QUALITATIVE ANALYSIS OF HOMOGENEOUS UNIVERSES

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ABSTRACT

We investigate the qualitative behaviour of cosmological models in two cases:

(i) Homogeneous and isotropic Universes containing viscous fluids in a stokesian non-linear regime;

(ii) Rotating expanding universes in a state in which matter is off thermal equilibrium.

I. INTRODUCTION

Recently the method of investigating qualitatively systems of differential equations wich describes certain special configurations of the gravitational field has attracted attention of many authors (2,3,4,6,7). The interest of such method is two fold: firstly, it gives a very good picture of the general behaviour of distinct solutions of a given set of differential equations and secondly, it help us in pointing into the direction in which the search of specific solutions should be undertaken.

It seems worth while to call attention to the fact that such qualitative analysis can be effectively made only in some restricted and very special circunstances,e.g., in the case the system of differential equations is reducible to an autonomous form of the type

$$\dot{x} = F(x,y)$$

$$\dot{y} = G(x,y)$$

a dot represent derivative with respect to a parameter, say the time t.

The right hand functions F and G are not explicit function of the time coordinate but may be any linear or non-linear function of variables x and y.

Astonishing enough, Einstein's set of gravity equations can be reduced to such planar autonomous system in some cases of real interest like, for instance, for homogeneous universes.

In the present work we will use such method to investigate two types of configurations: (i) Homogeneous and isotropic universe filled with a non-linear stokesian fluid; and (ii) rotating and expanding universes in a state in which matter is outside thermal equilibrium. In case (i) there are well known examples of explicit analytical solutions like Friedmann's models; in case (ii) an example has been exhibited recently by Novello and Rebouças[11] which generalizes the static rotating universe found by Gödel some years ago.

The influence of viscous phenomena in Cosmology has been examined by many authors[1,2,7,10] as a model of the cosmological fluid at the drastic regions near the singularity. Hitherto such viscous fluid has been treated only in the Cauchy linear case. One adds to the isotropic pressure p a term proportional to the expansion factor (bulk viscosity) or one introduces an anisotropic stress Π_{ij} linearly related to the shear σ_{ij}. The main reason for considering, as we do in the present work, a more general non-linear dependence of the pressure on the expansion rests on quantum effects.

Indeed, it has been suggested by many authors that the introduction of viscosity in the cosmical fluid is nothing but a phenomenological description of the effect of creation of particles by the non-stationary gravitational field of the expanding cosmos.

In ref.9 it is shown that the quantum corrections of the macroscopic stress energy tensor can be described by a polinomial function of the expansion factor Θ.

The presence of viscosity, through such polinomial dependence on Θ , changes radically the features of the Universe. For instance, G. Murphy has given recently a simple analytical model in which viscosity is even used to prevent a singularity region to occur. We remark that this is in no way in contradiction with the singularity theorems once the hypothesis required by these are not fulfilled by the viscous fluid.

In section II we present the main equations of the gravitational field for a viscous fluid in a non-linear Stokesian regime in an isotropic and homogeneous expanding Universe. Thus the modification introduced by viscosity can appear only as a change in the isotropic pressure p to $\tilde{p} = p$ + polinomial in Θ . We analyse the specific case of a quadratic regime $\tilde{p} = p + \alpha\Theta + \beta\Theta^2$. In section II we limit α and β to be constants. We associate such situation to the stationary case of a constant injection of new particles in the Universe inducing the viscous phenomena in a steady state regime. We make then some remarks in the general case of more complicated polinomial dependence of pressure on Θ. In section III we investigate the non stationary regime and allow for a non-constant quadratic coefficient β. Actually such β can depend only on the energy and we analyse a specif power law dependence. We compare our results with the linear case which have been examined previously.

Section IV deals with the qualitative analysis of rotating and expanding homogeneous universes. We present a class of models discussed by Novello and Rebouças and show that among this class there is only one geometry which attains Gödel's model assintotically.

II. Steady State Regime of Viscous Fluid

We start by considering a homogeneous and isotropic cosmological model. The fundamental lenght , in a comoving system of coordinates in which the field velocity is $V^{\alpha}=\delta^{\alpha}{}_{0}$, assumes the form

$$(1) \qquad ds^2 = dt^2 - A^2(t) \left[d\chi^2 + \sigma^2(\chi)(d\Theta^2 + \sin^2\Theta \, d\phi^2) \right]$$

in which $\sigma(\chi)$ may be χ, $\sin\chi$ or $\sinh\chi$.

Raychaudhuri's equation of the evolution of the expansion factor $\Theta = \frac{1}{\sqrt{-g}} (\sqrt{-g}\, V^{\alpha})_{|\alpha}$

$$(2) \qquad \dot{\Theta} + \frac{\Theta^2}{3} + \frac{\rho}{2} + \frac{3}{2}\tilde{p} = 0$$

The total pressure $\tilde{p}$ accounts for the isotropic pressure p plus viscous terms, which we will represent as a polinomial in Θ:

$$(3) \qquad \tilde{p} = p - \sum_{k=1}^{N} \alpha_K \Theta^K$$

In this section we will limit the α's to be constants. This should be interpreted as a steady state regime of permanent injection of new particles in the universe, following the suggestion of some authors which try to link the viscosity to the creation of particles mechanism.

From the conservation of energy we obtain

(4) $\dot{\rho} + (\rho+\tilde{p})\ \Theta = 0$

Equations (2)-(4) together with definition (3) constitute precisely a planar autonomous system. This very simple fact seems to be remarked by the first time only recently [(2)]. In order to this system to become equivalent to Einstein's equations we have to add the constraint condition

(5) $\rho - \frac{\Theta^2}{3} - \frac{3K}{A^2} = 0$

in which K is a constant that assumes the values 0, +1 or -1 depending on the function $\sigma(\chi)$.

The main consequence of the reduction of Einstein's equations to an autonomous planar system is the possibility of submitting such system to a qualitative investigation of the behaviour of the whole set of solutions without a complete knowledge of the analytical expression of a particular solution. This introduced great simplification and allows an investigation on such properties like the behaviour of solutions near singular points or on the stability, which should be hardly done by other means.

Belinski and Khalatnikov have examined qualitatively such system in case the viscous term is a linear function of the expansion. For the quadratic dependence, which will be the case discussed here, there are modifications on the phase plane (Θ, ρ) corresponding to behaviour of the universe which are not allowed to occur in the linear case.

We write equations (2)-(4) under the form

$$(\tilde{2}) \qquad \dot{\Theta} = P(\Theta,\rho)$$

$$(\tilde{4}) \qquad \dot{\rho} = L(\Theta,\rho)$$

and set $\tilde{p} = p - \alpha\Theta - \beta\Theta^2$. We obtain

$$(6a) \qquad P(\Theta,\rho) = -\frac{\rho}{2} - \frac{3}{2}\,p - \frac{\Theta^2}{3} + \frac{3\alpha}{2}\,\Theta + \frac{3}{2}\,\beta\,\Theta^2$$

$$(6b) \qquad L(\Theta,\rho) = -(\rho+p)\Theta + \alpha\Theta^2 + \beta\Theta^3$$

The singular points of the system are given by those values of Θ_o and ρ_o, in the phase plane, which annihilate simultaneously the right hand side of equations $(\tilde{2})$ and $(\tilde{4})$. We see imediately that there are only two singular points:

(i) The origin $0(0,0)$

(ii) The point $M(\Theta_o,\rho_o) = \left(\dfrac{-3\alpha}{3\beta-\gamma}\,,\ \dfrac{\Theta_o^2}{3}\right)$

in which we have set the equation of state $p = (\gamma-1)\rho$ with $1 \leq \gamma \leq 2$.

