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ON THE SOLUTION OF THE ELECTROMAGNETIC EQUATIONS FOR ROTATING SYSTEMS

by
I. Wolk and R. C. T. da Costa

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
Av. Wenceslau Braz, 71
RIO DE JANEIRO
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I. Wolk and R. C. T. da Costa
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro

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SUMMARY

We introduce the electromagnetic equations relative to a rotating system with the help of the space-time metric discussed by Kursonoglu, and investigate its solution in the case of an electric current flowing parallel to the rotation axis. We found that in the stationary case the rotation effects manifest themselves through a change in the argument of the Bessel function giving the radial solution; this change being absent for problems with cylindrical symmetry about the axis of rotation. However, in the time-dependent case even the symmetric solutions are modified, the Bessel functions being substituted by hypergeometric functions. The dependence in θ and z (the angular and axial coordinates) of the separated solutions is unaffected in all cases.

1 - INTRODUCTION

The introduction of the electromagnetic effects into the scheme of the General Relativity has followed, since the first developments of this theory, two different ways. The first one tries to take these effects into account through adequate changes in the energy-momentum tensor, whereas the second looks for a reformulation of the General Relativity, in which both gravitation and electromagnetism are taken, from the beginning, in the same footing (unified theories). However, the many difficulties met by these two techniques, both from the mathematical and physical point of view, gave rise to a third method of approach. This method consists in generalizing the usual electromagnetic equations by the simple introduction of the requirements of the General Relativity (covariance), assuming, at the same time, that the perturbations of the electromagnetic field in the geometry determined by the gravitation can be neglected. Although less ambitious than the former ones, this last approach method may prove to be very useful in giving the approximate interaction between the two fields, and in selecting the problems in which the application of the more elaborated theories is expected to be worthwhile. Among the problems discussed with the help of this method we may quote here Friedman's and Zitter's Universes, the quasi-uniform gravitational field and Schwarzchild's geometry[1,5].

More recently many papers have been devoted to the study of the rotating systems, but, so far as we are aware, these efforts have been directed only to the investigation of the properties of the gravitational field, without considering its interaction with the electromagnetic propagation, notwithstanding the interest this problem seems to deserve. This was the reason which led us to undertake the present work.

Among the papers dealing with this subject, we shall employ the results obtained by Kursonoglu[6], which will be briefly described in the next section.

2. FORMULATION OF THE PROBLEM

Kursonoglu's work refers to a fluid with cylindrical symmetry about the axis of rotation, the energy-momentum tensor of which is given by

$$T^{\mu\nu} = (C^2\rho - P)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + Pg^{\mu\nu}, \qquad (1)$$

where ρ is the density of the medium and P its pressure. The resulting geommetry is expressed by

$$ds^2 = (C^2 - \omega^2\rho^2)\,dt^2 - d\rho^2 - \frac{\rho^2\,d\theta^2}{1 - \omega^2\rho^2/C^2} - dz^2 , \qquad (2)$$

where the z-axis is chosen to be the axis of rotation and ω is considered as the angular velocity. From (2) we obtain

$$g^{oo} = \frac{1}{C^2 - \omega^2\rho^2}, \; g^{11} = -1, \; g^{22} = -\frac{1 - \omega^2\rho^2/C^2}{\rho^2}, \; g^{33} = -1, \qquad (3)$$

and the only $\Gamma^{\gamma}_{\alpha,\beta}$ different from zero are:

$$\Gamma^{o}_{10} = \Gamma^{o}_{01} = \frac{-\omega^2\rho}{C^2-\omega^2\rho^2}, \; \Gamma^{1}_{oo} = -\omega^2\rho, \; \Gamma^{1}_{22} = \frac{-\rho}{(1-\omega^2\rho^2/C^2)^2}, \; \Gamma^{2}_{12} = \Gamma^{2}_{21} =$$

$$= \frac{1}{\rho(1-\omega^2\rho^2/C^2)} \tag{4}$$

Maxwell equations for General Relativity are:

