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ABSTRACT - We discuss the construction of field strength copies without any gauge constraint. We give several examples, one of which is not only a field strength copy but also (at the same time) a "current copy".

The question of the existence of two or more potentials not connected by any gauge transformation giving rise to the same field strength, has lately received some attention. First, some examples were given [1,2] on those so called "field strength copies". Later, a necessary condition on a field for the existence of copies was also examined [2,3,4,5] and the known families of copies were also extended [6].

In a recent paper M. B. Halpern [7] discusses the general problem of the construction of all finite action gauge copies which can be simultaneously represented in the temporal gauge ($A_0=0$). Copies not belonging simultaneously to this gauge are seen from it only as "action copies" losing their property of having the same field(see reference [7]). He also stresses the importance of copies for "enhacement" of the corresponding action contribution to path integrals, over the non copied contributions.

In this paper we propose to pose and discuss the problem in a general way, so that we will be conceptually able to find all copies regardless of gauge conditions or finiteness of action. To give up the gauge condition is equivalent to find in the temporal gauge all potentials satisfying:

$$F_{\mu\nu}(A) = U^{-1} F_{\mu\nu} (\overline{A}) U$$

with U arbitrary element of the gauge group. All pairs A_μ, $\overline{A}_\mu (A_0 = \overline{A}_0 = 0)$ satisfying that equation are action copies in the temporal gauge. For the particular case $U \equiv 1$ they become field strength copies (in the same gauge).

Some actual solutions will be given, including as particular cases, all examples known to us.

For the potentials $A_\mu^{(1)}$ and $A_\mu^{(2)}$ to have the same field, they must satisfy [7]:

$$F_{\mu\nu}(\Delta) + \left[A_\mu^{(1)}, \Delta_\nu\right] - \left[A_\nu^{(1)}, \Delta_\mu\right] = 0 \qquad (1)$$

where

$$\Delta_\mu = A_\mu^{(2)} - A_\mu^{(1)} \qquad (2)$$

Instead of characterizing the copies by $A_\mu^{(1)}$ and $A_\mu^{(2)}$ we can equivalently take Δ_μ (eq.(2)) and

$$A_\mu = \frac{1}{2}\left(A_\mu^{(1)} + A_\mu^{(2)}\right) = A_\mu^{(1)} + \frac{1}{2}\Delta_\mu \qquad (3)$$

we have:

$$\Delta_{\nu,\mu} - \Delta_{\mu,\nu} + \left[A_\mu, \Delta_\nu\right] - \left[A_\nu, \Delta_\mu\right] = 0; \ (\Delta_{\nu,\mu} = \partial_\mu \Delta_\nu) \qquad (4)$$

We note first some general properties of eq. (4). I) If Δ_μ and A_μ are solutions, then $c\,\Delta_\mu$ and A_μ are also solutions (c=arbitrary constant). II) If $\Delta_\mu = \partial_\mu \Delta$, then $A_\mu = 0$ is a solution implying $A_\mu^{(1)} = -A_\mu^{(2)}$ (this is example 3 of reference [2]). III) When $\Delta_\mu = U^{-1}\partial_\mu U$ (pure gauge), then $A_\mu = \frac{1}{2}\Delta_\mu$ is a (trivial) solution, but from point I, we get the new solution $\Delta_\mu = c\,U^{-1}\partial_\mu U$, (quasi pure gauge [8]) $A_\mu = \frac{1}{2}U^{-1}\partial_\mu U$, which is example 1 of reference [2]. Example 2 *of the same reference is a gauge transform of this case (see* [6] *). IV) In SU(2) , when* $U = \sigma \equiv \underset{\sim}{\sigma} \cdot \underset{\sim}{n}(x)$, *the solution obtained in III is not a true copy* ($A^1 + \Delta$ *is a gauge transform of* A^1[6]*). In this case it is not difficult to see that a solution is obtained by taking* $A^{(1)}_\mu = \alpha\, \sigma\, \sigma_{,\mu}$ *and* $\Delta_\mu = \frac{1}{2}\, \sigma\, \sigma_{,\mu} + \sigma\, a_\mu$, *with* $a_{\nu,\mu} - a_{\mu,\nu} = -\left(\frac{1}{2} - \alpha\right)^2 \sigma(\sigma_{,\mu}\, \sigma_{,\nu} - \sigma_{,\nu}\, \sigma_{,\mu})$ *(see reference* [6]*). The example of reference* [1] *is a particular case for* $\underset{\sim}{n} = \underset{\sim}{r}/r$.