Then we examine the behaviour of the functions $P(\Theta,\rho)$ and $L(\Theta,\rho)$ in the neighborhood of the singular points.

The inportant elements of the analysis are given by the value of the determinant of the linear part of the expansion Δ and the trace σ of the matrix $\hat{\Delta}$:

$$\hat{\Delta}_o = \begin{pmatrix} \frac{\partial P}{\partial \Theta} & \frac{\partial P}{\partial \rho} \\ \frac{\partial L}{\partial \Theta} & \frac{\partial L}{\partial \rho} \end{pmatrix}$$

(at the singular point)

At the point B, we have

$$\Delta_B = \frac{3\alpha^2}{\gamma - 3\beta}$$

$$\sigma_B = \frac{\alpha}{3\beta - \gamma} \left(2 + \frac{3}{2}\gamma - \frac{9}{2}\beta\right)$$

Following [12] we conclude that point B is a saddle for the system if $\gamma < 3\beta$; and it is a two-tangent node if $\gamma > 3_\beta$. If $\beta = \frac{\gamma}{3} - \frac{4}{9}$ then B is a one-tangent node. The stability of the solution near the node can be known by simple inspection on the sign of the trace. For $\Theta_o > 0$, that is, $\gamma > 3\beta$ the node is stable; for $\Theta_o < 0$, the node is unstable. We assume α and β to be positive constants.

The characteristic roots which are the eigenvalue of matrix $\hat{\Delta}_B$ take the values $\lambda_1 = \frac{\Theta_o}{2}(3\beta - \gamma)$ and $\lambda_2 = -\frac{2}{3}\Theta_o$. The investigation of the behaviour of the solution for $t \to \pm\infty$ can be easily made in both cases (see graphs).

The examination of the integral curves near the origin 0 is somewhat more complicate due to the fact that 0 is a non-elementary point (that is, the corresponding determinant Δ_o vanishes identically at the origin). We will not give here all the long and tedious calculations that constitutes the analysis of the system at this point. Instead, we will present only the final results (see the graphs).

Although it is not our purpose here to proceed the analysis to higher than the quadratic dependence of the pressure on the expansion factor, let us make some comments for the case of higher power.

We set

$$\tilde{p} = p - \Psi\,\Theta^{n}$$

for $n > 0$. The singular point of the system, besides the origin, is given by the simultaneous solution of the equations

$$\rho_o = \frac{\Psi}{\gamma}\,\Theta_o^{n}$$

$$\Theta_o^{n-2} = \frac{\gamma}{3\Psi}$$

The determinant of the linearized matrix $\hat{\Delta}$ near the point $M(\Theta_o, \rho_o)$ reduces to

$$\Delta_M = \gamma\,\Theta_o^{2}\left(\frac{2-n}{3}\right)$$

We conclude that for any $n > 2$ point M is a saddle for the autonomous system. For n=2, there is no singularity other than the origin, unless the coefficient of viscosity Ψ and γ are related by the expression $\gamma = 3\Psi$.

Finally, for n=1, M is a node. For the analysis of the origin the whole features are very similar to the quadratic case.

Let us make some comments on these figures. We start by noting that we have drawn the integral curves in the whole domain of ρ , even for $\rho < 0$, although an universe filled with negative total energy in devoid of physical meaning, at least classically.

The character of the singular point B (node or saddle) dependes on the sign of Θ . Such situation does not occur in the linear case, once in that case the singularity can be located only at the first quadrant ($\Theta > 0$). Further, in the steady state linear case, the singular point B can be only a node. This makes a great difference between the linear case and the quadratic one.

Let us comment the graph 1. For the constant K=-1 we can distinguish two general behaviour.

(i) The universe starts at $t=-\infty$ with an infinite radius and negative Θ . The universe constracts from this dilute phase with no energy. Then as the universe constracts a negative energy starts to appear. Its absolute value increases until the contraction attends a minimum. Then the contraction begin desaccelerating and after a while it changes its sign and Θ becomes positive. The energy keeps negative. Now, the universe expands and after a certain (finite) time enters a region of positive energy which increases with the expansion. Finally, after a maximum of Θ is attained, a desacceleration of Θ occurs (although the sign of Θ does not change) until the solution enters the singularity B, in which once again the radius of the universe becomes infinity.

For an observer that see only the classical positive region, the universe starts with positive expansion $q_1, q_2 \ldots$ or q_n and zero energy. Then they follow the way from q_i to B.

In all these models, which represents only part of the integral curves limited by the requeriment of positivity of the energy, the universe starts abruptly with an arbitrary expansion Θ_i and zero energy and ends at B.

ii) The universe starts in the same conditions than in case(i). However after a certain time (finite) the negative energy decreases, until it attains once again the value zero after which it becomes positive. The energy increases and the contraction of the universe accelerates. After a certain finite time the energy attains a maximum and begins to decrease until it vanishes. The curve enters a region of negative energy, after which the behaviour of such universe follows the same lines as in the previous case (i). The universe has a classical meaning ($\rho > 0$) in the region $M\tilde{M}$, $N\tilde{N}$, and so on. A typical behaviour is: it starts at $(\Theta, \rho) = (M, 0)$ and ends, after a finite lapse of time at $(\tilde{M}, 0)$. During this brief period of time it experiences no singularity at all.

Let us turn now to the case in which the constant K=+1. The separatrix OR divides two regions. The region ROB contain solutions in which the Universe starts with an infinite expansion and infinite density. Then the expansion Θ decreases, the energy decreases until a minimum, after which it increases again and finally ends at the singularity B.

The region ROS contains solutions with the same behaviour as closed Friedmann model.

Finally, for K=0 we can have three solutions corresponding to the region OB, BL and OS, the interpretations of which are evident.

The graph 2 does not present any new features.

Graph 3 has a similar behaviour at negative values of Θ , but a different feature for $\Theta > 0$. This is due to the absence of the singular point B. Thus, all curves which ends at B in graphs 1,2 , now go to infinity. There is a region (BOR) with a saddle behaviour (actually, the origin is a saddle node). This region represents universes which starts with $(\Theta,\rho)=(\infty,\infty)$, the expansion decreases together with the density, attains a minimum and starts increasing again without limit.

Finally, graph 4 contains a combination of these previous models.

III. Quadratic Regime of Viscous Fluid (non-stationary case)

Let us now discuss a more realistic model of the viscous fluid by allowing the coefficients α and β to become functions of the total energy ρ . In order to examine the effects of the quadratic dependence without contamination of the linear factor, we set $\alpha=0$.

Assuming a power law dependence $\beta=M\rho^{\mu}$ (M and μ are constants) as in [2], we write

$$P(\Theta,\rho) = \rho - \frac{\Theta^2}{3} + \frac{3}{2} M \rho^{\mu} \Theta^2 - \frac{3}{2}\gamma\rho$$

$$L(\Theta,\rho) = M\Theta^3\rho^{\mu} - \gamma\rho\Theta$$

The singular points now (besides the origin) are doubled, appearing symmetrically with respect to an inversion of Θ. We will call these symmetric singular points $B_{(+)}$ (for $\Theta > 0$) and $B_{(-)}$ (for $\Theta < 0$).

