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DIFFERENTIAL GEOMETRY AND LIE SYMMETRY GROUPS *

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INTRODUCTION

The theory of continuous groups of transformations in the vicinity of the identity transformation, that is, the theory of the Lie groups, was applied to physical motions represented by ordinary differential equations by Hill [1]. By interpreting the possible motions as the solutions of a given differential equation, he has shown that associated to the differential equation there exists a Lie group of symmetry, that means, a group which transforms the whole family of possible motions into itself. This symmetry group will depend on the structure of the particular equation chosen. In other terms, each particular dynamical structure possess its own group of symmetry. Clearly, the situation may be inverted: we

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postulate a given symmetry group, for instance the Galilean group, and look for all possible equations which are consistent with this symmetry. This last way of looking at the problem is the way usually used in particle dynamics or in field theory. However, in this paper we do not use it. Rather, we investigate for each differential equation what is the symmetry group. In connection to the above interpretation used in field theory, our present way of looking at the problem allows to discover what are the extra symmetries beyond those who were postulated a priori.

In this paper we extend Hill's work to partial differential equations. This extension is equivalent to the study of the family of surfaces, associated to the differential equation, which are mapped onto itself by the transformations of the symmetry group. In this formulation a field is interpreted as a particular canonical form assumed by the equation of the surface. Consequently, the concept of field is a variant one with respect to the symmetry mappings.

The treatment of this problem, in this paper, is more extensively done for two-dimensions, that means, in the case where exists just two parameters for the motion of the system. This case corresponds formally to the situation treated specifically by Hill. However, in the last sections we also consider the extension of this method to four-dimensional situations. The differences which arise from the passage of two to four dimensions are discussed, and is shown that the derivation of the symmetry group is different for each circunstance, but the final result is formally the same, aside some specific singular results holding only in two-dimensions.

1. INVARIANCE OF THE DYNAMICAL EQUATIONS OF MOTION UNDER LIE GROUPS

In this section we will briefly review the results obtained by Hill. In this content several equations of motion will be considered and each particular class of motions which are obtained from these equations is subjected to the transformations of the Lie group, which acts on the plane where the motion takes place. Then, accordingly to the class of motions considered, we will obtain each particular representation of the Lie group, as a symmetry group for the motion.

This is mathematically obtained as follows: Consider the plane of the two variables x, t. In this plane let us introduce the constraint relations

$$\psi(x,t) = 0$$

and suppose we can write it under the canonical form

$$\psi \equiv x - f(t) = 0$$

Then, depending on the particular form of f(t) we will obtain each particular family of motions. The cases which are considered are,

$$f(t) = a \tag{1.1}$$

$$f(t) = at + b \tag{1.2}$$

$$f(t) = at^2 + bt + c \cdot \tag{1.3}$$

....

Corresponding to the differential equations

$$\frac{dx}{dt} = 0 \qquad (1.4)$$

$$\frac{d^2x}{dt^2} = 0 \qquad (1.5)$$

$$\frac{d^3x}{dt^3} = 0 \qquad (1.6)$$

The case (1.4) corresponds to a family of straight lines parallel to the t-axis. The case (1.5) corresponds to a family of straight lines starting at the point $x = b$ (for $t = 0$) and making the angle β with the x-axis,

$$a = 1/\mathrm{tg}\beta$$

Equation (1.6) corresponds to a family of parabolas.

We consider now the most general symmetry group of transformations on the plane (x,t) which transforms each of these family of motions into itself.

$$\begin{cases} x' = x + \xi(x,t) \\ t' = t + \eta(x,t) \end{cases} \qquad (1.7)$$

where $\xi(x,t)$ and $\eta(x,t)$ are first order infinitesimal functions. For (1.4) we get a symmetry Lie group depending on an infinite number of parameters, since the only restriction on these functions is

$$\xi_t = 0$$

This may be geometrically interpreted by the fact that the family of straight lines parallel to the axis t is invariant under the mappings

$$\begin{cases} x' = x + \xi(x) \\ t' = t + \eta(x,t) \end{cases}$$

But ξ cannot depend on t, since this dependence will imply that each element of the family of straight lines is mapped into a curve of arbitrary shape. For (1.5) and (1.6) we obtain a symmetry Lie group with a finite number of parameters, for (1.5) we obtain eight parameters and for (1.6) we get seven parameters. [1] These transformations may be, similarly to the previous case, interpreted geometrically as the possible symmetry transformations of the family of curves associated to each one of the equations (1.5) and (1.6).

Important sub-groups of these symmetry Lie groups are the Galilean group, the Poincaré group and the conformal group of the flat space-time, which possess respectivelly three, three and six parameters in the plane (x,t).

2. LIE GROUPS OF SYMMETRY OF PARTIAL DIFFERENTIAL EQUATIONS WHICH POSSESS x,t AS PARAMETERS

In this section we extend the previous conclusions to the case of the partial differential equations. This will be done as follows: We introduce another variable ϕ playing here the same role as x of the section (1), here (x,t) play the role of independent parameters, similarly to the role played by t in the previous section. Consequently the plane (x,t) of the section

(1) goes over the three-dimensional space (ϕ,x,t). Into this space we introduce the constraint relation

$$\psi(\phi,x,t) = 0$$

as before, we write this constraint condition under the canonical form

$$\psi \equiv \phi - f(x,t) = 0 \qquad (2.1)$$

and suppose that this function is solution of certain given partial differential equation, the analogous of the characteristic differential equation of Hill's work. The closest analogy is stablished by considering linear homogeneous partial differential equations. The reason for that, being as follows: in Hill's work all equations have to be homogeneous, or otherwise no invariance Lie group is obtained. As example, in the simplest situation of first order differential equations, the characteristic equation $dx/dt = 0$ possess a Lie invariance group with infinite number of parameters, displaying the geometric significance given in the section (1); however, the differential equation $dx/dt + A(t)\,x = 0$ has no Lie invariance group, since its invariance under the transformations (1.7) imply in vanishing ξ and η. To follow this situation as close as possible imply in considering only homogeneous partial differential equations. Nevertheless, here the situation may still be generalized so as to allow for the existence of non homogeneous partial differential equations, since now x is replaced by the field quantity and t by the pair x,t. Besides this, we have several important second order linear non-homogeneous partial differential equations in physics. Nevertheless, in order to simplify the treatment we shall stick to the case of homogeneity in the characteristic equations to be treated. The case of non homogeneous equations can be treated in similar way, with a little more of work.

Before introducing these characteristic equations, it is of some interest to determine the geometrical properties of the constraint surface given by (2.1). Its first fundamental form, element of area and second fundamental form are respectively given by

$$I \equiv d\vec{v}\cdot d\vec{v} = \eta_{AB} dv^A dv^B = dx^2 \pm dt^2 + d\phi^2$$

$$= (1 + \phi_x^2) dx^2 + (\pm 1 + \phi_t^2) dt^2 + 2\,\phi_x\,\phi_t\,dx\,dt$$

for

$$\eta_{AB} = \mathrm{diag}\,(1, \pm 1, 1),\ \vec{v} = (x, t, \phi)$$

$$d\sigma = \sqrt{1 + \phi_x^2 \pm \phi_t^2}\ dx\,dt$$

$$II_{(1)} \equiv -\,d\vec{v}\cdot d\vec{n} = (1 + \phi_x^2 + \phi_t^2)^{-\frac{1}{2}}\,(\phi_{xx}\,dx^2 + 2\,\phi_{xt}\,dx\,dt + \phi_{tt}\,dt^2)$$

$$II_{(2)} \equiv -\,d\vec{v}\cdot d\vec{n} = (1 + \phi_x^2 - \phi_t^2)^{-\frac{1}{2}}\,(\phi_{xx}\,dx^2 - \phi_{tt}\,dt^2)$$

$II_{(1)}$ refers to Euclidian three-space (x,t,ϕ), and $II_{(2)}$ to refers to a pseudo Euclidian space (x,t,ϕ). In both cases ϕ is taken as an spacelike axis. The $\vec{n}$ is the unit normal to the surface, with components for each choice of signature:

$$\vec{n} = (1 + \phi_x^2 \pm \phi_t^2)^{-\frac{1}{2}}\ (-\phi_x, -\phi_t, 1)$$

The metric on the surface of the space (x,t,ϕ) is then

$$g_{ij} = \begin{pmatrix} g_{xx} & g_{xt} \\ g_{xt} & g_{tt} \end{pmatrix} \begin{pmatrix} 1 + \phi_x^2 & \phi_x \phi_t \\ \phi_x \phi_t & \pm 1 + \phi_t^2 \end{pmatrix}$$

so that

$$d\sigma = \sqrt{\pm g}\ dx\ dt$$

The Christoffel symbols and the Riemannian curvature may be obtained for this surface immersed in Euclidian (or pseudo-Euclidian) three-space (x,t,ϕ) [2].

We may also suppose that the sub-space (x,t), the parameter space, has an intrinsic Riemannian curvature, case where we have:

$$I = \sum_{i=1}^{2} f_{ij}\, dx^i\, dx^j + d\phi^2$$

for f_{ij} the metric on the plane (x,t), and we suppose that the full metric on the space (x,t,ϕ) is of the form,

$$X_{AB} = \left(\begin{array}{c|c} f_{ij} & 0 \\ \hline 0 & 1 \end{array}\right)$$

We now introduce the characteristic partial differential equation, in any case the function given by (2-1) will be a solution for this equation. The first case is for first order equations, the similar of $dx/dt= = 0$ of reference (1). The most general first order partial differential equation which is linear and homogeneous, is of the form

$$A(x,t)\frac{\partial\phi}{\partial x} + B(x,t)\frac{\partial\phi}{\partial t} = 0 \qquad (2.6)$$

A particular situation which is of interest is for

$$\frac{\partial\phi}{\partial x} = 0 \qquad (2.7)$$

$$\frac{\partial\phi}{\partial t} = 0 \qquad (2.8)$$

formally this corresponds to the situation where $dx/dt = 0$, in the reference [1]. We then look for the invariance Lie group of these partial differential equations. The elements of this group are given in the three-space (ϕ,x,t) by the infinitesimal transformations

$$\begin{cases} \phi' = \phi + \xi(\phi,x,t) \\ x' = x + \eta(\phi,x,t) \\ t' = t + \lambda(\phi,x,t) \end{cases} \qquad (2.9)$$

Then,

$$\begin{cases} d\phi' = d\phi + \xi_\phi\, d\phi + \xi_x\, dx + \xi_t\, dt \\ dx' = dx + \eta_\phi\, d\phi + \eta_x\, dx + \eta_t\, dt \\ dt' = dt + \lambda_\phi\, d\phi + \lambda_x\, dx + \lambda_t\, dt \end{cases} \qquad (2.10)$$

Since we look for symmetry mappings for some partial differential equation, we must restrict to points in three-space which belong to the family of surfaces given by the general constraint relation (2.1). In this case the equations (2.10) give

$$\phi'_{x'}\, dx' + \phi'_{t'}\, dt' = \phi_x\, dx + \phi_t\, dt + \xi_\phi(\phi_x\, dx + \phi_t\, dt) +$$

$$+ \xi_x\, dx + \xi_t\, dt$$

or

$$\phi'_{x'}(dx + \eta_\phi(\phi_x\, dx + \phi_t\, dt) + \eta_x\, dx + \eta_t\, dt) +$$

$$+ \phi'_{t'}(dt + \lambda_\phi(\phi_x\, dx + \phi_t\, dt) + \lambda_x\, dx + \lambda_t\, dt) =$$

$$= \phi_x\, dx + \phi_t\, dt + \xi_\phi(\phi_x\, dx + \phi_t\, dt) + \xi_x\, dx + \xi_t\, dt$$

since we want to determine first order changing in the quantities ϕ_x and ϕ_t, we have that $\phi'_{x'} = \phi_x + \varepsilon$ and $\phi'_{t'} = \phi_t + \delta$; from the last equation we get

$$\phi'_{x'} = \phi_x - \eta_\phi\, \phi_x^2 + (\xi_\phi - \eta_x)\phi_x - \lambda_\phi\, \phi_x\, \phi_t - \lambda_x\, \phi_t + \xi_x \qquad (2.11)$$

$$\phi'_{t'} = \phi_t - \lambda_\phi\, \phi_t^2 + (\xi_\phi - \lambda_t)\phi_t - \eta_\phi\, \phi_t\, \phi_x - \eta_t\, \phi_x + \xi_t \qquad (2.12)$$

for the particular case where $\xi = 0$, $\eta_\phi = 0$ and $\lambda_\phi = 0$, we obtain the tensor law of transformation for a covariant vector in the plane (x,t). The choice $\xi = 0$ imply that in this case ϕ is a scalar function in the plane (x,t). For these same simplified transformations, a contravariant vector transforms as

$$V'_1 = V_1 + \eta_x\, V_1 + \eta_t\, V_2 \qquad (2.13)$$

$$V'_2 = V_2 + \lambda_x\, V_1 + \lambda_t\, V_2$$

An example of such vector is given by (dx, dt).

Before treating the invariance of the first order equation (2.6), it will be of interest to determine the variation on the second order derivatives of ϕ. Such variation will be needed for the study of the invariance presented by the second order partial differential equations. For obtaining this variation we differentiate the equations (2.11) and (2.12), and find after a very long but otherwise straightforward calculation:

$$\begin{aligned}\phi'_{x'x'} = {} & \phi_{xx} - 3\,\eta_\phi\,\phi_{xx}\,\phi_x - (2\eta_x - \xi_\phi)\phi_{xx} - \lambda_\phi(\phi_{xx}\,\phi_t + 2\phi_{xt}\,\phi_x) \\ & - 2\,\lambda_x\,\phi_{xt} - (2\,\eta_{\phi x} - \xi_{\phi\phi})\phi_x^2 + (2\,\xi_{\phi x} - \eta_{xx})\phi_x - 2\,\lambda_{\phi x}\,\phi_x\,\phi_t \\ & - \eta_{\phi\phi}\,\phi_x^3 - \lambda_{\phi\phi}\,\phi_x^2\,\phi_t - \lambda_{xx}\,\phi_t + \xi_{xx}\end{aligned} \tag{2.14}$$

$$\begin{aligned}\phi'_{t't'} = {} & \phi_{tt} - 3\,\lambda_\phi\,\phi_{tt}\,\phi_t - (2\lambda_t - \xi_\phi)\phi_{tt} - \eta_\phi(\phi_{tt}\,\phi_x + 2\phi_{xt}\,\phi_t) \\ & - 2\,\eta_t\,\phi_{xt} - (2\,\lambda_{t\phi} - \xi_{\phi\phi})\phi_t^2 + (2\xi_{\phi t} - \lambda_{tt})\phi_t - 2\,\eta_{\phi t}\,\phi_x\,\phi_t \\ & - \lambda_{\phi\phi}\,\phi_t^3 - \eta_{\phi\phi}\,\phi_t^2\,\phi_x - \eta_{tt}\,\phi_x + \xi_{tt}\end{aligned} \tag{2.15}$$

$$\begin{aligned}\phi'_{t'x'} = {} & \phi_{tx} - \eta_\phi(\phi_{xx}\,\phi_t + 2\,\phi_{tx}\,\phi_x) - \lambda_\phi(2\,\phi_{tx}\,\phi_t + \phi_{tt}\,\phi_x) - \\ & - \eta_t\,\phi_{xx} - \lambda_x\,\phi_{tt} - (\lambda_t + \eta_x - \xi_\phi)\,\phi_{xt} - (\eta_{\phi x} + \lambda_{\phi t} - \xi_{\phi\phi})\,\phi_t\phi_x \\ & - (\lambda_{xt} - \xi_{x\phi})\phi_t - (\eta_{xt} - \xi_{\phi t})\phi_x - \lambda_{\phi x}\,\phi_t^2 - \eta_{\phi t}\,\phi_x^2 \\ & - \eta_{\phi\phi}\,\phi_t\,\phi_x^2 - \lambda_{\phi\phi}\,\phi_x\,\phi_t^2 + \xi_{xt}\end{aligned} \tag{2.16}$$

Note that Equation (2.14) goes over (2.15), and reciprocally, under the interchanges.

$$x \longleftrightarrow t$$

$$\eta \longleftrightarrow \lambda$$

$$\phi \longleftrightarrow \phi$$

$$\xi \longleftrightarrow \xi$$

and (2.16) is symmetric under this inter-change of quantities. The same type of property is verified for (2.11) and (2.12) (they go one into the other by the above replacements).

