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GAUGE TRANSFORMATIONS AND THE RELATIVISTIC INVARIANCE OF ELECTRODYNAMICS*

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SUMMARY:

A relationship is shown to exist between some elements of the group of gauge transformations in electrodynamics and the ten dimensional Poincaré group. This result is local in structure, that is, it holds at some specified point in the four-dimensional Minkowski space. All properties presently obtained apply to free fields as well as to interacting fields. The generators of the gauge transformations which satisfy the above relationship are derived.

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I. INTRODUCTION

The gauge group of electrodynamics is an additive group of continuous transformations which mapps the space of all allowable four-vectors of potential into itself. By allowable here we mean those four-vectors which are solution of the field equations.

Although the field equations are form invariant under these transformations, the use of a theory where the field variables are gauge dependent conduct to well known difficulties. As example, (in this paper we treat only the classical formulation of electrodynamics) given the initial Cauchy data for the system at some time $t = t_0$, we cannot determine well prescribed values for all field variables throughout the region $t > t_0$. This type of difficulty may be overcomed by separating the set of field variables into two classes the first being independent of the choice of gauge and the second being gauge variant. However, we still have this second type of variables which are unphysical and which must be dropped out by the choice of some gauge condition.

Thus, in the framework of the Lorentz covariant field theory we can use a formulation of the electromagnetic field in terms of potentials and associated to this we have the possibility of describing the theory by means of variational principles, with the potentials playing the role of the configuration variables. As a consequence, variational conservation laws as given by the Noether theorems are obtained.

It is known that a similar situation exists in the theory of general relativity, where the invariance group is also a function group. However, in this theory we do not know how to eliminate the coordinate dependent variables in a well prescribed way (besides this, the coordinate independent variables, as given by Dirac's Hamiltonian formulation are not all independent). A variational principle exists for the theory but no conservation law is derived in a unique prescribed form since here the whole invariance group cannot be separate in a well prescribed fashion into a group with discrete parameters and a group involving arbitrary functions (as the gauge group of electrodynamics). Due to the fact that conservation laws are related to transformations depending on a set of discrete parameters, they are not to be uniquely determined in such formulation.

Thus, it is an important problem to study what should be the conditions under which it is possible to relate the function group to some discrete invariance group of the theory. In general relativity this is a complicated problem. In this paper we treat a somewhat simple problem, the relations which might exist between the gauge of electrodynamics and the Poincaré group. In spite of being a simple problem as compared with the above one, both are mathematically related.

Another way of looking at this problem is the construction of a Lorentz invariant theory written entirely in terms of gauge invariant quantities, in the case the electromagnetic field intensities $F_{\mu\nu}$. This method was treated in the literature [1], however it is not possible to derive a Lorentz invariant variational principle for this formulation, and thus we do not have the existence of symmetry principles which are associated to the existence of a Lagrangian density. Also, the correspondence of such method with general relativity is not obvious unless we restrict to the linear approximation of the general relativistic field equations.

Then, instead of proposing a formalism where the gauge group is dropped out from the outset, we try to formulate a type of weaker formalism which being compatible with the integral Noether theorems at the same time gives a correpondence between the gauge transformations and the discrete Poincaré transformations. As we have said before, this method is of interest for situations where we find more general continuous group of invariance.

It is shown in this paper that such type of relationship does exist for a sub-group of all possible gauge transformations. The problem is greatly simplified by considering only infinitesimal transformations in the space of the four-vectors of potential. In the case of general relativity this is equivalent to infinitesimal transformations both in the space of the $g_{\mu\nu}$ as well as in the coordinate space. Since all important quantities associated to conservation laws in differential form are local quantities, this kind of approximation is sufficient.

It will be shown that the method presently reported applies both for free fields as well as for interacting fields. In the last case we have only to exclude transformations at the point where there are particles which interact with the field.

II. THE GAUGE GROUP AND THE POINCARÉ GROUP FOR FREE FIELDS

The behavior of the electromagnetic potentials under the action of the Lorentz group uniquely characterizes those quantities as a four-vector in the four-dimensional flat space of special relativity, a result which comes from the relativistic invariance of the electromagnetic field equations.

To this discrete invariance group we have to add another invariance group of transformations, the gauge group, which transforms the manifold of all potentials, solutions of the field equations, into itself. This mapping is effected in a given fixed reference system, that is, without any change of coordinates in the four-dimensional space. In this sense, gauge transformations are what is sometimes called an "active transformations."

