 2\sigma)\ \phi^2\,\phi = 0\ . \tag{21}$$

We see that the role of the cosmological constant Λ (for $\Lambda>0$) is to reduce the value of the mass of ϕ to an effective mass m_{eff} given by:

$$(m_{eff})^2 \equiv m^2 - \frac{2}{3}\ \Lambda\ . \tag{22}$$

We recognize here the possibility of induction, by Λ, of the mechanism of spontaneous symmetry breaking. Let us briefly review the main points of this mechanism in flat space-time. In this case the equation for ϕ is

$$\Box\phi + m^2\phi - 2\sigma\phi^2\phi = 0\ . \tag{23}$$

If we look for the simplest solution $\phi = \phi_o$ = constant we obtain

$$\phi_o^2 = \frac{m^2}{2\sigma} . \tag{24}$$

This is possible only if the mass of the ϕ-field is imaginary (for $\sigma < 0$). This value for ϕ_o corresponds to the minimum of the energy which in this case is given by the minimum of the potential

$$V(\phi) = m^2\phi^2 - \sigma\phi^4. \tag{25}$$

We remark that in the fundamental state ϕ_o the system does not have the gauge symmetry (invariance under a change in the phase given by

$$\phi \to e^{i\alpha} \phi)$$

which is valid for the Lagrangian. Thus, the system breaks spontaneously the fundamental symmetry. Let us now pass to the curved geometry and ask for the modifications on this phenomenum introduced by the coupling with gravity. Equation (21) shows that the new constant solution, corresponding to (24) is:

$$\phi_o^2 = 2\left(\frac{2\Lambda - 3m^2}{m^2 - 12\sigma}\right). \tag{26}$$

We take as definition of the energy-momentum tensor of the ϕ-field the expression (Chernikov and Tagirov, 1968)

$$T_{\mu\nu}[\phi] = t_{\mu\nu} - \frac{\phi^2}{6}\left(R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}\right), \tag{27}$$

in which $t_{\mu\nu}$ is the tensor

$$t_{\mu\nu} \equiv \frac{1}{2}\left(\phi^*_{,\mu}\ \phi^*_{,\nu} + \phi^*_{,\nu}\ \phi_{,\mu}\right) - \frac{1}{2}\ g_{\mu\nu}\Big[\ \phi^*_{,\lambda}\ \phi_{,\varepsilon}\ g^{\lambda\varepsilon} -$$

$$-m^2\phi^*\phi + \sigma(\phi^*\phi)^2\Big] + \frac{1}{6}\Box\ (\phi^*\phi)g_{\mu\nu} - \frac{1}{6}(\phi^*\phi)_{,\mu;\nu}\ .$$

We can however re-write $T_{\mu\nu}$ in another form, using equation (17) for $g_{\mu\nu}$, with the inclusion of the σ-term.

Formally we have

$$T_{\mu\nu}[\phi] = t_{\mu\nu} - \frac{\phi^2}{6}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = t_{\mu\nu} + \frac{\phi^2}{6}\left[\frac{t_{\mu\nu}}{1-\frac{\phi^2}{6}} - \frac{\Lambda g_{\mu\nu}}{1-\frac{\phi^2}{6}}\right]$$

or

$$T_{\mu\nu}[\phi] = \frac{t_{\mu\nu}}{1-\frac{\phi^2}{6}} - \frac{\phi^2}{6-\phi^2}\ \Lambda\ g_{\mu\nu}\ . \qquad (27')$$

Thus, the energy $E(\phi)$ in the fundamental state ϕ_o is

$$E(\phi_0) = \frac{(3m^2-\Lambda)\phi^2 - 3\sigma\phi^4}{6-\phi^2}\ .$$

Let us add to E a constant just to renormalize the energy of the zero point, that is, $E(0) = -\Lambda$. We write $\tilde{E} \equiv E - \Lambda$.

The extremum points of $\tilde{E}$ are given by solutions of the equation

$$\sigma\phi_0^4 - 12\sigma\phi_0^2 + 6m^2 - 2\Lambda = 0\ . \qquad (29)$$

In order to the fundamental state (26) be an extremum it must satisfy simultaneously both equations (26) and (25). This is not possible in general, but it can occur if the cosmological constant Λ and the self-coupling constant σ are related by

$$\sigma = \frac{m^4}{8\Lambda} , \tag{30}$$

which implies, by (26), that

$$\phi_0^2 = \frac{m^2}{2\sigma} . \tag{31}$$

This is precisely the result expressed in equation (24) that we had obtained before in case of absence of gravity.

The interpretation we can give to such result is this: in general, the introduction of a cosmological constant to renormalize the value of E(0) does not allow the mechanism of spontaneous symmetry breaking. However if the constants σ and Λ are not completely independent but satisfy relation (30), then the constant fundamental solution ϕ_0 constitutes the ground state of our system. As a consequence of this, the gauge symmetry is broken. We call this an induced breaking symmetry mechanism. Remark that the extremun will be a minimun only if $3m^2 - 2\Lambda < 0$, in which case the effective mass is imaginary. We find here the same situation as in the flat space time. The role of the cosmological constant Λ is just to let the mass m of the ϕ field to be real and an imaginary effective mass to appear, which allows the existence of a ground state distinct from the trivial one ($\phi=0$). See figure (5.1).

