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MESON DEUTERON INELASTIC SCATTERING*

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ABSTRACT. The formal expression for the scattering operator for meson-deuteron inelastic scattering is expanded in terms of two-particle scattering operators. The terms of this expansion which represent first and second order processes are explicitly evaluated and discussed. The Coulomb interaction of two protons in the final state is treated. Formulae for the cross-sections for inelastic (without charge exchange) and charge-exchange meson-deuteron scattering are given.

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1. INTRODUCTION

It is not very easy to understand and interpret in terms of elementary interactions the experiments in which beams of particles are scattered by nuclear matter in general. Neutrons and protons are closely packed together to form the nucleus, so that the interaction of the incident particle with only one of the nucleons without the others strongly participating is almost impossible. During and after the interaction of the incident particle with one of the nucleons, this nucleon will interact strongly with the others. The incident particle itself will very likely interact with two or more nucleons at a time, or suffer multiple scattering, since the scattering centers (the nucleons) are so close to each other. On the other hand, the nucleons are not at rest inside the nucleus, and their motion should be known if properties of the elementary two-particle interactions are to be used or deduced. This knowledge is not available for most of the nuclei.

The deuteron is a rather special system among the nuclei. The two nucleons in the deuteron are separated by a relatively large distance, so that the incident particle may interact strongly with only one nucleon at a time. If the interaction between the incident particle and one nucleon lasts only a relatively short time, the presence of the second nucleon will not affect much the state of motion of the first nucleon during this interval of time, and the characteristics of the two-particle interaction will be approximately obeyed. Also, the deuteron possesses such particular features that may enable us to approximately describe scattering events in terms of two-particle interactions. These properties of the deuteron were first recognized by Chew [1] who introduced what is called the Impulse Approximation to treat the problem of scattering on deuterons. The conditions of applicability of the Impulse Approximation were qualitatively discussed by several authors [1 - 3]. Chew and Goldberger [4] expanded the formal expression for the transition probability for elastic scattering of a particle by a complexe nucleus in terms of two-particle scattering amplitudes and showed how the terms corresponding to the Impulse Approximation appear naturally in this case. The Impulse Approximation has been applied to pion-deuteron [5 - 6] and to K meson-deuteron scattering [7 - 8]. Gourdin and Martin [9] improved the calculation of inelastic K^+- deuteron scattering by taking into account the interaction of the two nucleons in the final state by analogy with the case of the photodisintegration of the deuteron.

We intend here to make a more complete quantitative analysis of the meson-deuteron inelastic scattering. We start by writing the formal expressions for the scattering amplitude (Sec. 2). This amplitude is then expanded in terms of two-particle scattering amplitudes (Sec. 3), and physical meaning is given to the terms in this expansion. The terms representing double scattering of the incident meson and those representing a meson-nucleon scattering followed by a nucleon-nucleon collision are explicitly written in Sec. 4. The analysis of the spin dependence is discussed in the Appendix. In Sec. 5 and 6 complete evaluation is made of the terms representing double scattering and nucleon-nucleon interaction in the final state. Expressions for the cross-sections for inelastic meson-deuteron scattering are written in Sec. 7. Charge exchange scattering and the Coulomb interaction of two protons in the final state are discussed in Sec. 8.

2. FORMAL DESCRIPTION OF INELASTIC MESON-DEUTERON SCATTERING

We first want to write the expression for the transition amplitude for inelastic scattering of mesons by deuterons in terms of the quantities describing the interactions between pairs of particles. We here consider only transitions from the initial state, which consists of a free meson incident on a deuteron, to a final state which is a system of three free particles, one meson and two nucleons. We do not consider processes in which particles are created or absorbed.

Let us call U the potential describing the nucleon-nucleon interaction. It is responsible for the formation of the bound neutron-proton system. Let K be the total kinetic energy (sum of the kinetic energies of the three particles), V_p and V_n the potentials responsible for the interactions between the meson and the proton and neutron respectively. The total hamiltonian of the interacting system is $H = K + U + V_p + V_n$. The initial state satisfies $(K+U)\psi_i = E_i\ \psi_i$ and the final state three free particles satisfies $K\phi_f = E_f\ \phi_f$. An outgoing scattering state $\psi_i^{(+)}$, which is an eigenstate of H corresponding asymptotically to a plane plus an outgoing wave of a free meson and a deuteron system, satisfies

$$\psi_i^{(+)} = \psi_i + (E_i - K - U + i\varepsilon)^{-1} (V_p + V_n)\psi_i^{(+)} =$$

$$= \psi_i + (E_i - K - U - V_p - V_n + i\varepsilon)^{-1} (V_p + V_n)\psi_i \,. \qquad (1)$$

An outgoing scattering state $\phi_f^{(+)}$ with asymptotic behaviour corresponding to plane plus outgoing waves of three free particles satisfies

$$\phi_f^{(+)} = \phi_f + (E_f - K + i\varepsilon)^{-1} (V_p + V_n + U)\phi_f^{(+)} =$$

$$= \phi_f + (E_f - K - V_p - V_n - U + i\varepsilon)^{-1} (V_p + V_n + U)\,\phi_f \,. \qquad (2)$$

Eq. (1) and (2) can be written in a single equation,

$$\varphi^{(+)} = \varphi + (E - H + i\varepsilon)^{-1} (H - E)\varphi = \Omega_+ \varphi \qquad (3)$$

where φ is any of the states of the set of "plane-wave states" of the system of three particles. This set includes the states in which we have a deuteron and a meson as two separate plane waves, and the states in which we have three separate plane waves. $\varphi^{(+)}$ represents the corresponding outgoing scattering states. As φ and $\varphi^{(+)}$ form two complete sets of states and there is a one-to-one correspondence between the elements of these sets, Eq. (3) defines completely the wave operator Ω_+. A wave operator Ω_- connecting

φ with the ingoing scattering states $\varphi^{(-)}$ can be defined in an analogous way.

We can write expressions for operators T^+ and T^-, which we call collision operators, related to these wave operators, and such that the square of their matrix elements between states of equal energies are proportional to the probability of transition between the two states. The elastic scattering has been treated by Chew and Goldberger (4). We now consider the case of inelastic meson-deuteron scattering. Gell-Mann and Goldberger (10) obtained that the expression for the transition amplitude from the meson-deuteron state to the three-free-particle state is given by $(T_{inel})_{fi} = \langle \phi_f | (U+V_p+V_n) | \psi_i^{(+)} \rangle$. By using the same method, another expression can be obtained, equivalent to the above for $E_i = E_f$, but written as a matrix element between states $\phi_f^{(-)}$ and ψ_i instead of between ϕ_f and $\psi_i^{(+)}$. We obtain $(T_{inel})_{fi} = \langle \phi_f^{(-)} | (V_p+V_n) | \psi_i \rangle$. We can then define the two operators

$$T^+_{inel} = (U+V_p+V_n) + (U+V_p+V_n)(E_i - K - U - V_p - V_n + i\varepsilon)^{-1}(V_p+V_n) \tag{4}$$

and

$$T^-_{inel} = (U+V_p+V_n)(E_f - K - U - V_p - V_n + i\varepsilon)^{-1}(V_p+V_n) + (V_p+V_n) \tag{5}$$

which are extensions of the the definition of T_{inel} to include off-the-energy shell matrix elements. Here the parameter E_i is the energy of the meson-deuteron state on the right of the operator T^+_{inel} and E_f is the energy of the free-particle state on the left of T^-_{inel}. Comparing Eqs. (4) and (5) we see that T^+_{inel} and T^-_{inel}

differ not only by the values E_i and E_f of the energy parameter in the denominator, but also by the extra U that appears in T^+_{inel}. We can prove that this difference disappears on the energy shell. In fact, since the integrals over closed surfaces of the flux of ϕ_f and ψ_i vanish, and K is hermitian, we have $\langle \phi_f | U | \psi_i \rangle =$ $= \langle \phi_f | - K + (U+K) | \psi_i \rangle = (E_i - E_f) \langle \phi_f | \psi_i \rangle$ which is zero for $E_f = E_i$.