Now we come to the general method.

The space components of (4) are:

$$\Delta_{j,i} - \Delta_{i,j} + [A_i, \Delta_j] - [A_j, \Delta_i] = 0 \qquad (5)$$

which can be written as:

$$B^a{}_k + \Delta^{ab}_{kj}\; A^b_j = 0 \qquad (6)$$

where

$$B^a{}_k = \epsilon_{ijk}\; \Delta^a_{j,i} \qquad (7)$$

and

$$\Delta^{ab}_{ki} = f^{abc}\, \epsilon_{ijk}\, \Delta^c{}_j \qquad (8)$$

(f^{abc} *are the structure constants of the gauge group).*

The other components of (4) can be written in the form:

$$\partial_0 \Delta^a_i = \partial_i \Delta^a_0 + f^{abc} (A^b_i \Delta^c_0 - A^b_0 \Delta^c_i) \tag{9}$$

(6) and (9) are the fundamental equations. They allow us to develope a constructive method for the solution of our problem. We consider (6) as an algebraic equation for the determination of A^a_i and (9) as an evolution equation. In fact, given the arbitrary functions $\Delta^a_i(\underset{\sim}{r}\ ;\ t_0)$ as an initial datae for the time t_0, and also the arbitrary functions $\Delta^a_0(\underset{\sim}{r}\ ,\ t)$, $A^a_0(\underset{\sim}{r}\ ,\ t)$, for all time t, then it is easy to show, by using repeatedly (6) and (9) that one can compute (in principle) all derivatives of Δ^a_i and A^a_i at $t=t_0$, thus allowing the construction of $\Delta^a_i(\underset{\sim}{r}\ ,\ t)$ and $A^a_i\ (\underset{\sim}{r}\ ,\ t)$. In particular if one choses $\Delta^a_0 \equiv A^a_0 = 0$, one obtains from (9), $\partial_0\ \Delta^a_i = 0$; i.e.: $\Delta^a_i\ (\underset{\sim}{r}\ ,\ t) = \Delta^a_i\ (\underset{\sim}{r}\ ,\ t_0)$ and of course $B^a_k(r,t)=B^a_k(r,t_0)$ Now, when $\mathcal{D}$ = Det $\Delta^{ab}_{ki} \neq 0$, eq. (6) has a unique solution A^b_j <u>independent</u> of time. Then it is either trivial (pure gauge) or the action can not be finite. So, when a finite action solution is wanted the determinant $\mathcal{D}$ must be zero. Then Halpern results follow [7] (in particular Δ_i is a pure gauge). Inversely, when Δ_μ is a pure gauge in SU(2), a coordinate system can be found in which $\Delta_0=0$, then if $\mathcal{D} \neq 0$ the solution found in point III is unique (infinite action). If $\mathcal{D}=0$ there exists a functional relation between the three parameters of the group. Then only two (or one) of them can be taken as independent: $U \equiv U(\alpha^1, \alpha^2)$, and

$$\Delta_i = U^{-1}\left\{ \frac{\partial U}{\partial \alpha^1}\ \alpha^1_{,i} + \frac{\partial U}{\partial \alpha^2}\ \alpha^2_{,i} \right\}$$

$$\Delta_i = M_{(1)}\ \alpha^{(1)}_{,i} + M_{(2)}\ \alpha^{(2)}_{,i} \tag{10}$$

(with obvious notation).

The matrix (8) is then (for SU(2)):

$$\Delta^{ab}_{ki} = \varepsilon^{abc} \quad \varepsilon_{ijk} \; (M^{c}_{(1)} \; \alpha^{(1)}_{,i} + M^{c}_{(2)} \; \alpha^{(2)}_{,i} \;) \tag{11}$$

As $\mathcal{D}=0$, *the solution of (6) are the zero eigenfunctions of (11), namely:*

$$M^{a}_{(1)} \; \alpha^{(2)}_{,k} \; ; \; M^{a}_{(2)} \; \alpha^{(1)}_{,k} \; ; \; M^{a}_{(1)} \; \alpha^{(1)}_{k} - M^{a}_{(2)} \alpha^{(2)}_{,k} \tag{12}$$