In the particular case of a linear mapping on the plane (x,t); that is, for $\xi = 0$, $\eta_\phi = \lambda_\phi = 0$ and $\eta_{ij} = 0$, $\lambda_{ij} = 0$, for $(i,j) = (x,t)$, we see that ϕ_{xx}, ϕ_{tt}, ϕ_{xt} transform as the components of a second order covariant tensor in the plane (x,t). For the situation where we allow the existence of quadratic terms in the mapping functions, that means, we take only $\xi = 0$, $\eta_\phi = \lambda_\phi = 0$, we have that ϕ_{xx}, ϕ_{tt}, ϕ_{xt} have the known transformation law, similar to an affinity in the plane (x,t).

The index notation greatly simplifies the notation, and the long calculation necessary for obtaining the variation in the second order derivatives is derived without much too work. Besides this, this notation is essential for carrying out the same analysis for a number of parameters greater than two, since then the explicit notation gets much too complicat ed. Due to this, we prove again the relations (2.11), (2.12) in this notation; calling $\eta^1 = \eta$, $\eta^2 = \lambda$; so that $x'^i = x^i + \eta^i(\phi, x^j)$ where $x^1 = x$, $x^2 = t$ (all indices run from 1 to 2), we have

$$d\phi' = d\phi + \xi_{,\phi}\, d\phi + \xi_{,i}\, dx^i$$

or,

$$\phi'_{,i'}\, dx'^{i} = \phi_{,i}\, dx^{i} + \xi_{,\phi}\, \phi_{,i}\, dx^{i} + \xi_{,i}\, dx^{i}$$

thus,

$$\phi'_{,i'}(dx^{i} + \eta^{i}_{,\phi}\, d\phi + \eta^{i}_{,j}\, dx^{j}) = \phi_{,i}\, dx^{i} + \xi_{,\phi}\, \phi_{,i}\, dx^{i} + \xi_{,i}\, dx^{i}$$

giving, to first order,

$$\phi'_{,i'}\, dx^{i} + \eta^{i}_{,\phi}\, \phi_{,i}\, \phi_{,j}\, dx^{j} + \eta^{i}_{,j}\, \phi_{,i}\, dx^{j} = \phi_{,i}\, dx^{i} + \xi_{,\phi}\, \phi_{,i}\, dx^{i} + \xi_{,i}\, dx^{i}$$

therefore

$$\phi'_{,j'} = \phi_{,j} - \eta^{i}_{,\phi}\, \phi_{,i}\, \phi_{,j} - \eta^{i}_{,j}\, \phi_{,i} + \xi_{,\phi}\, \phi_{,j} + \xi_{,j} \qquad (2.17)$$

this relation is equivalent to the two relations (2.11) and (2.12). For the variation in the second order derivatives, we have

$$\begin{aligned}\phi'_{,j'k'}\, dx'^{k} = {} & \phi_{,jk}\, dx^{k} - \eta^{i}_{,\phi\phi}\, \phi_{,k}\, \phi_{,i}\, \phi_{,j}\, dx^{k} - \eta^{i}_{,\phi k}\, \phi_{,i}\, \phi_{,j}\, dx^{k} - \\ & - \eta^{i}_{,\phi}\, \phi_{,ik}\, \phi_{,j}\, dx^{k} - \eta^{i}_{,\phi}\, \phi_{,i}\, \phi_{,jk}\, dx^{k} - \eta^{i}_{,j\phi}\, \phi_{,k}\, \phi_{,i}\, dx^{k} \\ & - \eta^{i}_{,jk}\, \phi_{,i}\, dx^{k} - \eta^{i}_{,j}\, \phi_{,ik}\, dx^{k} + \xi_{,\phi\phi}\, \phi_{,k}\, \phi_{,j}\, dx^{k} + \xi_{,\phi k}\, \phi_{,j}\, dx^{k} \\ & + \xi_{,\phi}\, \phi_{,jk}\, dx^{k} + \xi_{,jk}\, dx^{k} + \xi_{,j\phi}\, \phi_{,k}\, dx^{k}\end{aligned}$$

giving, to first order terms,

$$\begin{aligned}\phi'_{,j'k'} = {} & \phi_{,jk} - \eta^{i}_{,\phi}(\phi_{,ji}\, \phi_{,k} + \phi_{,ik}\, \phi_{,j} + \phi_{,i}\, \phi_{,jk}) - \eta^{i}_{,k}\, \phi_{,ji} - \\ & - \eta^{i}_{,j}\, \phi_{,ik} - \eta^{i}_{,\phi\phi}\, \phi_{,k}\, \phi_{,i}\, \phi_{,j} - \eta^{i}_{,\phi k}\, \phi_{,i}\, \phi_{,j} - \eta^{i}_{,\phi j}\, \phi_{,k}\, \phi_{,i} \\ & - \eta^{i}_{,jk}\, \phi_{,i} + \xi_{,\phi\phi}\, \phi_{,k}\, \phi_{,j} + \xi_{,\phi k}\, \phi_{,j} + \xi_{,\phi j}\, \phi_{,k} + \xi_{,\phi}\, \phi_{,jk} + \xi_{,jk}\end{aligned} \qquad (2.18)$$

This equation is equivalent to the three equations (2.14) through (2.16). Since this method is general, it applies directly to any number of parameters, and in particular to the case of four parameters x^i (for i going from 1 to 4) as is the situation in relativity.

However, in this section we will use the notation of components in order to have a direct analogy with the reference [1].

There exists a symbolic way for stablishing a comparison between the transformation relations of ϕ_x, ϕ_t with the corresponding equation of Hill's work, namely the quantity dx'/dt'. The same, of course, will hold for a comparison between the set $(\phi'_{x'x'}, \phi'_{x't'}, \phi'_{t't'})$ with $\frac{d^2x'}{dt'^2}$. As we have seen, we stablished initially the correspondence:

$$x \longrightarrow \phi$$

$$t \longrightarrow x,t$$

this implies

$$v = \frac{dx}{dt} \longrightarrow \begin{pmatrix} \phi_x \\ \phi_t \end{pmatrix}$$

$$v^2 \longrightarrow \begin{pmatrix} \phi_x \\ \phi_t \end{pmatrix} (\phi_x \; \phi_t) \begin{pmatrix} \phi_x^2 & \phi_x\phi_t \\ \phi_x\phi_t & \phi_t^2 \end{pmatrix}$$

$$\xi \longrightarrow \xi$$

$$\xi_x \longrightarrow \xi_\phi$$

$$\xi_t \longrightarrow \begin{pmatrix} \xi_x \\ \xi_t \end{pmatrix}$$

$$\eta \longrightarrow \begin{pmatrix} \eta \\ \lambda \end{pmatrix}$$

$$\eta_x \longrightarrow \begin{pmatrix} \eta_\phi \\ \lambda_\phi \end{pmatrix}$$

$$\eta_t \longrightarrow \begin{pmatrix} \eta_x & \lambda_x \\ \eta_t & \lambda_t \end{pmatrix} \equiv \begin{pmatrix} \partial_x \\ \partial_t \end{pmatrix} (\eta, \lambda)$$

This in matrix notation, using that,

$$\Psi = \begin{pmatrix} \phi_x \\ \phi_t \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \eta \\ \lambda \end{pmatrix}$$

$$\frac{d}{dt} \longrightarrow \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} = \partial$$

gives

$$\frac{dx}{dt} = v \longrightarrow \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} \phi = \partial\phi = \Psi$$

$$\left(\frac{dx}{dt}\right)^2 = v^2 \longrightarrow \Psi \cdot \Psi^T = (\partial\phi) \cdot (\partial\phi)^T$$

$$\frac{\partial\eta}{\partial t} = \eta_t \longrightarrow \partial \cdot \Sigma^T = \begin{pmatrix} \partial_x \\ \partial_t \end{pmatrix} \overset{(\eta,\ \lambda)}{\begin{pmatrix} \eta_x & \lambda_x \\ \eta_t & \lambda_t \end{pmatrix}}$$

$$\frac{\partial\xi}{\partial t} = \xi_t \longrightarrow \partial \cdot \xi$$

$$\frac{\partial\eta}{\partial x} = \eta_x \longrightarrow \partial_\phi \Sigma \equiv \Sigma_\phi$$

Then, the Hill's formula for the variation of dx/dt,

$$v' \cong v + \xi_x v + \xi_t - \eta_x v^2 - \eta_t v$$

will be formally extended to

$$\partial'\phi' = \partial\phi + \xi_\phi\, \partial\phi + \partial\xi - (\partial\phi) \cdot (\partial\phi)^T \cdot \Sigma_\phi - (\partial\Sigma^T) \cdot \partial\phi$$

a simple algebraic calculation shows that they are our relations (2.11) and (2.12). Here they are written in a compact and elegant form. The same kind of notation is applied to the variations on the second partial derivatives, as compared to Hill's variation in $\frac{d^2x}{dt^2}$.

Returning to the original problem of determination of the invariance group of the first order equation (2.6), we use the transformation equations (2.11) and (2.12). However, more information is still necessary. We have to know how the coefficients in (2.6) transform. Three possibilities are open:

i) $A'(x', t') = A(x, t)$; $B'(x', t') = B(x, t)$ - that is, A and B are scalars in the plane (x, t).

ii) A and B transform as a covariant vector on the plane (x, t).

iii) A and B transform as a contravariant vector on the plane (x, t) - that is, according to the condition (2.13).

As is clear, we have to use condition (iii) due to the form of transformation of the derivatives of ϕ. We obtain,

$$A'\phi'_x + B'\phi'_t = A\phi_x + B\phi_t + \xi_\phi(A\phi_x + B\phi_t) - \eta_\phi(A\phi_x^2 + B\phi_x\phi_t)$$

$$- \lambda_\phi(B\phi_t^2 + A\,\phi_x\,\phi_t) + A\,\xi_x + B\,\xi_t \qquad (2.19)$$

Equation (2.6) will be covariant if

$$\eta_\phi = \lambda_\phi = 0$$
$$\xi_x = \xi_t = 0$$

Giving for the symmetry group,

$$\begin{cases} \phi' = \phi + \xi(\phi) \\ x' = x + \eta(x,t) \\ t' = t + \lambda(x,t) \end{cases}$$

Note that for this case the presence of the factor ξ_ϕ implies that in the transformation law for (ϕ_x, ϕ_t) we have extra terms not included in the tensorial transformation law on the plane (x,t):

$$\begin{cases} \phi'_{x'} = (1 + \xi_\phi)\phi_x - \eta_x \phi_x - \lambda_x \phi_t \\ \phi'_{t'} = (1 + \xi_\phi)\phi_t - \eta_t \phi_x - \lambda_t \phi_t \end{cases}$$

In the particular case of the equations (2.7) and (2.8), we get only the two conditions,

$$\xi_x = 0, \ \xi_t = 0 \qquad (2.20)$$

Which formally correspond to the condition $\xi_t = 0$ of reference [1]. In both situations, the increment of the independent variable does not depend on the parameters. The geometrical interpretation of this conclusion here follows similar to the case for total differential equations: the conditions (2.20) preserve the mapping of a plane perpendicular to the axis -ϕ. In other words, (2.20) allow the mappings of such family of planes on itself. Thus, the symmetry group for the surface ϕ = const. is,

$$\begin{cases} \phi' = \phi + \xi(\phi) \\ x' = x + \eta(\phi,\ x,\ t) \\ t' = t + \lambda(\phi,\ x,\ t) \end{cases}$$

The geometrical interpretation of the invariance group of (2.6), and the parallel interpretation of the family of surfaces associated to this equation is not so simple as was for the previous particular case. Assuming that $A(x,t)$ has no zeros within a certain domain D, we may write

$$\frac{\partial\phi}{\partial x} = - \frac{B(x,t)}{A(x,t)} \frac{\partial\phi}{\partial t}$$

and use this formula, along with their partial derivatives, for writing the Taylor series expansion of $\phi(x,t)$ around the origin, as function of x,t and of the constants $\phi(0,0)$, $A(0,0)$, $B(0,0)$ and $\left(\frac{\partial\phi}{\partial t}\right)_o$ together with their derivatives at the origin. The family of surfaces will then be the locus of points on the three-space (ϕ,x,t) which satisfy this series expansion:

$$\begin{aligned} \Psi \equiv \phi - f(0,0) &+ \left(\frac{\partial f}{\partial t}\right)_o \left[t + xC + xt\,D + \frac{x^2}{2} F + \frac{x^2 t}{2} M + \ldots\right] \\ &+ \left(\frac{\partial^2 f}{\partial t^2}\right)_o \left[\frac{t^2}{2} + xt\,C + \frac{x^2 t}{2} F + \frac{x^2}{2} G + \ldots\right] + \\ &+ \left(\frac{\partial^3 f}{\partial t^3}\right)_o \left[\frac{x^2 t}{2} G + \ldots\right] + \ldots = 0 \end{aligned} \qquad (2.21)$$

for

$$C = -\left(\frac{B}{A}\right)_o,\ D = -\left(\partial_t \frac{B}{A}\right)_o,\ F = -\left(\partial_x \frac{B}{A}\right)_o + \left(\frac{B}{A}\partial_t \frac{B}{A}\right)_o$$

$$G = \left(\frac{B^2}{A^2}\right)_o,\ M = \left(-\partial_{xt}\frac{B}{A} + \partial_t\left[\frac{B}{A}\partial_t\frac{B}{A}\right]\right)_o$$

in the particular case of the solutions of (2.7) and (2.8), which are trivial solutions of (2.6), we have only the first two terms on the expansion (2.21).