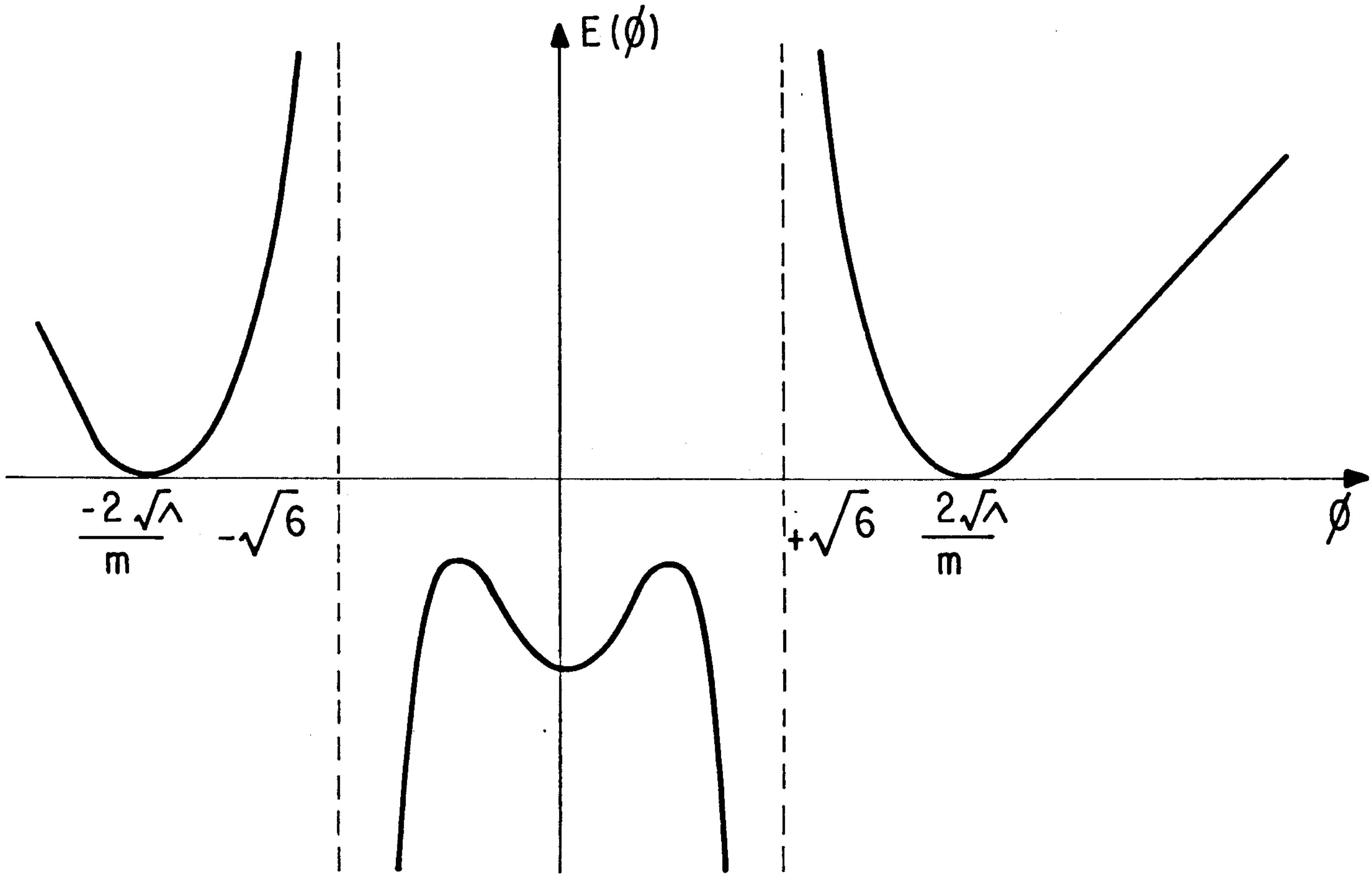

Fig. 5.1 - Case in which $2\Lambda > 3m^2 > \Lambda > \frac{3}{2} m^2$.
(See the text)

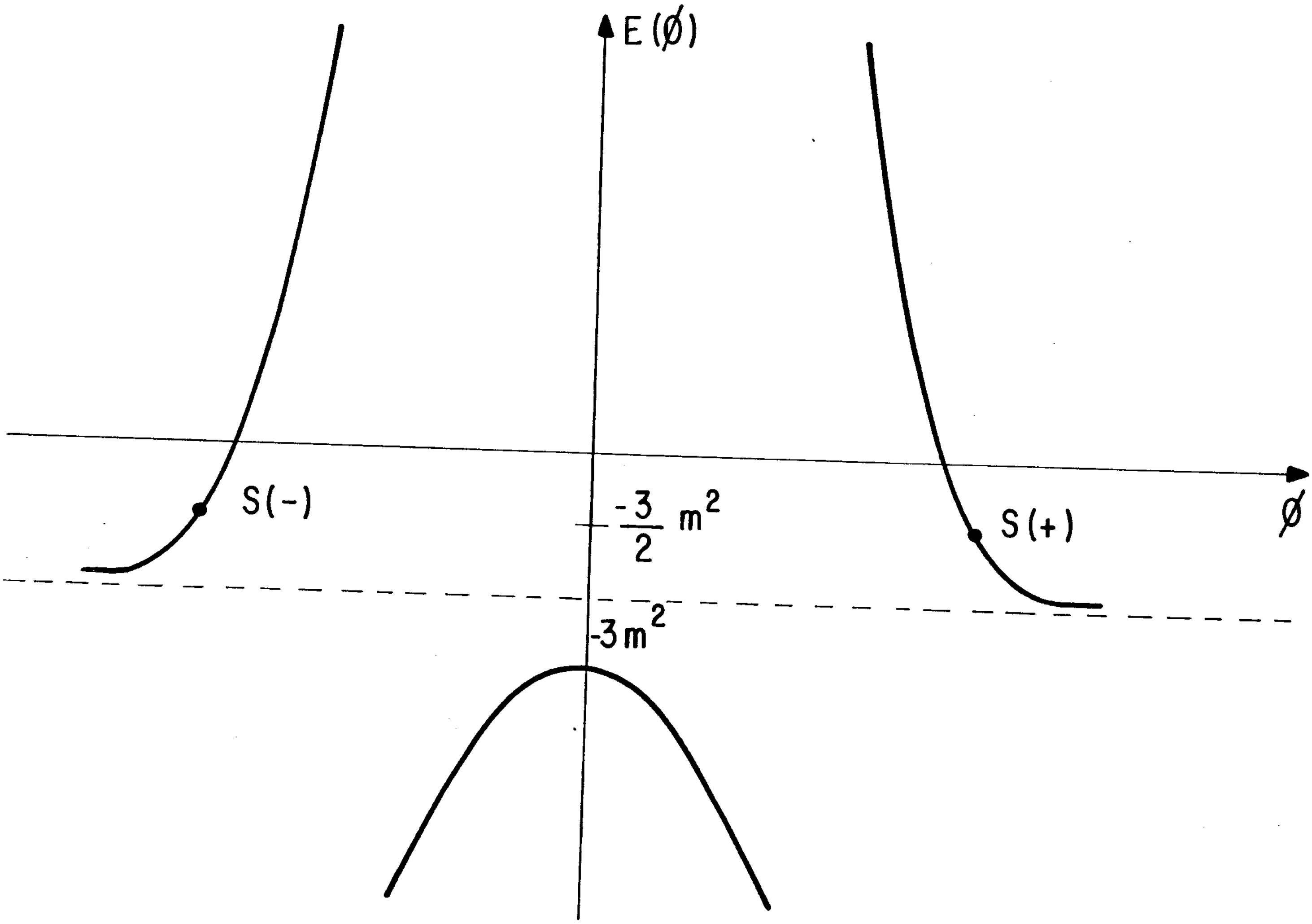

Fig. 5.2 - Case in which $\sigma = 0$.
$S_{(+)}$ and $S_{(-)}$ are De Sitter Universes

What is the role of the other constant σ which appears in this theory? In order to understand this let us set $\sigma = 0$ and see what are the consequences of this.

