NONLINEAR SCALAR FIELD IN GENERAL RELATIVITY

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ABSTRACT

A nonlinear scalar field S of short range is considered, in the lights of general relativity. The pseudovirial theorem of Rosen is found to be related to the principle of equivalence, in the weak field limit. A self-interaction additive term λS^4 is included in the Lagrangean, which permits nonsingular and source-free systems. A family of static, spherically symmetric solutions is obtained, with a nonlinear spectrum of masses. The gravitational potentials tend to the Schwarzschild ones at infinity, while the scalar field presents Yukawa-type asymptotic behaviour.

1. INTRODUCTION

The idea of explaining the structure of elementary particles in terms of general relativity has a long history[1]. Such a classical approach has been almost abandoned after the discovery of the quantum properties of particles. However, one may still hope that some essential quantities which characterize elementary particles, such as baryon number, can be related to topological properties of spacetime.

In this connection, recent studies of nonlinear fields brought a light on some interesting possibilities for the structure of elementary particles[2,3]. Thus it seems worthwhile to get back to the original idea and look for the possibility of explaining these structures as a result of mutually interacting fields in a curved spacetime.

On this line of research, it is now customary not to introduce any rest mass term in the description of the system[4]. On can also try to eliminate sources of fields; however, it was recently shown[5] that the simple minimum coupling of, say, a Klein-Gordon field and the gravitational field does not allow an equilibrium configuration of nonsingular systems.

In this paper we consider a self-interacting scalar field S alone, in the framework of general relativity. In § 2 we formulate the theory and investigate the static, spherically symmetric nonsingular solutions of the equations. In the weak gravitation limit, we show that the pseudovirial theorem of Rosen[6] is related to the principle of equivalence. As shown by Rosen, this theorem does not permit nonsingular solutions if the additive potential term to the otherwise free Lagrangean is positive definite. For several types of potential, we verified numerically that this is true even for the strong gravitation limit, as far as there is no singularity in metrics. In § 3 we then consider the negative λS^4 term as the self-interaction potential, and show explicitly some examples of nonsingular solutions. In § 4 we discuss the stability of these solutions, and in § 5 some comments about the obtained mass spectrum are made.

2. FIELD EQUATIONS

We start from the Lagrangean

$$\kappa \mathcal{L} = (-g)^{1/2} \left[\frac{1}{2} R + S_{,\alpha} S_{,\beta}\, g^{\alpha\beta} - V(S^2) \right] , \quad \kappa = 8\pi G/c^4 , \tag{2.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, R is the scalar curvature, S a scalar field, and a subscripted comma means ordinary derivative with respect to coordinates. V is a potential only depending on the modulus of the field S.

Einstein's equations are obtained from the invariance of the action integral upon variations of the metric potentials,

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -\, 2S_{,\mu} S_{,\nu} + (S_{,\alpha} S^{,\alpha} - V) g_{\mu\nu} , \tag{2.2}$$

while the invariance upon variations of the scalar field yields

$$S^{;\mu}_{;\mu} + S\, dV/dS^2 = 0 , \tag{2.3}$$

where a semicolon means covariant derivative. This equation is indeed the Bianchi identity associated to (2.2).

We now consider a static, spherically symmetric system; we write the line element in the form

$$ds^2 = e^{2\eta} (dx^0)^2 - e^{2\alpha} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 , \tag{2.4}$$

with the potentials η , α , and S depending only on r. From (2.2) and (2.3) one then obtains the coupled equations

$$\eta_1 = r S_1{}^2 - \alpha_1 , \tag{2.5}$$

$$2r\alpha_1 = r^2 S_1{}^2 + 1 - (1 - r^2 V) e^{2\alpha} , \tag{2.6}$$

$$S_{11} + (\eta_1 - \alpha_1 + 2/r) S_1 - S\, e^{2\alpha}\, dV/dS^2 = 0 , \tag{2.7}$$

where a subscript 1 means d/dr. Non-general relativistic theory is recuperated by taking the weak field limit of the system, in which the potentials η and α are of the order $S^2 << 1$.