$$F^{\nu\mu}{}_{;\mu} = j^{\nu}, \quad F_{\mu\nu} = \phi_{\nu;\mu} - \phi_{\mu;\nu}, \quad \phi_{\mu;\mu} = 0 , \tag{5}$$

which can be written as:[2]

$$j_\mu = g^{\nu\rho}\left[\phi_{\mu;\rho\nu} - \phi_{\rho;\mu\nu} + \phi_{\rho;\nu\mu}\right] , \tag{6}$$

with

$$\begin{aligned} \phi_{\alpha;\beta} &= \frac{\partial\phi_\alpha}{\partial x^\beta} - \Gamma^k_{\alpha\beta}\phi_k , \\ \phi_{ik;\ell} &= \frac{\partial\phi_{ik}}{\partial x^\ell} - \Gamma^m_{i\ell}\phi_{mk} - \Gamma^m_{k\ell}\phi_{im} , \end{aligned} \tag{7}$$

Using (4) and (7) we obtain an explicit form for (6)

$$\begin{aligned} j_o &= \frac{1}{C^2-\omega^2\rho^2}\frac{\partial^2\phi_o}{\partial t^2} - \frac{\partial^2\phi_o}{\partial\rho^2} - \frac{C^2+\omega^2\rho^2}{C^2-\omega^2\rho^2}\frac{1}{\rho}\frac{\partial\phi_o}{\partial\rho} - \frac{\partial^2\phi_o}{\partial z^2} - \left(1-\frac{\omega^2\rho^2}{C^2}\right)\frac{1}{\rho^2}\frac{\partial^2\phi_o}{\partial\theta^2} + \\ &\quad + 2\frac{\omega^2\rho}{C^2-\omega^2\rho^2}\frac{\partial\phi_1}{\partial t} , \\ j_1 &= \frac{1}{C^2-\omega^2\rho^2}\frac{\partial^2\phi_1}{\partial t^2} - \frac{\partial^2\phi_1}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\phi_1}{\partial\rho} - \frac{\partial^2\phi_1}{\partial z^2} - \left(1-\frac{\omega^2\rho^2}{C^2}\right)\frac{1}{\rho^2}\frac{\partial^2\phi_1}{\partial\theta^2} + \\ &\quad + 2\frac{\omega^2\rho}{(C^2-\omega^2\rho^2)^2}\frac{\partial\phi_o}{\partial t} + \frac{2}{\rho^3}\frac{\partial\phi_2}{\partial\theta} + \frac{1}{\rho^2}\phi_1 , \\ j_2 &= \frac{1}{C^2-\omega^2\rho^2}\frac{\partial^2\phi_2}{\partial t^2} - \frac{\partial^2\phi_2}{\partial\rho^2} + \frac{C^2+\omega^2\rho^2}{C^2-\omega^2\rho^2}\frac{1}{\rho}\frac{\partial\phi_2}{\partial\rho} - \frac{\partial^2\phi_2}{\partial z^2} - \left(1-\frac{\omega^2\rho^2}{C^2}\right)\frac{1}{\rho^2}\frac{\partial^2\phi_2}{\partial\theta^2} - \\ &\quad - \frac{2}{1-\omega^2\rho^2/C^2}\frac{1}{\rho}\frac{\partial\phi_1}{\partial\theta} , \end{aligned} \tag{8}$$

$$j_3 = \frac{1}{C^2-\omega^2\rho^2}\frac{\partial^2\phi_3}{\partial t^2} - \frac{\partial^2\phi_3}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\phi_3}{\partial\rho} - \left(1 - \frac{\omega^2\rho^2}{C^2}\right)\frac{1}{\rho^2}\frac{\partial^2\phi_3}{\partial\theta^2} - \frac{\partial^2\phi_3}{\partial z^2}$$