This case is represented in figure (5.2). We see that the point of constant solution ϕ_0 cannot be an extremum for any value of Λ. Instead of this, we recognize the instability of the constant solution which corresponds to $E(\phi_0) = -\frac{3}{2}m^2$.

The equation for the metric in this state, reduces to

$$R^{\mu}{}_{\nu} - \frac{1}{2}R\delta^{\mu}{}_{\nu} = \frac{3}{2}m^2\,\delta^{\mu}{}_{\nu} \quad , \tag{31}$$

which corresponds to a negative effective cosmological constant $\Lambda_{eff} = -\frac{3}{2}m^2$. In case of homogeneous cosmological models it corresponds to closed de Sitter Universe. Figure (5.2) shows that this is a highly unstable situation, the whole system decaying to the assymptoticaly stable case $E(\pm\infty) = -3m^2$.

Thus we recognize that the role of the self-coupling term of the ϕ field is just to increase the stability of its fundamental symmetry beaking constant solution.

5.4 - REPULSIVE GRAVITY OF MASSLESS PARTICLES INDUCED BY THE SCALAR FIELD

In this section we present a consequence of the existence of the symmetry breaking mechanism on the properties of the gravitational field. From figure (5.1) we can conclude that in the ϕ_o-states the scalar field does not contribute to the energy which is responsible for curving the space-time. This is due to the fact that $E(\phi_o)=0$. Let us see the consequences of this in case there are other sources of energy present.

In this case, equation (17) has an additional term of energy:

$$(1-\frac{1}{6}\phi^2)(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}) = -E_{\mu\nu}-T_{\mu\nu} \; , \tag{32}$$

in which $E_{\mu\nu}$ is given by the right hand side of (17) and $T_{\mu\nu}$ represents the energy momentum tensor of the rest of the matter. In the fundamental state equation (32) takes the form

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu} = -\frac{3m^2}{3m^2-2\Lambda}T_{\mu\nu} \; . \tag{33}$$

Remark that in order of the value (24) of ϕ_o to remain valid in case $T_{\mu\nu}\neq 0$, equation (18) must hold and this imposes that the energy-momentum tensor must be trace-free ($T\equiv T_{\mu\nu}g^{\mu\nu}=0$).

From equation (33) we obtain that the net consequence of the existence of the scalar field in the fundamental state ϕ_o is just to renormalize the gravitational constant K (which we take as unity in the system we are using) to an effective constant given by:

$$K_{REN} = \frac{3m^2}{3m^2 - 2\Lambda} K \quad . \qquad 34)$$

The sign of the value of the renormalized gravitational constant depends crucially on the mass of ϕ and on the cosmological constant Λ.

Thus, if the mass of the scalar field is such that $3m^2 - 2\Lambda < 0$, then K_{REN} becomes negative. We then conclude that the gravitational field created by a radiation gas is repulsive in the fundamental state ϕ_o of the scalar field.

Let us point out that this mechanism does not change any property of photons and neutrinos, but only create a medium in which the gravitational field generated by these particles becomes repulsive.

The application of the above mechanism to our Universe could lead us to an alternative explanation of its actual expanding era. Indeed, the dominant energy at early times comes from massless particles (this is a direct consequence of the conservation law, which states that for a perfect fluid with equation of state relating the pressure p to the density of energy ρ by $p=\lambda\rho$, in a Friedmann Universe, we have $\rho = \rho_o A^{-3(1+\lambda)}$, see section 3.2).

If the scalar field has a very small mass $m^2 < \frac{2}{3}\Lambda$ (that is $m \lesssim 10^{-34}$ MeV) and if the system exists in the fundamental state ϕ_o, then the conditions are fulfilled in order to the above result of cosmic repulsion be applied.

5.5 - MORE THAN ONE SCALAR FIELDS COUPLED NON-MINIMALLY TO GRAVITY

Let us consider now the situation in which there are two scalar fields $\phi(x)$ and $\psi(x)$ conformally coupled to gravity and without any direct interaction. We set for the Lagrangian

$$L = \sqrt{-g}\Big[R + \partial_\mu\psi^*\partial_\nu\psi g^{\mu\nu} - M^2\ \psi^*\psi - \frac{1}{6}R\,\psi^*\psi + \partial_\mu\phi^*\partial_\nu\phi\ g^{\mu\nu} - $$

$$- m^2\ \phi^*\phi - \frac{1}{6}\ R\phi^*\phi + \alpha(\phi^*\phi)^2 + \beta(\psi^*\psi)^2 + 2\Lambda\Big]. \tag{35}$$

The equations of motion for ϕ, ψ and $g_{\mu\nu}$ are:

$$\Box\phi + (m^2 + \frac{1}{6}\ R)\phi - 2\alpha(\phi^*\phi)\phi = 0 \tag{36}$$

$$\Box\psi + (M^2 + \frac{1}{6}\ R)\psi - 2\beta(\psi^*\psi)\psi = 0 \tag{37}$$

$$(1 - \frac{1}{6}\ \phi^2 - \frac{1}{6}\ \psi^2)(R_{\mu\nu} - \frac{1}{2}\ Rg_{\mu\nu}) = -E_{\mu\nu}(\phi,\psi). \tag{38}$$

in which $E_{\mu\nu}(\phi,\psi)$ is given by a straightforward generalization of expression (27). Remark that the energy of the coupled system ϕ, ψ in the curved space is not the sum of its individual energies. This is due to the fact that we have to take into account the interaction energy of ψ and ϕ through the mediator role of gravitational field.

Taking the trace of (38) we obtain

$$R = m^2\phi^2 + M^2\psi^2 - 4\Lambda. \tag{39}$$

We remark that R has the important property of being independent of the value of the self-coupling constants α and β and depends on the scalar fields only through mass-dependent terms. Let us consider the case in which one of the field is massless, say m=0. Then, using (39) into (36), (37) we obtain

$$\Box\phi - \frac{2}{3}\Lambda\phi - 2\alpha\phi^3 + \frac{1}{6}M^2\psi^2\phi = 0 \qquad (40)$$

$$\Box\psi + (M^2 - \frac{2}{3}\Lambda)\psi + (\frac{M^2}{6} - 2\beta)(\psi^*\psi)\psi = 0. \qquad (41)$$

These equations present a very curious property.At first sight it seems that we have constructed a system which violates the action-reaction principle. Indeed, equation (40) has a term $\frac{1}{6}M^2\psi^2\phi$ which represents the dependance of the equation of motion of the ϕ field on ψ. In the equation of ψ (41) there is no correspondent term which represents the reaction effect of ϕ on ψ. However this is only an apparent puzzle, since the action and reaction effects are manifested by the mediation of the gravitational field. Remark that, as locally we can reduce $\Box$ to the flat Minkowskii operator $\overset{(o)}{\Box}$, the puzzle appears as a manifestation of local violation of the action-reaction principle.