Usually $V(S^2)$ is required to be positive definite, normalizing the value of $S(r)$ to zero for $r \to \infty$. Such a field S only has singular solutions. This result is not altered even in the limit of strong gravitational field, provided the metric coefficients do not have any singularity. In order to obtain a class of regular solutions, we must release the restriction of positive definiteness of $V(S^2)$, as is shown below.

We look for solutions whose asymptotic gravitational behaviour is of the Schwarzschild type:

$$\eta(r) = -\alpha(r) \quad , \quad r \to \infty \quad . \tag{2.8}$$

Since in the weak field limit one has[7]

$$\eta(r) = -\frac{1}{2}\kappa r^{-1}\int_0^\infty (2T_0^0 - T)\, r^2\, dr \quad , \quad r >> 1 \quad , \tag{2.9}$$

$$\alpha(r) = \frac{1}{2}\kappa r^{-1}\int_0^\infty T_0^0\, r^2\, dr \quad , \quad r >> 1 \quad , \tag{2.10}$$

one finds that (2.8) implies

$$<2T_0^0 - T> = <T_0^0> \quad , \tag{2.11}$$

where < > means total space integration, and T is the trace of the energy momentum tensor T^μ_ν . In the weak field limit, this tensor is given by

$$\kappa T^\mu_\nu = ({S_1}^2 + V)\delta^\mu_\nu - 2{S_1}^2\, \delta^\mu_1\, \delta^1_\nu \quad , \tag{2.12}$$

then (2.11) implies the pseudovirial theorem of Rosen[6] ,

$$\langle S_1{}^2\rangle + 3\langle V\rangle = 0 \quad . \tag{2.13}$$

One observes in this equation that $V(S^2)$ must give negative contributions to the volume integral $\langle V\rangle$. For various types of potential, we have verified numerically that the above conclusion is correct even in the strong gravitation limit, by seeing how the asymptotic behavior of solutions depends on the values of parameters which describe the boundary conditions at $r = 0$.

When a singularity of the metric potential $\alpha(r)$ is admitted, we can show that even when the potential $V(S^2)$ is taken to be positive definite, there exists a consistent solution of the scalar field S, which can be considered as a "kink" in the Schwarzschild geometry[8].

Here we investigate the case where the restriction of the positive definiteness of $V(S^2)$ is released. Then we start from the positive potential corresponding to the Klein-Gordon field ($V_{KG} = \mu^2 S^2$) and add a negative term, in the form of

$$V(S^2) = \mu^2 S^2 \left(1 - \frac{1}{2} f^2 S^2\right) \quad , \tag{2.14}$$

where f is a dimensionless constant factor.

We anticipate now a result which will only become apparent in § 5. It is sufficient for our purposes to consider the cases where the metric potentials η and α are both of order $S^2 \ll 1$. The field equations then become

$$(r\alpha)_1 = \frac{1}{2} r^2 \left[S_1{}^2 + \mu^2 S^2\left(1 - \frac{1}{2} f^2 S^2\right)\right] \quad , \tag{2.15}$$

$$S_{11} + 2S_1/r - \mu^2(1 - f^2 S^2)S = 0 \quad , \tag{2.16}$$

and (2.5). We find that the equations decouple themselves, in this order of approximation; we initially solve (2.16) for $S(r)$, then $\alpha(r)$ and $\eta(r)$ are obtained from (2.15) and (2.5), consecutively.

3. SOLUTIONS

In view of difficulty in obtaining analytic solutions, we integrate (2.16) numerically. The two parameters μ and f are conveniently absorbed in the form

$$x = \mu r \quad , \quad y = fS \quad , \tag{3.1}$$

so that the function y and variable x are proportional to the potential S and radial coordinate r, respectively. From (2.16) one then obtains

$$y'' + 2y'/x = y(1 - y^2) \quad , \tag{3.2}$$

where a prime means d/dx. We impose, as boundary conditions to the nonlinear equation (3.2), that $y(x)$ be nonsingular everywhere, and that it tends to zero at infinity. These conditions, taken together, form an eigenvalue problem for the central value $y(0)$. The first five eigenvalues are given in Table 1, and the eigensolutions $y(x)$ corresponding to the first three eigenvalues are reproduced in Fig. 1.

