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ON THE CONSISTENCY OF WAVE EQUATIONS IN DE-SITTER SPACE

A. Vidal
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro - Brazil

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In describing higher spin particles interacting with gravitational or electromagnetic fields, formalisms have been used that lead to inconsistent wave equations.

The presence of a gravitational field, as Buchdahl [1] has shown, results in the incompatibility of the wave equations, unless the space be Riemannian with constant curvature. A realization of that space is a De-Sitter one. Analogously, if electromagnetic interactions are introduced in the higher spin equations by applying the usual Dirac rule of minimal coupling, one arrives to the same difficulty [2]. In fact, additional subsidiary conditions arise, so the wave functions no longer describe the spin required. To avoid that one commonly uses the Lagrangian method originated by Fierz and Pauli [2]. Despite this general situation, the spin-1 case is an exception. The introduction of the minimal electromagnetic interaction in spin-1 equations in a correct way, implies the consistency of the resulting equations. This is true also in curved spaces.

The following is concerned with an example of the last case when a De-Sitter space is considered and the procedure to obtain consistent spinorial equations with interaction is similar to that developed elsewhere [5, 9]. A tensor formulation of those equations is also obtained which is the extension to the De-Sitter space of the Proca equations minimally coupled.

As is known, there are several free-force equations for particles with spin in the De-Sitter space having in common to reduce the ordinary ones when this space goes over into usual 4-dimensional flat space. One of those equations, due to Gursey and Lee [3], is a reformulation in an invariant form of the spin 1/2 equation established, the first time, by Dirac [4]. In order to describe spin-1 particles, one starts from a natural generalization [5] of the Gursey-Lee equation, which may be written simply [6]

$$\left[\left(\beta^i \frac{\partial}{\partial x^i} - \frac{2}{\rho}\beta^5\right) + m\right]_{a'\,a} \psi_{ab..} = 0 \qquad (1)$$

where $\psi_{ab..}$ is a symmetric spinor, ρ is equal to the radius of the De-Sitter pseudosphere and m is the mass of the particle. Equations (1) are the spin 1/2 Gursey-Lee equation in each of the spinor indices (a,b,..). They are an extension to the De-Sitter space of the Bargmann-Wigner equations for arbitrary spin, and become the ordinary ones when this space is restricted to be the usual space-Time [5].

The generalization of equations (1) to those with electromagnetic interaction merely by performing the gauge invariant replacement [6] $\partial_\mu \longrightarrow D_\mu$ in (1), comes out with inconsistencies, leading to

$$\left[\beta^{i}_{a'a} D_i , \beta^{j}_{b'b} D_j\right] \psi_{ab..} = - ieF_{ij}\, \beta^{i}_{a'a}\, \beta^{j}_{b'b}.\ \psi_{ab..} \tag{2}$$

which is an identity. It can be seen that, because of the wave equations, the commutator in (2) vanishes while **the right hand** side remains different from zero. This ambiguity is encountred in several formalisms [7].

Taking into account that equations (1) for spin 1 may be rewritten in the form

$$\left(D_i\, \Gamma^{i} - \frac{2}{\rho}\, \Gamma^{5} \right) \psi + 2\, m\, \psi = 0 \tag{3}$$

where ψ is a ten-component symmetric spinor,

$$\Gamma^{\mu} = \frac{1}{2}\left(\beta^{\mu}_{(1)} + \beta^{\mu}_{(2)}\right)$$
$$2\,\Gamma^{5} = \left(\beta^{5}_{(1)} + \beta^{5}_{(2)}\right) \tag{4}$$

and applying on (3) the operator

$$D_j\, \Gamma^{j}_{(-)} \equiv D_j \left(\beta^{j}_{(1)} - \beta^{j}_{(2)}\right) \tag{5}$$

where the two sets of β^{μ} matrices, given by the direct product

$$\beta^{\mu}_{(1)} = \beta^{\mu}\, I\ , \quad \beta^{\mu}_{(2)} = I\, \beta^{\mu}$$

act on two separate suffices of ψ_{ab}, one finds

$$D_\nu\, \Gamma^{\nu}_{(-)}\, \psi = \frac{1}{4\,m} \left(ieF_{\mu\nu}\, \Gamma^{\nu}_{(-)}\, \Gamma^{\mu} - \frac{4}{\rho}\, \Gamma^{5} \right) \psi \tag{6}$$