We can obtain more general expressions for operators T^+ and T^- which apply to both elastic and inelastic scattering. With the definitions of φ, $\varphi^{(+)}$, $\varphi^{(-)}$, Ω_+, Ω_- given above we have that T^+_{inel} and T^+_{el} can be written in a single expression

$$T^+ = (H - E_f) + (H - E_f)(E_i - H + i\varepsilon)^{-1}(H - E_i) = (H - E_f)\Omega_+(E_i) \tag{6}$$

and T^-_{inel} and T^-_{el} can both be given the form

$$T^- = (H - E_i) + (H - E_f)(E_f - H + i\varepsilon)^{-1}(H - E_i) = \Omega^*_-(E_f)(H - E_i) \tag{7}$$

3. EXPANSION OF THE COLLISION OPERATORS FOR MESON-DEUTERON INELASTIC SCATTERING IN TERMS OF TWO-PARTICLE OPERATORS

The purpose of our analysis is to obtain (approximate) expressions relating quantities which can be obtained from the experiments, i.e. quantities like matrix elements of collision operators. Since almost no information is available at the present about the potentials V_p and V_n, it is in terms of two-particle collision operators that the meson deuteron scattering is to be analysed.

For the meson-proton system we define [4] the collision operators $t_p^+ = V_p + V_p\,(E_i - K - V_p + i\varepsilon)^{-1}\,V_p$ and $t_p^- = V_p + V_p\,.\,.\,(E_f - K - V_p + i\varepsilon)^{-1}\,V_p$ where E_i is the energy of the plane wave state on the right and E_f that of the plane-wave state on the left when these operators are taken between two states. These are extensions for regions off-the-energy shell of the usual operators of the scattering theory. We need these extended definitions because we shall be concerned with off-the-energy shell matrix elements. For the meson-neutron and the nucleon-nucleon interactions we define analogous expressions t_n^+, t_n^-, t_u^+, t_u^-.

These definitions given for the t^+ operators (we drop for a while the indices n, p, u) assume that they are to be taken between states such that on the right one has an eigenstate of the kinetic energy operator, with energy E_i. This definition is not complete for our purpose, in the sense that it does not tell how the operator acts on an arbitrary state, i.e. on a superposition of free particle states ϕ_ℓ (with $K\,\phi_\ell = E_\ell\,\phi_\ell$). t^+ must have the property

$$t^+ \sum_\ell c_\ell\,\phi_\ell> = \sum c_\ell \left[V + V\,(E_\ell - K - V + i\varepsilon)^{-1}\,V\right]\,\phi_\ell>$$

We can find an explicit expression for such an operator, namely

$$t^+ = \sum_j \left[V + V\,(E_j - K - V + i\varepsilon)^{-1}\,V\right]\,\phi_j><\phi_j$$

Similarly the complete definition of the operator t^- is

$$t^- = \sum_j \phi_j><\phi_j \left[V + V\,(E_j - K - V + i\varepsilon)^{-1}\,V\right]$$

For brevity we shall supress the br[illegible]ket and summation symbols when writing the expressions for the t-operators, but keeping in mind these complete expressions when operating with t^+ and t^- on packets of free waves. For the calculation of cross-sections we shall be interested in the on-the-energy shell matrix elements of the collision operators for meson-deuteron scattering. So we take $E_f = E_i = E$, $T^+_{inel} = T^-_{inel} = T_{inel}$. Our task in this section is to expand the expression for T_{inel} in such a way that the two-particle collision operators t_p, t_n, t_u appear in the most important terms instead of the potentials V_p, V_n, U.

The initial state consisting of a deuteron and a free meson, can be represented at a given time by a superposition of plane-wave states of three free free particles $|\psi_i\rangle = \sum_\ell c_\ell |\phi_\ell\rangle$, where c_ℓ is determined by our knowledge of the structure of the deuteron, namely the deuteron wave-function in momentum space. By applying the operator identity $A^{-1} = B^{-1} + A^{-1}(B - A) B^{-1}$ to the expression defining t^+_p after multiplication on the left by $(E_\ell - K + i\varepsilon)^{-1}$ we obtain

$$(E_\ell - K - V_p + i\varepsilon)^{-1} V_p \mid \phi_\ell\rangle = (E_\ell + i\varepsilon - K)^{-1} t^+_p \mid \phi_\ell\rangle \quad (8)$$

where E_ℓ is the kinetic energy of the particles in the free state ϕ_ℓ. Analogously we can obtain

$$\langle \phi_\ell \mid V_p (E_\ell - K - V_p + i\varepsilon)^{-1} = \langle \phi_\ell \mid t^-_p (E_\ell - K + i\varepsilon)^{-1} \quad (8')$$

These are very useful relations. We can write analogous expressions for V_n and U.

By using the above mentioned operator identity, and relations like (8) and (8') we obtain

$$T_{inel}|\psi_i\rangle = \sum_\ell c_\ell \Big\{(U+V_p+V_n)+(U+V_p+V_n)\Big[(E_\ell+i\varepsilon-K-V_p)^{-1} V_p +$$

$$+ (E_\ell+i\varepsilon-K-V_n)^{-1} V_n + (E+i\varepsilon-K-U-V_p-V_n)^{-1} (E_\ell-E+U+V_n) \,.$$

$$. \,(E_\ell+i\varepsilon-K-V_p)^{-1} V_p + (E+i\varepsilon-K-U-V_p-V_n)^{-1} (E_\ell-E+U+V_p) \,.$$

$$. \,(E_\ell+i\varepsilon-K-V_n)^{-1} V_n \Big]\Big\} \,|\, \phi_\ell\rangle = \sum_\ell c_\ell \Big\{U+t_p^+ + t_n^+ + (U+V_p+V_n).$$

$$. \,(E+i\varepsilon-K-U-V_p-V_n)^{-1} (E_\ell-E)\,(E_\ell+i\varepsilon-K)^{-1} (t_p^+ + t_n^+) \;+$$

$$+ \Big[(U+V_n)+(U+V_p+V_n)(E+i\varepsilon-K-U-V_p-V_n)^{-1} (U+V_n)\Big](E_\ell+i\varepsilon-K)^{-1} t_p^+ +$$

$$+ \Big[(U+V_p)+(U+V_p+V_n)(E+i\varepsilon-K-U-V_p-V_n)^{-1}(U+V_p)\Big](E_\ell+i\varepsilon-K)^{-1} t_n^+\Big\}|\phi_\ell\rangle$$