One observes in Fig. 1 the maximum concentration of the field in the innermost regions, in all solutions. One also observes that the i^{th} solution presents $i - 1$ zeros in finite regions, and that all solutions have the usual Yukawa-type asymptotic behaviour. Higher order states are strongly peaked in the center.

The gravitational potential $\alpha(x)$ is obtained from the integration of

$$(x\alpha)' = \frac{1}{2} f^{-2} x^2 (y'^2 + y^2 - \frac{1}{2} y^4) \quad . \tag{3.3}$$

It is proportional to f^{-2}, and is regular everywhere. Fig. 2 represents the solution for $\alpha(x)$ corresponding to the first eigenvalue $y(0)$. One remarks the hyperbolic behaviour of $\alpha(x)$ for $x \to \infty$, what is characteristic of Schwarzschild-type systems. Indeed, one obtains from (3.3), (2.13) and (3.2)

$$\alpha(r) = G\, m/(c^2 r) \quad , \quad r \to \infty \quad , \tag{3.4}$$

where the mass m of the system (Table 1) is given for each solution by

$$f^2 \mu\, G\, m/c^2 = \int_0^\infty y^2 x^2\, dx = \frac{1}{4} \int_0^\infty y^4 x^2 dx = \frac{1}{3} \int_0^\infty y'^2 x^2 dx \quad . \tag{3.5}$$

The solutions for $\alpha(x)$ corresponding to higher eigenvalues of $y(0)$ are similar: all start from the value $\alpha(0) = 0$, next assume negative values and later positive values, and finally vanish in the Schwarzschild form (3.4).

The gravitational potential $\eta(x)$ plays the role of Newtonian potential. It is obtained from the integration of

$$\eta' = f^{-2} x\, y'^2 - \alpha' \quad , \tag{3.6}$$

and is also proportional to f^{-2}. It starts from a negative value at the center of symmetry,

$$\eta(0) = - f^{-2} \int_0^\infty y'^2 x\, dx \quad , \tag{3.7}$$

and increases monotonically outwards. In the asymptotic regions it presents the usual Schwarzschild behaviour,

$$\eta(r) = - G\, m/(c^2 r) \quad , \quad r \to \infty \quad . \tag{3.8}$$

The solution for $\eta(x)$ corresponding to the first eigenvalue of $y(0)$ is presented in Fig. 2; the solutions corresponding to higher eigenvalues behave similarly.

4. INSTABILITY

We now study the behaviour of our solutions under small radial perturbations, and we obtain an expression for the corresponding characteristic time parameter.

Let $y(x)$ be one of the solutions of the static equation (3.2), and define the time-dependent, radially perturbed solution

$$y^*(x,t) = y(x) + \delta y(x,t) \quad . \tag{4.1}$$

From (2.3), (2.14) and (3.1) one then finds that y^* satisfies, in the weak field limit, the equation

$$\ddot{y}^* = \nabla^2 y^* - y^* + y^{*3} \quad , \tag{4.2}$$

where a dot means $\partial/\partial(\mu c t)$ and $\nabla^2 = \partial^2/\partial x^2 + 2x^{-1}\,\partial/\partial x$. For small perturbations one obtains, from (4.1) and (4.2) ,

$$\delta\ddot{y} + \left[- \nabla^2 + 1 - 3y^2 \right] \delta y = 0 \quad . \tag{4.3}$$

Now consider a set of functions $f_i(x)$ satisfying

$$\left[-\nabla^2 + 1 - 3y^2\right] f_i = \rho_i f_i \quad , \quad \rho_i = \mathrm{const} \quad . \tag{4.4}$$

The boundary condition for $f_i(x)$ is that it is finite everywhere. If negative eigenvalues ρ_i exist for a given static solution $y(x)$, one finds from (4.3) that the corresponding eigenmodes $f_i(x)$ develope exponentially in time. One then associates to each of those eigenmodes a characteristic time parameter given by

$$\tau_i^{-1} = \mu c(-\rho_i)^{1/2} \quad . \tag{4.5}$$

This quantity does not depend on the parameter f.