They are given by the conditions

$$\rho_o = \frac{\Theta_o^2}{3}$$

$$3M\rho_o^{\mu} = \gamma \qquad .$$

Developping $P(\Theta,\rho)$ and $L(\Theta,\rho)$ in the neighborhood of these points we obtain

$$\begin{pmatrix} P(\Theta,\rho) \\ \\ L(\Theta,\rho) \end{pmatrix} \simeq \begin{pmatrix} -\frac{3}{2}\Theta_o + \gamma\Theta_o & 1 + \frac{3}{2}\gamma(\mu-1) \\ \\ \frac{2}{3}\gamma\Theta_o^2 & \gamma\Theta_o(\mu-1) \end{pmatrix} \begin{pmatrix} \Theta \\ \\ \rho \end{pmatrix} +$$

$$+ \text{ higher powers of } \Theta,\rho \text{ .}$$

Thus, the determinant Δ of the linear part is given by

$$\Delta = -\frac{2\gamma}{3}\mu\,\Theta_o^2$$

and its trace σ :

$$\sigma = \Theta_o\left(\gamma\mu - \frac{2}{3}\right)$$

Thus, we obtain the results:

If $\mu > 0$ then point B is a saddle

If $\mu < 0$ and $4\Delta - \sigma^2 < 0$, point B is a two-tangent node.

If $\mu = -\frac{2}{3\gamma}$, point B is a one-tangent node.

Furthermore, if $\Theta_o > 0$ the node is stable and if $\Theta_o < 0$ the node is unstable.

Let us make some comments on these figures. For K=-1 , there are models which start at $B^{(-)}$ and ends at $\tilde{\Theta}_i$. They represents universes which starts with infinite radius and finite energy. It contracts until the energy annihilate at $\tilde{\Theta}_i$. If we follow this integral curve into the negative region $\rho < 0$, then we see that the model into the singular point at the origin with zero expansion, and zero energy. A symmetric situation occurs for curves going into B(+) with K=-1.

In the figure 8 the eliptic sectors characterizes the integral curves of our system for $\rho < 0$. These curves represents unphysical configuration of universes which start at t= $-\infty$ with zero expansion and zero energy. The universe has at its begining an infinite radius and enters an accelerating era until it attains the epoch of maximum contraction, after which the contraction becomes desaccelerated. By the middle of its life, it enters a region of expansion and keeps expanding (with encreasing Θ) until it arrives at a maximum value Θ_{max}. After that, its expansion starts to be desaccelerating until it comes back to the original state ($\Theta=\rho=0$).

Let us now turn to the physical region ($\rho > 0$). The behaviour of the integral curves for expanding universes in the quadratic viscous regime has almost the same features as in the linear case. The singular point $B_{(+)}$ is a saddle which distinguish four region of distinct behaviour:

region I from $(\Theta,\rho)=(+\infty, 0)$ to $(0,0)$;

region II from $(+\infty,0)$ to $(+\infty, +\infty)$;

region III from $(0,+\infty)$ to $(+\infty, +\infty)$;

region IV from $(0,+\infty)$ to values of negative Θ.

All these regions are equally presented in the linear case and has been discussed previously by Belinski and Khalatnikov. Let us turn to the case of negative Θ. Here the situation changes drastically. The existence of a new singular point $B_{(-)}$ which turns to be a saddle (fig.8) introduces an infinite barrier represented by the separatrix $A\ B_{(-)}M$. Thus contrary to the linear case in which any curve which pass through points near the origin of the ρ axis goes to $(-\infty,+\infty)$ here, in the quadratic case, due to the existence of the boundary $AB_{(-)}M$ these curves can only end with an infinite contraction and vanishing total energy. This represents universes which starts with zero expansion, zero energy and infinite radius $(A \to \infty)$. After that, the energy increases, attains a maximum (near the saddle point $B_{(-)}$) and diminishes indefinitely. The curves from region IV , crossing the ρ-axis, go just near $B_{(-)}$ and then are repelled by the saddle. Such models represents a cosmos that starts from a highly condensed phase $(A \to 0)$ with an infinite energy. Then as the universe expands (slowly) the energy decreases, until a minimum value ρ_{min} (different for each model). Beyond that point the sign of the Θ function changes: the universe enters a contracting era and keeps contracting forever, increasing indefinitely the value of the energy.

IV. Qualitative Analysis of Rotating Homogeneous Universes

Recently, Novello and Rebouças (NR) [11] presented a cosmological model which has expansion, shear and rotation. Such solution can describe a previous era of Gödel's cosmos in which the galactic fluid was not yet thermalized, allowing for heat exchange among its parts.

The fundamental form of the geometry is given by

$$ds^2 = dt^2 + 2A(x,t)\,dy\,dt + \frac{1}{2}\,A^2(x,t)\,dy^2 - H^2(t)\,dz^2 - F^2(t)\,dx^2$$

Novello and Rebouças geometry is obtained by setting

H = constant

F = constant

$A(x,t) = A_0\, e^{cx}(\Theta_0\, t + 1)$

in which A_0, c and Θ_0 are constants which specifies the amount of total (heat) energy.

The energy-momentum tensor in the comoving system is given by

$$T_{\mu\nu} = \rho\, V_\mu V_\nu - p(g_{\mu\nu} - V_\mu V_\nu) + q_\mu V_\nu + q_\nu V_\mu$$

q_μ is the (four-vector) heat flux.

In the Novello-Rebouças solution the heat flux q^α rests in the plane orthogonal to the vorticity vector (actually, q^μ which is a space-like vector orthogonal to $V^\alpha = \delta^\alpha{}_0$, constitutes jointly with the vorticity ω^μ and the acceleration a^μ a triad in which each vector is orthogonal to the others).

Einstein's equations impose

$$\rho = \frac{-2}{1-\varepsilon}\Lambda$$

$$\Lambda = -\frac{c^2}{2}\frac{1-\varepsilon}{1+\varepsilon}$$

The model has an equation of state $p=\varepsilon\rho$ and Λ is the cosmological constant. The total amount of heat flux is given by $L = L_0(\Theta_0 t + 1)^{-2}$. We will now perturb such solution by letting F to be a non-constant function of t.

Einstein's equations $T^1_1 = R^1_1 - \frac{1}{2}R + \Lambda\,\delta^1_1$ and $R^2_2 = R^1_1$ are no more automatically satisfied but instead they give

$$\frac{\ddot{F}}{F} + \frac{\dot{A}}{A}\frac{\dot{F}}{F} - \frac{\dot{F}^2}{F^2} = 0$$

$$\frac{\ddot{A}}{A} - \frac{\ddot{F}}{F} + \frac{5\dot{F}^2}{F^2} - \frac{\dot{A}}{A}\frac{\dot{F}}{F} = 0$$

in which now $A(x,t) = e^{cx}A(t)$.

The remaining set of Einstein's equations gives the same relations as before and fix the relation between the density of energy and Λ.

Define two new functions ϕ and χ by setting:

$$\phi \equiv \frac{\dot{F}}{F}$$

$$\chi \equiv \frac{\dot{A}}{A}$$

We then obtain

$$\dot{\phi} = - \phi \chi$$

$$\dot{\chi} = - \chi^2 - 4 \phi^2$$

Thus, the system of Einstein equations are reduced to a homogeneous (second order) planar autonomous system. Let us now concentrate our attention in the exam of this system. We note that there is only one singular point: the origin $(\phi,\chi)=(0,0)$. The fact that the system is homogeneous makes easy the drawn of the integral curves of the system (see figure 9).

Let us make some remarks on this graph. Point (0,0) represents Gödel's stationary solution. Thus there are only two integral curves which attain effectively Gödel era. Both solutions have F=constant, and they are distinguished only by the sign of the expansion factor. These two solutions are precisely the cases presented in the Novello-Rebouças solution. If the universe is expanding, the solution tends to Gödel cosmos in the future infinite; if the universe is contracting the solution started from Gödel's model in the infinite past. All others solutions (F≠constant) can become very near Gödel's but cannot attain really a stationary era.