The origin of this is the non-symmetrical role of the scalar fields in Lagrangian (35) in case one of the fields is massless.

Equations (40) and (41) show that the fundamental constant solution $\psi = \psi_o$ can occur independently of ϕ, but the possibility of a fundamental solution ϕ_o depends on the prior existence of ψ in its fundamental state. Let us see what are the conditions for this to occur.

The existence of solution ψ_o is simply related to the condition $3M^2 - 2\Lambda < 0$. In this case,

$$\psi_o = 2 \frac{(3M^2-2\Lambda)}{12\beta - M^2} .$$

It turns out that as a consequence of this, the effective mass of the ϕ field becomes imaginary, which is the condition the field ϕ must fulfil in order to admit a fundamental solution ϕ_o.

Indeed, in this case,

$$m^2_{eff}(\phi) = \frac{M^4-8\beta\Lambda}{12\beta-M^2} < 0 .$$

Thus, the existence of the fundamental solution ψ_o induces the existence of the fundamental solution ϕ_o. We call such situation a cascade process, once it can occur with a series of scalar fields, the existence of the fundamental state for a massive field being the cause of the existence of fundamental states for massless fields.

5.6 - BROKEN SYMMETRY IN AN EXTERNAL GRAVITATIONAL FIELD

In section 5.3 we have examined how the curvature of space-time created by a scalar field ϕ can affect the mechanism of symmetry breaking in the fundamental state of ϕ. Here, we intend to analyse two simple examples of this mechanism in a curved external background geometry. This means that ϕ has to be considered as a test-field, the energy of which is so low that we can neglect the disturbances it provokes in the geometry.

(i) Gödel Geometry

We will show now that there is no non-trivial fundamental solution in the static rotating Gödel's cosmos.

The background geometry in a cylindric coordinate system takes the form

$$ds^2 = dt^2 - dr^2 - dz^2 + g(r)d\psi^2 + 2h(r)d\psi\, dt\ , \tag{42}$$

with

$$g(r) \equiv 2\sinh^4 r - \sinh^2 r \cosh^2 r$$
$$h(r) \equiv \sqrt{2}\ \sinh^2 r\ .$$

A scalar field conformally coupled to metric (42) and with a quartic self-interaction takes the form

$$\phi + (m^2 - \frac{\Omega^2}{3})\phi - 2\sigma\phi^3 = 0\ , \tag{43}$$

in which we used the fact that

$$R = -2\Omega^2. \tag{44}$$

The constant Ω measures the rotation of the fundamental observers co-moving with the source of the geometry. A constant solution, ϕ_o = constant, of equation (43) exists which is distinct from the trivial one ($\phi=0$).

We have

$$\phi_o^2 = \frac{1}{2\sigma}\left[m^2 - \frac{\Omega^2}{3}\right] . \qquad (45)$$

Remark that, as $\sigma < 0$, the solution exists only in the case $3m^2 < \Omega^2$. From (27), the value of ϕ which is an extremum of the energy $E(\phi)$ is given by solution of the equation

$$\phi^4 - 12\phi^2 + \frac{6m}{\sigma} = 0 . \qquad (46)$$

A little algebra proves that (45) and (46) are not compatible, thus showing that the fundamental constant solution ϕ_o does not define the fundamental ground state of the ϕ field in Gödel's background.

(ii) Friedmann Geometry

Once Friedmann geometry is not static and the scalar R depends on time there is no possibility of having a non-trivial constant solution for the scalar field. However, it is possible to find a solution which has only one degree of freedom $\phi = \phi(t)$ and which allows for the mechanism of symmetry breaking.

Let us write the geometry of the (open) Friedmann cosmos in the conformal form

$$ds^2 = A^2(\eta)\left[d\eta^2 - d\chi^2 - \sinh^2\chi(d\theta^2 + \sin^2\theta d\psi^2)\right] . \qquad (47)$$

In this coordinate system we have

$$R = \frac{6}{A^2} \left(\frac{A''}{A} - 1 \right), \qquad (48)$$

in which a dash (') means derivative with respect to η.

Setting $\phi = \phi(\eta)$ the equation for ϕ takes the form

$$\phi'' + 2 \frac{A'}{A} \phi' + (m^2 - \frac{2}{3} \Lambda) A^2 \phi + (\frac{m^2}{6} - 2\sigma) A^2 \phi^3 = 0 \,. \qquad (49)$$

Introducing a new variable f by means of the definitions [Melnikov et al (1979)]

$$\phi = \sqrt{\frac{+1}{2|\sigma|}} \frac{f}{A} \,, \qquad (50)$$

equation (49) yields:

$$f'' + f(m^2A^2 - 1) + f^3 = 0 \,. \qquad (51)$$

Let us examine the limiting case in which, when $\eta \to 0$ the radius of the Universe goes to A_0 (which can be taken as 0).(Following Melnikov and others (1979).

Equation (51) takes the form

$$f'' + f(m^2 A_o^2 - 1) + f^3 = 0 \,. \qquad (51)'$$

We set

$$f' = y \qquad (52a)$$

$$y' = (1 - m^2 A_0^2) f - f^3 \qquad (52b)$$

Thus we have reduced equation for f in a planar autonomous system. There are three singular points for this system which are

$$B_0 = (0,0)$$

$$B_1 = (\sqrt{1-m\ A_0^2}\ ,\ 0)$$

$$B_2 = (-\ \sqrt{1-m^2 A_0^2}\ ,\ 0)\ .$$

In case A_0 vanishes, points B_1 and B_2 coalesce to B_0. Using the results of section 2.12 we infer that:

(i) if $m^2 A_0^2 > 1$ then points B_1, B_2 are stable (center) which implies no symmetry breaking.

(ii) if $m^2 A_0^2 < 1$ then points B_1, B_2 are saddle (unstable) points.