Since the electromagnetic equations, as stated in (8), are too involved to be solved in the general case, we shall only consider some particular situations of interest. The simplest of them is obtained by taking $j_o = j_1 = j_2 = \phi_o = \phi_1 = \phi_2 = 0$, corresponding to a current flowing in the direction of the z-axis. Assuming the separated solution

$$\phi_3(\rho, \theta, z, t) = R(\rho)\ \Theta(\theta)\ Z(z)\ T(t)\ , \qquad (9)$$

we obtain for points outside the current ($j_3 = 0$):

$$T'' = -\Omega^2 T,\ Z'' = k_z^2 Z,\ \Theta'' = -n^2\ \Theta \quad \text{and} \quad (10)$$

$$R'' + \frac{1}{\rho}R' + \left[k_z^2 - (1-\omega^2\rho^2/C^2)\frac{n^2}{\rho^2} + \frac{\Omega^2}{C^2-\omega^2\rho^2}\right]R = 0\ .$$

By taking $x = \omega\rho/C$ the radial equation is transformed into

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left[\frac{C^2k_z^2}{\omega^2} - \frac{1-x^2}{x^2}n^2 + \frac{(\Omega/\omega)^2}{1-x^2}\right]R = 0\ . \qquad (11)$$

Two particular cases of (11) will be considered here:

2.1 - The Stationary Solution

Putting $\Omega = 0$ in (11) we get

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + \left[\left(\frac{C^2k_z^2}{\omega^2} + n^2\right)x^2 - n^2\right]R = 0\ , \qquad (12)$$

Introducing the change of variables $\xi = \sqrt{\frac{C^2k_z^2}{\omega^2} + n^2}\ x$ equation (12) is transformed into

$$\xi^2 \frac{d^2R}{d\xi^2} + \xi \frac{dR}{d\xi} + (\xi^2 - n^2) R = 0 , \qquad (13)$$

the solution of which is:

$$R(\xi) = Jn(\xi) = Jn\left(\sqrt{k_z^2 + \frac{\omega^2 n^2}{c^2}}\, \rho\right) , \qquad (14)$$

where $Jn(\xi)$ is the Bessel function of order n. From (14) we see that for problems with cylindrical symmetry ($n = 0$) the usual solution is not modified by the rotation. For $n \neq 0$, in order to obtain the values of the field on the z-axis independent of ω, the arbitrary constants appearing in the solution must be functions of ω in the way given below:

$$R(\rho) = A'(\omega)\, J_n\left(\sqrt{k_z^2 + \left(\frac{\omega n}{c}\right)^2}\, \rho\right) + B'(\omega)\, Y_n\left(\sqrt{k_z^2 + \left(\frac{\omega n}{c}\right)^2}\, \rho\right) , \qquad (15)$$

where $A' = A\left(1 + \left(\frac{\omega n}{Ck_z}\right)^2\right)^{-n/2}$, $B' = B\left(1 + \left(\frac{\omega n}{Ck_z}\right)^2\right)^{n/2}$ and Y_n is the singular solution of the Bessel equation as defined by Hankel [7].

When we employ in (10) $Z'' = -k_z^2 Z$ as the part of the solution depending on z, the new radial solution for $\frac{\omega n}{k_z C} \neq 1$ is

$$R(\rho) = {}_n\left(\sqrt{-k_z^2 + \left(\frac{\omega n}{c}\right)^2}\, \rho\right) , \qquad (16)$$

which is a modified Bessel function only for $\frac{\omega n}{k_z C} < 1$. When $\frac{\omega n}{k_z C} = 1$ we have

$$x^2 \frac{d^2R}{dx^2} + x \frac{dR}{dx} - n^2 R = 0 , \qquad (17)$$

the solution of which is

$$R = Ax^n + Bx^{-n} . \qquad (18)$$

For problems not depending on z we have

$$R(\rho) = 2^n\, n!\left(\frac{C}{\omega n}\right)^n A\, J_n\left(\frac{\omega n}{C}\,\rho\right) - \frac{(\omega n/C)^n}{2^n (n-1)!}\, B\, Y_n\left(\frac{\omega n}{C}\,\rho\right), \tag{19}$$

which gives $A\rho^n + B\rho^{-n}$ when $\omega \longrightarrow 0$. Let us now compare the behaviour of the regular solutions $2^n\, n!\left(\frac{C}{\omega n}\right)^n J_n\left(\frac{\omega n}{C}\,\rho\right)$ and ρ^n for large values of n. Using Carlini's expression for $J_n(nx)$ when $n \longrightarrow \infty$ and $0 < x < 1$ [8].