Still another general example involving first order equations, may be given by means of

$$P \frac{\partial \phi}{\partial x} + Q \frac{\partial \phi}{\partial t} = R$$

for P, Q and R three continuous differentiable functions of (ϕ,x,t) on a domain D of the three-space (ϕ,x,t) [3]. Since this example seems much too abstract for having interest, we will not treat it here.

In the remaining of this section we treat the situation for second-order partial differential equations, which deserve special attention since they are the more important for mathematical physics.

Consider the general second-order homogeneous partial differential equation (the case for non-homogeneous equations will be considered later, in an example of direct interest in physics)

$$A \frac{\partial^2 \phi}{\partial u^2} + B \frac{\partial^2 \phi}{\partial u \, \partial v} + C \frac{\partial^2 \phi}{\partial v^2} + D \frac{\partial \phi}{\partial u} + E \frac{\partial \phi}{\partial v} = 0$$

where A, B, C, D and E are functions of the parameters u, v. This equation can be transformed into the forms

$$\frac{\partial^2 \phi}{\partial u \, \partial v} - A \frac{\partial \phi}{\partial u} - B \frac{\partial \phi}{\partial v} = 0 \qquad (2.22)$$

$$\frac{\partial^2 \phi}{\partial u^2} - A' \frac{\partial \phi}{\partial u} - B' \frac{\partial \phi}{\partial v} = 0 \qquad (2.23)$$

by a real or imaginary substitution of the parameters [4]. Thus, we may consider only these last two equations. An special case of interest is given for A = B = 0 in (2.22),

$$\frac{\partial^2 \phi}{\partial u\, \partial v} = 0 \tag{2.24}$$

This equation assumes the form of the two-dimensional wave equation in the plane (x,t) if we identify the parameters u, v by the relations:

$$\begin{cases} u = x + t \\ v = x - t \end{cases}$$

or

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2.25}$$

The solutions of this equation define a family of translation surfaces on the space (ϕ, x, t),

$$\phi = f(u) + \psi(v)$$

From (2.14) and (2.15) we find for the transformation of the left hand side of (2.25), under the mapping given by (2.9),

$$\begin{aligned} \phi'_{x'x'} - \phi'_{t't'} &= \phi_{xx} - \phi_{tt} + \eta_\phi(\phi_{tt}\, \phi_x + 2\, \phi_{xt}\, \phi_t - 3\, \phi_{xx}\, \phi_x) \\ &- \lambda_\phi(\phi_{xx}\, \phi_t + 2\, \phi_{xt}\, \phi_x - 3\, \phi_{tt}\, \phi_t) - 2\, \eta_x\, \phi_{xx} + 2\, \lambda_t\, \phi_{tt} + \\ &+ \xi_\phi(\phi_{xx} - \phi_{tt}) - 2(\lambda_x - \eta_t)\phi_{xt} - (2\eta_{\phi x} - \xi_{\phi\phi})\, \phi_x^2 + \\ &+ (2\, \lambda_{t\phi} - \xi_{\phi\phi})\phi_t^2 + (2\, \xi_{\phi x} - \eta_{xx} + \eta_{tt})\phi_x - \\ &- (\lambda_{xx} + 2\, \xi_{\phi t} - \lambda_{tt})\phi_t - 2(\lambda_{\phi x} - \eta_{\phi t})\phi_x \phi_t - \\ &- \eta_{\phi\phi}\, \phi_x^3 - \phi_t^2 \phi_x) - \lambda_{\phi\phi}(\phi_x^2 \phi_t - \phi_t^3) + \xi_{xx} - \xi_{tt} \end{aligned}$$

Invariance of the equation (2.25) then implies in the conditions:

$$\eta_\phi = 0 \qquad (2.26\text{-}1)$$

$$\lambda_\phi = 0 \qquad (2.26\text{-}2)$$

$$\eta_x = \lambda_t \qquad (2.26\text{-}3)$$

$$\eta_t = \lambda_x \qquad (2.26\text{-}4)$$

$$2\,\xi_{\phi x} - \eta_{xx} + \eta_{tt} = 0 \qquad (2.26\text{-}5)$$

$$\lambda_{xx} + 2\,\xi_{\phi t} - \lambda_{tt} = 0 \qquad (2.26\text{-}6)$$

$$\xi_{xx} - \xi_{tt} = 0 \qquad (2.26\text{-}7)$$

$$2\,\eta_{\phi x} - \xi_{\phi\phi} = 0 \qquad (2.26\text{-}8)$$

$$2\,\lambda_{t\phi} - \xi_{\phi\phi} = 0 \qquad (2.26\text{-}9)$$

Of these equations, the numbers 3 and 4 are characteristic not only of the Lorentz transformations but also of the conformal transformations in flat spacetime. The solution of the system of conditions (2.26) is of the form,

$$\begin{cases} \xi = a\phi + r(u) + s(v) & (2.27\text{-}1) \\ \eta = b + h(u) + g(v) & (2.27\text{-}2) \\ \lambda = c + h(u) - g(v) & (2.27\text{-}3) \end{cases}$$

for a, b and c three infinitesimal constants. Thus, we obtain an infinite parameter group of symmetry for (2.25). For obtaining the usual form for the Poincaré group and the special conformal group in the plane (x,t), we expand the functions h(u) and g(v) in Taylor series around the origin.

$$h(u) = h(0) + u\,h'(0) + u^2/2 \cdot h''(0) + \ldots \qquad (2.28\text{-}1)$$

$$g(v) = g(0) + v\,g'(0) + v^2/2 \cdot g''(0) + \ldots \qquad (2.28\text{-}2)$$

Up to linear terms in the parameters u, v we have the sub-group

$$\begin{cases} \eta = b + e.x + d.t \\ \lambda = c + d.x + e.t \end{cases} \qquad (2.29)$$

(note that for symplifying the notation we have denoted by the same symbol the constant factors in (2.27-2.3) even after the series expansion). In (2.29) we used the notation,

$$e = h'(0) + g'(0)$$
$$d = h'(0) - g'(0)$$

Thus, retaining the linear terms in the previous series expansion we obtain the Poincaré group in the plane (x,t) (with three parameters, the quantities b, c and d), and the scale transformations with parameter e. [5]

Going to the second order terms in the expansion (2.28), we get

$$\begin{cases} \eta = b + e.x + d.t + \frac{1}{2} w(x^2 + t^2) - k.xt \\ \lambda = c + d.x + e.t - \frac{1}{2} k(x^2 + t^2) + w.xt \end{cases} \qquad (2.30)$$

where,

$$w = h''(0) + g''(0)$$
$$-k = h''(0) - g''(0)$$

which represents the special conformal group with six parameters in the plane (x,t). The interpretation of the parameters being as usually:

translations: b, c
relative velocity parameter: d
scale parameter: e
relative acceleration parameter: w
timelike acceleration parameter: k

Regarding subsequent references to this group, we will use the notation C_o for denoting it. As is known, in four dimensions each element of C_o contains fifteen parameters.

An extension to third order power of u, v in the previous series expansion, give the following structure:

$$\begin{cases} \eta = b + ex + dt + \frac{1}{2} w(x^2 + t^2) - kxt + \frac{1}{2} n \left(\frac{x^3}{3} + xt^2 \right) + \frac{1}{2} p\left(\frac{t^3}{3} + x^2 t\right) \\ \lambda = c + dx + et - \frac{1}{2} k(x^2 + t^2) + wxt + \frac{1}{2} p \left(\frac{x^3}{3} + xt^2 \right) + \frac{1}{2} n\left(\frac{t^3}{3} + x^2 t\right) \end{cases} \quad (2.31)$$

where,

$$n = h'''(0) + g'''(0)$$

$$p = h'''(0) - g'''(0)$$

these transformations add two more parameters to the previous case, the quantities n, p. Therefore, here we have an eight parameter group.

With respect to the dimensions, we are considering c = 1, therefore space and time have the same dimension. Thus, the parameters e and d have no dimension (the later representing a velocity); the parameters b, c have the dimension of length. The parameters w and k have the dimension of the inverse of length, and finally n and p have the dimension of the inverse of the square of a length (a super-acceleration).

The present analysis of considering further approximations of the exact transformation equations (2.27) may be continued, but we will not do this here, since this process is a straightforward continuation of what we did up to now.

A different type of symmetry group arises in the event that we consider the identity mapping onto the plane (x, t), that is, $\eta = 0$ and $\lambda = 0$; with the further restriction that $a = 0$. For this case, the transformation equations are simply

$$\begin{cases} \phi'(x) = \phi(x) + r(u) + S(v) \\ x' = x \\ t' = t \end{cases}$$

For this situation, the infinitesimal function $\xi(x, t)$ satisfies the wave equation

$$\xi_{xx} - \xi_{tt} = 0$$

consequently, this particular symmetry transformation just express the known superposition property for solutions of a linear differential equation, namely, it mapps a given translation surface $\phi = f(u) - \Psi(v) = 0$ into another nearby similar surface, as : $\phi' - f'(u) - \Psi'(v) = \phi - \xi(x, t) - f(u) - \Psi(v)=0$, for $\xi = r(u) + s(v)$. This symmetry mapping may be regarded as a gauge transformation for the scalar [6] wave function $\phi(x,t)$. In passage we note that here we have no gauge invariant quantities involving first order derivatives of ϕ, as is the case for a vector wave equation. The only gauge invariant quantity which can be formed with ϕ is given by the left hand side of the wave equation

$$\Box\phi' = \Box\phi = 0$$

and thus, it vanishes over all region of the parameters x, t.

Turning back to the general second order differential equation written on page 21, we see that for the choice of coefficients as,

$$A = 1,\ B = D = E = 0,\quad C = -1$$

(and writing $u = x$, $v = t$) we get the hyperbolic second order differential equation which we called as the wave equation. Its symmetry group was determined subsequently. Another possible choice might be

$$A = 1,\ B = D = E = 0,\ C = 1$$

In this case we obtain an elliptic second order differential equation. If we still keep the same notation of using x, t as the parameters, the equation is the Laplace equation

$$\phi_{xx} + \phi_{tt} = 0 \tag{2.32}$$

It should be observed, a priori, that here t displays the character of a spacelike direction, similar to that of x, as compared with the case for the hyperbolic differential equation. For getting this conclusion, we use the property that in special relativity the metric is indefinite, and thus favours the hyperbolic character of the linear second order differential equation. Equivalently, an hyperbolic second order linear differential equation has the Poincaré group as the symmetry group in the linearization of its full symmetry group. These results will be mathematically stated by the determination of the symmetry Lie group for the elliptic second order linear equation (2.32). A calculus similar to that made for the corresponding hyperbolic equation (see the calculations preceeding the Eq. (2.26)) gives the following conditions for the symmetry mapping functions ξ, η and λ,

$$\eta_\phi = 0 \tag{2.33-1}$$

$$\lambda_\phi = 0 \tag{2.33-2}$$

$$\eta_x = \lambda_t \tag{2.33-3}$$

$$\lambda_x = -\eta_t \tag{2.33-4}$$

$$2\,\eta_{\phi x} - \xi_{\phi\phi} = 0 \tag{2.33-5}$$

$$2\,\lambda_{t\phi} - \xi_{\phi\phi} = 0 \tag{2.33-6}$$

$$2\,\xi_{\phi x} - \eta_{xx} - \eta_{tt} = 0 \tag{2.33-7}$$

$$2\,\xi_{\phi t} - \lambda_{xx} - \lambda_{tt} = 0 \tag{2.33-8}$$

$$\lambda_{\phi x} + \eta_{\phi t} = 0 \tag{2.33-9}$$

$$\xi_{xx} + \xi_{tt} = 0 \tag{2.33-10}$$

working out these conditions we arrive at the following simplified relations:

$$\xi = a\phi + \Psi(x, t) \tag{2.34}$$

for Ψ a solution of

$$\Psi_{xx} + \Psi_{tt} = 0$$

and

$$\eta_{xx} + \eta_{tt} = 0 \tag{2.35}$$

$$\lambda_{xx} + \lambda_{tt} = 0 \tag{2.36}$$

where η and λ satisfy the conditions (2.33-3) and (2.33-4). A trivial solution of the Laplace's equation which satisfy these last two conditions is the linear representation

$$\begin{cases} x' = x + \alpha x + \beta t + \gamma \\ t' = t - \beta x + \alpha t + \delta \end{cases} \tag{2.37}$$

Under this transformation, a point with coordinates $\binom{x}{t}$ is transformed by the matrix (do not consider here the parameters γ and δ)

$$U = (1 + \alpha)\cdot I + \Lambda$$

where I is the 2 x 2 unit matrix, and Λ is the skew symmetric matrix

$$\Lambda = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

in the event that the scale factor α is zero, the matrix U is orthogonal and thus β represents a rotation parameter. For $\alpha \neq 0$ evidently U is not orthogonal. Then, the linear symmetry mappings are here the scale transformations and the rotation in the plane of the parameters (x, t). This makes clear that here x and t are similar quantities, this fact is a characteristic of a positive definite metric. For the case of the hyperbolic equation the rotation is not present since we dispose only of one space-like direction. Evidently, in a general example with n dimensions, or in particular with four dimensions as is the situation in relativity, the rotations will be part of the symmetry group.

Now we extend the symmetry group to quadratic terms in the parameters x, t. Writing

$$\begin{cases} \eta = \eta_L + \frac{1}{2} ax^2 + bxt + \frac{1}{2} st^2 \\ \lambda = \lambda_L + \frac{1}{2} ex^2 + fxt + \frac{1}{2} gt^2 \end{cases}$$

where η_L and λ_L refer to the linear transformations given by (2.37). We impose that these mapping functions satisfy the Eqs. (2.35) and (2.36), as result we get the conditions:

$$a + s = 0$$
$$e + g = 0$$

so that

$$\begin{cases} \eta = \eta_L + \frac{1}{2} ax^2 + bxt - \frac{1}{2} at^2 \\ \lambda = \lambda_L + \frac{1}{2} ex^2 + fxt - \frac{1}{2} et^2 \end{cases}$$

Imposing further that (2.33-3) and (2.33-4) are satisfied, we obtain that $a = f$, $b = -e$. Consequently, the correct transformation equation up to quadratic terms in x,t will be:

$$\begin{cases} \eta = \eta_L + \frac{1}{2} ax^2 + bxt - \frac{1}{2} at^2 \\ \lambda = \lambda_L - \frac{1}{2} bx^2 + axt + \frac{1}{2} bt^2 \end{cases} \qquad (2.38)$$

This group also contains six parameters in similarity with the special conformal group seen before. However, it cannot be made similar to the conformal group since the Lorentz transformations are not present (they are replaced by the rotations). Nevertheless, since in some sense there exists a connection between an hyperbolic and an elliptic equation by the process of considering the non relativistic limit of the first type of equation, we might expect that in this process we can obtain a similar limit for both symmetry groups. This is indeed the case, but for showing this we have to take first of all the two following steps:

i) the parameter β has to be put equal to zero (the previous symmetry between x and t has to be broken, since now t will be made an absolute quantity, modulo the scaling factors);

ii) we have to use explicitly the constant c (speed of light in vacuum).