after using the relation (2). A comparison between equations

(3) and (6) leads to

$$D_\nu \, \beta^\nu_{(1)} \psi = m\psi + \frac{1}{\rho} \Gamma^5 \psi + \frac{1}{8m} \left(ieF_{\mu\nu} \, \Gamma^\nu_{(-)} \, \Gamma^\mu - \frac{4}{\rho} \Gamma^5 \right) \psi \quad (7.1)$$

$$D_\nu \, \beta^\nu_{(2)} \psi = m\psi + \frac{1}{\rho} \Gamma^5 \psi - \frac{1}{8m} \left(ieF_{\mu\nu} \, \Gamma^\nu_{(-)} \, \Gamma^\mu - \frac{4}{\rho} \Gamma^5 \right) \psi \quad (7.2)$$

which are the spin-1 spinorial equations in De-Sitter space when an electromagnetic field is present. These equations, equivalent to the Proca ones with minimal coupling in that space, are a set of consistent equations. In flat space [5], they reduce to those of Belinfante [10] which are equivalent to the interacting Proca equations [9].

On working out (7) with the help of appropiate operators and using [5, 8, 9]

$$\psi_{ab} = \left[(\gamma^\mu C) \, \phi_\mu - \tfrac{1}{2} (\gamma^\mu \gamma^\nu C) \, F_{\mu\nu} \right] \quad (8)$$

constructed with the ten Dirac symmetric matrices, one arrives to an extension to the De-Sitter space of the Proca equations with minimal coupling. The reciprocal is true. The consistency between (7) and the symmetry condition on ψ_{ab} is also maintained. Therefore, this is a correct way to introduce interactions in the present case, not in the De-Sitter space only but in curved spaces.

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2. M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).
3. F. Gursey and T. D. Lee, Proc. Nat. Acad. Sci. **49**, 179 (1963).
4. P. A. M. Dirac, Ann. Math. **36**, 657 (1935).
5. A. Vidal, Notas de Física **15**, nº 6 (1969) (to be published).
6. Greek indices take the values 1, 2, 3, 4, 5, latin indices 1, 2, 3, 4. All repeated indices are to be summed over. Here $\hbar = c = 1$, and

$$\beta^\mu = \frac{\partial x^\mu}{\partial \xi^\nu}\gamma^\nu, \quad \beta^5 = \frac{1}{\rho}\,(\xi_\mu \gamma^\mu)$$

$$\beta^5\beta^i + \beta^i\beta^5 = 0$$

$$\beta^i\beta^j + \beta^j\beta^i = g^{ij}$$

$$g^{ij} = \left(\frac{\partial x^i}{\partial \xi^\mu}\right)\left(\frac{\partial x^j}{\partial \xi_\mu}\right)$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = (1,1,1,-1,1)_{\text{diagonal}}$$

$$\gamma^\mu_{(1)}\gamma^\nu_{(2)} = \gamma^\nu_{(2)}\gamma^\mu_{(1)}$$

where the γ^μ are five constant 4 x 4 Hermitian matrices. Also one uses

$$\partial_\mu = \frac{\partial x^i}{\partial \xi^\mu}\frac{\partial}{\partial x^i}, \quad D_\mu = \frac{\partial x^i}{\partial \xi^\mu}\left(\frac{\partial}{\partial x^i} - ieA_i\right)$$

where A_μ is a given external electromagnetic field.
7. See for example, M. M. Bakri, Nuovo Cimento **51A**, 864 (1967).

8. The matrix C is such that

$$\gamma^{\mu^T} = - C^{-1} \gamma^\mu C$$

$$C^T = - C \qquad (T = \text{transpose})$$

9. J. Leite Lopes, Lectures on Relativistic Wave Equations, Rio de Janeiro, 1960; A. Kalnay, Doctoral Thesis, Buenos Aires, Argentina, 1964; A. Vidal, Equações Relativistas para Partículas com Spin, Rio de Janeiro, 1965.

10. J. F. Belinfante, Physics 6, 870 (1939).

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