Let us evaluate the contribution from one of the terms in the brackets. In order to do this, we apply the operators inside the brackets to the final state $\langle\phi_f|$ on the left

$$\langle\phi_f|\Big[(U+V_n)+(U+V_p+V_n)(E+i\varepsilon-K-U-V_p-V_n)^{-1} (U+V_n)\Big].$$

$$(E_\ell+i\varepsilon-K)^{-1} t_p^+ \,|\, \psi_i\rangle = \langle\phi_f \,|\, \Big\{ t_u^- + t_n^- + t_u^- (E+i\varepsilon-K)^{-1} (V_p+V_n) \,.$$

$$(E + i\varepsilon - K - U - V_p - V_n)^{-1} U + t_n^- (E + i\varepsilon - K)^{-1} (U + V_p) \cdot$$

$$\cdot (E + i\varepsilon - K - U - V_p - V_n)^{-1} V_n + V_p (E + i\varepsilon - K - U - V_p - V_n)^{-1} (U + V_n) +$$

$$+ U(E + i\varepsilon - K - U - V_p - V_n)^{-1} V_n + V_n (E + i\varepsilon - K - U - V_p - V_n)^{-1} U \Big\}$$

$$(E_\ell + i\varepsilon - K)^{-1} t_p^+ | \varphi_i >$$

We thus obtain

$$\langle \phi_f | T_{inel} | \psi_i \rangle = \sum_\ell c_\ell \langle \phi_f | t_p^+ + t_n^+ + t_u^- (E_\ell + i\varepsilon - K)^{-1} t_p^+ + t_n^- (E_\ell + i\varepsilon - K)^{-1} t_p^+ +$$

$$+ t_u^- (E_\ell + i\varepsilon - K)^{-1} t_n^+ + t_p^- (E_\ell + i\varepsilon - K)^{-1} t_n^+ + \text{remainder} \mid \phi_\ell \rangle \qquad (9)$$

where

$$\text{remainer} = T_{inel} (E + i\varepsilon - K - U)^{-1} (E_\ell - E)(E_\ell + i\varepsilon - K)^{-1} (t_p^+ + t_n^+) +$$

$$+ \text{terms of higher order} \qquad (10)$$

The purpose of this series of transformations has been that of eliminating the "unobservable" potentials and introducing the "observable" collision operators. We have thus expanded T_{inel} in products of the type $t\, G_o\, t'\, G_o\, t'' \ldots$ (where $G_o = (E - K + i\varepsilon)^{-1} =$ = free particle propagator) of higher and higher orders, which correspond to multiple scattering processes, together with terms which represent corrections to them coming from essentially three-body effects. By "terms of higher order" in (10) we mean terms which when expressed in the form of products of collision operators and propagators will consist of a product of two or more propagators and three or more collision operators.

It is easy to understand the meaning of the terms in Eq. (9). These with t_p^+ and t_n^+ alone correspond to single scattering by the proton and by the neutron, respectively. The collision operator contains a δ-function of momentum variables as a factor so that the momentum is conserved in each collision. However, energy cannot be conserved in the two-particle collision, i.e. $\langle \phi_f | t_p^+ | \phi_\ell \rangle$ and $\langle \phi_f | t_n^+ | \phi_\ell \rangle$ in Eq. (9) are necessarily off-the-energy shell matrix elements. Since we impose conservation of energy in the whole process, $E_f = E_i$ = incident meson energy + deuteron mass. But E_ℓ = incident meson energy + proton mass + neutron mass + proton kinetic energy + neutron kinetic energy, so that $E_\ell > E_f$ for any ℓ.

Terms like $\sum_\ell c_\ell \langle \phi_f | t_n^-(E_\ell - K + i\epsilon)^{-1} t_p^+ \mid \phi_\ell \rangle$ represent double scattering processes. Here the incident meson collides with a proton of a certain momentum labelled ℓ, the system of three free particles of energy E_ℓ then propagates until there is scattering of the meson by the neutron, leading the system to the specified final state. There is conservation of momentum, but not necessarily of energy, in each of these two collisions. The terms $t_u^- (E_\ell + i\epsilon - K)^{-1} t_p^+$ and $t_u^- (E_\ell + i\epsilon - K)^{-1} t_n^+$ represent the collision of the meson with one of the nucleons followed by a collision of the two nucleons, and are the simplest form of "potential corrections" to the multiple scattering model. The terms of first and second order are diagrammatically represented in Fig. 1, where nucleons are represented by heavy lines, mesons by dotted lines.

The "remainder" represents multiple scattering processes of higher order, and essentially-three-body processes which cannot be expressed in terms of two-particle collision operators. They bring out the fact, for example, that the two nucleons are not free, and strictly speaking cannot be considered as such during the interval of time during which collision processes occur. For example, pions can be exchanged between the two nucleons while interactions with the incident meson are taking place. These effects are illustrated in the diagrams of Fig. 2.

We can obtain a different expansion for T_{inel} in the following way. We first apply T_{inel} to $\langle\phi_f|$ from the right, obtaining

$$\langle\phi_f|T_{inel} = \langle\phi_f| \Big\{ t_p^- + t_p^- (E+i\epsilon-K)^{-1} \Big[V_n + (U+V_n)(E+i\epsilon-K-U-V_p-V_n)^{-1} \times$$

$$\times (V_p+V_n)\Big] + t_n^- + t_n^-(E+i\epsilon-K)^{-1}\Big[V_p+(U+V_p)(E+i\epsilon-K-U-V_p-V_n)^{-1} (V_p+V_n)\Big] +$$

$$+ t_u^-(E+i\epsilon-K)^{-1}\Big[(V_p+V_n)+(V_p+V_n)(E+i\epsilon-K-U-V_p-V_n)^{-1} (V_p+V_n)\Big] \Big\}$$

By applying this expression from the left on $|\Psi_i\rangle = \sum_\ell c_\ell |\phi_\ell\rangle$ we obtain, collecting terms up to second order,

$$\langle\phi_f|T_{inel}|\Psi_i\rangle = \sum_\ell c_\ell \langle\phi_f| t_p^- + t_n^- + t_p^-(E+i\epsilon-K)^{-1} t_n^+ + t_n^-(E+i\epsilon-K)^{-1} t_p^+ +$$

$$+ t_u^- (E+i\epsilon-K)^{-1} (t_n^+ + t_p^+) + \text{remainder} \; |\phi_\ell\rangle \qquad (11)$$

where

$$\langle\phi_f|\text{remainder}|\psi_i\rangle = \sum c_\ell \langle\phi_f|\, t_u^- \,(E+i\varepsilon-K)^{-1}\, T_{el}(E-K-U+i\varepsilon)^{-1}(E_\ell-E+U)\,\times$$

$$\times\,(E_\ell-K+i\varepsilon)^{-1}\,(t_p^+ + t_n^+)|\,\phi_\ell\rangle + \text{terms of the same and higher orders.} \tag{12}$$

Comparing the two expansions (9) and (11) we see that they differ, firstly in the single scattering terms by the fact that in one case we have t_p^+ and t_n^+ and in the other we have t_p^- and t_n^- which are different for off-the-energy shell matrix elements. Secondly, in (9) the energy parameter in the propagators of the second order terms is E_ℓ, while in (11) we have E. The "remainder" for (11) is of one "order" higher than that for (9). This could suggest that (11) is a better expansion than (9). However, the contributions coming from these remainder terms are difficult to estimate. Also, we do not know anything about the behaviour of off-the-energy shell matrix elements of collision operators. Thus we could not really justify preference for one or other of the two expansions. We can expect that their difference is smaller than the error involved in neglecting the residual terms of the expansions.