$$J_n(nx) \approx \frac{x^n \exp\left\{n\sqrt{1-x^2}\right\}}{\sqrt{2\pi n}\,(1-x^2)^{1/4}\,(1+\sqrt{1-x^2})^n}, \tag{20}$$

we obtain for the ratio of the two solutions:

$$\left(\frac{C}{\omega\rho n}\right)^n 2^n\, n!\; J_n\left(\frac{\omega\rho n}{C}\right) \simeq \frac{\exp\left\{n\,(\sqrt{1-x^2}-1)\right\}}{(1-x^2)^{1/4}\left(\frac{1+\sqrt{1-x^2}}{2}\right)^n}, \tag{21}$$

which goes to zero when $n \longrightarrow \infty$, for $x \neq 0$. It is easy to show that the two solutions above are of the same order only in the neighbourhood of the axis of rotation for which $\omega\rho/C \lesssim \underline{O}(n^{-\frac{1}{2}})$. Therefore, the difference between the two solutions becomes increasingly more pronounced for large values of n.

2.2 - The Time-Dependent Solution

The simplest type of time-dependent solution is obtained from (11) by taking $k_z = n = 0$:

$$R'' + \frac{1}{x}\,R' + \frac{(\Omega/\omega)^2}{1-x^2}\,R = 0\,. \tag{22}$$

Using the change of variables $x = \sqrt{\xi}$ we obtain

$$\xi(\xi-1)\,\frac{d^2R}{d\xi^2} + (\xi-1)\,\frac{dR}{d\xi} - \left(\frac{\Omega}{2\omega}\right)^2 R = 0\,, \tag{23}$$

which is an hypergeometric equation. Its regular solution at the origin is given by

$$R_1(\xi) = F(\alpha,\beta,\gamma,\xi) \;, \tag{24}$$

with $\alpha = \Omega/2\omega$, $\beta = -\Omega/2\omega$ and $\gamma = 1$. The solution, singular at the origin, is obtained from (23) by putting [9]

$$R_2(\xi) = F(\alpha, -\alpha, 1, \xi)\, \ln\xi + \sum_{n=1}^{\infty} C_n\, \xi^n \,, \tag{25}$$

Which finally yields:

$$C_n = \frac{2\alpha^2}{(n!)^2} \prod_{s=0}^{n-1} (s^2-\alpha^2) \left[\sum_{p=1}^{n-1} \frac{1/p}{p^2-\alpha^2} - \frac{1}{n\alpha^2} \right] . \tag{26}$$

For the cases of high or low frequencies (according to ω/Ω is respectively much smaller or much larger than one) expression (24) can be easily simplified, its contribution to the magnetic field being written as

$$H = -\frac{\Omega}{c}\left[J_1\left(\frac{\Omega\rho}{c}\right) + \left(\frac{\omega}{\Omega}\right)^2 \sum_1^{\infty} (-1)^{n+1} \frac{2+1/n}{3\cdot 2^{2n}(n-1)!^2} \left(\frac{\omega\rho}{c}\right)^{2n+1} \right] + \underline{O}\left(\frac{\omega}{\Omega}\right)^4 , \tag{27}$$

$$H = -\left(\frac{\Omega}{\omega}\right)^2 \sum_1^{\infty} \frac{\omega}{2nc} \left(\frac{\omega\rho}{c}\right)^{2n-1} + \underline{O}\left(\frac{\Omega}{\omega}\right)^4 . \tag{28}$$

Similar approximations can also be found for the singular solution (26), as the reader may readily verify.

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