Equation (4.4) resembles the Schrödinger equation for a particle in a central force potential

$$U(x) = 1 - 3y^2(x) \quad . \tag{4.6}$$

For the case where $y(x)$ is the first eigensolution of (3.2) one finds numerically that the lowest eigenvalue of (4.4) is negative ($\rho_i = -15.29$). This subject is further discussed in § 5.

Higher order eigensolutions of (3.2) present a more negative potential $U(x)$ in the central regions, as can be seen from Fig. 1 and Eq. (4.6). It is then reasonable to expect that small radial perturbations of these solutions also evolve exponentially in time.

5. DISCUSSIONS

Our general relativistic treatment of the nonlinear field model gives a very simple interpretation of the pseudovirial theorem of Rosen[6]: it results from the equivalence of gravitational source and energy, in the limit of weak gravitational field.

This theorem does not permit localized, nonsingular, static and spherically symmetry solutions if the potential term $V(S^2)$ is positive definite. We released this restriction and obtained numerically a family of solutions. Our potential $V(S^2)$ depends on two parameters, f and μ . These parameters are absorbed in the field variable ($y = f\,S$) and radial variable ($x = \mu\,r$), in such a way that the eigenvalue problem for $y(x)$ has a parameter-independent form.

One finds, from (3.1) and (3.2), that f^{-1} measures the overall intensity of the scalar field S. Our solutions are then only valid for large values of f^2. This is confirmed in the equations (3.3) and (3.6), which define the gravitational potentials α and η . The inverse length parameter μ gives an estimate of the dimensions of the system, as is deduced from Fig. 1. The diameters are of order of magnitude μ^{-1} .

To each given solution it corresponds a different mass m, whose value is obtained from the asymptotic behaviour of the gravitational potentials. The values of m are presented in Table 1, in terms of the dimensionless quantity $M = (f^2\mu)(G/c^2)m$.

Our naive physical model is certainly not able to represent the large number of properties of elementary particles. However, if we take for the length parameter μ^{-1} a value of order 10^{-13}cm, which is the range of strong interactions, we obtain for the characteristic time parameter τ_i a value $\sim 10^{-24}$sec, which corresponds to the usual width of mesonic resonances. Furthermore, if we take $f^2 \sim 10^{40}$, to adjust our lowest energy state to that of a pion, then the second state has the mass $\sim$ 900 MeV, and the third $\sim$ 2.7 GeV. This is a reasonable level spacing, considering the masses of η' meson and the cluster found in high energy multiple π-meson production[9,10]. One should also remark that such a large value of f^2 completely justifies the linearized approximation taken in § 2.

As shown in § 4, our solutions do not present an oscillatory behaviour against small perturbations around the equilibrium, so they are not convenient for the usual semi-classical quantization procedure[2]. A further study is then naturally suggested, of the mechanism of instability in the strong field limit of general relativity, together with possible interactions with other fields.