We can try to write (9) or (11) in terms of two-particle scattering states. Let us consider (11). We can group the "single scattering" and "potential correction" terms into the form

$$\langle\phi_f|\left[1 + t_u^-\,(E + i\varepsilon - K)^{-1}\right](t_p^+ + t_n^+) = \langle\phi_{fu}^{(-)}\,|(t_p^+ + t_n^+)$$

where $|\,\phi_{fu}^{(-)}\rangle$ represents a free meson plane wave and an ingoing-wave scattering state of the two-nucleon system. It is a solution of the Schrodinger equation with hamiltonian $K+U$ and specified asymptotic behaviour. Eq. (11) then becomes

$$\langle\phi_f|T_{inel}|\psi_i\rangle = \langle\phi_{fu}^{(-)}|(t_p^+ + t_n^+)|\psi_i\rangle + \langle\phi_f|\Big\{ t_p^- (E+i\varepsilon-K)^{-1} t_n^+ + $$

$$+ t_n^- (E+i\varepsilon-K)^{-1} t_p^+ + \text{remainder} \Big\}|\psi_i\rangle \quad (13)$$

The first term on the right hand side of (13), with its ingoing-wave scattering state in the left hand side of the matrix element, resembles the usual form of the Final State Interaction Theory [11].

The double scattering terms can also be expressed in terms of meson-nucleon scattering states. Since ϕ_f is a three-free--particle state with energy E,

$$\langle\phi_{fp}^{(-)}| = \langle\phi_f|\left[1 + t_p^-(E-K+i\varepsilon)^{-1}\right] = \langle\phi_f|\left[1 + V_p(E-K+i\varepsilon-V_p)^{-1}\right]$$

is a solution of the Schrodinger equation, with hamiltonian $K+V_p$ representing a free neutron and an ingoing-wave scattering state of the meson-proton system. We have an analogous expression for $\phi_{fn}^{(-)}$, the meson-neutron scattering state. Thus Eq. (11) can be written

$$\langle\phi_f|T_{inel}|\psi_i\rangle = \langle\phi_{fu}^{(-)}|(t_p^+ + t_n^+)|\psi_i\rangle + \langle\phi_{fp}^{(-)}|t_n^+|\psi_i\rangle +$$

$$+\langle\phi_{fn}^{(-)}|t_p^+|\psi_i\rangle - \langle\phi_f|(t_p^+ + t_n^+)|\psi_i\rangle + \text{remainder} \quad (14)$$

The approximation whereby one assumes that $\langle\phi_f|T_{inel}|\psi_i\rangle =$

$= \langle \Phi_f | (t_p + t_n) | \Psi_i \rangle$ is usually called Impulse Approximation. Much has already been said in the litterature [1-3] on the conditions under which multiple scattering of the meson and the effects of the nucleon-nucleon interaction are small compared to single scattering processes. The meson being fast and its interaction with the nucleon being of short range, we expect that during the short interval of time in which the meson nucleon interaction takes place, the nucleon-nucleon binding has small effects. The short range of the meson-nucleon interaction as compared to an average internucleonic distance in the deuteron, would cause double scattering processes to be much less important than the single scattering ones. The importance of the second order processes will depend, among other things, on the value of the matrix elements of the collision operators t_p, t_n, t_u, i.e. on peculiarities of the particular system studied. If we keep all the second order terms we may have under certain conditions a good approximation to the meson-deuteron inelastic scattering.

4. DYNAMICAL VARIABLES

In section 3 we expressed formally the collision operator for the inelastic meson-deuteron scattering in terms of the several two-particle collision operators. We now introduce explicitly the dynamical variables describing the system and show how the main terms of the expansion we have obtained depend on these variables and on the quantities describing two-particle processes more directly.

Let us define the following symbols. $\vec{q}$ - meson momentum in lab system; E_q - meson total energy in lab; $\vec{p}$, $\vec{n}$, E_p, E_n - proton and neutron lab momenta and total energies; $\vec{K}$ - total momentum of the proton-neutron system; $\vec{\ell}$ - proton neutron relative momentum (momentum of proton relative to centre of mass of the neutron-proton system). Indices o and f will indicate the values of the variables in the initial and final state respectively. M will indicate the nucleon mass and m the meson mass. We have the relations $\vec{K} = \vec{p} + \vec{n}$, $\vec{\ell} = (E_n\, p - E_p\, n)/(E_n + E_p)$. Let us call $\vec{k}_p$ and $\vec{k}_n$ the meson-proton and meson-neutron relative momenta, defined by $\vec{k}_p = (E_p\,\vec{q} - E_q\,\vec{p})/(E_p + E_q)$ and $\vec{k}_n = (E_n\,\vec{q} - E_q\,\vec{n})/(E_n + E_q)$.

We use δ-function normalization for the plane waves. The momentum space representation for the initial state is $\langle \vec{\ell}, \vec{q}, \vec{k} | \psi_i(\vec{q}_o, \vec{k}_o) \rangle = \delta(\vec{q} - \vec{q}_o)\, \delta(\vec{k} - \vec{k}_o) \psi_D(\ell)$ where $\psi_D(\ell)$ is the (normalized to one) deuteron wave function in momentum space. We keep the spin variables implicit for a while to avoid unnecessary complications.

As t_p does not act on the neutron and contains as a factor a δ-function responsible for conservation of momentum in the collision involved, we can write

$$\langle \vec{p}', \vec{n}', \vec{q}' | t_p | \vec{p}, \vec{n}, \vec{q} \rangle = -\delta(\vec{n}' - \vec{n})\, \delta(\vec{p}' + \vec{q}' - \vec{p} - \vec{q}) \langle \vec{k}_p' | r_p | \vec{k}_p \rangle \quad (15)$$

thus defining the operator r_p. Analogously can be defined operators r_n and r_u.

For the term in the expansion of $\langle \phi_f | T_{inel} | \psi_i \rangle$ representing single scattering by the proton we obtain

$$<\phi_f(\vec{q}_f, \vec{n}_f, \vec{p}_f)|t_p|\psi_i(\vec{q}_o, \vec{K}_o)> =$$

$$= -\delta(\vec{P}_f - \vec{P}_i)\psi_D(n_f)< \frac{M\vec{q}_f - E_{q_f}\vec{p}_f}{M + E_{q_f}} |r_p| \frac{M\vec{q}_o + E_{q_o}\vec{n}_f}{M + E_{q_o}} > \qquad (16)$$

where $\vec{P}_f = \vec{K}_f + \vec{q}_f$ and $\vec{P}_i = \vec{K}_o + \vec{q}_o$ are the total momenta in the final and initial state respectively. The nucleons have been treated as non-relativistic. Analogous expression can be written for the term representing single scattering by the neutron.