Taking this into consideration, we rewrite (2.38) as

$$\begin{cases} x' = x + \alpha x + \gamma + \frac{1}{2}\,\frac{a}{c^2}\,x^2 + bxt - \frac{1}{2}\,\frac{a}{c^2}\,(c^2\ t^2) \\ t' = t + \alpha t + \delta - \frac{1}{2}\,\frac{b}{c^2}\,x^2 + \frac{a}{c^2}\,xt + \frac{1}{2}\,bt^2 \end{cases}$$

note that b has dimension T^{-1}, similarly to the k used before. Taking the non relativistic limit of these transformation equations, we get

$$\begin{cases} x' \approx x + \alpha x + \gamma + bxt - \frac{1}{2}\,at^2 \\ t' \approx t + \alpha t + \delta + \frac{1}{2}\,bt^2 \end{cases}$$

Now, the Galilean transformation is obtained by noting that since t is absolute, modulo the scalling factors, the translation factor γ may be replaced by an usual rigid translation plus the translation term $- v_0 t$,

$$\gamma \longrightarrow \gamma - v_0 t$$

and we get:

space translations: $\gamma - v_0 t - \frac{1}{2}\,at^2$

(the parameter a is an acceleration)

space scale factor: $(\alpha\ + bt)$

(note the existence of a scale factor depending linearly on t)

time translation: δ

time scale factor: $\alpha + \frac{1}{2}\,bt$

(note the existence of a scale factor depending linearly on t).

But this is exactly the same structure which is obtained in the non relativistic limit of the special conformal group derived for the hyperbolic second order differential equation.

Independently of this conclusion, we will show in the next section that the transformations (2.38) act as a conformal transformation in a space (x,t) with positive definite Euclidian metric. From the viewpoint of physics this group has no interest, unless its non relativistic limit is taken, according to the process of taking $\beta = 0$. In this case it goes over the non relativistic conformal group (the Galilean special conformal group).

For closing up this section, we consider the symmetry Lie group for the Klein-Gordon equation. It should be noted, that since the beginning of this section we are considering just one field component ϕ. This is done by two reasons: first, it allows a direct geometrical interpretation of the formalism in terms of a certain family of surfaces on the three-space (ϕ, x, t). Second, it represents the simplest generalization of the formalism treated by Hill for total differential equations. Obviously, the analytical process here developed may be extended for a field with several components, and this will be treated in a next section.

Consider the quantity

$$\mathscr{D}(\phi) \equiv \phi_{xx} - \phi_{tt} - m^2 \phi^2 = 0 \qquad (2.39)$$

As is well known, the conformal invariance of this equation is reached only after taking into account a variation in the mass m,

$$m' = m + \delta m$$

Due to this, we a priori consider m as variant under the Lie transformations. The same process of transforming the left hand side of (2.39) as was done before for the cases considered, gives the following conditions for the ξ, η, λ and δm, which hold in order to have the invariance of the equation.

$$\xi = 0$$

$$\frac{\delta m}{m} = - \eta_x$$

$$\eta_x - \lambda_t = 0$$

$$\lambda_x - \eta_t = 0$$

$$\lambda_\phi = \eta_\phi = 0$$

Clearly, the solution of these equations is similar to that obtained for the wave equation, with the only modification that now m varies according to the previous relation. Thus, here η and λ have the form given by (2.27-2) and (2.27-3). It is interesting to get the explicit variation in m up to cubic powers of x,t in the transformation function η. Using (2.31) we get,

$$- \frac{\delta m}{m} = e + wx - kt + \frac{1}{2} (nx^2 + nt^2) + pxt \qquad (2.40)$$

in this formula only the conformal coefficients appear (the e, w and k), besides this, there is also contribution of the coefficients of the higher powers in (x,t), the quantities n and p. This already shows that the terms in third power of (x,t) in the mapping functions are also related to conformal mappings.

3. THE CHANGE IN THE LINE ELEMENT UNDER THE SYMMETRY LIE MAPPINGS

In this section we study the variation of the line element on the parameter plane (x,t). Since on this plane the equations considered up to now are all associated to a flat spacetime geometry, we start with the two possible situations:

i) a pseudo-Euclidian geometry;

ii) an Euclidian geometry.

The first case holding for the equations of the hyperbolic structure, and the last case for the elliptic equations. The first situation, which is the more important , is treated first. We have

$$\begin{cases} dx' = dx + \eta_x\, dx + \eta_t\, dt \\ dt' = dt + \lambda_x\, dx + \lambda_t\, dt \end{cases}$$

using the conditions (2.26-3) and (2.26-4), we get

$$\begin{cases} dx' = dx + \eta_x\, dx + \eta_t\, dt \\ dt' = dt + \eta_t\, dx + \eta_x\, dt \end{cases}$$

Up to first order terms, the variation on the line element $ds^2 = dx^2 - dt^2$, will be,

$$ds'^2 = (1 + 2\,\eta_x)ds^2 \tag{3.1}$$

Going up to cubic powers in the transformation law, we get for this conformal changing in ds^2:

$$ds'^2 = (1 + 2(e + wx - kt + 1/2(x^2 + t^2)n + pxt))ds^2 \tag{3.2}$$

The several terms on this expression may be written in compact notation as follows: call the pseudo-Euclidian metric by η_{ij} = diag. (1, -1), x^i = = (x,t), a^i = (w, k) where the last component is timelike. Then, the covariant a_i has components (w, -k), and

$$\eta_{ij}\, a^i\, x^j = a_j\, x^j = wx - kt$$

Introducing the matrix

$$b_{ij} = \frac{1}{2}\begin{pmatrix} n & p \\ p & n \end{pmatrix}; \qquad \eta^{ij}\, b_{ij} = 0$$

so that

$$b_{ij}\, x^i\, x^j = 1/2(x^2 + t^2)\, n + pxt$$

we have,

$$ds'^2 = (1 + 2(e + \eta_{ij}\, a^i\, x^j + b_{ij}\, x^i x^j))ds^2 \qquad (3.3)$$

This shows that the cubic powers on x, t on the transformation (2.31) are also conformal mappings. Note that the variation in the mass in the Klein-Gordon equation is given in terms of the same expression which appears in the variation of ds^2. This is a well known result.

For the case of Euclidian geometry, we have from (2.33-3) and (2.33-4)

$$\begin{cases} dx' = dx + \eta_x\, dx + \eta_t\, dt \\ dt' = dt - \eta_t\, dx + \eta_x\, dt \end{cases}$$

Therefore, to first order terms we obtain,

$$ds'^2 = (1 + 2\eta_x)\, ds^2$$

for, $ds^2 = dx^2 + dt^2$.

Then, the interpretation of the conformal mappings is similar to the previous case. As we have shown both groups coincide at a proper non relativistic limit.

4. GENERALIZATION FOR MORE THAN ONE INDEPENDENT VARIABLE

Presently we treat the case where there exists several quantities similar to the ϕ considered previously. Following the original analogy with particle dynamics, this corresponds to several motions representing the dynamics of a system of particles. Calling by $x^i(t)$ the coordinate x of the particle "i" at time t (here we also take "one-dimensional motions"), we have the correspondence:

$$x^i \longrightarrow \phi^i$$

$$t \longrightarrow (x,t)$$

In general i may run from 1 to n; for forming up a vector we take n = 2. It should be observed that this analogy is only formal as compared with the previous case of a scalar field, since now each x^i will correspond to a ϕ^i which is not a scalar, but is combined with all remaining ϕ^j for forming some geometrical object, for instance a vector.

A first look on the possible generalization of the transformation equations (2.9) for this situation should be to take:

$$\begin{cases} \phi'^i = \phi^i + \xi^i(\phi^j, x, t) \\ x' = x + \eta(\phi^j, x, t) \\ t' = t + \lambda(\phi^j, x, t) \end{cases} \tag{4.1}$$

for i, j = 1, 2. However, this will not conduct to the proper solution for a given characteristic equation, as for instance the vector wave equation.

$$(\partial^2_{xx} - \partial^2_{tt})\,\phi^i = 0$$

For obtaining the correct solution of the transformation symmetry group we have to add to ξ^i a dependence on the quantities $\eta^i_{,j}(\eta^1 = \eta,\ \eta^2 = \lambda)$.

$$\begin{cases} \phi'^i = \phi^i + \xi^i(\phi^j, \eta^\ell_{,m}, x^r) \\ x'^i = x^i + \eta^i(\phi^j, x^r) \end{cases} \tag{4.2}$$

A discussion of this question in done on the appendix, where we treat in details a more simple but mathematically related problem: the case for ϕ which transforms as an scalar density.

As before, the relations (4.2) are supplemented by the constraint conditions

$$\chi^i \equiv \phi^i - f^i(x, t) = 0 \tag{4.3}$$

which exist on the abstract four-spaced ϕ^1, ϕ^2, x, t. That is, we introduce into this space two surfaces by considering that the four-space splits into two spaces of the type (ϕ, x, t) each one with its family of surfaces of the form (4.3).

5. GENERALIZATION FOR A FOUR DIMENSIONAL PARAMETER SPACE

Here we consider the situation for an unique independent variable ϕ, but introduce four quantities x^i (latin indices, in this section indicate a variation from 1 to 4) as parameters. Since this, is the case, in general, for relativity, this situation is of special interest. The Lie transformations are,

$$\begin{cases} x'^i = x^i + \eta^i(\phi, x^m) \\ \phi' = \phi + \xi(\phi, x^m) \end{cases}$$

The invariance of the wave equation (the metric η_{ij} has signature +2)

$$\eta^{ij} \phi_{;ij} = 0$$

under these transformations, imply in the conditions:

$$\eta^1_{,1} = \eta^2_{,2} = \eta^3_{,3} = \eta^4_{,4} \tag{5.1}$$

$$\eta^{\alpha,\beta} + \eta^{\beta,\alpha} = 0, \qquad \alpha \neq \beta \tag{5.2}$$

$$\eta^{\alpha,4} + \eta^{4,\alpha} = 0 \tag{5.3}$$

$$\eta^i_{,\phi} = 0 \tag{5.4}$$

$$\xi_{,\phi\phi} = 0 \quad {}^{7} \tag{5.5}$$

$$\eta^{ij} \xi_{,ij} = 0 \tag{5.6}$$

$$\eta^{ij} \eta^r_{,ij} - 2\eta^{ri} \xi_{,\phi i} = 0 \tag{5.7}$$

(in this section greek indices go from 1 to 3). Independently of the possible solutions for these equations, we can check directly how the transformations behave in the space of parameters (x^i). For doing this we need only to know that the mapping functions η^i satisfy the conditions (5.1) through (5.4). The change into the line element

$$ds = \eta_{ij}\, dx^i\, dx^j$$

is given by

$$ds'^2 = \eta_{ij}\, dx'^i\, dx'^j = ds^2 + (\eta_{\alpha,\gamma} + \eta_{\gamma,\alpha})\, dx^\alpha\, dx^\gamma +$$

$$+ 2\, \eta_{\alpha,4}\, dx^\alpha\, dx^4 - 2\, \eta^4{}_{,\alpha}\, dx^\alpha\, dx^4 - 2\, \eta^4{}_{,4}(dx^4)^2$$

Using (5.1) through (5.3), we get

$$ds'^2 = ds^2 + 2\, \eta^4{}_{,4}\, ds^2$$

Denoting η^1 by η, so as to accord for the notation used previously, and using again the equation (5.1), we obtain a conformal variation similar to that obtained in two dimensions (see Eq. (3.1)).

$$ds'^2 = (1 + 2\, \eta_x) ds^2 \,.$$

We now study the solutions for the equations (5.4) through (5.7). First of all, we will see that in four dimensions the conformal transformations of C_0 do not satisfy the equation $\Box\eta^i = 0$, verified in two dimensions (here we also use the fact η^i does not depend on ϕ, as is seen from (5.4)). Indeed, from the general formula (A.12) of the appendix, we have to first order,

$$\eta^i = -\frac{1}{2} a^i \eta_{jk} x^j x^k + \eta_{jk} a^j x^k x^i$$

(note that Eqs. (5.1) through (5.4) are satisfied), then:

$$\Box\eta^i = -a^i \eta^{jk} \eta_{jk} + 2 a^i ; \quad \eta^{jk} \eta_{jk} = \delta^j_j = n$$

which shows that for two dimensions (n=2) this expression vanishes, but for n=4 it will not vanish, instead, it takes the value

$$\Box\eta^i = - 2 a^i \tag{5.8}$$

Therefore, the solutions for the conditions (5.4) through (5.7), here, will be different from those corresponding to the same situation in two dimensions. The solutions have the form,

$$\xi = A(x^i)\phi + b\phi + c + U(x^i) \tag{5.9}$$

such that

$$\Box\xi = 0$$

and η^i is given as the solution of the equation

$$\Box\eta^i = 2 \eta^{ij} A_{,j} \tag{5.10}$$

besides this, η^i, the solution of (5.10), is restricted by the conditions (5.1) through (5.3). The quantities b and c are infinitesimal constants, and $A(x^i)$ is an infinitesimal function of the parameters x^i, the same holding for $U(x^i)$.

The quantity $A(x^i)$ will determine the structure of the symmetry group for the wave equation in this case. Again it should be noted the difference of this situation with the corresponding one for two dimensions. For this last situation the equations determining the descriptors η^i may be separated completely from the equation which determines ξ. In this process the determination of the full conformal group on the plane (x,t) follows its own pattern independently of what happens for ξ, and consequently any "spin" with respect to this group on the plane (x,t) has to be introduced initially in the descriptor ξ as was done in section (4).