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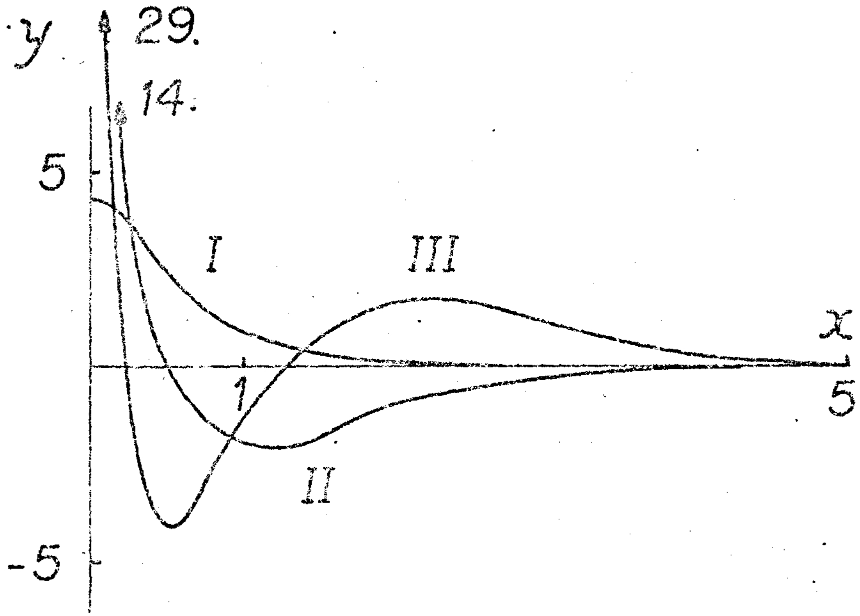

Fig.1 - First three eigenstates y(x) of the scalar field $S = y/f$, against radial coordinate $x = \mu r$.

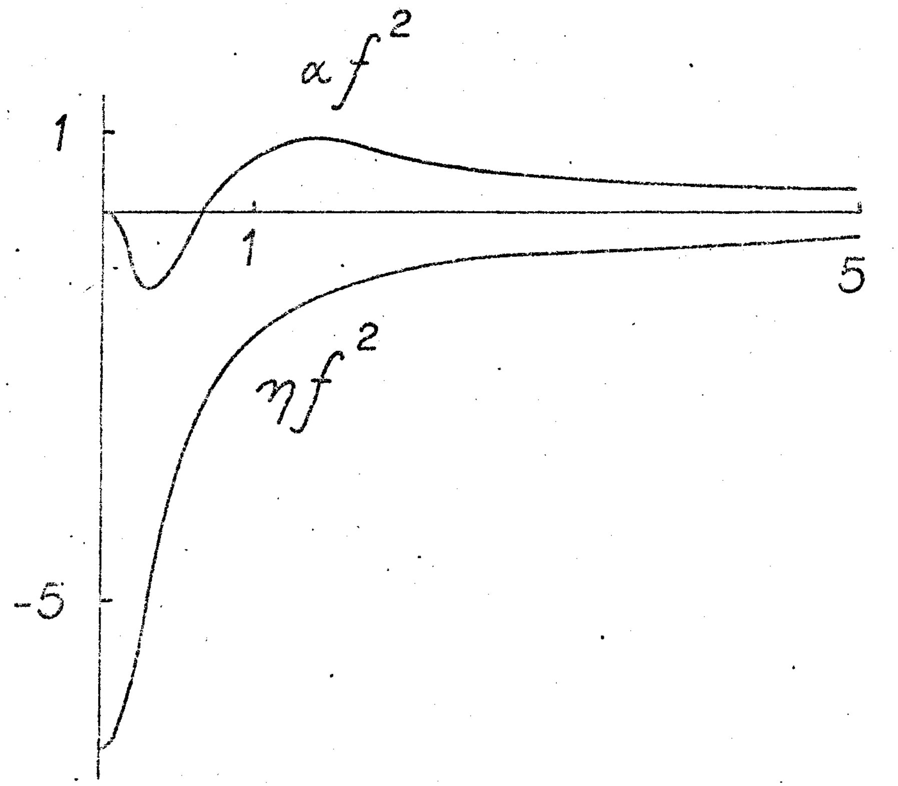

Fig. 2 - Gravitationals potentials $\eta(x) = \frac{1}{2} \log g_{00}$ and $\alpha(x) = \frac{1}{2} \log (-g_{rr})$ of the first state, against radial coordinate $x = \mu r$.

Table 1

First five eigenvalues $y(0) = fS(0)$ and masses $M = (f^2 \mu G/c^2)m$.

States	y(0)	M
I	4.3378	1.50
II	14.104	9.48
III	29.143	28.7
IV	51.385	68.3
V	77.766	128.