We then conclude that the non-singular open cosmological model (with $A_0 \neq 0$) can stabilize the vacum of the ϕ field if the mass of ϕ and the minimum radius of the Universe are such that they satisfy the condition

$$m^2\ A_0^2\ > 1\ . \tag{53}$$

5.7 - NON LINEAR PHOTONS

The investigation of the general properties of non-linear electrodynamics is not new. The idea seems to be ancient (Mie, 1912) but it gained new interest after the contribution of Born (1933) and Born and Infeld (1934, 1935). We do not intend here to present a review of this subject. We suggest the reading of the very attractive and complete review made by Plebansky (1968). Here we present an analysis of non-linear electrodynamics induced by non-minimal coupling with gravity. As we saw in the previous section, this coupling can be responsible of drastic modifications in the metric properties mainly related to the problem of singularities. As a particular and important example we will show here that non linear photons generate a Friedmann-like cosmos with a minimum radius. Let us limit our analysis here to the gauge-dependent theory described by the Lagrangian (massive photon model):

$$L = \sqrt{-g}\ \{\frac{1}{k}(1+\lambda W_\mu W_\nu g^{\mu\nu})R - \frac{1}{4}\ F_{\mu\nu}F^{\mu\nu}\} + L_m \tag{54}$$

in which

$$F_{\mu\nu} \equiv W_{\mu;\nu} - W_{\nu;\mu} = W_{\mu,\nu} - W_{\nu,\mu} \ .$$

λ is a constant with the same dimensionality as Einstein's coupling constant $K[\ (\text{energy})^{-1}(\text{length})]$.

The equation of motion for $g_{\mu\nu}$ and $F_{\mu\nu}$ obtained from (54) are:

$$(1+\lambda W^2)(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}) - \lambda\Box W^2 g_{\mu\nu} + \lambda W^2{}_{,\mu;\nu} + \lambda R W_\mu W^\nu = -kE_{\mu\nu} - k\ T^{(m)}_{\mu\nu} \tag{55a}$$

$$F^{\mu\nu}{}_{;\nu} = - \frac{\lambda}{k} RW^{\mu} , \tag{55b}$$

in which $T_{\mu\nu}$ represents the stress-energy tensor of the matter and $E_{\mu\nu}$ is the Maxwell's tensor

$$E_{\mu\nu} = F_{\mu\alpha} F^{\alpha}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} . \tag{56}$$

In case there are charged particles we have to add to the right-hand side of (55b) a current J^{μ}. Taking the divergence of this equation yields

$$J^{\mu}{}_{;\mu} = \frac{\lambda}{k} (RW^{\mu})_{;\mu} . \tag{57}$$

If the divergence of RW^{μ} does not vanish then charge is not conserved. In this case the number of created particles depends on the value of the scalar of curvature through equation (57). Note that creation of change in this model can occur only in those regions in which the scalar of curvature does not vanish. This is not a sufficient condition, of course, but it is a necessary one.

The effect of a breakdown of charge conservation on cosmological scale was analysed, some years ago, by Lyttleton and Bondi (1959) and criticized by Hoyle (1959). The essential idea of the Lyttleton-Bondi (LB) analysis rests on the observation that a slight difference in the magnitude of the electric charges of the proton and the electron could give rise to a repulsive force. On cosmic scale, the result could be an alternative explanation of the observed expansion of the Universe. The modification suggested by LB consists in adding a mass term $\epsilon W_\mu W_\nu g^{\mu\nu}$ to Maxwell's Lagrangian, allowing for a non-null divergence of the potential vector W^μ. Then they construct a cosmological solution of an universe filled with such massive photons.The result is a steady state (de Sitter type) cosmological configuration. In a subsequent paper Hoyle (1959) has shown that LB model is equivalent to the introduction of a fluid with negative energy that could be generated by a scalar field (see section 5.3). As a consequence, the equation of motion which gives the behavior of LB electrodynamics in an expanding steady-state homogeneous and isotropic universe is similar to the equation of Hoyle's C-field, which is responsible for matter creation. Thus, the effect of the proposed modification of electrodynamics through the Lyttleton-Bondi hypothesis is indistinguishable — with respect to cosmic effects — from Hoyle's model of continuous creation of matter.

Although there is a point of contact with the Lyttleton--Bondi scheme of modified electrodynamics, the model we discuss here is very distinct from their proposal. The crucial difference is contained in the introduction of nonlinearities through the dependence of the mass term on the scalar of curvature. Actually, many new features appear in this model which have no equivalent in LB's. For instance, it does not admit a cosmological steady-state configuration. Such a solution, which is a typical property of the Lyttleton-Bondi model, is indeed the main point of contact of the LB model and Hoyle's version of continuous creation of matter.

Let us come back now to equation (55a). Taking the trace of this equation, we find

$$R = kT - 3\lambda \Box W^2, \tag{58}$$

where T is the trace of the stress-energy tensor. Thus, we obtain from eq. (55b)

$$F^{\mu\nu}{}_{;\nu} = \frac{3\lambda^2}{k} (\Box W^2) W^\mu - \lambda T W^\mu + J^\mu, \tag{59}$$

which explicitly exhibits the nonlinear character of the model. It seems worthwhile to remark here that such a type of nonlinearity behavior can be introduced in an equivalent way without making an appeal to nonminimal coupling with gravitation. Indeed, if we consider a Lagrangian of the form

$$L_N = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sigma (W^\mu W_{\mu;\lambda})^2,$$

a straighforward calculation shows that the equation of motion obtained from such L_N is precisely eq.(59) (without the trace term, of course).

The wave equation for the potential vector W^μ is given by

$$\Box W^\mu + R^\mu{}_\alpha W^\alpha - (W^\nu{}_{;\nu})^{;\mu} = \frac{3\lambda^2}{k} (\Box W^2) W^\mu, \tag{60}$$

in the absence of currents and matter. The first two terms of this equation are nothing but de Rahm's wave operator in curved space. The third term is proportional to the gradient of the scalar of curvature in the W^{μ} direction.

Let us now turn to the following question: assuming the existence of an Universe filled with such nonlinear photons,what are the global properties of this cosmos? We will now show that there is a solution of the above set of equations (55) which represents a homogeneous and isotropic universe.

As there is no privileged direction in space, in which the electric and the magnetic vectors could point, we conclude that both vectors must vanish. From Eq.(55b), the scalar of curvature must vanish, too,

$$R = 0 . \tag{61}$$

As a consequence, charge is conserved. Equation (61) may be written equivalently,

$$\Box W^2 = 0 . \tag{62}$$

Let us define a function $\Omega \equiv 1 + \lambda W^2$. Then our set of eqs. can be written in the form

$$R_{\mu\nu} = - \frac{\Omega_{,\mu;\nu}}{\Omega} \tag{63}$$

$$\Box \Omega = 0 . \tag{64}$$

Let us look for a solution of this set of equations in which the infinitesimal element of length has the form

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta \; d\psi^2)\right]. \tag{65}$$

After some simple calculations,we obtain the equations for a(t) and $\Omega(t)$.