Now for the double scattering terms. Introducing complete sets of free-particle states between operators and propagators, using expression like Eq. (15) and the explicit representation of final and initial states, we obtain

$$<\phi_f(\vec{q}_f, \vec{n}_f, \vec{p}_f)|t_n(E - K + i\varepsilon)^{-1} t_p|\psi_i(\vec{q}_o, \vec{K}_o = 0)> = \delta(\vec{P}_f - \vec{P}_i) \times$$

$$\times \int d_3\vec{\ell} < \frac{M\vec{q}_f - E_{q_f}\vec{p}_f}{M + E_{q_f}} |r_p| \frac{M(\vec{q}_o - \vec{\ell} - \vec{n}_f) + E_{q_m}\vec{\ell}}{M + E_{q_m}} > (E - E_m + i\varepsilon)^{-1} \quad .$$

$$. < \frac{M\vec{q}_m - E_{q_m}\vec{n}_f}{M + E_{q_m}} |r_p| \frac{M\vec{q}_o + E_{q_o}\vec{\ell}}{M + E_{q_o}} > \psi_D(\ell) \qquad (17)$$

where $\vec{p} = -\vec{n} = \vec{\ell}$, $\vec{q}_m = \vec{q}_o - \vec{\ell} - \vec{n}_f$, and $E_m = 2M + (\ell^2 + n_f^2)/(2M) + \sqrt{m^2 + q_m^2}$.

For the double-scattering term in which the first collision is on the neutron we obtain an expression similar to Eq. (17), with the proper changes of the roles of the neutron and proton variables.

For the term $t_u(E-K+i\varepsilon)^{-1}\ t_p$ which occurs in the expansion of T_{inel} we obtain

$$\langle\phi_f(\vec{q}_f,\ \vec{p}_f,\ \vec{n}_f)|\,t_u(E-K+i\varepsilon)^{-1}\ t_p|\psi_i(\vec{q}_o,\ \vec{K}_o=0)\rangle = \delta(\vec{P}_f-\vec{P}_i)\ I_{up} =$$

$$= \delta(\vec{P}_f-\vec{P}_i)\int d_3\vec{\ell}\,\langle\vec{\ell}_f|r_u|\vec{\ell}_m\rangle(E-E_m+i\varepsilon)^{-1}\langle\vec{k}_{p_m}|r_p|\vec{k}_p\rangle\psi_D(\ell) \qquad (18)$$

where $\vec{p}_m = \vec{\ell} + \vec{q}_o - \vec{q}_f$ and $E_m = E_{q_f} + 2M + \ell^2/(2M) + (\vec{\ell}+\vec{q}_o-\vec{q}_f)^2/(2M)$. Analogously for the term $t_u(E-K+i\varepsilon)^{-1}\ t_n$.

We thus see that all the terms contributing to the expansion of T_{inel} have $\delta(\vec{K}_f+\vec{q}_f-\vec{K}_o-\vec{q}_o)$ as a factor, as they should. Let us introduce the operator R_{inel} such that

$$(T_{inel})_{fi} = -\delta(\vec{P}_f-\vec{P}_i)\ (R_{inel})_{fi} \qquad (19)$$

The meson-deuteron inelastic scattering cross-section is given by

$$d\,\sigma_{inel} = \int\Sigma\,\frac{(2\pi)^4}{v}\ \delta(E_f-E_i)|(R_{inel})_{fi}|^2\ d_3\vec{\ell}_f\,d_3\vec{q}_f \qquad (20)$$

where Σ represents the appropriate sum and average over the final and initial states and v is the velocity of the incident meson in the rest system of the deuteron.

5. THE DOUBLE SCATTERING TERMS

We now proceed to the explicit evaluation of the terms contributing to the matrix element of the collision operator T_{inel}. The terms corresponding to single scattering of the incident meson are already given explicitly by Eq. (16) and analogous. To obtain the contributions coming from double scattering and potential correction terms we have to evaluate integrals as those in Eqs. (17) and (18).

We first notice in the integrands the presence of the matrix elements of r_p, r_n, r_u with arguments which depend on the variable of integration. In Eq. (17) the dependence of the arguments on $\vec{\ell}$ is explicitly exhibited. The values of ℓ that contribute to the integral are those available in the deuteron wave-function, i.e. those which make $\ell^2 \psi_D(\ell)$ large. These values of ℓ lie between zero and about 150 MeV/c. As $\vec{\ell}$ varies in modulus and direction within this range of values, the relative momentum of the two colliding particles and the scattering angle vary. If q_o is not small the relative momentum of the meson-nucleon system will vary within a not very wide solid angle and the scattering angle will correspondingly not have a large fluctuation.

Using the relation $(E - E_m \mp i\epsilon)^{-1} = P[1/(E - E_m)] \pm i\pi\, \delta(E - E_m)$ where P means principal value, we can separate the integrals representing the second order processes into two parts, one taking into account the contributions coming from values of E_m on the energy shell $E_m = E$, and the other involving values of E_m which are different from E. In Eqs. (17) and (18), $E_m = E$ is the energy shell for the second interaction in the double scattering processes represented by them. On-the-energy shell matrix elements of the collision operator for the second correspond to off-the-energy shell matrix elements in the first scattering. The larger the value of ℓ, the farther from the shell $E_\ell = E$ is the matrix element for the first interaction. This is so if we use the expansion (11) for T_{inel}, as we did in Sec. 4. If, instead, we adopt the expansion (9) the strong contribution will come from values of ℓ such that $E_m = E_\ell$, which is the energy shell for the first interaction in the second order processes.

So, strictly speaking, a knowledge of the behaviour of the off-the-energy shell matrix elements of the collision operators of the meson-nucleon systems is essential in our problem. This knowledge is not available at the present, however, and we shall then have to assume some sort of behaviour of these matrix elements off-the-energy shell, perhaps that they have a constant value. As the deuteron wave-function contains momenta up to a value which is not very large, only matrix elements which are not very far from the energy shell will have important contributions, and it may be not so bad to assume a constant value.

Let us consider the process in which the meson is scattered by a neutron and then rescattered by the proton. The matrix element is given by Eq. (17), where if the intermediate meson is non-relativistic we have $E_m = 2M + (n_f^2 + \ell^2)/(2M) + m + (\vec{q}_o - \vec{n}_f - \vec{\ell})^2/(2m)$. Let us call

$$\vec{K} = (\vec{q}_o - \vec{n}_f)\mu/m$$

$$C = \mu\left(E_{q_f} + p_f^2/(2m) - m - K^2/(2m)\right) \tag{21}$$

where $\mu = Mm/(M+m)$. The integral in Eq. (17) is singular in a spherical surface $k_m^2 = (\vec{\ell} - \vec{K})^2 = K^2 + 2C = k_f^2$ where $\vec{k}_f$ and $\vec{k}_m$ are the momenta of the meson in the final and intermediate states, relative to the centre of mass of the meson-proton system.