V. CONCLUSION

The purpose of the present paper is to make use of the method of qualitative analysis of planar autonomous system of differential equation, in order to investigate some homogeneous cosmological models. We have discussed in section II and III the case of homogeneous and isotropic cosmos filled with a stokesian fluid in a quadratic regime. We have shown how the quadratic term can deviates the configurations of the cosmos from the usual models, in some cases very drastically. We have presented these new features in a self explanatory series of graphs. Then we have turned our discussion to rotating cosmos and compared the configuration of these models with Gödel's solution. We show that Novello-Rebouças solution is the unique solution (with heat flux) which admits Gödel's cosmos as an assimptotic era.

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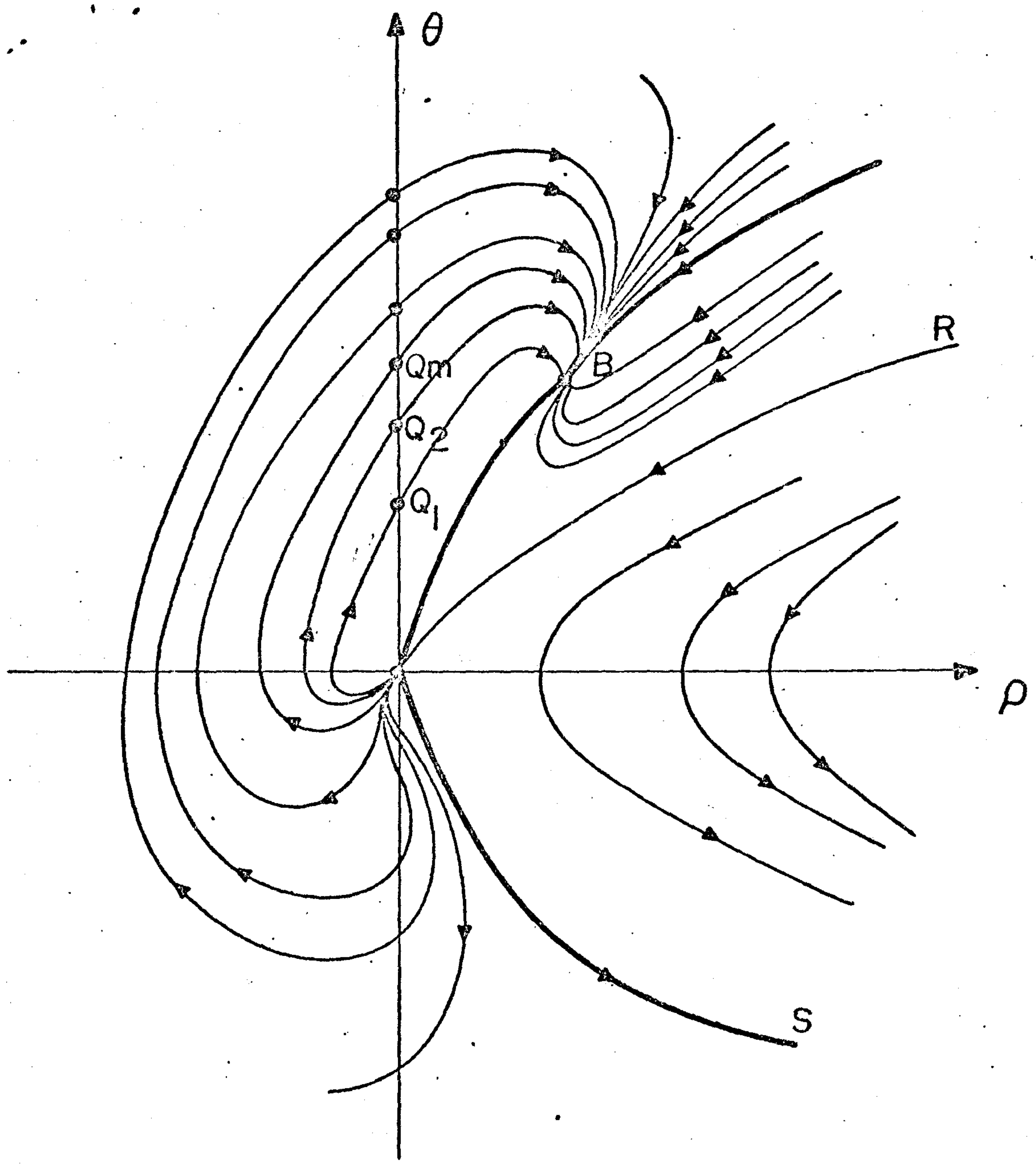

Fig.1 - Case in which α and β are constants. Point B is a two-tangent node. The curve is draw for $\frac{\gamma}{3} - \frac{4}{9} < \beta < \frac{\gamma}{3}$.

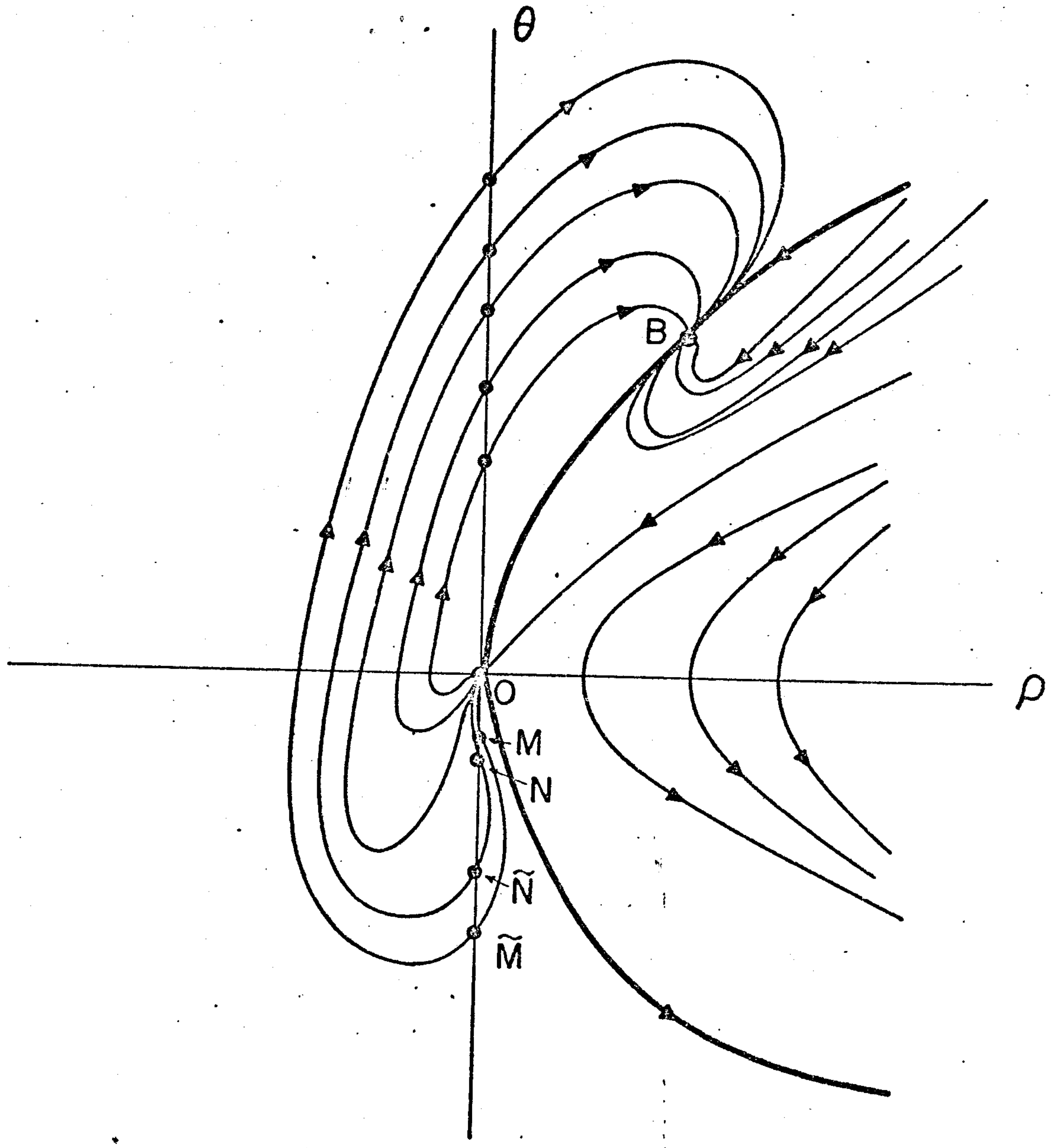

Fig.2 - α and β are constants. Point B is a one tangent node.
$\beta = \frac{\gamma}{3} - \frac{4}{9}$.