It is also known that besides this type of active transformations we still have the invariance of the field equations under conformal transformations. The conformal transformations are also active transformations but they are distinguished from the gauge transformations since they depend on a finite number of parameters [2]. This result holds only in special relativity.

Thus, we conclude that for particles with zero rest mass we have a larger number of invariance groups than those existent for particles of finite rest mass. Besides this, these extra invariance groups which are necessarily represented by active transformations include two distinct types of transformations, one represented by a function group, the other by a discrete group, the fifteen parameter conformal group.

Presently we consider the relationship which may exist between the gauge group, or at least between some sub-group of gauge transformations, and the Poincaré group. In doing so we are obtaining a description of the zero rest mass particles which is much more similar to all other particles, in the sense that we may drop out the extra invariance presented by the gauge transformations if we restrict to the transformations satisfying the above relationship.

The first step towards a correspondence between a passive transformation like the Poincaré transformation and an active transformation as the gauge transformation is to consider the space of all four-vectors of potential. In this space the gauge transformations are a mapping of this space into itself.

$$A'_{\mu}(x) = A_{\mu}(x) + \Lambda_{,\mu}(x) \quad (1)$$

We may interpret, and this holds in general for more abstract function groups, this group as a set of transformations depending on an infinite number of discrete parameters, which may be represented by the coefficients of the several terms of the power series expansion of the function $\Lambda(x)$ around the origin.

$$\Lambda(x) = \Lambda(0) + x^{\alpha}(\Lambda_{,\alpha})_0 + \frac{1}{2} x^{\alpha} x^{\beta} (\Lambda_{,\alpha\beta})_0 + \ldots \quad (2)$$

The function $\Lambda(x)$ satisfies the scalar wave equation

$$\Box \Lambda = 0 \,. \tag{3}$$

In this paper only infinitesimal transformations will be considered, the function $\Lambda(x)$ is continuous, with continuous partial derivatives up to any order. This last requirement may be weakened if eventually we have to cut off the series at some term, case where we need the partial derivatives only up to that order. However, we will show that a solution of (3) may be found in the form given by (2), and satisfying all the necessary requirements without the necessity of any cut off. Therefore, all coefficients which appear in (2) are first order infinitesimals.

The transformation (1) with Λ given by (2) is assumed to be regular at all points x^{α}. For free fields far away from its sources which is the situation presently considered, there is no further condition imposed on Λ in order to maintain this property. We now write the Poincaré transformations in the form of an active transformation [3] by expanding in power series of x^{α} all point dependent terms which appear in the transformation law of the four-vector A_{μ},

$$\tilde{A}_{\mu}(x) - A_{\mu}(x) = - \varepsilon^{\alpha}\left(\frac{\partial A}{\partial x^{\alpha}}\right)_{0} - \varepsilon^{\alpha}{}_{\mu}\, A_{\alpha}(0) - x^{\nu}\left\{\varepsilon^{\alpha}{}_{\nu}\left(\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)_{0} + \varepsilon^{\alpha}{}_{\mu}\left(\frac{\partial A_{\alpha}}{\partial x^{\nu}}\right)_{0} + \varepsilon^{\alpha}\left(\frac{\partial^{2} A_{\mu}}{\partial x^{\alpha}\partial x^{\nu}}\right)_{0}\right\} -$$

$$- \frac{x^{\nu}x^{\lambda}}{2}\left\{\varepsilon^{\alpha}{}_{\nu}\left(\frac{\partial^{2} A_{\mu}}{\partial x^{\alpha}\partial x^{\lambda}}\right)_{0} + \varepsilon^{\alpha}{}_{\lambda}\left(\frac{\partial^{2} A_{\mu}}{\partial x^{\alpha}\partial x^{\nu}}\right)_{0} + \varepsilon^{\alpha}{}_{\mu}\left(\frac{\partial^{2} A_{\alpha}}{\partial x^{\nu}\partial x^{\lambda}}\right)_{0} + \varepsilon^{\alpha}\left(\frac{\partial^{3} A_{\mu}}{\partial x^{\alpha}\partial x^{\lambda}}\right)_{0}\right\} - \dots \tag{4}$$

the ε^{α} and $\varepsilon^{\alpha}{}_{\beta}$ are the ten infinitesimal parameters of this element of the Poincaré group.

$$\varepsilon^{\alpha\mu} = g^{\mu\beta}\,\varepsilon^{\alpha}{}_{\beta} = -\varepsilon^{\mu\alpha}$$