Let us assume that $\langle |r_p| \rangle$ and $\langle |r_n| \rangle$ in Eq. (17) are constants equal to a_p and a_n respectively, so as to be extracted from inside the integral signs. Using the Hulthén wave-function $\psi_D(r) = N\left[\exp(-\alpha r) - \exp(-\beta r)\right]/r$ for the deuteron, we obtain for the on-the-energy shell part

$$i a_p a_m J_{pn}^{(1)} = \int d_3\vec{\ell} \langle \vec{q}_f, \vec{p}_f | r_p | \vec{q}_m, \vec{\ell} \rangle (-i\pi)\delta(E-E_m) \langle \vec{q}_m, \vec{n}_f | r_n | \vec{q}_o, -\vec{\ell} \rangle \psi_D(\ell) =$$

$$= a_p a_n \, \mu (2\pi)^{3/2} \frac{N}{K} \frac{i}{2} \left[\ln \frac{\alpha^2 + (K - k_f)^2}{\beta^2 + (K - k_f)^2} - \ln \frac{\alpha^2 + (K + k_f)^2}{\beta^2 + (K + k_f)^2} \right] \tag{22}$$

In the integration over the sphere $E = E_m$ the relative meson-to-proton momentum varies only in direction, its modulus being constant. So, extracting $\langle |r_p| \rangle$ from the integrand means only to assume that it is independent of the scattering angle (that is, that it has an S-wave-like behaviour). On the other hand, the energy shell $E_\ell = E_m$ for the scattering by the neutron is a sphere with centre at the point $-\vec{K}(M+m)/(M-m)$ and radius $|M\vec{q}_o - m\vec{n}_f|(M-m)^{-1}$. This sphere does not cross the surface $E = E_m$, and so only off-the-

-energy shell matrix elements of r_n are involved in the integration in Eq. (22).

For the principal part of the integral in Eq. (17) we obtain, by considering the matrix elements as constant

$$a_p a_n J_{pn}^{(2)} = P \int d_3 \vec{\ell} \langle \vec{q}_f, \vec{p}_f | r_p | \vec{q}_m, \vec{\ell} \rangle (E-E_m)^{-1} \langle \vec{q}_m, \vec{n}_f | r_n | \vec{q}_o, -\vec{\ell} \rangle \psi_D(\ell) =$$

$$= a_p a_n \, \mu (2\pi)^{3/2} \frac{N}{K} \left[\tan^{-1}\left(\frac{\beta K}{C + \frac{1}{2}\beta^2}\right) - \tan^{-1}\left(\frac{\alpha K}{C + \frac{1}{2}\alpha^2}\right) \right] \tag{23}$$

Two of the three quantities C, K, k_f have a certain freedom of variation with respect to each other, which is only restricted by the energy conservation in the whole process, $E_f = E_i$. We have evaluated numerically Eqs. (22) and (23) for the case of incident K-mesons of momentum q_o = 200 MeV/c for several values of the momenta of the particles in the final state, trying to cover all the spectra of possible values. We obtained that the two parts, $J_{pn}^{(1)}$ and $J_{pn}^{(2)}$, are in general of the same order of magnitude, one or other predominating in the different regions of the spectrum.

It is instructive to study the way in which the integral in Eq. (23) is formed. It is particularly interesting to know whether or not important contributions to this integral come from values near the energy shell. Using the variable $\vec{k}_m = \vec{\ell} - \vec{K}$ we can write

$$J_{pn}^{(2)} = (2\pi)^{\frac{1}{2}} \left(\frac{N}{K}\right) \int_0^\infty \frac{k_m \, d\, k_m}{(k_m+k_f)(k_m-k_f)} \left[\ell n \frac{\beta^2+(k_m-K)^2}{\alpha^2+(k_m-K)^2} - \ell n \frac{\beta^2+(k_m+K)^2}{\alpha^2+(k_m+K)^2} \right] \tag{24}$$

Studying in detail the integrand we obtain that the important contributions to the integral come from values of k_m which are not very far from the energy shell. If instead of having the deuteron wave-function (which gave rise to the two subtracting logarithm functions) we had a flat wave-function in momentum space (which would correspond to the deuteron having a small radius), there would be stronger cancellation of the contributions coming from the neighbourhood of the pole, and more important contributions would come from large values of k_m, i.e. from regions far from the energy shell.

Thus we conclude that keeping $\langle |r_p| \rangle$ constant in the integration is possibly not a bad approximation, since the most important contributions come from values not very far from the energy shell.

For the double scattering process in which the meson first hits the proton and then is scattered by the neutron, we obtain results of the same form as Eqs. (22) and (23) with the roles of proton and neutron exchanged in the definitions in Eq. (21).

We now compare the magnitudes of the contributions of the single and double scattering processes to the transition amplitude. For a final state in which the momenta of the particles are $\vec{q_f}$, $\vec{p_f}$, $\vec{n_f}$ we have that the scattering amplitude for single scattering by the proton is proportional to $-a_p \Psi_D (n_f)$ and if the scattering is by the neutron it is proportional to $- a_n \Psi_D(p_f)$. The values of these amplitudes vary along the spectra of possible values of p_f, n_f, but to have a value characteristic of the important part of the spectrum, we may take the value of the deuteron wave-function at $\ell = 0$, which is of about $(2\pi)^{\frac{1}{2}}\ N/\alpha^2$. The magnitudes of the expressions that multiply $a_p a_n (2\pi)^{3/2}\ N\mu/K$ in Eqs. (22) and (23) have intervals of variation which are inside the interval $(0, \pi)$. For small values of K these expressions tend to zero so that the factor 1/K outside is cancelled. Assuming $a_p \sim a_n$ we then obtain

$$\frac{\text{2nd order}}{\text{1st order}} \simeq (2\pi)^2\ a\mu\alpha \ = \alpha(\sigma/4\pi)^{\frac{1}{2}} \approx \frac{1}{50}\ \left[\sigma(\text{mb})\right]^{\frac{1}{2}} \qquad (25)$$

where σ(mb) is the total cross-section for meson-nucleon scattering measured in milibarns.

6. THE "POTENTIAL CORRECTION" TERMS

Let us now consider the second order process in which the meson collides with the proton, which recoils and is then scattered by the neutron. The integrand in Eq. (18) is singular on the surface of radius ℓ_f and centre at the point $-\frac{1}{2}(\vec{q}_o - \vec{q}_f)$. This sphere is the energy shell for $\langle \vec{\ell}_f | r_u | \vec{\ell}_m \rangle$, while the energy shell for $\langle \vec{k}_p | r_p | \vec{k}p_e \rangle$ is a plane ortogonal to $(\vec{q}_o - \vec{q}_f)$. This plane does not cross the sphere if we impose conservation of energy in the whole process. The integral in Eq. (18) is similar to that in Eq. (17), and we could make considerations about the behaviour of the integrand near and far from the energy shell analogous to those we made for the double scattering terms.