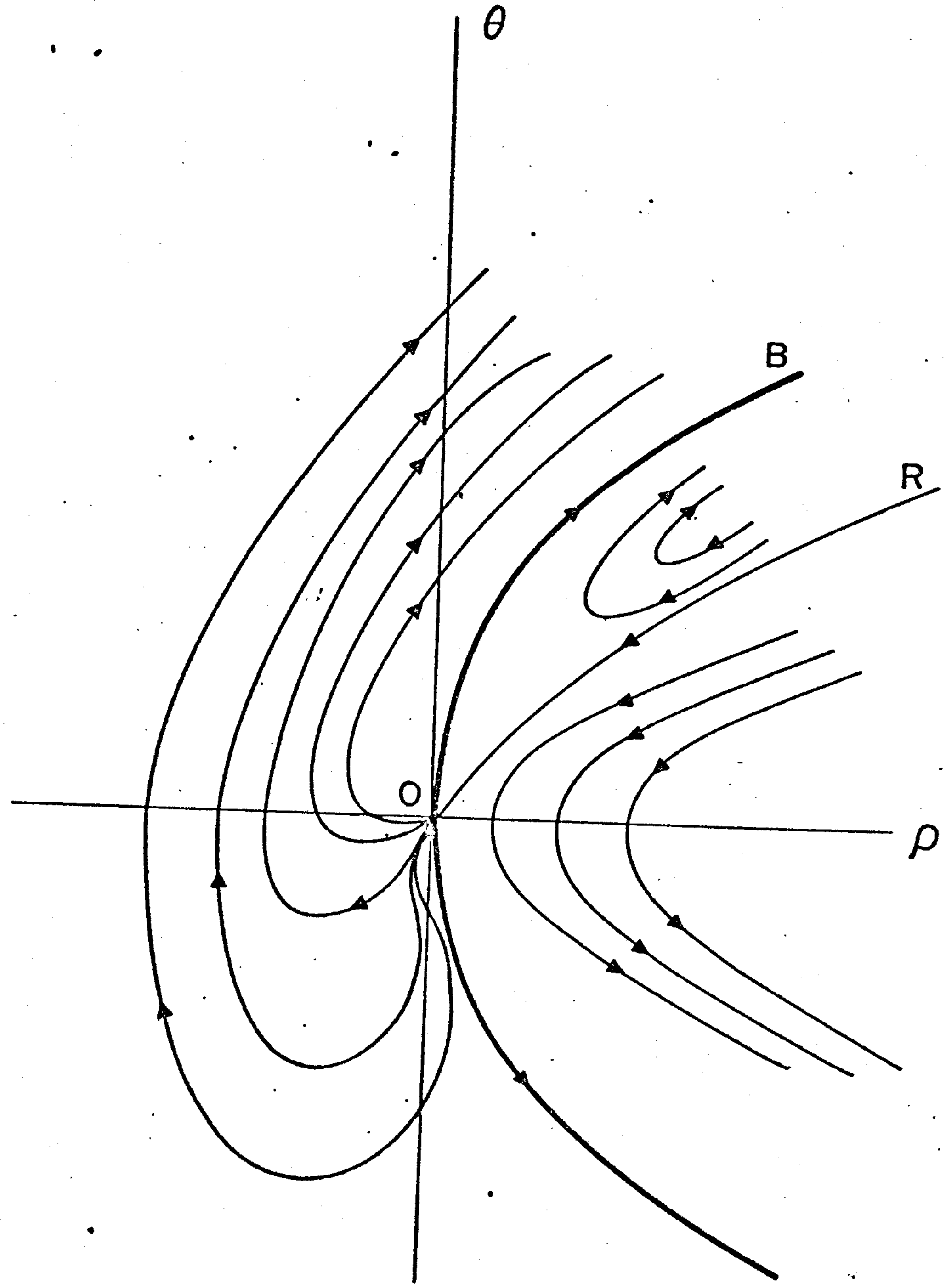

Fig.3 - α and β are constants. The case in which $\beta=\gamma/3$ there is only one singular point at the origin 0.

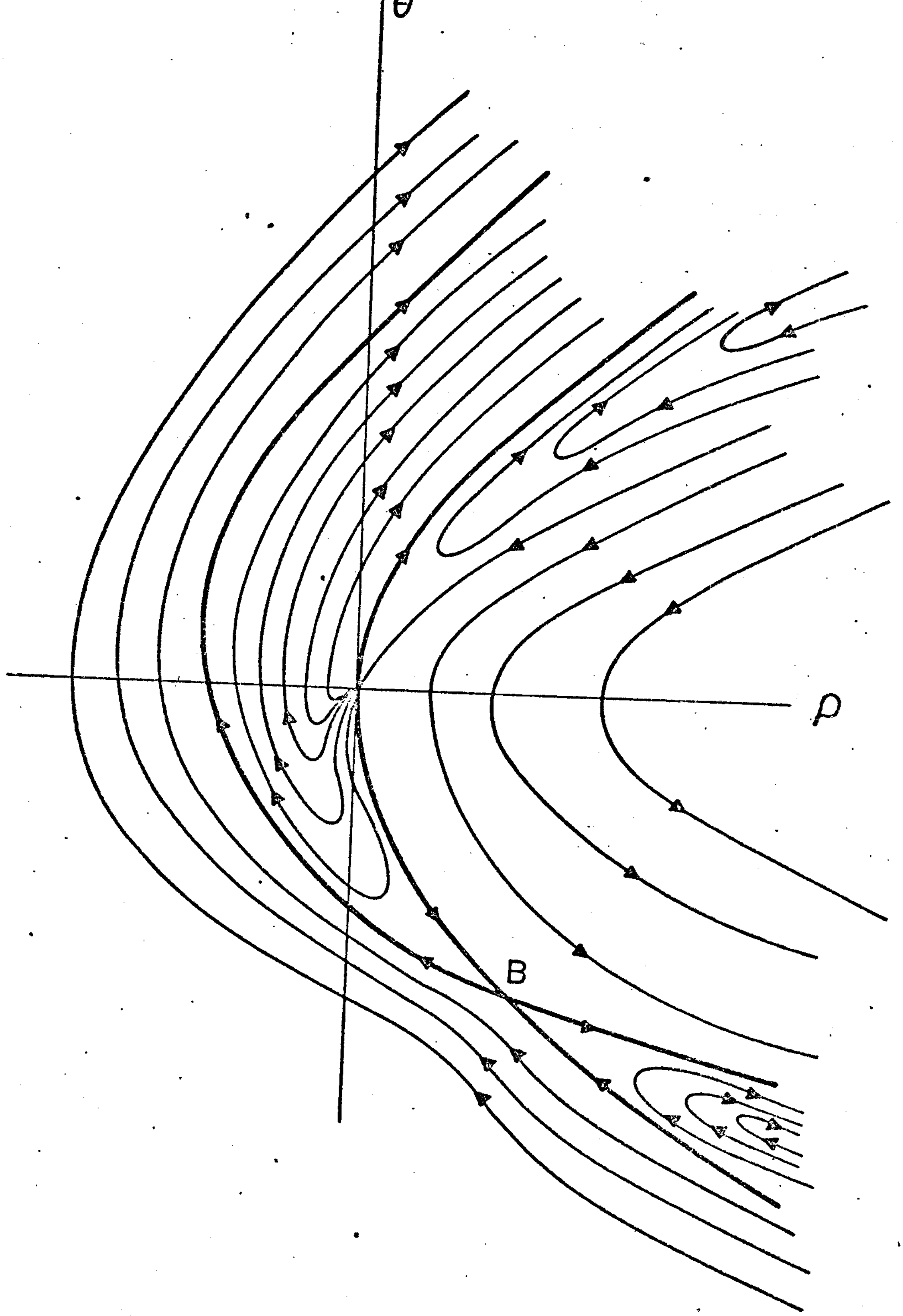

Fig.4 – α and β are constants. Point B is a saddle. This case occurs for $\beta > \gamma/3$.