As it stands, the relativistic transformation law (4) is also a mapping of the space of all four-vectors into itself. Indeed, by applying the invariant D'Alembertian operator on $\tilde{A}_{\mu}$ we find,

$$\Box\,\tilde{A}_{\mu}(x) = \Box\,A_{\mu}(x) + \Box\,\Gamma_{\mu}(x) \qquad (5)$$

(for simplyfying the notation we called the infinite power series expansion present in the right hand side of (4) by $\Gamma_{\mu}(x)$), but

$$\Box\,\Gamma_{\mu} = -\varepsilon^{\alpha}{}_{\mu}\,\Box A_{\alpha} - \varepsilon^{\alpha}\frac{\partial}{\partial x^{\alpha}}\,\Box A_{\mu} - \varepsilon^{\alpha}{}_{\nu}\,x^{\nu}\frac{\partial}{\partial x^{\alpha}}\,\Box\,A_{\mu} \qquad (6)$$

which vanish for free fields, thus proving that $\tilde{A}_{\mu}$ is also a solu tion of the field equation.

Looking for a similarity with the geometrical Poincaré group, we try to obtain a representation of the transformations (1) and (2) which depends on some finite set of parameters. These transformations have to satisfy (3) [4], so that we look for solutions of the scalar wave equation which depend on some finite set of parameters.

For obtaining this we first expand $\Box\Lambda$ in power series of x^{α}

$$\Box\Lambda = (\Box\Lambda)_0 + x^{\alpha}(\partial_{\alpha}\Box\Lambda)_0 + \frac{1}{2}x^{\alpha}x^{\beta}(\partial^2_{\alpha\beta}\Box\Lambda)_0 + \dots \qquad (7)$$

The equation (3) implies that

$$(\Box\Lambda)_0 = 0 \ , \tag{8-1}$$

$$(\partial_\alpha \Box\Lambda)_0 = 0 \ ; \tag{8-2}$$

$$(\partial^2_{\alpha\beta} \Box\Lambda)_0 = 0 \ . \tag{8-3}$$

....

It may be verified that a function $\Lambda(x)$ satisfying all the conditions (8), and represented by an expansion like (2) is of the form

$$(\partial_\alpha \Lambda)_0 = K_\alpha \ , \tag{9}$$

and for all $p > 1$,

$$\left(\partial^p_{\mu_1\cdots\mu_p}\Lambda\right)_0 = -\frac{1}{p}\left\{K^\lambda{}_{\mu_1} S^{(o)}_{\lambda\mu_2\cdots\mu_p} + \cdots K^\lambda{}_{\mu_p} S^{(o)}_{\lambda\mu_1\cdots\mu_{p-1}}\right\} \tag{10}$$

where K_μ and $K_{\mu\nu}$ are a set of ten infinitesimal parameters.

$$K_{\mu\nu} = g_{\mu\lambda} K^\lambda{}_\nu = -K_{\nu\mu} ,$$

The $S^{(o)}_{\mu_1\cdots\mu_k}$ which appear in the equation (10) are

$$S^{(o)}_{\mu_1\cdots\mu_k} = \sum_{(\mu_1\cdots\mu_k)} \left(\frac{\partial^{k-1} A_{\mu_k}}{\partial x^{\mu_1}\cdots\partial x^{\mu_{k-1}}}\right)_0 \tag{11}$$

where the symbol $\sum_{(\mu_1\cdots\mu_k)}$ indicates a sum over all permutations of the indices $\mu_1\cdots\mu_k$.

The proof that the power series expansion of Λ with coefficients given by (9) and (10) is a solution of the conditions (8) follows from the relation

$$\left(\partial^{n}_{\mu_1\cdots\mu_n}\Box\Lambda\right)_o = -\frac{1}{n+2}\left\{K^{\lambda}{}_{\mu_1}\,S^{(o)}_{\mu_2\cdots\mu_n\lambda\alpha}{}^{\alpha} + \cdots K^{\lambda}{}_{\mu_n}\,S^{(o)}_{\mu_1\cdots\mu_{n-1}\lambda\alpha}{}^{\alpha}\right\}$$

which was obtained from (10) and from the fact that the $K_{\mu\nu}$ are skew-symmetric. The above relation holds for all $n \geqslant 1$. Now, it may be easily seen that the contracted quantities $S^{(o)}_{\mu_1\cdots\mu_k\alpha}{}^{\alpha}$ vanish as consequence of the field equations plus the Lorentz condition,