Let us call

$$\vec{\Delta} = \vec{q}_f - \vec{q}_o \qquad (26)$$

Considering the matrix elements of r_p and r_u in Eq. (18) as constants respectively equal to a_p and a_u we obtain for the on-the--energy shell part of the integral

$$ia_p a_u I_{up}^{(1)} \equiv a_p a_u (M/\Delta)(2\pi)^{3/2}\, N(1/2)\left[\ln \frac{\alpha^2+(\frac{1}{2}\Delta-\ell_f)^2}{\beta^2+(\frac{1}{2}\Delta-\ell_f)^2} - \ln\frac{\alpha^2+(\frac{1}{2}\Delta+\ell_f)^2}{\beta^2+(\frac{1}{2}\Delta+\ell_f)^2}\right] \tag{27}$$

and for the principal part

$$a_u a_p I_{up}^{(2)} \equiv a_p a_u\, (M/\Delta)(2\pi)^{3/2}\, N\left[\tan^{-1}\left(\frac{\beta\Delta}{\ell_f^2-\frac{1}{4}\Delta^2+\beta^2}\right)-\tan^{-1}\left(\frac{\alpha\Delta}{\ell_f^2-\frac{1}{4}\Delta^2+\alpha^2}\right)\right] \tag{28}$$

with $I_{up} = a_u a_p(iI_{up}^{(1)} + I_{up}^{(2)})$. If the first collision is with the deuteron we obtain expressions similar to Eqs. (27) and (28), the only change being that a_p is substituted by a_n.

The nucleon-nucleon system has a larger cross-section than the meson-nucleon systems. So the approximation of assuming a constant value for the matrix elements in the principal-part integrals might be not so good in the case of nucleon-nucleon interaction as it was assumed to be in the case of the meson-nucleon interactions. The introduction of some sort of cut-off may be necessary. On the other hand, for meson incident momenta over a certain value, the recoil energies of the nucleons will be such that the nucleon-nucleon interaction in the final state will occur rather strongly in S, P and higher waves. It thus seems important to take into account the finite range of the nuclear forces and to include higher waves in our treatment of the nucleon-nucleon interaction. Both these tasks can be more easily accomplished if we write the expression in Eq. (18) in configuration space. Assuming that $\langle \vec{k}'|r_p|\vec{k}\rangle$ is a constant a_p, and introducing the Fourier transforms of the quantities in the integrand, we obtain

$$\langle f|t_p|i\rangle = -\delta(\vec{P}_f - \vec{P}_i)\int \langle f|\vec{r}\rangle\, e^{-i\frac{\Delta}{2}\cdot\vec{r}}\, \psi_D(r)\, d_3\vec{r}$$

where $\psi_D(r)$ is the deuteron wave function in configuration space, and

$$\langle\vec{r}|f\rangle = \langle\vec{r}|(E-K-i\varepsilon)^{-1}\, t_u|\phi_f\rangle = -(2\pi)^{-3/2}\int e^{i\vec{\ell}'\cdot\vec{r}}(E-E_{\ell'}-i\varepsilon)\langle\vec{\ell}'|r_u|\ell_f\rangle d_3\vec{\ell}'$$

is the configuration space representation of the scattered waves in the nucleon-nucleon interaction. $\vec{r}$ is the relative proton-to--neutron coordinate.

For the cases of S-waves we substitute $\langle\vec{\ell}'|r_u|\vec{\ell}_f\rangle$ by a_u (a constant) and obtain

$$\langle\vec{r}|f_s\rangle = -a_u(M/2)\ell_f\, 4\pi^2(2\pi)^{-3/2}\left[i j_o(\ell_f r) + n_o(\ell_f r)\right]$$

The first part, $j_o(\ell_f r)$ comes from the on-the-energy shell part of the integral, while $n_o(\ell_f r)$ comes from the principal part. Now, this is a valid solution of the Schrodinger equation only for values of r that are outside the range of the nuclear forces. For $r \to 0$, $n_o(\ell_f r)$ tends to infinity. To avoid this we have to introduce a cut-off in $\langle\vec{\ell}'|r_u|\vec{\ell}_f\rangle$ so that it tends to zero as $(\ell' - \ell_f)$ increases. This would affect the part $n_o(\ell_f r)$, transforming it in a function which converges as $r \to 0$, leaving the part $j_o(\ell_f r)$ as it is. The

The best way to introduce this effect is directly in the result above: we can either cut-off for distances smaller than the range of the nuclear forces, or introduce a convenient convergence factor, for example $(1-e^{-Zr})$ where Z is a parameter related to the range of the forces [9].

Analogously, the contribution coming from the P-wave interaction is obtained substituting $\langle \vec{\ell}' | r_u | \vec{\ell}_f \rangle$ by $b_u \cos(\vec{\ell}', \vec{\ell}_f)$. We obtain

$$\langle \vec{r} | f_p \rangle = -b_u (M/2) \ell_f \, 4\pi^2 (2\pi)^{-3/2} i \, P_1 \left[\cos(\vec{\ell}_f, \vec{r}) \right] \Big\{ i j_r(\ell_f r) +$$

$$+ \frac{2}{\pi} \left[j_1(\ell_f r) \, Ci(\ell_f r) + n_1(\ell_f r) \, Si \, (\ell_f r) \right] \Big\}$$

Outside the range of the forces we have the simpler expression

$$\langle \vec{r} | f_p \rangle = -b_u (M/2) \ell_f \, 4\pi^2 (2\pi)^{-3/2} \, i \, P_1 \left[\cos(\vec{\ell}_f, \vec{r}) \right] \Big\{ i j_1(\ell_f r) + n(\ell_f r) \Big] \Big\}$$

This can be transformed in a function which converges as $r \to 0$ by introducing in the part $n_1(\ell_f r)$ a convergence factor $(1-e^{-Zr})^2$. This rule can be extended (9) to any ℓ-wave.

We shall then have, taking only S and P waves in the nucleon-nucleon interaction,

$$I_{up} + I_{un} = M \ell_f (2\pi)^{3/2} a_u (a_p + a_n) \int_0^\infty \left[-i j_o(\ell_f r) + n_o(\ell_f r)(1-e^{-Zr}) \right] \cdot$$

$$\cdot \, j_o \left(\frac{\Delta}{2} r \right) \psi_D(r) r^2 dr + M \ell_f (2\pi)^{3/2} b_u (a_n - a_p) \cos(\vec{\ell}_f, \vec{\Delta}) \, .$$

$$\cdot \int_0^\infty \left[-ij_1(\ell_f r) + n_1(\ell_f r)(1-e^{-Zr})^2\right] j_1\left(\frac{\Delta}{2} r\right) \psi_D(r)\, r^2\, dr \tag{29}$$