The simplest choice:

$$A(x^i) = 0, \; b = \text{scale factor}$$

gives the linearized transformations of the Poincaré group plus scale transformations on the parameter space (x^r),

$$\xi = b\phi + c + U(x^i), \; \Box U = 0$$

$$\eta^i = (\eta, \omega, \rho, \lambda) = (\eta^\alpha, \eta^0)$$

$$\begin{cases} \eta = c_1 + \epsilon x + \alpha y + \gamma z + d_1 t \\ \omega = c_2 - \alpha x + \epsilon y + \nu z + d_2 t \\ \rho = c_3 - \gamma x - \nu y + \epsilon z + d_3 t \\ \lambda = c_4 + \vec{d}\cdot\vec{x} + \epsilon t \end{cases}$$

where we have used the conditions (5.1) through (5.3). The four parameters $(c_0, \vec{c})$ represent the translations, the (α, γ, ν) the three possible rotations, the d represents the relative velocity parameter (with a minus sign) and finally, ϵ is the scale parameter,

The next possible choice is

$$A(x^i) = - a_r x^r, \quad b = \text{scale factor}$$

From (5.10) we have,

$$\Box\eta^i = - 2 a^i \tag{5.11}$$

which shows from (5.8) that this case corresponds to incorporate the special conformal transformations of C_0 in the symmetry group.

The general solution of (5.11) is of the form

$$\eta^i = \overset{(o)}{\eta}{}^i + \overset{(1)}{\eta}{}^i \tag{5.12}$$

for $\overset{(o)}{\eta}{}^i$ the general solution of the homogeneous equation, in the case, the descriptors of the Poincaré plus scale transformations; and $\overset{(1)}{\eta}{}^i$ a solution of (5.11), in the case, the descriptors of the special conformal transformations. Therefore, for the present choice of $A(x^i)$ we conclude that C_0 is the symmetry group for the wave equation in four dimensions.

From (5.9) we have

$$\xi = -(a_i\, x^i)\phi + b\phi + c + U(x^i)$$

where here U is a solution of the equation

$$\Box U = 2\, a^{\ell}\, \phi_{,\ell}$$

writing,

$$b = -e$$

(note that we are using the notation of the previous sections, of indicating the scale parameter by the letter e) we have,

$$\xi = -(e + a_i\, x^i)\phi + c + U(x^i) \tag{5.13}$$

Noting that here (5.12) corresponds to the transformations (2.30), now put in four dimensional form, we have from (3.3), which holds for this case,

$$\eta_x = e + a_i\, x^i = \frac{1}{4}\, \eta^s_{\ ,s}$$

Then, (5.13) takes the form

$$\xi = -\frac{1}{4}\eta^{s}_{,s}\,\phi + c + U$$

Consequently, the wave equation in four dimensions, displaying invariance under C_0 will be the wave equation for an scalar density of weight 1/4 with respect to the conformal transformations of C_0 (an scalar for the Poincaré transformations) [8].

This conclusion was not necessarily verified in two-dimensions, since in this situation we may have an scalar wave equation invariant under C_0.

The process of giving values to $A(x^i)$ may be continued. Due to the special interest into the next term, we give here the next possible choice. Writing

$$A(x^i) = -a_r x^r + \alpha b_{rs} x^r x^s + \beta\, b\, \eta_{rs} x^r x^s \qquad (5.14)$$

where b is the trace of b_{rs} which in four-dimensions does not have to vanish, and α, β are two numerical constants to be determined. Then, the Equation (5.10) takes the form

$$\Box\eta^i = -2\,\eta^{ij}(a_j - 2\alpha\, b_{js} x^s - 2\beta b\, \eta_{js} x^s) \qquad (5.15)$$

For matching this result with the four-dimensional conformal transformation containin super-acceleration coefficients, we use the general formula written on the appendix which generalizes (A.12). However, this formula was determined in connection with the two-dimensional problem (which was the problem more extensively treated in this paper), where b = 0. Thus, we still can add to this formula a term like $c.b\, x^i$ to the point dependent translation factor $\alpha^i(x)$ of the appendix. Since from the beginning we do not know what should be the numerical factor in front of this new term, we put it as a numerical factor <u>c</u> also to be determined. Linearization of the general formula for the conformal transformation then gives:

$$\eta^i = -\frac{1}{2} a^i \eta_{jk} x^j x^k + \eta_{jk} a^j x^k x^i + \frac{2}{3} b_{jk} x^j x^k x^i - \frac{1}{3} \eta_{\ell m} b^i_j x^j x^\ell x^m$$
$$+ c.b\, x^i \eta_{jk} x^j x^k$$

computing its D'Alembertian, (in two-dimensions this gives $\Box\eta^i = 0$)

$$\Box\eta^i = -2 a^i - \frac{4}{3} b^i_k x^k + \left(\frac{4}{3} + 12\, c\right) b x^i \qquad (5.16)$$

Comparison of (5.15) with (5.16) gives, $\alpha = -\frac{1}{3}$; $\beta = \frac{1}{3} + 3c$. Then, we have for (5.14),

$$A(x^i) = -a_r x^r - \frac{1}{3} b_{rs} x^r x^s + \frac{1}{3} b\, \eta_{rs} x^r x^s + 3c.b\, \eta_{rs} x^r x^s$$

and from (5.9), [9]

$$\xi = (-e + A(x^i))\phi + c' + U \qquad (5.17)$$

Finally, the constant <u>c</u> is adjusted in order that this variation in ϕ be the correspondent to a variation of an scalar density with weight $\frac{1}{4}$ for the conformal transformations. This is equivalent to require that

$$-e + A(x^i) = -\frac{1}{4} \eta^i_{,i}$$

for the previous η^i involving a^i, b_{ij}, b and also the scale factor <u>e</u>. This is reached for $c = -\frac{1}{12}$, and at the same time we have to recalibrate the constants b_{ij} and b as: $b_{ij} \to \frac{2}{5} b_{ij}$, $b \to \frac{2}{5} b$.

As before the term U in (5.17) is such that $\Box\xi = 0$. Thus, (5.15) or (5.16) will have the form

$$\Box\eta^i = -2a^i - \frac{4}{3}\tilde{b}^i{}_k x^k + \frac{1}{3}\tilde{b}x^i$$

and its general solution is of the form,

$$\eta^i = \overset{(0)}{\eta}{}^i + \overset{(1)}{\eta}{}^i$$

for $\overset{(1)}{\eta}{}^i$ the descriptor containing acceleration and super-acceleration parameters, that is

$$\overset{(1)}{\eta}{}^i = -\frac{1}{2} a^i \eta_{jk} x^j x^k + \eta_{jk} a^j x^k x^i + \frac{2}{3}\tilde{b}_{jk} x^j x^k x^i - \frac{1}{3}\eta_{\ell m}\tilde{b}^i_j x^j x^\ell x^m$$
$$- \frac{1}{12}\tilde{b}\,\eta_{jk} x^i x^j x^k$$

where $\tilde{b}_{jk} = \frac{2}{5} b_{jk}$, $\tilde{b} = \frac{2}{5} b$. However, this formula still has to be consistent with the conditions (5.1) through (5.3). This implies that $\tilde{b}_{ij} = q\,\eta_{ij}$, for q an infinitesimal parameter. Substitution of this value for $\tilde{b}_{ij}$ in the above formula shows that the terms in $\tilde{b}_{ij}$ and $\tilde{b}$ cancel out. Therefore, there is no contribution of "super-acceleration" terms for the symmetry group in four-dimensions. Eventually, it may happen that certain combinations involving fourth-order powers of the coordinates may contribute to $\overset{(1)}{\eta}{}^i$, but we do not enter into further details, which involve a very long calculation. Then, we conclude that the relevant symmetry group in this case is C_0.

6. THE VARIATION IN THE FLAT SPACETIME GEOMETRY GENERATED BY THE LIE TRANSFORMATIONS AND THE CURVATURE

We may interpret the variation on the line element given by (3.1) which holds for two or four dimensions as a change on the metric tensor of the flat spacetime geometry. Then,

$$ds'^2 = (1 + 2\,\eta_x)\,\eta_{ij}\,dx^i\,dx^j$$

gives a new metric,

$$g_{ij} = (1 + 2\eta_x)\,\eta_{ij} \qquad (6.1)$$

the inverse to this conformally flat metric is,

$$g^{ij} = (1 - 2\,\eta_x)\,\eta^{ij}$$

The Christoffel symbols resulting from this metric and from its inverse are given by

$$\Gamma^i_{jk} = \delta^i_j\,\eta_{,x^1x^k} + \delta^i_k\,\eta_{,x^1x^j} - \eta^{im}\,\eta_{jk}\,\eta_{,x^1x^m}$$

It should be noted that the only part of the mappings which do not contribute to deviations from the Minkowskian geometry are the Lorentz transformations. For these transformations $\eta_x = 0$.

Now, given two arbitrary metrics $\overline{g}_{ij}$ and g_{ij} which are related by a conformal transformation,

$$\overline{g}_{ij} = e^{2\sigma}\,g_{ij}$$

$$\overline{g}^{ij} = e^{-2\sigma}\,g^{ij}$$

the corresponding curvature tensors are related by [10]

$$e^{-2\sigma}\overline{R}_{hijk} = R_{hijk} + g_{hk}\,\sigma_{ij} + g_{ij}\,\sigma_{hk} - g_{hj}\,\sigma_{ik} - g_{ik}\,\sigma_{hj} +$$

$$(g_{hk}\,g_{ij} - g_{hj}\,g_{ik})\Delta_1\sigma \qquad (6.2)$$

where,

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\,\sigma_{,j}:\ \sigma_{,ij} = (\sigma_{,i})_{|j} = (\sigma_{,j})_{|i}$$

$$\Delta_1\sigma = g^{ij}\,\sigma_{,i}\,\sigma_{,j}$$

and a vertical bar means covariant differentiation. The change in the components of the Ricci tensor is given by

$$\overline{R}_{ij} = \overline{g}^{hk}\,\overline{R}_{hijk} = R_{ij} + (n-2)\sigma_{ij} + g_{ij}\left[\Delta_2\sigma + (n-2)\Delta_1\sigma\right] \qquad (6.3)$$

where n is the dimension of the manifold (all latin indices go from 1 to n), and

$$\Delta_2\sigma = g^{ij}\,\sigma_{,ij} = g^{ij}\left(\frac{\partial^2\sigma}{\partial x^i\,\partial x^j} - \frac{\partial\sigma}{\partial x^k}\left\{{k \atop ij}\right\}\right)$$

Now we use the following notation: we denote the pair $(g_{ij}, \overline{g}_{ij})$ by (η_{ij}, g_{ij}) in order to keep the notation used in the beginning of this section. Therefore, originally we use cartesian coordinates in flat space time and consequently the affinity is zero, and

$$R_{hijk} = 0$$

according to the above notation, we denote $\overline{R}_{hijk}$ by R_{hijk}. To first order we have,

$$\sigma = \eta_x$$

Then,

$$\sigma_{,ij} = \frac{\partial^2 \eta_X}{\partial x^i \, \partial x^j} - \left\{ {\ell \atop ij} \right\} \frac{\partial \eta_X}{\partial x^\ell} = \eta_{,x^1 x^i x^j} \, ,$$

$$\Delta_1 \sigma \approx 0 \, ,$$

$$\sigma_{ij} \approx \sigma_{,ij} \, ,$$

$$\Delta_2 \sigma \approx \eta^{ij} \frac{\partial^2 \eta_X}{\partial x^i \, \partial x^j} = \Box \eta_X$$

And from (2.6) we obtain

$$R_{hijk} = \eta_{hk} \, \eta_{,x^1 x^i x^j} + \eta_{ij} \, \eta_{,x^1 x^h x^k} - \eta_{hj} \, \eta_{,x^1 x^i x^k} - \eta_{ik} \, \eta_{,x^1 x^h x^j} \quad (6.4)$$

For the Ricci tensor we have

$$R_{ij} = (n-2) \eta_{,x^1 \, x^i \, x^j} + \eta_{ij} \, \Box \eta_X \, . \quad (6.5)$$

In two dimensions this tensor vanishes, since in this case $\Box \eta_X = 0$,

$$R_{ij} = 0$$

Note that for two dimensions there will exist just one independent component for the Ricci tensor R_{ij}, since then $R_{12} = 0$, $R_{11} = - R_{22} = \Box \eta_X$. The index "2" use here corresponds to "4" of the four-dimensional case. The curvature tensor for two dimensions also has just one component, the quantity R_{1212}, with value

$$R_{1212} = \Box \eta_X = 0$$

This component is related to the scalar of curvature R, by $R_{1212} = R/2$. In this case R/2 is the Gaussian curvature in the parameter plane (x,t). Thus, all these quantities vanish for two dimensions.

The Riemann tensor in four-dimensions is given by (6.4) for all indices going from 1 to 4. Since in this case (for the four-dimensional hyperbolic equation) C_0 is the symmetry group, containing at most quadratic terms in the coordinates, we have that R_{hijk} vanishes.

Thus, the space-time geometry remains flat for all types of non-linear symmetry groups considered.

7. NOTE ON THE FIRST ORDER CONSISTENCY CONDITIONS OBTAINED FOR THE ELLIPTIC EQUATION

Consider the plane of the complex variables $Z = x + it$. Let $\omega = f(z)$ be a single valued and analytic function on the Z-plane. Write, $\omega = x' + it'$ Then, from (2.9), (2.33-1), (2.33-2) we have

$$\omega = x' + it' = x + \eta + i(t+\lambda) = u(x, t) + iv(x, t)$$

Since f(z) is analytic on the Z-plane, it satisfies the Cauchy-Riemann differential conditions,

$$u_x = v_t, \qquad u_t = - v_x$$

or

$$\eta_x = \lambda_t, \qquad \eta_t = - \lambda_x$$

which are the conditions (2.33-3), (2.33-4) obtained as part of the conditions for invariance of the elliptic second order equation. From a known result of the theory of complex functions, the transformation $\omega = f(z)$ is conformal if $f(z)$ is analytic on the Z-plane, and $f'(z) \neq 0$. Thus, the (2.33-3), ((2.33-4) just state the fact that the Lie symmetry transformations for the elliptic second order equation contain conformal transformations. Formally, this conclusion is extended for the hyperbolic equation by writing $t \to x_0 = = it$.

8. NON LINEAR PARTIAL DIFFERENTIAL EQUATION IN TWO DIMENSIONS AND THE SYMMETRY GROUP

For simplifying the treatment, we shall restrict to the case of an unique independent variable ϕ. In the previous sections we have considered only differential equations in flat spacetime, and the reason is that those equations usually do not contain further independent variables besides the ϕ (as for instance the wave equation, the Klein-Gordon equation etc.), and consequently they render a simple analysis. Evidently, the present method may be used for treating with several independent variables (as was already done on the section (4)), and thus, may be extended for curved spacetime, where we treat the metric g_{ij} and ϕ as independent variables, and the x,t as parameters. We chose the particular case of non-linear equations for treating by the first time with curved spacetimes.