$$\Box A^{\mu} = 0, \quad \partial_{\mu} A^{\mu} = 0 .$$

Thus, a function Λ given by

$$\Lambda(x) = \Lambda(0) + x^{\alpha} K_{\alpha} - \cdots \frac{(p+1)}{p(p+1)!}\, x^{\mu_1}\cdots x^{\mu_p} \cdot \left\{K^{\lambda}{}_{\mu_1}\,S^{(o)}_{\lambda\mu_2\cdots\mu_p} + \cdots K^{\lambda}{}_{\mu_p}\,S^{(o)}_{\lambda\mu_1\cdots\mu_{p-1}}\right\} - \cdots \qquad (12)$$

is a solution of the scalar wave equation, and depends on ten first order parameters. Evidently this is not the unique possible solution of this equation, which means that the gauge transformations which have a function Λ of the above form belong to a sub-group of all possible gauge transformations. We call this sub-group by G_s. For this sub-group the relationships which have been discussed before will be determined. First of all we see that each element of G_s depends on ten infinitesimal parameters and posses a transformation law which is similar to that of the Poincaré group (given by the equation (4)). These two transformations are not entirely identical since we know that the electromagnetic field strenghts do not vary under G_s, but changes as an antisymmetric second order tensor under the Lorentz transformations. In order to have a quantity which varies under both types of transformations, we introduce the symmetric second rank tenso

$$S_{\mu\nu} = \partial_\nu \ A_\mu + \partial_\mu A_\nu$$

Under a Lorentz transformation we have

$$\tilde{S}_{\mu\nu}(x) - S_{\mu\nu} = -\varepsilon_\mu^\alpha S_{\alpha\nu}(x) - \varepsilon_\nu^\alpha S_{\mu\alpha}(x) - \varepsilon_\lambda^\alpha x^\lambda \partial_\alpha S_{\mu\nu} \quad (13)$$

and for a transformation belonging to G_S

$$S'_{\mu\nu}(x) - S_{\mu\nu}(x) = 2\Lambda_{,\mu\nu}(x) , \quad (14)$$

where Λ is given by (12). The equations (13) and (14) are still not the same. However, we will show that a relationship does exist, or in more proper words, may be imposed in a very natural way between the transformations (13) and (14). First we consider the variations in A due to these two transformations

$$\tilde{A}_\mu(x) - A_\mu(x) = -\varepsilon_\mu^\alpha A_\alpha(x) - \varepsilon^\alpha \partial_\alpha A_\mu - \varepsilon_\beta^\alpha x^\beta \partial_\alpha A_\mu \quad (15)$$

$$A'_\mu(x) - A_\mu(x) = \Lambda_{,\mu}(x) . \quad (16)$$

These variations are carried out on all points x^μ belonging to the domain of definition of Λ. We consider the effect of this variation at the origin $x^\mu = 0$, which is a point arbitrarily chosen inside the domain of definition of x^μ. We get

$$\tilde{S}_{\mu\nu}(0) - S_{\mu\nu}(0) = -\varepsilon_\mu^\alpha S_{\alpha\nu}(0) - \varepsilon_\nu^\alpha S^{(0)}_{\mu\alpha} , \quad (17)$$

$$S'_{\mu\nu}(0) - S_{\mu\nu}(0) = -K_\mu^\lambda S_{\lambda\nu}(0) - K^\lambda_\nu S^{(0)}_{\lambda\mu} , \quad (18)$$

$$\tilde{A}_\mu(0) - A_\mu(0) = -\varepsilon_\mu^\alpha A_\alpha(0) - \varepsilon^\alpha(\partial_\alpha A_\mu)_0 , \quad (19)$$

$$A'_\mu(0) - A_\mu(0) = K_\mu . \quad (20)$$

Since these transformations are similar in form, it is natural to take

$$K^{\lambda}_{\mu} = A\,\varepsilon^{\lambda}{}_{\mu}, \tag{21}$$

$$K_{\mu} = -B\,(\varepsilon^{\alpha}{}_{\mu}\,A_{\alpha}(0) - \varepsilon^{\alpha}(\partial_{\alpha}\,A_{\mu})_{0})\ . \tag{22}$$

We may further consider the two constants A and B as equal to one. We may give here a more complete definition of G_s, as the sub-group of gauge transformations with a generating function of the form given by (12) and with parameters given by the relations (21) and (22) for A and B both equal to one. Thus, the transformations belonging to G_s and the Poincaré transformations can be made equal, for all gauge variant quantities, at the point where the observer is located (the origin of the coordinate system). This equality of these transformations is strictly a local property, whenever measurements are made on points outside the origin, we will obtain different results. This property resembles the fact that locally a Riemannian space may be assimilated to a local flat space, in the sense that locally the unique independent transformation is the Poincaré transformation.