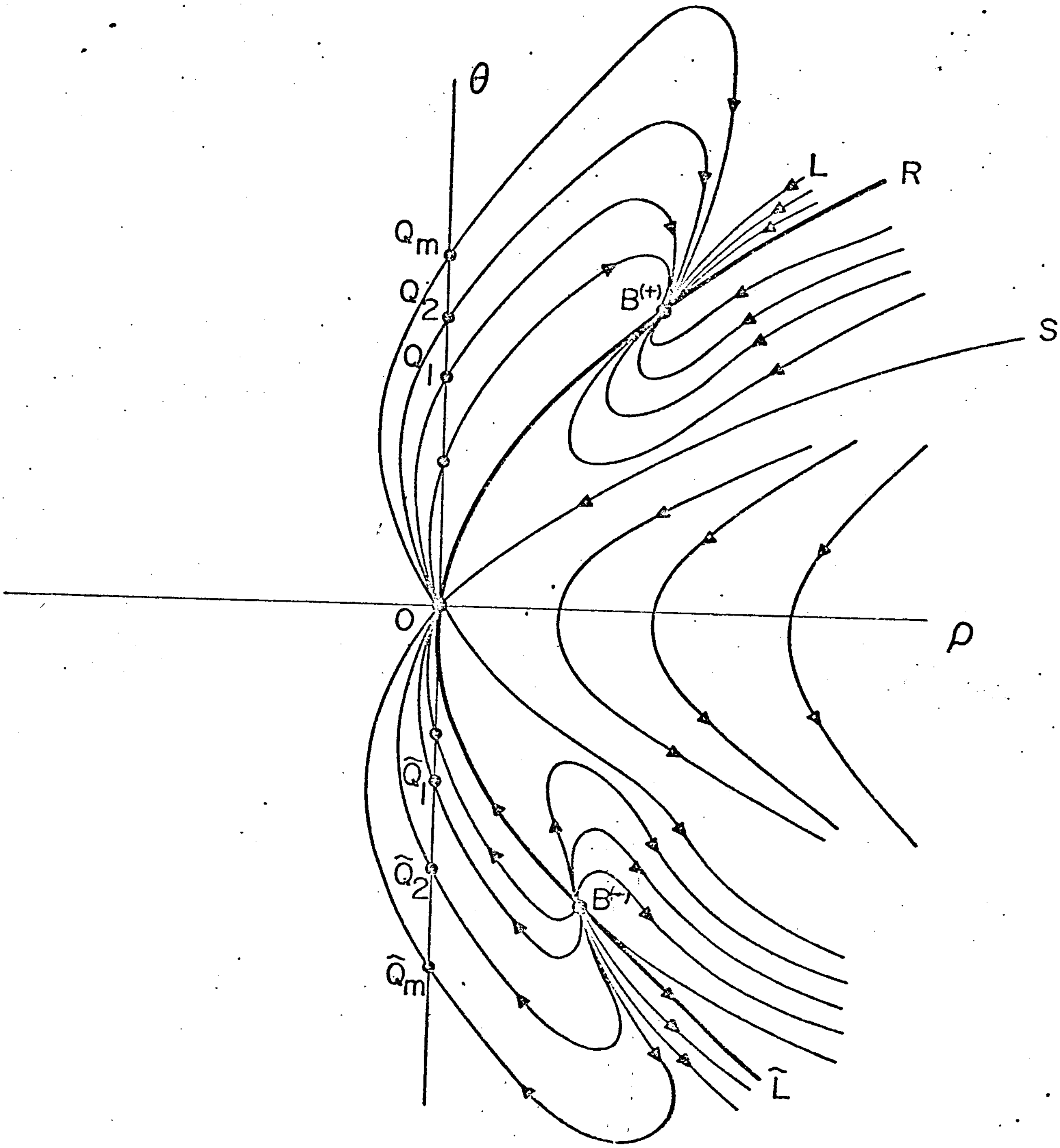

Fig.5 – $\tilde{p}=p-M\ \rho^{\mu}\ \theta^2$, M and μ are constants. The figure is draw for the case in which $-\frac{2}{3\gamma}<\mu<0$. $B_{\pm}$ are two-tangent nodes.