The values of the curvature are

$$R^0{}_0 = 3\frac{\ddot{a}}{a},$$

$$R^1{}_1 = \frac{\ddot{a}}{a} + 2\,\frac{\dot{a}}{a^2} - \frac{2}{a^2}\,\frac{\sigma''}{\sigma}\;, \tag{66}$$

$$R^2{}_2 = R^3{}_3 = \frac{\ddot{a}}{a} + 2\,\frac{\dot{a}^2}{a^2} - \frac{1}{a^2}\left(\frac{\sigma''}{\sigma} + \frac{\sigma'^2 - 1}{\sigma^2}\right),$$

in which a dot means time derivative.

The covariant derivatives of Ω are given by

$$\Omega^0{}_{;0} = \ddot{\Omega}$$

$$\Omega^1{}_{;1} = \Omega^2{}_{;2} = \Omega^3{}_{;3} = -\frac{\dot{a}}{a}\,\dot{\Omega}\;.$$

From this, we obtain the result that the 3-curvature $^{(3)}R$ must be a constant.

Let us define $\epsilon \equiv -\frac{1}{6}\,{}^{(3)}R$. Then ϵ may assume the values 0, +1, -1. Correspondingly, the function $\sigma(\chi)$ may be χ, $\sin\chi$ or $\sinh\chi$. The solution is easily obtained:

$$a(t) = (-\epsilon t^2 + bt + c)^{1/2}, \tag{68}$$

$$\Omega = \frac{\Omega_0}{a}\,(-2\epsilon t + b)\,, \tag{69}$$

in which band c are constants.

Let us make some comments on these solutions. First of all, we remark that a simple explicit form for the function a(t) is available when the cosmos is filled with such nonlinear photons.

Constants b, c, and Ω_0 are not completely arbitrary. They have to satisfy a constraint which is linked to the definition of Ω. As in the isotropic world there is not a privileged direction, the vector W^{μ} must be of the form $W^{\mu}=(\dot{\phi},0,0,0)$. We have set a derivative on ϕ just to recall that W^{μ} is a gradient. Thus, we have

$$1 + \lambda\dot{\phi}^2 = \Omega_0(-2\epsilon t + b)(-\epsilon t^2 + bt + c)^{-1/2} . \qquad (70)$$

Let us examine this relation for the three possible values of ϵ separately. In the case of $\epsilon = 0$, then $\lambda\phi^2 = \Omega_0 b/a - 1$. If λ is negative, then $\Omega_0 b$ must be negative too.

In the closed universe, $\lambda\dot{\phi}^2 = (\Omega_0/a)(-2t + b) - 1$. In the case of a negative λ, then Ω_0 must be positive and b negative. Finally, for the open model, if λ is negative, b must be positive and Ω_0 negative. Now let us turn to the function a(t). The possibility of a real solution is dominated by the sign of

$$\Delta \equiv b^2 + 4\epsilon c .$$

Remark that in case $\Delta<0$ the radius a(t) does not vanish. This means that in these cases the singularity is avoided. The reason for this is precisely the dependence of the curvature of the non-linear photons. Thus, we see once more that non minimal coupling may lead to non singular cosmological models. We leave to the reader to recognize that this result does not contradict the famous singularity theorems by Hawking, Ellis, Penrose and others.

[see Hawking and Ellis (1973)].

The above cosmological solution is stable against a small perturbation generated by the introduction of a small quantity of matter. Actually, this property does not depend on our specific model but is a consequence of the absence of density of matter in the expanding background.

As a consequence of the energy-balance equation, and owing to the absence of electric and magnetic fields, the stress--energy tensor of the matter must be conserved. Let us consider a fluid (dust) with an energy-momentum tensor given by $T_{\mu\nu} = (\delta\rho)V_\mu V_\nu$, where $\delta\rho$ is a small density.

We choose the comoving frame in order to set the fluid velocity V^μ to have the value $V^\mu = \delta^\mu_0$. Conservation of $T^{\mu\nu}$, projected in the V^μ direction, gives

$$(\delta\rho)^{\bullet} + (\delta\rho)\theta = 0.$$

In the case of the Euclidean section, using the results obtained above, the expansion θ equals $\frac{3}{2} b\,(bt + c)^{-1}$. A direct integration yields

$$\delta\rho = (\delta\rho)_0\,(bt + c)^{-3/2}.$$

Thus, as time goes on the total perturbation decreases showing the stability of the model under a small injection of matter in our nonlinear-photon cosmos.

Actually, one can show a result stronger than this,e.g., that our model universe cannot share the bending of space--time with a finite density of matter. This can be seen by a direct inspection om equations $R = 0$ and $\Omega = 0$. These two equa tions specify the functions $a(t)$ and $\Omega(t)$, giving no possibili ty of inserting another function $p(t)$ in our equations.

5.8 - KASNER ERA

Lifshitz and Khalatnikov (1980) showed that the general behavior of the metric of the Universe in its early epochs is of a Kasner type. This means that the structure of Einstein's equations is such that it admits a Friedmann like Universe as an asymptotic limit of a higher anisotropic regime characterized by stochastic oscilations of the axis of anisotropy. In other words, the directional homogeneity of the cosmos is related to the alternance of anisotropic axis throughout the whole space-time. Lifshitz and Khalatinov have shown that the main basis for their analysis depends on the fact that in Kasner-like geometries the behavior of the metric, in the neighborhood of the primordial global singularity, is not dictated by the matter terms. This means simply that a comparison of the Einstein tensor ($G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2} R\delta^{\mu}{}_{\nu}$) with the energy tensor $T^{\mu}{}_{\nu}$ shows that for $t \to 0$, we have $\frac{|G^{\mu}{}_{\nu}|}{|T^{\mu}{}_{\nu}|} \to \infty$, for different values of μ and ν.

This vacum stage allows the existence of a chaotic era of alternative mixing axis of anisotropy, which is the main property of the Kasner regime. Indeed, as it is well-known, a perfect fluid can not be responsible for the curvature of a Kasner Universe. One is then faced with the question: is it possible to find an alternative behavior of the source of the geometry which could avoid the different treatment which leads from Kasner regime to a Friedmann era? The answer to this is related to the question: could one find a stress-energy tensor which can accomodate both kinds of geometries? Let us show now that the non-linear photons introduced in the last section can do this job.