We must note that at all points x different from the origin the gauge transformations belonging to G_s are entirely determined from the knowledge of the ten parameters $\varepsilon_{\lambda\mu}$ and ε_{λ} of the Poincare group. This means that in spite of both transformations being not identical at all points in space-time, the Poincaré transformations play the fundamental part of the invariance group of the theory since the remaining transformations are determined as consequence of the relativistic invariance of the theory.

According to the above considerations we may construct a formalism which uses the potentials as field variables in the Lagrangian, and thus displays all important features which have been referred before, and at the same time the gauge transformations are treated as a secondary transformation, the fundamental role being taken over by the geometrical Poincaré group.

It should be noted that Λ plays a dual role, first it may be interpreted as an operator associated to the generator of the gauge transrormations. Second, it may be interpreted as a geometrical quantity, in the case an scalar, with respect to the transformations of the Poincaré group, case where we obtain the following variation,

$$\Lambda'(x') = \Lambda(x') = \Lambda(x) + (\varepsilon^{\alpha} + \varepsilon^{\alpha}_{\ \lambda} x^{\lambda}) \frac{\partial \Lambda}{\partial x^{\alpha}}$$

due to the fact that the derivatives of Λ are first order infinitesimals, we neglect these terms when multiplied by the parameters of the Poincaré transformation, this gives

$$\Lambda'(x') = \Lambda(x) .$$

This means that up to first order terms the function Λ is not only an scalar, but has a stronger geometrical property, is an invariant.

III. THE GAUGE AND POINCARÉ GROUPS FOR INTERACTING FIELDS

So far we have considered the case of free fields, that is, the field very far away from its sources. The method presented in the last section may be extended to the case where there are interactions of the field with point charges or currents.

In this case the domain of definition of Λ is further restricted by the requirement that the vicinity of the external point charges or currents are excluded from the domain of definition of Λ. Indeed, the field at the position of the charges is singular and there is no reason why the transformation itself should not be singular too at those points. In particular, the origin of the coordinate system must be chosen in a point where there is no charge. Under these requirements it is simple to verify that all previous results apply equally to the case of interactions.

In short, we have interacting fields but the gauge transformations are effected as if we had free fields, however, the potential A_μ on which the gauge transformations operate, is the total potential obtained by superposition of all parts of the system. Excluding the points where this potential is singular from the domain of definition of Λ, we can transform the problem for interactions in a form which is similar to that for free fields by choosing the origin conveniently.

IV. THE GENERATORS OF G_s

We have seen that G_s depends on ten infinitesimal parameters which are given in terms of the parameters of the Poincaré group by means of the equations (21) and (22), for A and B equal to one. It is evidently interesting to determine the generators of G_s which will be too in the total number of ten. For the determination of these generators we use the Hamiltonian formulation and Poisson brackets since presently we treat classical systems. Due to this we first take the decomposition of all quantities associated to the field in their spatial and temporal parts. Latin indices will indicate quantities with indices running from one to three.

Since for the Maxwell Lagrangian density the canonical momentum density conjugate to the scalar potential vanishes, only the vector potential A_i is to be considered as a configuration type variable [5], satisfying the fundamental Poisson bracket relation

$$\left[A_i(x^k, x^o), \pi^j(x'^k, x^o)\right] = \delta_i^j \, \delta(\vec{x} - \vec{x}'),$$

π^j is given by,

$$\pi^j = F^{jo} . \qquad (23)$$

The generato $\mathcal{G}$ for the gauge transformations is such that

$$\delta A_i(x) = \left[A_i(x), \mathcal{G}\right] = \Lambda_{,i}(x) . \qquad (24)$$

We look for a relation of the form

$$\mathcal{G} = K_\mu \Phi^\mu + K_{\mu\nu} \Phi^{\mu\nu}, \tag{25}$$

giving $\mathcal{G}$ in terms of the ten partial generators Φ^μ and $\Phi^{\mu\nu}$.
From (24) it follows that $\mathcal{G}$ must have the form

$$\mathcal{G} = \int d_3x \, \pi^k(x) \, \Lambda_{,k}(x) \, . \tag{26}$$

Using (23) we write (26) as

$$\mathcal{G}(x^o) = \int d_3x \, F^{\mu o}(x) \Lambda_{,\mu}(x) = \int_\Sigma d\sigma_\lambda \, F^{\mu\lambda}(x) \, \Lambda_{,\mu}(x) \tag{27}$$

where the integration on the right hand side is extended over the hyperplane x^o = const.