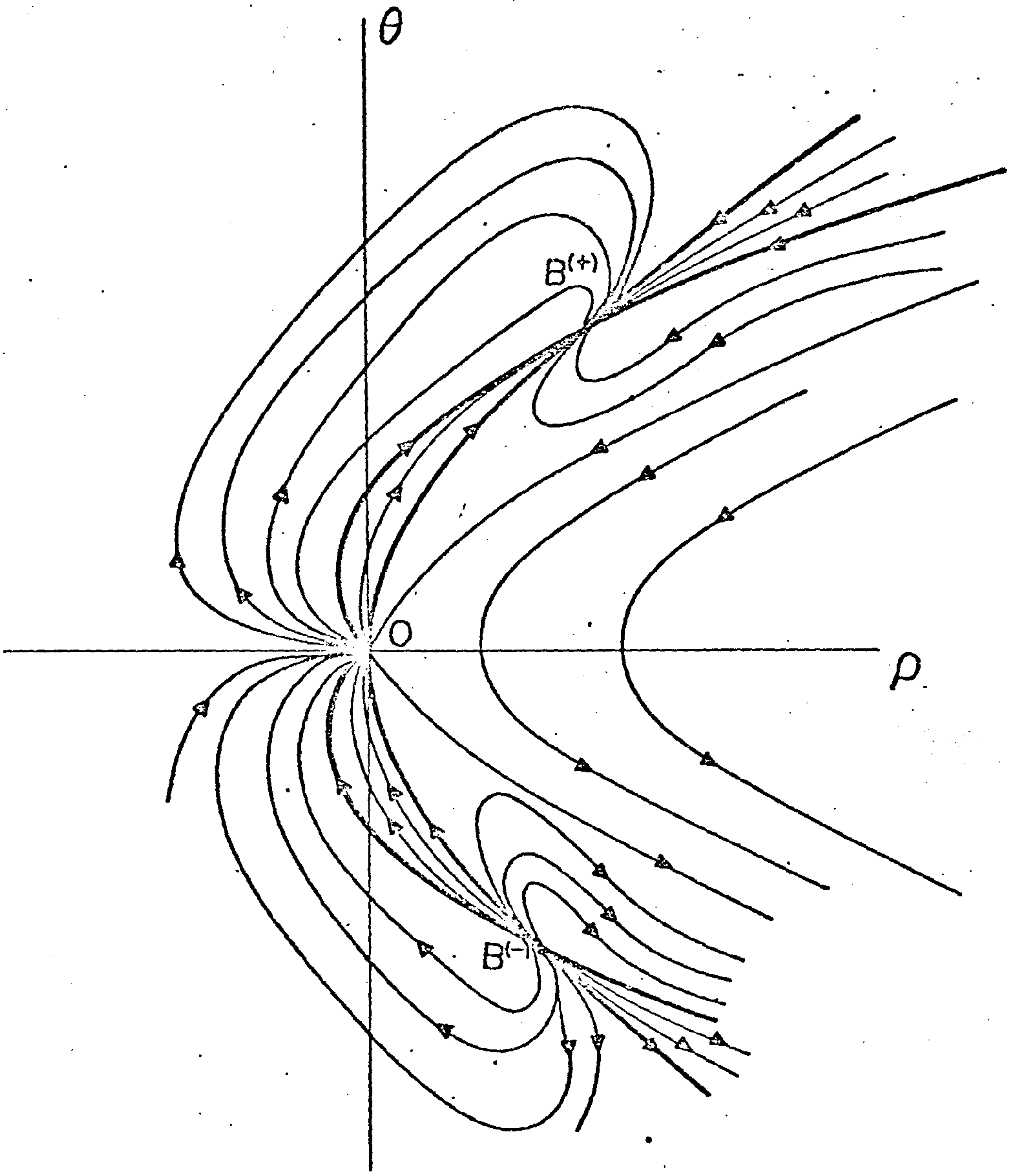

Fig.6 – $\tilde{p}=p - M\,\rho^{\mu}\,\theta^{2}$. M and μ are constants. The figure shows the case in which $\mu < -\frac{2}{3\gamma}$. $B_{\pm}$ are two-tangent nodes

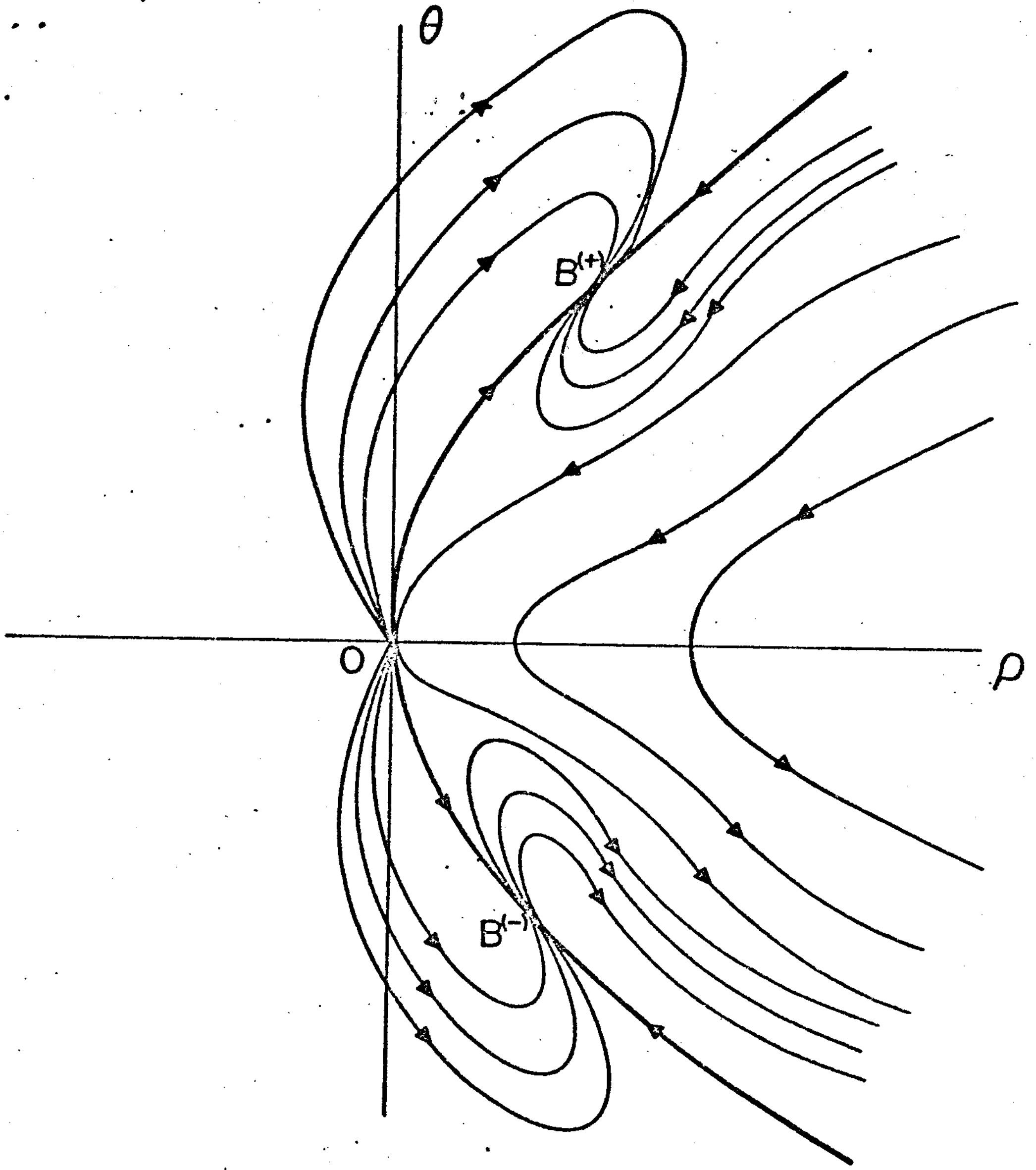

Fig. 7 – $\tilde{p}=p - M\rho^{\mu}\Theta^2$; M and μ are constants. The figure shows the case $\mu = -\frac{2}{3\gamma}$. $B_{\pm}$ are one-tangent nodes.