We shall consider the following problem: we write a non-linear equation for ϕ in flat spacetime and determine its symmetry group. After this we pass to a curved spacetime and see what is the variation in the symmetry group, due to the transition $\eta_{ij} \rightarrow g_{ij}$ on the metric.

We take the equation,

$$\eta^{i\ell}\,\eta^{jk}\,\phi_{,ij}\,\phi_{,\ell k} = 0 \tag{8.1}$$

since the indices here go from 1 to 2, this equation is of the form

$$\phi_{xx}^2 + \phi_{tt}^2 - 2\,\phi_{xt}^2 = 0 \qquad (8\text{-}2)$$

From the formulas (2.14), (2.15) and (2.16) we get to first order:

$$\phi'^2_{x'x'} + \phi'^2_{t't'} - 2\,\phi'^2_{x't'} = \phi^2_{xx} + \phi^2_{tt} - 2\,\phi^2_{xt} + 2\,\xi_\phi(\phi^2_{xx} + \phi^2_{tt} - 2\,\phi^2_{xt}) +$$

$$+ \eta_\phi(-6\,\phi^2_{xx}\,\phi_x - 2\,\phi^2_{tt}\,\phi_x - 4\,\phi_{tt}\,\phi_{xt}\,\phi_t + 4\,\phi_{xx}\,\phi_{xt}\,\phi_t + 8\,\phi^2_{xt}\,\phi_x)$$

$$+ \lambda_\phi(-2\,\phi^2_{xx}\,\phi_t - 4\,\phi_{xx}\,\phi_{xt}\,\phi_x - 6\,\phi^2_{tt}\,\phi_t + 8\,\phi^2_{xt}\,\phi_t + 4\,\phi_{tt}\,\phi_{xt}\,\phi_x)$$

$$+ 4\,\eta_x(\phi^2_{xt} - \phi^2_{xx}) + 4\,\lambda_t(\phi^2_{xt} - \phi^2_{tt}) + 4(\eta_t - \lambda_x)\,\phi_{xx}\,\phi_{xt} +$$

$$+ 4\,(\lambda_x - \eta_t)\phi_{tt}\,\phi_{xt} - 2\,\eta_{\phi\phi}(\phi_{xx}\,\phi^3_x + \phi_{tt}\,\phi^2_t\,\phi_x - 2\,\phi_{xt}\,\phi^2_x\,\phi_t)$$

$$- 2\,\lambda_{\phi\phi}(\phi_{xx}\,\phi^2_x\,\phi_t + \phi_{tt}\,\phi^3_t - 2\,\phi_{xt}\,\phi_x\,\phi^2_t) + 2(2\,\eta_{\phi x} - \xi_{\phi\phi})(-\phi_{xx}\,\phi^2_x +$$

$$+ \phi_{xt}\,\phi_t\,\phi_x) + 2\,(2\lambda_{t\phi} - \xi_{\phi\phi})(-\phi_{tt}\,\phi^2_t + \phi_{xt}\,\phi_t\,\phi_x) +$$

$$+ 2(\xi_{\phi x} - \eta_{xx})\phi_{xx}\,\phi_x + 2(\xi_{\phi t} - \lambda_{tt})\phi_{tt}\,\phi_t + 2\,\lambda_{xx}\,\phi_{xx}\,\phi_t -$$

$$- 2\,\eta_{tt}\,\phi_{tt}\,\phi_x + 4(\lambda_{xt} - \xi_{x\phi})\phi_{xt}\,\phi_t + 4(\eta_{xt} - \xi_{\phi t})\phi_{xt}\,\phi_x$$

$$+ 4\,\lambda_{\phi x}(\phi_{xt}\,\phi^2_t - \phi_{xx}\,\phi_x\,\phi_t) + 4\,\eta_{\phi t}(\phi_{xt}\,\phi^2_x - \phi_{tt}\,\phi_x\,\phi_t)$$

$$+ 2\,\xi_{xx}\,\phi_{xx} + 2\,\xi_{tt}\,\phi_{tt} - 4\,\xi_{xt}\,\phi_{xt}$$

giving the following independent conditions,

$$\eta_\phi - 0, \qquad \lambda_\phi = 0$$

$$\eta_x = \lambda_t$$

$$\eta_t = \lambda_x$$

$$\lambda_{xx} = \eta_{xx} = \lambda_{tt} = \eta_{tt} = 0$$

$$\xi_{\phi\phi} = \xi_{\phi t} = \xi_{\phi x} = \xi_{xt} = \xi_{xx} = \xi_{tt} = 0$$

with solutions,

$$\begin{cases} \xi = k + bx + ct & (8.3\text{-}1) \\ \eta = a + dx + ft & (8.3\text{-}2) \\ \lambda = e + fx + dt & (8.3\text{-}3) \end{cases}$$

The mappings on the plane (x,t) are here the Poincaré transformations and scale transformations (the Poincaré transformations have to appear in any case, since we have taken initially the differential equation in relativistic form). The transformation on ϕ given by the first equation (8.3) is here a trivial symmetry transformation since the differential equation is of second order. Therefore, we see that the non-linearity of the differential equation in flat spacetime has the effect of a very large restriction on the symmetry group.

Now, we go over a curved spacetime, with a metric g_{ij} which is not a field of inertia. In two dimensions, g_{ij} has three independent factors, the g_{11}, g_{12} and g_{22}. The equation (8.1) takes the form,

$$g^{i\ell} g^{jk} \phi_{,ij} \phi_{,\ell k} - 2 g^{i\ell} g^{jk} \{^{r}_{\ell k}\} \phi_{,ij} \phi_{,r} + g^{i\ell} g^{jk} \{^{m}_{ij}\}\{^{r}_{\ell k}\}\phi_{,m}\phi_{,r} = 0 \qquad (8.4)$$

Here, the independent variables are the g_{ij} and ϕ. However, we take g_{ij} as given, since otherwise the equation has to be sypplemented by the field equation for the g_{ij}, and the discussion would be much too complicated. According to our method, the full Lie group now acts on a six dimensional embedding space where the axis are represented by g_{11}, g_{12}, g_{22}, ϕ, x, t. The transformations on this space in infinitesimal form are,

$$\begin{cases} g'_{ij} = g_{ij} + \omega_{ij}(g_{\ell k}, \phi, x, t) & (8.5\text{-}1) \\ \phi' = \phi + \xi(g_{\ell k}, \phi, x, t) & (8.5\text{-}2) \\ x' = x + \eta(g_{\ell k}, \phi, x, t) & (8.5\text{-}3) \\ t' = t + \lambda(g_{\ell k}, \phi, x, t) & (8.5\text{-}4) \end{cases}$$

The "constraint surface" in three-space (ϕ, x, t) given by (2.1) for the family of solutions of (8.4) is determined in principle by the knowledge of the further constraints [11]

$$\chi_{ij} \equiv g_{ij} - \varphi_{ij}(x,t) = 0$$

On the region of the six dimensional embedding space where the four constraint conditions $\Psi = 0$, $\chi_{ij} = 0$ hold, we can compute the variation on the first derivatives of g_{ik} by an extension of the same calculation done before for $\phi_{,i}$. We find, [12]

$$g'_{ij,s'} = g_{ij,s} + \omega_{ij}{}^{,\underline{\ell k}}\, g_{\ell k,s} + \omega_{ij,\phi}\, \phi_{,s} + \omega_{ij,s} -$$

$$- \eta^{r,\underline{\ell k}}\, g_{\ell k,s}\, g_{ij,r} - \eta^{r}_{,\phi}\, g_{ij,r}\, \phi_{,s} - \eta^{r}_{,s}\, g_{ij,r} \qquad (8.6)$$

for,

$$\omega_{ij}{}^{,\underline{\ell k}} = \frac{\partial \omega_{ij}}{\partial g_{\ell k}}, \quad \eta^{r,\underline{\ell k}} = \frac{\partial \eta^r}{\partial g_{\ell k}}$$

The Equation (8.6) is reducible to the usual variation of the derivatives of a Riemannian metric under coordinate mappings in the case where

$$\eta^r_{,\phi} = 0, \ \eta^{r,\underline{\ell k}} = 0, \quad \omega_{ij,\phi} = 0$$

$$\omega_{ij} = - \eta^s_{,j}\, g_{is} - \eta^s_{,i}\, g_{sj}$$

Similarly, on this region we can determine the expressions for the variations of $\phi_{,i}$ and $\phi_{,ij}$. These expressions are the generalizations of the formulas (2.17) and (2.18), for the case of a curved parameter plane (x,t) (where exists a metric g_{ij} which is a dynamical quantity). The Equations (2.17) and (2.18) are associated to a flat parameter plane (x, t). The results are,

$$\phi'_{,j'} = \phi_{,j} - \eta^{i}_{,\phi}\,\phi_{,i}\,\phi_{,j} - \eta^{i}_{,j}\,\phi_{,i} + \xi_{,\phi}\,\phi_{,j} + \xi^{,\underline{\ell k}}\,g_{\ell k,j} - \eta^{s,\underline{\ell k}}\,\phi_{,s}\,g_{\ell k,j} + \xi_{,j} \tag{8.7}$$

where,

$$\xi^{,\underline{\ell k}} = \frac{\partial \xi}{\partial g_{\ell k}}$$

and,

$$\begin{aligned}
\phi'_{,r'm'} = {} & \phi_{,rm} - \eta^{i}_{,\phi}\left[\phi_{,i(r}\,\phi_{,m)} + \phi_{,i}\,\phi_{,rm}\right] - \eta^{i}_{,(r}\,\phi_{,m)i} - \\
& - \eta^{i}_{,\phi\phi}\,\phi_{,i}\,\phi_{,r}\,\phi_{,m} - \eta^{i}_{,\phi(r}\,\phi_{,m)}\,\phi_{,i} - \eta^{i}_{,rm}\,\phi_{,i} \\
& + \xi_{,\phi\phi}\,\phi_{,r}\,\phi_{,m} + \xi_{,\phi(r}\,\phi_{,m)} + \xi_{,\phi}\,\phi_{,rm} + \xi_{,rm} \\
& + \xi^{,\underline{\ell k}}\,g_{\ell k,rm} - \eta^{s,\underline{\ell k}}\left[g_{\ell k,(m}\,\phi_{,r)s} + g_{\ell k,rm}\,\phi_{,s}\right] \\
& + g_{\ell k,(r}\,\xi^{,\underline{\ell k}}_{,m)} + \xi'^{\underline{\ell k}}_{,\phi}\,g_{\ell k,(m}\,\phi_{,r)} - \eta^{s,\underline{\ell k}}_{,\phi}\,\phi_{,s}\,g_{\ell k,(r}\phi_{,m)} \\
& - g_{\ell k,(m}\,\eta^{s,\underline{\ell k}}_{,r)}\,\phi_{,s} + \xi'^{\underline{\ell k}\,\underline{ns}}\,g_{\ell k,r}\,g_{ns,m} - \\
& - \eta^{s,\underline{\ell k}\,\underline{np}}\,\phi_{,s}\,g_{\ell k,r}\,g_{np,m}
\end{aligned} \tag{8.8}$$

where round brackets mean symmetrization over the indices, without the numerical factor 1/2. It should be noted that in spite of our present work on two dimensions, the formulas (8.5) through (8.8) are valid for any dimension.

Writing,

$$\phi'_{,i'j'} = \phi_{,ij} + \tau_{ij}$$

$$\phi'_{,i'} = \phi_{,i} + \chi_i$$

$$\left\{ {r \atop \ell k} \right\}' = \left\{ {r \atop \ell k} \right\} + \Omega^r_{\ell k}$$

where these infinitesimal increments are given respectively by (8.8), (8.7) and combinations of (8.6). We can obtain the following variation for the l.h.s. of the fieldequation (8.4), which for abbreviate the notation we denote by $\mathscr{D}$,

$$\mathscr{D}' = (1-\omega)\mathscr{D} + 2\left(-\tau^{ij} \left\{ {m \atop ij} \right\} + \chi_r\, g^{i\ell} g^{jk} \left\{ {m \atop ij} \right\} \left\{ {r \atop \ell k} \right\}\right) \phi_{,m}$$

$$+ 2\left(\tau^{ij} - \chi_r\, g^{i\ell} g^{jk} \left\{ {r \atop \ell k} \right\}\right) \phi_{,ij} + 2\, \Omega^s_{\ell k}\left(g^{\ell i} g^{jk} \left\{ {m \atop ij} \right\} \phi_{,m} \phi_{,s}\right.$$

$$\left. - g^{i\ell} g^{jk} \phi_{,ij} \phi_{,s}\right) \tag{8.9}$$

where we have explicitly used the fact that here we work on two dimensions. The ω is the trace of ω_{ij}, $\omega = g^{ij}\, \omega_{ij}$. Thus, the condition for invariance of the equation $\mathscr{D} = 0$, will be

$$2(-\tau^{ij}\left\{{m \atop ij}\right\} + \chi_r\, g^{i\ell}\, g^{jk}\left\{{m \atop ij}\right\}\left\{{r \atop \ell k}\right\})\ \phi_{,m} +$$

$$+ 2(\tau^{ij} - \chi_r\, g^{i\ell}\, g^{jk}\left\{{r \atop \ell k}\right\})\phi_{,ij} + 2\ \Omega^{s}_{\ell k}\ (g^{\ell i}\ g^{jk}\left\{{m \atop ij}\right\}\phi_{,m}\ \phi_{,s}$$

$$- g^{i\ell}\ g^{jk}\ \phi_{,ij}\ \phi_{,s}) = 0 \qquad (8.10)$$

A possible solution of these conditions is obtained for the usual symmetry group of general relativity, the M.M.G., which here is given as

$$\xi = 0$$

$$\eta^{i}_{,\phi} = 0,\ \eta^{i,\underline{\ell k}} = 0,\ \ \omega_{ij,\phi} = 0,\ \ \omega_{ij} = -\eta^{s}_{,j}\, g_{is} - \eta^{s}_{,i}\ \ g_{sj}$$

$$\chi_i = -\eta^{s}_{,i}\ \phi_{,s}\ ,\quad \tau_{ij} = -\eta^{s}_{,j}\ \phi_{,is} - \eta^{s}_{,i}\ \phi_{,js} - \eta^{s}_{,ij}\ \ \phi_{,s}$$

and where $\Omega^{r}_{\ell k}$ is given by the usual variation in the Christoffel symbols induced by point to point mappings of the M.M.G.

Presently we will not enter into further details of other possible solutions for the conditions (8.10), except for an special situation which will be discussed in the following.

There exists a situation which within the context of this work is of special interest: the Weyl group of conformal transformations of the metric tensor g_{ij}. This group will be denoted as C_g.[13] We try to see if C_g may be a symmetry group for (8.4).