We set the geometry under the form:

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \qquad (71)$$

Using the same function Ω as in the previous isotropic case the basic equations of the theory reduce to the same set of equations (63) and (64).

Using (71) we find that it yields:

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\ddot{\Omega}}{\Omega} = 0 \qquad (72)$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c} + \frac{\dot{\Omega}}{\Omega}\right) = 0 \qquad (73a)$$

$$\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c} + \frac{\dot{\Omega}}{\Omega}\right) = 0 \qquad (73b)$$

$$\frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{\Omega}}{\Omega}\right) = 0. \qquad (73c)$$

Following Kasner we try a solution in the form: $a = a_0 t^{p_1}$, $b = b_0 t^{p_2}$, $c = c_0 t^{p_3}$ and $\Omega = \Omega_0 t^{p_4}$. Substituting this ansatz into eq.(72) and (73) we find that the numbers p_1, p_2, p_3 and p_4 must satisfy the relations

$$p_1 + p_2 + p_3 + p_4 = 1 \qquad (74)$$

$$(p_1)^2 + (p_2)^2 + (p_3)^2 + (p_4)^2 = 1. \qquad (75)$$

We recognize here that this set of numbers represents a Kasner solution of a five-dimensional Eintein's theory (in vacuum) in which $\Omega(t)$ plays the role of the expansion factor of the fifth axis. This is not astonishing, as the theory of the

the non-linear photons in the high symmetrical geometry we are analysing has a unique scalar function which is left free and it then turns out to be in perfect analogy with theories of five-dimensional structure for the space-time [Belinsky and Khalatnikov (1973)].

The next step is to show that during the evolution of this cosmos we can pass from this Kasner regime to the Friedmann solution. This is left to the reader.

APPENDIX 1: ELECTRIC AND MAGNETIC GRAVITATIONAL MONOPOLES

We have shown in section 2 that in the absence of sources Eintein's equations of gravity reduces to the set

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = 0 \cdot \tag{1}$$

This implies that the equation for the metric field $g_{\mu\nu}(x)$ is invariant under a dual rotation [eq.(75), section 2]. We are then led to ask about the possibility of extending this symmetry to the motion of particles. A free particle in a gravitational field follows a geodesics. In the geodesic equation of motion the presence of the gravitational field is given by the metrical connection, which is coordinate dependent(locally, it can be made to vanish) and, in this sense, cannot be considered as a true observable. The corresponding observable quantity is the vector η^α which connects in a congruence of geodesics, two points of neighbouring curves with the same value of the affine parameter. The connecting vector η^α satisfies the Jacobi equation

$$D^2\eta^\alpha/DS^2 = W^\alpha_{\beta\mu\nu}\, V^\nu\, V^\beta\, \eta^\mu , \tag{2}$$

in which $V^\mu = dX^\mu/dS$ is the tangent vector to the geodesics $X^\mu(s)$. s is an affine parameter. We can thus characterize electric (E)-poles as those particles that move, under the influence of gravitational forces, on curves such that their connecting vector satisfies the Jacobi equation (2). This way of describing the behavior of particles in a given gravitational field itself suggests that we must look for the generalization of the Jacobi equation in order to introduce a new feature in the motion of particles in curved space. The symmetric properties of the Riemann tensor give a unique way of constructing such an equation. Indeed, let $y^{\alpha}(s)$ be a congruence of curves on the space-time Riemannian manifold such that their connecting vector Π^{α} satisfies the equation

$$\frac{D^2 \Pi^{\alpha}}{Ds^2} = f \overset{*}{W}{}^{\alpha}{}_{\beta\mu\nu} V^{\beta} V^{\nu} \Pi^{\mu}; \qquad (3)$$

f is a constant characteristic of each particle. We will call a magnetic (H)-pole any particle that moves on curves $y^{\alpha}(s)$ such that their connecting vector of the congruence, Π^{α}, satisfies equation (3).

The reason for not having a term analogous to the constant f in equation (2) reflects the constancy of the ratio of inertial to gravitational mass - and is indeed the main reason for geometrizing gravitational interaction. The new particles do not follow geodesic lines but , as we will see, curves of forced motion. In other words, H poles are not minimally coupled with gravitation.

Using definitions (4a,b) of section 2 we can write these equations in the form:

$$D^2\eta^{\alpha}/Ds^2 = E^{\alpha}{}_{\mu}\, \eta^{\mu} \qquad (4)$$

$$D^2\Pi^{\alpha}/Ds^2 = f\, H^{\alpha}{}_{\mu}\, \Pi^{\mu}. \qquad (5)$$

The origin of the terms E pole and H pole now becomes transparent: they unambigously denote particles that couple, through two types of tidal forces, with the electric and magnetic parts of the Weyl tensor, respectively.

The equations of motion for E poles are geodesics and the corresponding equations of motion for H poles are curves of forced motion. The acceleration effect on H poles is a completely new phenomenon that has no equivalence in Newtonian theory. So it is a typical effect of the curvature of space-time. We will give here some properties of H-pole trajectories.

Let $y^{\alpha}(s)$ be the curve under discussion and consider a real parameter s on it. The equation for $y^{\alpha}(s)$ will be written as

$$\frac{d^2}{ds^2} y^{\alpha}(s) + \{^{\alpha}_{\mu\nu}\} \frac{dy^{\mu}}{ds} \frac{dy^{\nu}}{ds} = F^{\alpha} , \tag{6}$$

where F^{α} is the forced motion term which induces the deviation of $y^{\alpha}(s)$ from the equation of a geodesic. Then, we construct a family of curves that generate a congruence $y^{\alpha}(s,v)$ in which v distinguishes different curves and s is a parameter on each curve. Next we impose equation (5) on the connecting vector Π^{α} (that can be defined as the derivative of $y^{\alpha}(s,v)$ with respect to the v variable). As a consequence, the force F^{α} must be a solution of the equation

$$F_{\alpha,\mu} - \{^{\lambda}_{\alpha\mu}\} F_{\lambda} = E_{\alpha\mu} + f H_{\alpha\mu} . \tag{7}$$