The general zero eigenfunction is given by a linear combination of these solutions, the three coefficients being arbitrary functions. For example, when $\Delta_{i} = \sigma \; \partial_{i}\sigma$ $(\sigma = \underset{\sim}{\sigma} . \underset{\sim}{n}\; ;\; \underset{\sim}{n}^{2} = 1)$, *the three zero eigenfunctions can be written as(not summed!)* $\sigma^{j} \; n^{j}_{,i} - \sigma \; n^{j} \; n^{j}_{,i}$ $(j=1,2,3)$. *The family of point III takes then the form* $(\Delta = c \; \sigma \; \partial_{i} \; \sigma\;)$

$A_{i} = \frac{1}{2} \sigma \; \sigma_{,i} + \sum_{j} (\sigma^{j} \; \phi^{j} \; n^{j}_{,i} - \sigma \, n^{j} \; \phi^{j} n^{j}_{,i})$ $(\phi^{j}$ *are three arbitrary functions).*

When U depends on only one function, $U \equiv U\;(\alpha)$, *eq. (10) reduces to*

$$\Delta_{i} = M \; \alpha_{,i} \tag{13}$$

Then the matrix (8) has the zero eigenfunctions:

$$M^{a} \; a_{i} \quad ; \quad N^{a} \quad \alpha_{,i} \tag{14}$$

$(a_{i}$ *= arbitrary vector field;* N^{a} *arbitrary functions)*
The general zero eigenfunction is a combination of the functions in (14)

It can be directly verified that the four dimensional form of (10) , (12) and (13) , (14) are solutions of equation (1). The latter solution, which we write in the form:

$$A_\mu^{(1)} = a_\mu(x)\ M\ (\alpha\ (x)\ ) + \alpha_{,\mu}(x)\ N(x) \tag{15}$$

has some interesting properties:

1^{st}: For each choice $M(\alpha)$, $\alpha(x)$, $a_\mu(x)$ and $N(x)$, the potential (15) has an infinite set of copies given by:

$$\bar{A}_\mu^{(2)} = (a_\mu + f(\alpha)\ \alpha_{,\mu}\ )\ M + \alpha_{,\mu}\ N \tag{16}$$

where $f(\alpha)$ is an arbitrary function.

2^{nd}: If a_μ is chosen as a fixed vector, $a_\mu \equiv n_\mu$; $\alpha \equiv \alpha(\varepsilon \cdot x)$ with ε_μ also a fixed vector, and finally N is taken equal to $M(\alpha)$, then we obtain example 4 of reference [2].

3^{rd}: If a_μ is set equal to zero, then (15) reduces to

$$A_\mu^{(1)} = \alpha_{,\mu}\ N(x) \tag{17}$$

and the copies are given by

$$\bar{A}_\mu^{(2)} = \alpha_{,\mu}(N(x) + L\ (\alpha)\ ) \tag{18}$$

where $L(\alpha)$ is an arbitrary matrix function of its argument. This family of copies has as many arbitrary functions (of α) as there are generators of the gauge group (three for $SU(2)$).

4^{th}: If $a_\mu \equiv 0$, and $N(x) = \Psi(x)\ M(\alpha)$ then (17),(18) take the form:

$$A_{\mu}^{(1)} = \Psi \alpha_{,\mu} M \tag{19}$$

$$A_{\mu}^{(2)} = (\Psi + f(\alpha)) \alpha_{,\mu} M \tag{20}$$

from them we can find the field:

$$F_{\mu\nu} = (\Psi_{,\mu}\alpha_{,\nu} - \Psi_{,\nu} \alpha_{,\mu}) M \equiv g_{\mu\nu} M \tag{21}$$

and the source

$$j_{\nu} = \partial_{\mu} F_{\mu\nu} + [A_{\mu}, F_{\mu\nu}]$$

$$j_{\nu} = g_{\mu\nu,\mu} M + g_{\mu\nu} \alpha_{,\mu} M' \tag{22}$$

Now, whatever the form of $f(\alpha)$, the potentials (19), (20) have not only the same field strenght but also the same current. Note that there is no possibility of (19) and (20) to be connected by a gauge transformation (except in degenerate cases) as M^2 can be taken to be unity, then for such a transformation to exist it must commute with M (as (21) is invariant) and also with M' (as (22) is also invariant) and that is impossible. Note also that (12) and (19), when the $\alpha^{(i)}$ are functions of only one variable, contain as particular cases the examples given in reference (7). On the other hand, if the α, s are function of the four variables then those copies are not included in Halpern's construction.

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