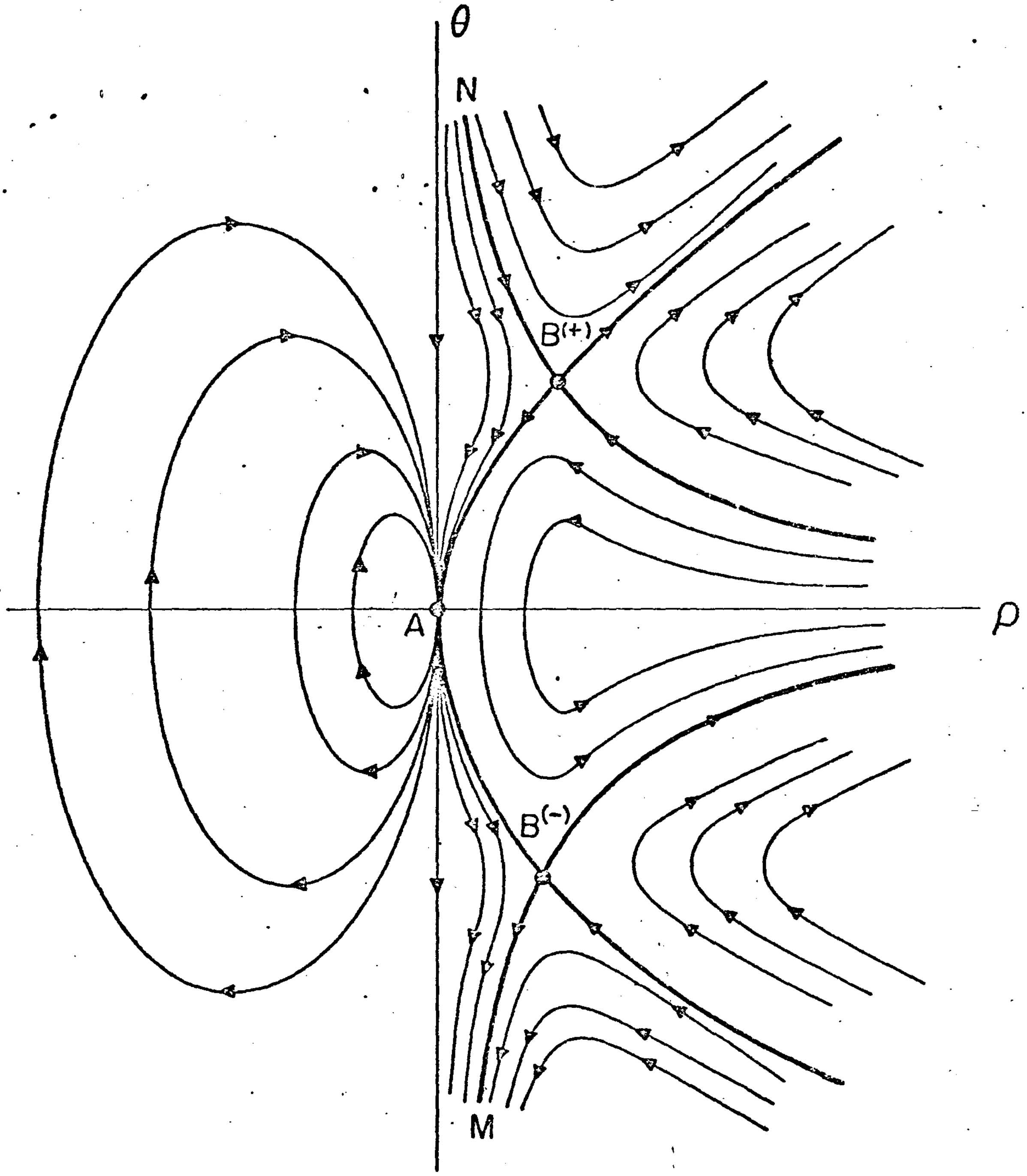

Fig. 8 - Case $\tilde{p}=p - M\rho^{\mu}\theta^{2}$; M and μ are constants, with $\mu > 1$.
Points $B_{\pm}$ are saddle.

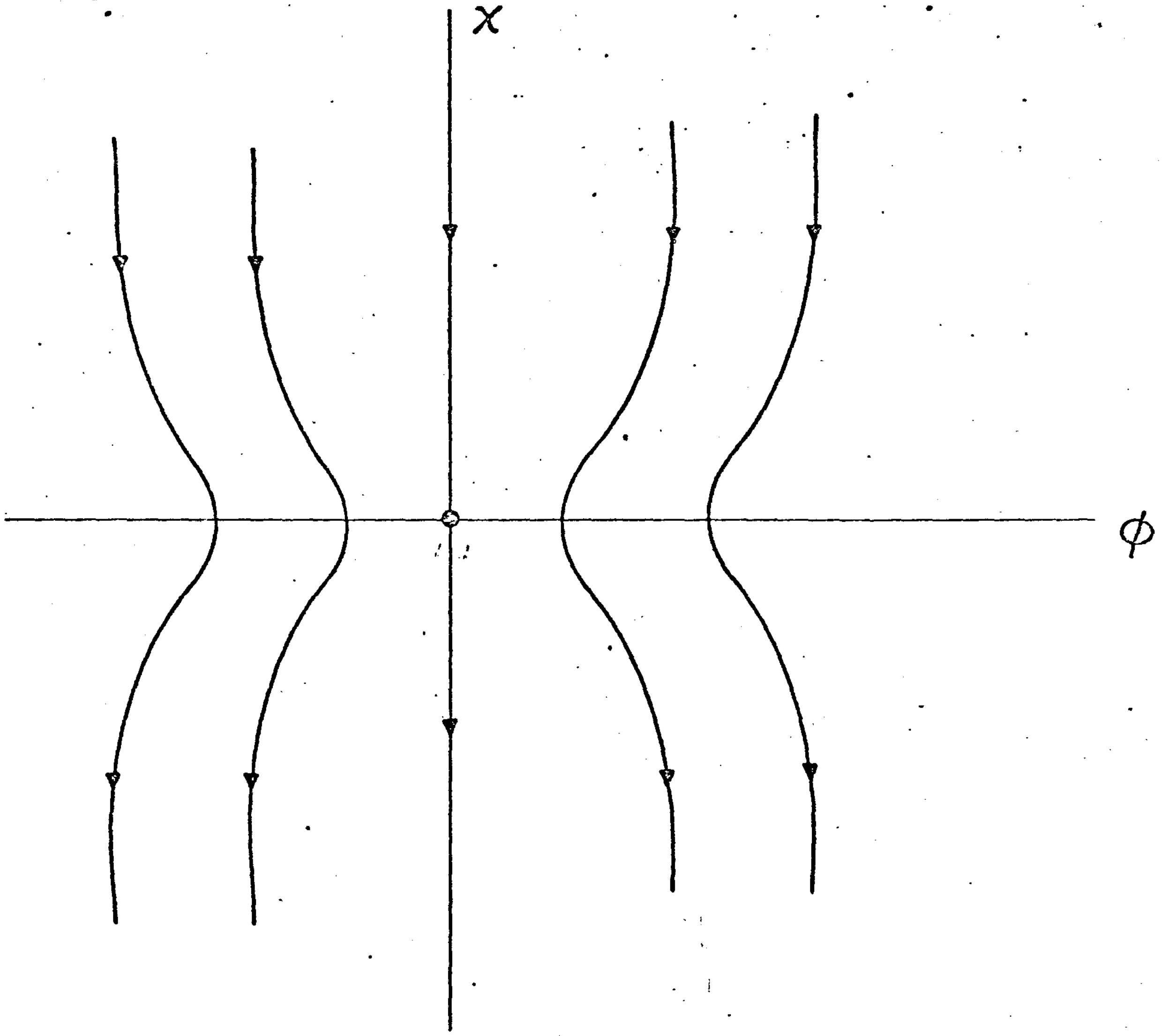

Fig. 9 - Qualitative analysis of rotating cosmological models.

CAPTION FOR FIGURES AND TABLE

Fig. 1 Phase diagram of the bond-dilute Z(4) model in square lattice (the point E is here located according to the results obtained through the present approximations; it is however possible that the exact p_0 equals 1/2). B (I_1, I_2 and I_3) is (are) the pure Potts (Ising) critical point(s); the line BD (I_1G and I_2D) corresponds to bond-dilute Potts (Ising) model(s). P, F and I denote the para-, ferromagnetic and intermediate phases.

Fig. 2 Fixed p sections of the phase diagram of Fig. 1. (a) p=1; (b) p=0.8; (c) p=0.7; (d) p=0.6; (e) p =0.53. The line BD corresponds to the bond-dilute Potts model.

Fig. 3 Fixed K_2^0/K_1^0 ratio sections of the phase diagram of the bond-dilute Z(4) model in square lattice. (a) $K_2^0/K_1^0=0.5$ (Potts); (b) $K_2^0/K_1^0=0.3$; (c) $K_2^0/K_1^0=0.25$; (d) $K_2^0/K_1^0=0$ (Ising); (e) $K_2^0/K_1^0=-0.3$, P and F denote the para- and ferromagnetic phases.

Table 1 Relevant quantities (calculated through the t-, τ - and σ- conjectures) associated to the phase diagram represented in Fig. 1 (where the point E is located at $p = p_0$). See the text for the values followed by (?). (a) Wu and Lin 1974; (b) Sykes and Essam 1963; (c) Baxter 1973; (d) Southern and Thorpe 1979; (e) Kramers and Wannier 1941; (f) Domany 1978; (g) Harris 1974.