The variation in the Lagrangian density under a gauge transformation is (we consider here free fields)

$$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \Lambda_{,\mu} \right)_{,\nu} \tag{28}$$

which vanishes as consequence of the field equations plus the skew symmetry of $\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}$. The variation in the Action integral is therefore,

$$\delta I = - \frac{1}{16\pi} \int d_4x \, (F^{\mu\nu} \Lambda_{,\mu})_{,\nu} = - \frac{1}{16\pi} \oint d\sigma_\nu \, F^{\mu\nu} \Lambda_{,\mu} = 0$$

which may be written as

$$\delta I = - \frac{1}{16\pi} \left\{ \int_\Sigma d\sigma_\nu \, F^{\mu\nu} \Lambda_{,\mu} - \int_{\Sigma'} d\sigma_\nu \, F^{\mu\nu} \Lambda_{,\mu} \right\} = 0 \, .$$

Thus, we conclude that $\mathcal{G}$ independs on the choice of the hyperplane of integration (is a constant of the motion).

We use this result for taking the $\mathcal{G}$ which stands in (24) on the hyperplane x^o = constant, where x^o is the instant of time in which A_i is given. Thus,

$$\left[A_i(\vec{x}, x^o), \mathcal{G}(x^o)\right] = \Lambda_{,i}(\vec{x},x^o) + \int d_3x' \; \pi^k(\vec{x}', x^o)\left[A_i(\vec{x},x^o), \Lambda_{,k}(\vec{x}',x^o)\right] \tag{29}$$

Using (12) we obtain for the above integral the value,

$$- \int d_3x' \; \pi^k(\vec{x}',x^o) \sum_{p\geq 2} \frac{1}{(p+1)!} x'^{\mu_1}\ldots x'^{\mu_p} \left\{ K^\lambda{}_k \left[A_i(\vec{x},x^o), S^{(o)}_{\lambda\mu_1\ldots\mu_p}\right] + \right.$$

$$\left. \ldots + K^\lambda{}_{\mu_p} \left[A_i(\vec{x},x^o), S^{(o)}_{\lambda\mu_1\ldots\mu_{p-1}k}\right] \right\}. \tag{30}$$

Expanding $A_i(x, x^o)$ in power series of x^o, we easily verify that the equation (30) vanish, and therefore $\mathcal{G}$ is the correct generator.

With the end of calculations, like in the equation (24), is sufficient to take the integration on volume present in (27) over a finite region enclosing the point x in which A_i is given.

Substituting (12) into (27),

$$\mathcal{G} = K_\mu \int F^{\mu o} d_3x - K^\lambda_\mu \sum_p \frac{1}{(p+1)!} S^{(o)}_{\lambda\mu_1\ldots\mu_p} \sum_{(\mu\mu_1\ldots\mu_p)} \int d_3x F^{\mu o} x^{\mu_1}\ldots x^{\mu_p}, \tag{31}$$

which is of the form (25) with

$$\Phi^\mu(x^o) = \int F^{\mu o} d_3x \; , \tag{32}$$

$$\Phi^{\lambda\mu}(x^o) = - \underset{\lambda\mu}{A} \sum_p \frac{1}{(p+1)!} S^{(o)\lambda}{}_{\mu_1\ldots\mu_p} \sum_{(\mu\mu_1\ldots\mu_p)} \int d_3x \; F^{\mu o} x^{\mu_1}\ldots x^{\mu_p}, \tag{33}$$

where $\underset{\lambda\mu}{A}$ indicates antisymmetrization on the indices λ, μ.

Both Φ^{μ} and $\Phi^{\lambda\mu}$ are constant of the motion, since presently we consider free systems.

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3. A simple example that a passive transformation is equivalent to some element of an active transformation is given in the case of rotations in the three-dimensional Euclidian space.

4. For the Poincaré transformations this condition is satisfied by consequence of the field equations $\Box A = 0$ in the initial representation (see the relation (6)).

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