The weyl group is obtained from our general transformations (8.5) by taking there,

$$\eta = 0$$

$$\lambda = 0$$

and by convenience, writing

$$\xi = 0$$

and trying to see if the symmetry group for (8.4) may have under these conditions a ω_{ij} of the form,

$$\omega_{ij}(x^r) = \Lambda(x^r) g_{ij}$$

that means, a ω_{ij} such that, for arbitrary $\Lambda(x^r)$,

$$\omega_{ij,\phi} = 0$$

$$\omega_{ij} \frac{\ell k}{} = \frac{1}{2} \Lambda(x^r)(\delta_i^\ell \, \delta_j^k + \delta_i^k \, \delta_j^\ell)$$

For this situation, we get

$$\chi_j = 0$$

$$\tau_{ij} = 0$$

$$\Omega^r_{\ell k} = \frac{1}{2} g^{rm}(\varepsilon_{\ell mk} + \varepsilon_{mk\ell} - \varepsilon_{\ell km}) - \Lambda g^{rm} \Gamma_{\ell k,m}$$

for,

$$\varepsilon_{\ell mk} = \frac{\Lambda}{2} (\delta^r_\ell \, \delta^s_m + \delta^s_\ell \, \delta^r_m) g_{rs,k} + \Lambda_{,k} \, g_{\ell m}$$

From (8.10) it follows that in this case there will be no solution for Λ different of the trivial solution $\Lambda = 0$. Therefore C_g is not a symmetry group for (8.4). This conclusion should be expected since for the Weyl formulation (in the sense of a conformal invariant theory in curved spacetime) the affinities have to be corrected. In other words, we may recover C_g as a symmetry group for (8.4) if we introduce compensating fields into the affinity. By this process we go from the Einstein affinity to the Weyl affinity which is conformal invariant under the Weyl group.

CONCLUSION

By an extension of a method proposed by Hill for determining the symmetry transformations for any given, isolated, dynamical system, we determine the symmetry transformations associated to free fields. By a free field we understand any isolated field. Thus, we may extend the present treatment also for interacting fields, case where we have the "free system" composed by the two interacting fields. In general, the symmetry transformations associated to a given field may be modified by the presence of another field, that is, by an interaction. As example, the second kind gauge symmetry of the coupled Dirac and Maxwell fields is not a symmetry of the free Dirac field.

The mathematical theory used by Hill was the Lie theory for ordinary differential equations, which presently is extended for partial differential equations. As compared with the usual treatment of field theory, the present approach has several different peculiarities. Fundamentally, we may trace the different ideas underlining both treatments, already to the situation treated by Hill in particle dynamics: Any chosen isolated dynamical system possess its own Lie transformations of symmetry, independently if they have a Lagrangian or not. Then, we do not look for all possible dynamical systems symmetric under a given transformation, for instance the Galilean transformation. In this last case, these systems are directly derived from a set of Lagrangians which present invariance under the Galilean transformations. Presently, rather, we consider each separated dynamical system (dynamical in the sense of the field equation) and look for its characteristic symmetries.

Field theory is based on the fundamental concept that the coordinate symmetry group is the Poincaré group. Our present method is not against this concept, since usually the Poincaré group is a symmetry group for a very large class of partial differential equations (those of hyperbolic character). Eventually, we may obtain its non-relativistic version (the Galilean group) depending on the type of equation under consideration. Clearly, we have to say that also the order of the differential equation is of importance. A first order partial differential equation has an infinite parametric Lie group of symmetry, which is much more general than the Poincaré group; however, no Lagrangian does exist for those equations.

The more important idea behind the present method, is that each isolated system "builds up" its own symmetry group. Now, depending on the existence, or not, of a Lagrangian (presenting invariance under these symmetry transformations), we may translate this into the known language of conservation laws (or not). If no Lagrangian does exist, such symmetries will not be observed as conservation laws. As is known, general relativity is a theory of this type: the gravitational field has its own symmetries, those of a curved spacetime, possessing (or not) eventually some number of isometries. In this sense, the gravitational field is the first example of a field which is separated from all the other (the coordinate symmetry group is more general than the Poincaré group).

Other important result which is obtained, is that the general Lie transformations has descriptors, associated to the coordinates, which may depend on the field functions. This is already a generalization of the Poincaré transformations. The existence, or not, of such generalized Lie transformations will depend, according to our method, of the characteristic field equation chosen. In this work, we have proven that no generalized transformations of this sort will appear if the characteristic equation is an usual field equation, that means, a Poincaré-covariant equation.

However, it is possible to generalize such "Poincaré-covariant" equation in such form that it presents a generalized symmetry of this form. A more detailed treatment of this problem will be done in a future paper.

Regarding this last result, of the existence of generalized point transformations, we may consider that a point in the manifold is completely determined by its coordinates plus the value of the field for these coordinates. We may then consider a point to point mapping of the generalized transformation as a transformation which mapps this type of "point" into another similar point. This is different from the point to point mappings of the Poincaré group, which does not depend on the field acting on the manifold. The existence of such generalized point to point mappings was already considered in the literature [14], more directly in connection to the case of general relativity [15].

According to our present method, we see that we may also consider the existence of such generalized coordinate mappings even in flat space-time. Paralelly to this last conclusion, it is interesting to note that the usual gauge transformations (electromagnetic, isotopic, etc.) may be formally generalized to gauge transformations where the gauge function depends on the potentials (electromagnetic, Yang-Mills, etc.) [16].

* * *

PPENDIX

1. Here we show in a direct way how to prove that the conditions (2.26) are the correct conditions for the invariance of the wave equation under the general mappings considered. We shall do this prove in the case where ϕ is an scalar, $\xi = 0$, and the mappings affect only the coordinate plane (x,t), that is $\eta_\phi = 0$, $\lambda_\phi = 0$. In this case the usual procedures for handling with tensors hold, and they are directly used for proving the invariance of the wave equation. Thus, under the mappings

$$\begin{cases} x' = x + \eta(x,t) \\ t' = t + \lambda(x,t) \end{cases}$$

with inverse,

$$\begin{cases} x = x' - \eta(x', t') \\ t = t' - \eta(x', t') \end{cases}$$

the left hand side of the flat-two-dimensional D'Alembertian varies as

$$\phi'_{x'x'} - \phi'_{t't'} = \phi_{xx} \frac{\partial x}{\partial x'} + \phi_{xt} \frac{\partial t}{\partial x'} - \phi_x \frac{\partial^2 \eta'}{\partial x'^2} - \phi_{xx} \frac{\partial \eta'}{\partial x'} \frac{\partial x}{\partial x'}$$

$$- \phi_{xt} \frac{\partial \eta'}{\partial x'} \frac{\partial t}{\partial x'} - \phi_t \frac{\partial^2 \lambda'}{\partial x'^2} - \phi_{xt} \frac{\partial \lambda'}{\partial x'} \frac{\partial x}{\partial x'}$$

$$- \phi_{tt} \frac{\partial \lambda'}{\partial x'} \frac{\partial t}{\partial x'} + \phi_{xx} \frac{\partial \eta'}{\partial t'} \frac{\partial x}{\partial t'} + \phi_{xt} \frac{\partial \eta'}{\partial t'} \frac{\partial t}{\partial t'}$$

$$+ \phi_x \frac{\partial^2 \eta'}{\partial t'^2} - \phi_{xt} \frac{\partial x}{\partial t'} - \phi_{tt} \frac{\partial t}{\partial t'} + \phi_t \frac{\partial^2 \lambda'}{\partial t'^2}$$

$$+ \phi_{xt} \frac{\partial \lambda'}{\partial t'} \frac{\partial x}{\partial t'} + \phi_{tt} \frac{\partial \lambda'}{\partial t'} \frac{\partial t}{\partial t'}$$

to first order this gives,

$$\phi'_{x'x'} - \phi'_{t't'} = \phi_{xx} - \phi_{tt} - 2 \phi_{xx} \eta'_{x'} - 2 \phi_{xt} \lambda'_{x'} + 2 \phi_{xt} \eta'_{t'} +$$

$$+ 2 \phi_{tt} \lambda'_{t'} - \phi_x(\eta'_{x'x'} - \eta'_{t't'}) + \phi_t(\lambda'_{t't'} - \lambda'_{x'x'})$$

Therefore, the conditions for the invariance of the wave equation are:

$$\lambda'_{x'} = \eta'_{t'}$$

$$\lambda'_{t'} = \eta'_{x'}$$

$$\Box' \eta' = 0$$

$$\Box' \lambda' = 0$$

Since the primes may be dropped due to the fact that the difference between $\eta' = \eta(x', t')$ and $\eta(x, t)$ is a second order term, we arrive at the conditions (2.26) for this type of approximation (neglecting ξ and the derivatives η_ϕ and λ_ϕ).

2. *Generalization for an scalar density of weight W, solution of the wave equation*

Here we generalize the scalar wave equation in order to obtain the same equation for an scalar density. That is, we have

$$(\partial_{xx} - \partial_{tt})\, \phi(x, t) = 0$$

but allow ϕ to transform as an scalar density under the restricted mappings on the plane (x, t). An example of such quantity is given if we consider the integral (here we use four dimensions

$$I(x) = \int \partial(x)\, d_4 x$$

originally we are in flat spacetime, but due to the non-linear [15] form assumed by the conformal part of the transformations, we have that

$$d_4 x' = \left| \frac{\partial x'}{\partial x} \right| d_4 x$$

for $\left| \frac{\partial x'}{\partial x} \right| \neq 1$, consequently ϕ has to be an scalar density with weight +1.

Presently we consider two dimensions because this simplifies the calculations, and all conclusions are directly extended to four dimensions.

For this case we have to modify the transformation equation (2.9), [16] since it implies that for a ϕ solution of the wave equation

$$\phi'(x'^i) = \phi(x^i) + a\phi(x^i) + U(x,t); \quad \Box U = 0,$$

and by expansion on the l.h.s.,

$$\phi'(x^i) = \phi(x^i) + a\, \phi(x^i) + U(x^i) - \eta^i\, \phi_{,i} \tag{A-1}$$

for a constant. But an scalar density with weight W changes as

$$\phi'(x^i) = \phi(x^i) - W\, \eta^s_{,s}\, \phi - \eta^i\, \phi_{,i} \tag{A-2}$$

which cannot be taken similar to (A-1) since $\eta^s_{,s}$ is not constant. Note that for the Lorentz transformations $\eta^s_{,s} = 0$.

The necessary modification on (2.9) is obtained by imposing that now ξ depends on the $\eta^s_{,s}$,

$$\begin{cases} \phi' = \phi + \xi(\phi, \eta^s_{,s}, x^i) \\ x'^i = x^i + \eta^i(\phi, x^s) \end{cases} \tag{A-3}$$

(the indices here go from 1 to 2). The formulas (2.17) and (2.18) now assume the form.

$$\phi'_{,i'} = \phi_{,i} + \xi_{,\phi}\, \phi_{,i} + \xi_\psi\, \eta^s_{,si} + \xi_{,i} - \eta^r_{,\phi}\, \phi_{,r}\, \phi_{,i} - \eta^r_{,i}\, \phi_{,r} \tag{A-4}$$

$$\begin{aligned} \phi'_{,r'm'} = {} & \phi_{,rm} - \eta^i_{,\phi}\left[\phi_{,i(r}\, \phi_{,m)} + \phi_{,i}\, \phi_{,rm}\right] - \eta^i_{,(r}\, \phi_{,m)i} \\ & - \eta^i_{,\phi\phi}\, \phi_{,i}\, \phi_{,r}\, \phi_{,m} - \eta^i_{,\phi(r}\, \phi_{,m)}\, \phi_{,i} - \eta^i_{,rm}\, \phi_{,i} \\ & + \xi_{,\phi\phi}\, \phi_{,r}\, \phi_{,m} + \xi_{,\phi(r}\, \phi_{,m)} + \xi_{,\phi}\, \phi_{,rm} + \xi_{,rm} \\ & + \xi_{,\phi\psi}\, \eta^s_{,s(r}\, \phi_{,m)} + \xi_{,\psi(r}\, \eta^s_{,m)} + \xi_\psi\, \eta^s_{,srm} \end{aligned} \tag{A-5}$$

From (A-5) we get the following conditions for the invariance of the wave equation:

$$\eta^{i}_{,\phi} = 0 \qquad \text{(A-6-1)}$$

$$\eta_x = \lambda_t, \; \eta_t = \lambda_x \qquad \text{(A-6-2)}$$

$$\eta^{k(j}\,\eta^{i)}_{,\phi k} - \eta^{ij}\,\xi_{,\phi\phi} = 0 \qquad \text{(A-6-3)}$$

$$\Box\eta^{r} - 2\,\eta^{ir}\,\xi_{,\phi i} + 2\,\xi_{\phi\psi}\,\eta^{s}_{,si}\,\eta^{ir} = 0 \qquad \text{(A-6-4)}$$

$$\Box\xi + 2\,\xi_{,\psi i}\,\eta^{s,i}_{,s} + \xi_{\psi}\,\Box\eta^{s}_{,s} = 0 \qquad \text{(A-6-5)}$$

In the formulas (A-4) through (A-6) we used the notation, $\psi = \eta^{s}_{,s}$. Clearly, in the first order approximation, ξ has to be linear in the quantity Ψ. Therefore we can write,

$$\xi(\phi, \Psi, x^{i}) = F(\phi, x^{i})\Psi + G(\phi, x^{i}) \qquad \text{(A-7)}$$

The solutions for the equations (A-6) together with the condition (A-7) is of the form

$$\Box\eta^{i} = 0 \qquad \text{(A-8)}$$

$$\xi = -2\,W\,\eta_x\phi + a\phi + U(x^{i}) \qquad \text{(A-9)}$$

for W̲ and a̲ two constants and,

$$\Box U = 0, \quad \Psi = 2\,\eta_x$$

Since here the Jacobian of the transformations in the plane (x, t) which satisfy (A-6-1) is (also use (A-6-2)).

$$\left| \frac{\partial x'}{\partial x} \right| = 1 + \eta^{s}_{,s} = 1 + 2\,\eta_x$$

we conclude that (A-9) describes an scalar density with weight W (note that W is a finite constant, in other terms, is a number).

For W=0 we recover the previous result given by (2-27-1). From the formal mathematical analogy between the equations of transformations (4.2-1) and (A.3-1), as well as, from the mathematical similarity of the transformation laws for an scalar density and a vector under infinitesimal mappings with $\eta^{i}_{,\phi} = 0$:

$$\phi'(x^r) = \phi(x^r) - W\,\eta^{s}_{,s}\,\phi - \eta^{s}\,\phi_{,s}$$

$$\phi'_i(x^r) = \phi_i(x^r) - \eta^{s}_{,i}\,\phi_s - \eta^{s}\,\phi_{i,s}$$

we can infer directly that the ξ_i for a field ϕ_i transforming as a vector under infinitesimal mappings with $\eta^{i}_{,\phi} = 0$, will be the natural extension of (A.9) for this case:

$$\begin{aligned} \xi_1 &= -\eta_x\,\phi_1 - \lambda_x\,\phi_2 + a\,\phi_1 + U_{,x} \\ \xi_2 &= -\eta_x\,\phi_2 - \lambda_x\,\phi_1 + b\,\phi_2 + U_{,t} \end{aligned} \qquad \text{(A.10)}$$

The terms which do not mix the components are the gauge contribution for the vector field.