In order to arrive at this condition for F^{α} a lot of work is saved if we note that the second absolute derivative of Π^{α} with respect to the s parameter can be written as

$$\frac{D^2\Pi^{\alpha}}{Ds^2} = \frac{d^2\Pi^{\alpha}}{ds^2} + \left\{{\sigma \atop \sigma\lambda}\right\}\Big|_{\epsilon} \frac{\partial y^{\lambda}}{\partial s}\frac{\partial y^{\epsilon}}{\partial s}\Pi^{\alpha} + \left\{{\alpha \atop \epsilon\sigma}\right\}\left\{{\sigma \atop \mu\lambda}\right\}\frac{\partial y^{\lambda}}{\partial s}\frac{\partial y^{\epsilon}}{\partial s}\Pi^{\mu} + \left\{{\alpha \atop \mu\nu}\right\}\Pi^{\mu}\frac{\partial^2 y^{\lambda}}{\partial s^2}$$

$$+ 2\left\{{\alpha \atop \sigma\lambda}\right\}\frac{\partial y^{\lambda}}{\partial s}\frac{d\Pi^{\alpha}}{ds}, \qquad (8)$$

where

$$\frac{Dt^{\alpha}}{Ds} \equiv t_{;\beta}\frac{\partial y^{\alpha}}{\partial s} = \left(t^{\alpha}_{,\beta} + \left\{{\alpha \atop \epsilon\beta}\right\}t^{\epsilon}\right)\frac{\partial y^{\beta}}{\partial s}$$

and

$$\frac{dt^{\alpha}}{ds} = t^{\alpha}_{,\beta}\frac{\partial y^{\beta}}{\partial s} \equiv \frac{\partial t^{\alpha}}{\partial y^{\beta}}\frac{\partial y^{\beta}}{\partial s}.$$

Equation (7) for F^{α} seems, at first sight, to be highly involved. In order to know the motion of H poles, we have to know the force F^{α}. To obtain F^{α} we must solve equation (7) in which the electric and magnetic parts of the Weyl tensor are obtained by projection onto the direction of motion of the H pole. It seems like a 'bootstrap' situation. Fortunately, due to the symmetric properties of H poles we will show that this is only an apparent situation - we can deal very directly with this new kind of motion. The ultimate reason for this simplification rests on the conformal behaviour of this theory.

Before showing this we remind the reader that F^{α} must fulfill the following special properties in order to be a solution of expression (7).

(i) F^{α} must be a gradient. Indeed, using the symmetric properties of $E_{\alpha\beta}$ and $H_{\alpha\beta}$ we have

$$F_{\alpha;\beta} - F_{\beta;\alpha} = 0 \ .$$

So, due to the symmetric properties of the metric connection $\{^{\alpha}_{\mu\nu}\}$ this implies $F_{\alpha} = \phi_{,\alpha}$.

(ii) ϕ is a solution of the wave equation ($\Box\phi = 0$) in the given background metric. This can be easily seen by writing equation (7) in terms of the ϕ field and making use of the trace-free property of the Weyl tensor.

(iii) F^{α} is a constant of motion for H poles. Indeed, the absolute derivative of F^{α} gives

$$\frac{D\phi,^{\alpha}}{Ds} = \phi^{,\alpha}_{;\beta} \frac{\partial y^{\beta}}{\partial s} = (E^{\alpha}_{\beta} + f\, H^{\alpha}_{\beta}) \frac{\partial y^{\beta}}{\partial s} = 0$$

The last equality comes from the fact that $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are orthogonal to $\partial y^{\alpha}/\partial s$. The above property implies that H poles travel on curves of constant acceleration. Consider now a Riemannian manifold V_4 containing a metric $g_{\mu\nu}(x)$ and a set of non-null geodesics, characterized by a generic tangent vector $u^{\alpha}(s)$, where s is an affine parameter. Let us then project the Weyl tensor and its dual into the u^{α} directions, in order to define its electric and magnetic parts.

An arbitrary conformal mapping of V_4 into $\tilde{V}_4$ generated by a function ψ will be given by setting

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}(x) = e^{2\psi(x)} g_{\mu\nu}(x) \qquad (9a)$$

$$g^{\mu\nu} \rightarrow \tilde{g}^{\mu\nu}(x) = e^{-2\psi(x)} g_{\mu\nu}(x) \ . \qquad (9b)$$

As a consequence of this mapping, the quantities $E_{\alpha\beta}$ and $H_{\alpha\beta}$ and the properties of the congruence generated by u^{α} change accordingly

$$\begin{aligned}
u^{\alpha} &\rightarrow \tilde{u}^{\alpha} = e^{-\psi} u^{\alpha} \\
E_{\alpha\beta} &\rightarrow \tilde{E}_{\alpha\beta} = E_{\alpha\beta} \\
H_{\alpha\beta} &\rightarrow \tilde{H}_{\alpha\beta} = H_{\alpha\beta} \\
\theta &\rightarrow \tilde{\theta} = e^{-\psi}\theta - 3(e^{-\psi})_{,\alpha} u^{\alpha} \\
\sigma_{\mu\nu} &\rightarrow \tilde{\sigma}_{\mu\nu} = e^{\psi}\sigma_{\mu\nu} \\
\omega_{\mu\nu} &\rightarrow \tilde{\omega}_{\mu\nu} = e^{\psi}\omega_{\mu\nu} ,
\end{aligned} \tag{10}$$

in which we have used the invariance property

$$\eta_{\alpha\beta}^{\mu\nu} \rightarrow \tilde{\eta}_{\alpha\beta}^{\mu\nu} = \eta_{\alpha\beta}^{\mu\nu} . \tag{11}$$

We would like to call attention to the fact that it is not possible to change the shear-free and/or the rotation-free properties of a congruence of geodesics by a conformal transformation. This is not the case for the acceleration vector. The geodesics $u^{\alpha}(s)$ are mapped into accelerated curves $\tilde{u}^{\alpha}(\tilde{s})$ of equations of motion given by

$$D\tilde{u}^{\alpha}/D\tilde{s} = \frac{1}{2}(e^{-2\psi})_{,\beta} g^{\alpha\beta} . \tag{12}$$

From the whole class of functions ψ let us select the set $\{\psi \equiv \phi\}$ such that obeys the equation

$$\frac{1}{2}\, e^{-2\phi}{}_{,\alpha;\tilde{\sigma}} = e^{-2\phi}(E_{\alpha\sigma} + f\, H_{\alpha\sigma}) , \tag{13}$$

in which the tilde over σ on the left-hand side of this expression means that the covariant derivative is taken in the conformally transformed metric $\tilde{g}_{\mu\nu}(x)$. By making this choice of functions, we map the class of geodesics $u^{\alpha}(s)$ into the class of accelerated curves $\tilde{u}^{\alpha}(\tilde{s})$ defined by equation (6). In other words, we map the trajectories of E poles into trajectories of H poles. We remark that the right-hand side of equation (13) can be evaluated without reference to the curves of H poles. This is a simple direct consequence of the transformation properties of the Weyl tensor under a conformal mapping. This shows the way of circumventing the bootstrap situation we seemed to be faced with before.

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