The equations (A.10) hold for a ϕ_i which is a solution of the wave equation. The conditions (A.8) are verified here, and:

$$\eta_x = \lambda_t,\ \eta_t = \lambda_x,\ \ \Box U = 0$$

also a and b are two infinitesimal constants.

3. *Note on the determination of the conditions (2.26) for the parameter plane (x,t).*

In the determination of the solutions (2.27) to the equations (2.26), we used the property that the conditions (2.26-3) and (2.26-4) imply into wave equations for η and λ, and consequently the general form for the solution of these conditions is to set η and λ equal to:

$$\eta = b + h(u) + g(v)$$

$$\lambda = c + h(u) - g(v)$$

Then, by series expansion we arrive at (2.31) up to cubic powers in $\underline{x}$ and $\underline{t}$. However, we may obtain a similar expansion without using the equations

$$\Box\eta = 0, \quad \Box\lambda = 0$$

We use only the two fundamental conditions

$$\eta_x = \lambda_t$$

$$\eta_t = \lambda_x$$

Writing,

$$\eta_x = \lambda_t = \varphi(x, t)$$

$$\eta_t = \lambda_x = \Psi(x, t)$$

we get,

$$d\eta = \varphi(x,t)dx + \Psi(x,t)dt$$

$$d\lambda = \Psi(x,t)dx + \varphi(x,t)dt$$

therefore

$$\eta = a + \int \varphi(x,t)dx + \int \Psi(x,t)dt$$

$$\lambda = b + \int \Psi(x,t)dx + \int \varphi(x,t)dt$$

Expanding the functions $\varphi(x,t)$ and $\Psi(x,t)$ in power series of x and t, and performing the integrations, we obtain η and λ up to any desired order of approximation. Going up to cubic powers on x and t by this process, we arrive exactly at the formula (2.31).

4 - *Determination of the finite form for the transformation with descriptors given by (2.31).*

Lie proved that the transformations of the group $C^{(o)}$, the group which we have denoted as the "special conformal group", may be presented as the product of an inversion by a translation followed by a product of another inversion [7]. In this concern, scale transformations are not included. Since usually in the literature these last transformations (without scale) are called as the special conformal transformations, the result of Lie applies to these transformations.

We give a brief review of the concepts involved:An inversion through the origin, or transformation by reciprocal radii is defined by

$$x' = P_o(x) = \frac{x}{x \cdot x}, \quad P_o P_o = I$$

where x is a fourvector and $x \cdot x$ denotes the scalar product of x with itself. Performing a translation by amount -a,

$$x' = T_a(x) = x - a$$

It should be observed that all translations here are rigid translations, forming part of the Poincaré mappings. We have:

$$P_o\, T_a(x) = P_o(x-a) = \frac{x-a}{(x-a)\cdot(x-a)} = P_a(x)\,,$$

that is, as result we get an inversion through x = a.

Taking the product

$$C_a(x) = P_aP_o(x)=P_oT_aP_o(x) = P_oT_a\left(\frac{x}{x\cdot x}\right) = P_o\left(\frac{x}{x\cdot x} - a\right) = \frac{\frac{x}{x\cdot x} - a}{\left(\frac{x}{x\cdot x} - a\right)\cdot\left(\frac{x}{x\cdot x} - a\right)}$$

for a = 0 we get the identity transformation. An easy calculation gives:

$$x' = C_a(x) = \frac{x - a\,(x\cdot x)}{1-2a\,x + (a\cdot a)(x\cdot x)} \qquad \text{(A.11)}$$

This formula by the appropriate series expansion retaining only first powers of the constant four-vector $\underline{a}$ gives the infinitesimal conformal transformation involving the parameters of acceleration.

For our present choice on the constants in the equation (2.30), we have to choose the acceleration terms with a numerical factor 1/2:

$$a \to 1/2\ a$$

Thus, the (2.30) are the linearization of an expression like (A.11) there making $a \to 1/2\ a$:

$$x' = \frac{x - \frac{1}{2}\,a(x.x)}{1 - a.x + \frac{1}{4}\,(a.a)(x.x)} \qquad \text{(A.12)}$$

Now, it is not difficult to see directly what should be the modification on (A.12) for obtaining the similar formula for the transformations (2.31) involving the first term which is a deviation from $C^{(o)}$. We have to drop out the concept of kiping only rigid translations; considering the generalized point dependent translation [18].

$$x'^{\mu} = \tau_{\alpha}(x^{\mu}) = x^{\mu} - \frac{1}{2} a^{\mu} - \frac{1}{3} b^{\mu}{}_{\lambda} \; x^{\lambda} = x^{\mu} - \alpha^{\mu}(x)$$

Then, by a similar process as before, we write the generalized transformation

$$\mathcal{C}_{\alpha}(x^{\lambda}) = P_o \, \tau_{\alpha} \; P_o(x^{\lambda}) = P_o \, \tau_{\alpha} \left(\frac{x^{\lambda}}{x \cdot x}\right) = P_o \left(\frac{x^{\lambda}}{x \cdot x} - \alpha^{\lambda}\right) =$$

$$= \frac{\dfrac{x^{\lambda}}{x \cdot x} - \alpha^{\lambda}}{\eta_{\sigma\tau} \left(\dfrac{x^{\tau}}{x \cdot x} - \alpha^{\tau}\right) \cdot \left(\dfrac{x^{\sigma}}{x \cdot x} - \alpha^{\sigma}\right)}$$

After some calculations we arrive at

$$x'^{\lambda} = \mathcal{C}_{\alpha}(x^{\lambda}) = \frac{x^{\lambda} - \frac{1}{2} a^{\lambda} \, (x.x) - \frac{1}{3} b^{\lambda}{}_{\tau} \; x^{\tau} \, (x.x)}{1 - a\,x + \frac{1}{4}\,(a.a)(x.x) - \frac{2}{3}\, b_{\mu\rho} \; x^{\mu}x^{\rho} + \frac{1}{3}\left[b_{\mu\rho} a^{\mu}x^{\rho} + \frac{1}{3}\, b_{\sigma\rho} \; b^{\sigma}_{\beta} x^{\rho} x^{\beta}\right](x.x)}$$

which by linearization gives the transformation equations (2.31).

This process may be extended for any approximation in power series expansion on (x,t) for our general solutions (2.27-2) and (2.27-3) for the wave equation (or Klein Gordon equation). Its finite form for the conformal part involving the acceleration parameter and all subsequent generalizations is obtained by the generalization:

$$T_a(\text{rigid translation}) \rightarrow \tau_\Sigma : \Sigma^\mu = \frac{1}{2} a^\mu + \frac{1}{3} b^\mu{}_\rho x^\rho + \gamma^\mu{}_{\nu\rho} x^\nu x^\rho + \ldots$$

the generalized conformal transformation will then be

$$x'^\lambda = \mathcal{C}_\Sigma(x^\lambda) = P_o \, \tau_\Sigma \, P_o(x^\lambda)$$

The presence of non-rigid translations is directly connected with the fact that these conformal mappings introduce a curvature on the geometry of spacetime. As we have seen, the final metric tensor being conformally flat has in general a curvature.

5. *Note on the proof of the group property of the infinitesimal transformations (2.27-2) and (2.27-3).*

Here we do not take into consideration the (2.27-1) since we are not directly interested in this transformation. We consider only the mappings on the parameter plane (x,t), which are those of direct interest in the study of the symmetries of the coordinates. Of course, the present proof may be easily extended also for the full transformations (2.27).

Let an element of the transformations (2.27-2,3) be

$$\eta_1(x,t) = h_1(u) + g_1(v), \quad \lambda_1(x,t) = h_1(u) - g_1(v)$$

which maps a point (x,t) into another point (x', t').

$$\begin{cases} x' = x + \eta_1(x,t) \\ t' = t + \lambda_1(x,t) \end{cases}$$

Take another element of the set of transformations,

$$\eta_2(x,t) = h_2(u) + g_2(v), \quad \lambda_2(x,t) = h_2(u) - g_2(v)$$

such that it transforms (x', t') into (x", t"),

$$\begin{cases} x'' = x' + \eta_2(x', t') \\ t'' = t' + \lambda_2(x', t') \end{cases}$$

then, to second order in the descriptors,

$$\begin{cases} x'' = x + \eta_1(x,t) + \eta_2(x,t) + \eta_1(x,t) \dfrac{\partial \eta_2(x,t)}{\partial x} + \lambda_1(x,t) \dfrac{\partial \eta_2(x,t)}{\partial t} \\[2ex] t'' = t + \lambda_1(x,t) + \lambda_2(x,t) + \eta_1(x,t) \dfrac{\partial \lambda_2(x,t)}{\partial x} + \lambda_1(x,t) \dfrac{\partial \lambda_2(x,t)}{\partial t} \end{cases}$$

Consider now the transformation with descriptors η_2, λ_2 acting on (x,t):

$$\begin{cases} \bar{x}' = x + \eta_2(x,t) \\[1ex] \bar{t}' = t + \lambda_2(x,t) \end{cases}$$

and η_1, λ_1 on $(\bar{x}', \bar{t}')$:

$$\bar{x}'' = \bar{x}' + \eta_1(\bar{x}', \bar{t}')$$

$$\bar{t}'' = \bar{t}' + \lambda_1(\bar{x}', \bar{t}')$$

similarly as before,

$$\begin{cases} \bar{x}'' = x + \eta_2(x,t) + \eta_1(x,t) + \eta_2(x,t) \dfrac{\partial \eta_1(x,t)}{\partial x} + \lambda_2(x,t) \dfrac{\partial \eta_1(x,t)}{\partial t} . \\[2ex] \bar{t}'' = t + \lambda_2(x,t) + \lambda_1(x,t) + \eta_2(x,t) \dfrac{\partial \lambda_1(x,t)}{\partial x} + \lambda_2(x,t) \dfrac{\partial \lambda_1(x,t)}{\partial t} \end{cases}$$

then, the difference between the two coordinate points (x'', t''), $(\bar{x}'', \bar{t}'')$ will be

$$\bar{x}'' - x'' = \eta_2(x,t)\,\frac{\partial\eta_1(x,t)}{\partial x} + \lambda_2(x,t)\,\frac{\partial\eta_1(x,t)}{\partial t} - \eta_1(x,t)\,\frac{\partial\eta_2(x,t)}{\partial x} - \lambda_1(x,t)\,\frac{\partial\eta_2(x,t)}{\partial t}$$

$$\bar{t}'' - t'' = \eta_2(x,t)\,\frac{\partial\lambda_1(x,t)}{\partial x} + \lambda_2(x,t)\,\frac{\partial\lambda_1(x,t)}{\partial t} - \eta_1(x,t)\,\frac{\partial\lambda_2(x,t)}{\partial x} - - \lambda_1(x,t)\,\frac{\partial\lambda_2(x,t)}{\partial t}$$

which is the commutator for the transformations under consideration. Now, from the definition of the form of the functions η and λ we get,

$$\frac{\partial\eta_k(x,t)}{\partial t} = \frac{dh_k(u)}{du} - \frac{dg_k(v)}{dv}$$

$$\frac{\partial\eta_k(x,t)}{\partial x} = \frac{dh_k(u)}{du} + \frac{dg_k(v)}{dv}$$

$$\frac{\partial\lambda_k(x,t)}{\partial t} = \frac{dh_k(u)}{du} + \frac{dg_k(v)}{dv}$$

$$\frac{\partial\lambda_k(x,t)}{\partial x} = \frac{dh_k(u)}{du} - \frac{dg_k(v)}{dv}$$

for $k = 1, 2$. Therefore,

$$\begin{cases} \bar{x}'' - x'' = 2\left\{h_2\,\dfrac{dh_1}{du} - h_1\,\dfrac{dh_2}{du} + g_2\,\dfrac{dg_1}{dv} - g_1\,\dfrac{dg_2}{dv}\right\} \\[2ex] \bar{t}'' - t'' = 2\left\{h_2\,\dfrac{dh_1}{du} - h_1\,\dfrac{dh_2}{du} - g_2\,\dfrac{dg_1}{dv} + g_1\,\dfrac{dg_2}{dv}\right\} \end{cases}$$

Writing,

$$h_2 \frac{dh_1}{du} - h_1 \frac{dh_2}{du} = \frac{1}{2} \Phi(u)$$

$$g_2 \frac{dg_1}{dv} - g_1 \frac{dg_2}{dv} = \frac{1}{2} \chi(v)$$

we have

$$\begin{cases} \bar{x}'' - x'' = \Phi(u) + \chi(v) \\ \bar{t}'' - t'' = \Phi(u) - \chi(v) \end{cases}$$

This shows that the commutator of two mappings of the form considered is another similar mapping. Then, the set of transformations with descriptors

$$\eta_k(x,t) = h_k(u) + g_k(v), \quad \lambda_k(x,t) = h_k(u) - g_k(v)$$

form a group.

REFERENCES:

1. E. L. Hill, Phys. Rev. 84, 1165 (1951).

2. D. J. Struik, Lectures on class. diff. geom., Addison Wesley (1950), page 113.

3. R. S. Burington, C. C. Torrance, Higher Mathematics, Mc Graw Hill Inc. page 750 (1939).

4. See reference 3 on the page 109

5. Explicitly, we may write $h(u) = \lambda\varphi(u)$, $g(v) = \lambda\psi(v)$, where $\varphi(u)$ and $\psi(v)$ are arbitrary functions and λ is an infinitesimal parameter of first order. Then, $e = \lambda(\varphi'(0) + \psi'(0))$, $d = \lambda(\varphi'(0) - \psi'(0))$ etc., are first order constants.

6. The term scalar here, have to be used with proper care since no coordinate mapping is considered. We mean by "scalar" the quantity ϕ with just one component.

7. This condition comes from $\eta^{ik}\eta^{j}_{,\phi k} + \eta^{jk}\eta^{i}_{\phi k} - \eta^{ij}\xi_{,\phi\phi} = 0$, which corresponds to the (2.26-8,9), by using the equation (5.4).

8. In this case, when $a^i = 0$, $e = 0$, we can always put $U = -c$, giving $\xi = 0$.

9. In this formula write c' in place of c which appears as a trivial additive factor.

10. L. P. Eisenhart, Riemannian geometry, Princeton Univ. Press, page 89, 1949.