The integral involving $j_o(\ell_f r) j_o\left(\frac{\Delta}{2} r\right)$ gives Eq. (27), the only change being that we now have $(a_p + a_n)$ instead of only a_p. The integral with $n_o(\ell_f r) j_o(\frac{\Delta}{2} r)$ gives the result of Eq. (28) minus the same expression where we substitute $\alpha \to \alpha + Z, \beta \to \beta + Z$ (this is due to the cut-off factor that has been introduced). For the part of Eq. (29) corresponding to on-the-energy shell P-waves we obtain

$$(a_n - a_p) b_u\, i\, L_u^{(1)} \cos(\vec{\ell}_f, \vec{\Delta}) \equiv b_u (a_n - a_p) \cos(\vec{\ell}_f, \vec{\Delta})(M/\Delta)(2\pi)^{3/2} i\, N \times$$

$$\times \left\{ (\beta^2 + \ell_f^2 + \frac{1}{4}\Delta^2) / (2\ell_f \Delta)\ \ln \frac{\beta^2 + (\ell_f + \frac{\Delta}{2})^2}{\beta^2 + (\ell_f - \frac{\Delta}{2})^2} - \right.$$

$$\left. - (\alpha^2 + \ell_f^2 + \frac{1}{4}\Delta^2)/(2\ell_f \Delta) \ln \frac{\alpha^2 + (\ell_f + \frac{\Delta}{2})^2}{\alpha^2 + (\ell_f - \frac{\Delta}{2})^2} \right\} \tag{30-}$$

and for the integral involving $n_1(\ell_f r) j_1(\frac{\Delta}{2} r)$ we obtain

$$(a_n - a_p) b_u\, L_u^{(2)} \cos(\vec{\ell}_f, \vec{\Delta}) \equiv (a_n - a_p) b_u \cos(\vec{\ell}_f, \vec{\Delta})(M/\Delta)\, (2\pi)^{3/2}\, N \times$$

$$\times \left\{ (\beta^2 + \ell_f^2 + \frac{1}{4}\Delta^2) \Big/ (\Delta \ell_f)\ \tan^{-1} \left[\frac{\beta \Delta}{(\beta^2 + \ell_f^2 - \frac{1}{4}\Delta^2)} \right] - \right.$$

$$\left. - (\alpha^2 + \ell_f^2 + \frac{1}{4}\Delta^2) \Big/ (\Delta \ell_f) \left[\tan^{-1} \frac{\alpha \Delta}{(\alpha^2 + \ell_f^2 - \frac{1}{4}\Delta^2)} \right] \right\} \tag{31}$$

minus twice this same expression with $\alpha \to \alpha + Z$, $\beta \to \beta + Z$ plus the same expression with the substitution $\alpha \to \alpha + 2Z$, $\beta \to \beta + 2Z$.

For $\Delta \to 0$, all these expressions contributing to $I_{up} + I_{un}$ tend to zero, in spite of the presence of Δ in the denominator. For $\ell_f \to 0$, the S-wave parts and also the P-wave on-the-energy shell part, Eq. (30), remain finite, but Eq. (31) increases like $1/\ell_f$, and the matrix element diverges if we consider b_u as a constant. This can be modified by noticing that the P-wave scattering amplitude b_u must tend rapidly to zero with the relative momentum of the two colliding particles ($b_u \sim \ell_f^3$ for low energies).

To compare the magnitudes of these nucleon-nucleon interaction effects with the first order terms, we can do as we did in the discussion of the double scattering terms: verify that the functions that appear can only vary inside a limited interval, and then take typical values for them, as well as for the first order terms. We obtain

$$\frac{\text{2nd order}}{\text{1st order}} \simeq \frac{\alpha}{\Delta} \sqrt{\sigma_{NN} \text{ (barns)}}$$

where $\alpha = 45.7$ MeV/c and σ_{NN}(barns) is the nucleon-nucleon total cross-section measured in barns. For low energy n-p scattering in the triplet state we have $\sigma_{NN} \simeq 2.4$ barns, and the relation above indicates that the effect of the nucleon-nucleon interaction in the final state can be very strong. For example, for incident K mesons of momentum q_o = 200 MeV/c a momentum transfer of 150 or 200 MeV/c is "typical", and the ratio between 2nd and 1st order matrix elements can then be of about 1/3 or 1/2. We thus expect that cross-sections are strongly affected by the nucleon-nucleon interaction in the final state.

7. EVALUATION OF CROSS-SECTIONS

We have obtained explicit expressions for all terms, corresponding to first and second order processes, that contribute to the transition amplitude for meson-deuteron inelastic scattering. We have now to square this transition amplitude and make the necessary sums over spin variables.

We here assume that there is no spin-dependence in the meson-nucleon interactions. As the deuteron has spin 1, only triplet final state will occur. Using the decomposition of the nucleon-nucleon triplet P-wave in terms of eigenstates of J as given in the Appendix, we have

$$(R_{inel})_{fi} = \langle f| \Big\{ A + P_t \int d_3\vec{\ell}_i \Big[(b_o-b_2)(1/12)(\vec{\sigma}_p-\vec{\sigma}_n) \cdot \frac{\vec{\ell}_i}{\ell_i}(\sigma_p-\sigma_n) \cdot \frac{\vec{\ell}_f}{\ell_f} +$$

$$+ (b_1-b_2)(1/8)(\vec{\sigma}_p+\vec{\sigma}_n) \cdot \frac{\vec{\ell}_i}{\ell_i}(\vec{\sigma}_p+\vec{\sigma}_n) \cdot \frac{\vec{\ell}_f}{\ell_f} + b_2 \cos(\vec{\ell}_f,\vec{\ell}_i) \Big] Q(\vec{\ell}_i) \Big\} |i\rangle$$

where b_o, b_1, b_2 are parameters describing the scattering in the $J = 0, 1, 2$ states respectively. Rearranging the terms we can put this in a form convenient for use of the formulae given in the Appendix. We obtain

$$\Sigma|(R_{inel})_{fi}|^2 = A^*A + \frac{1}{9}(b_o+3b_1+5b_2)\, B^*A + \frac{1}{9}(b_o^*+3b_1^*+5b_2^*)A^*B +$$

$$+ \frac{1}{3}\Big[\tfrac{1}{2}\,|b_1+b_2|^2 + \frac{1}{9}\,|b_o+2b_2\Big]^2 B^*B \qquad (32)$$

where A includes the single and double scattering effects, and the potential correction terms involving S-waves in the nucleon-nucleon interaction, and B includes the processes with nucleon-nucleon interaction in P-waves. It is interesting to notice that the parameters b_o, b_1, b_2 appear in the last term of Eq. (32) in a combination different from the combination with statistical weights $b_o + 3b_1, + 5b_2$ which is the only one that occurs in the scattering of unpolarized nucleons. This is, of course, due to the correlation of the spins of the two nucleons in the deuteron.

For the case of the inelastic K^+-deuteron scattering, we have the possibility of a double exchange process in which the K^+ hits the neutron giving K^o and proton, and then the K^o hits the proton producing K^+ and neutron. We then have

$$A = a_p \psi_D(n_f) + a_n \psi_D(p_f) - a_p a_n \left[i J_{pn}^{(1)} + J_{pn}^{(2)} \right] - (a_p a_n + a_{ex}^2) \left[i J_{np}^{(1)} + J_{np}^{(2)} \right] -$$

$$- (a_p + a_n)\, a_u \left[i I_{up}^{(1)} + I_{up}^{(2)} \right] \tag{33}$$

where $I_{up}^{(1)}$, $I_{up}^{(2)}$, $J_{pn}^{(1)}$, $J_{pn}^{(2)}$ are given by Eqs. (27), (28), (22), (23). $J_{np}^{(1)}$ and $J_{np}^{(2)}$ are obtained from $J_{pn}^{(1)}$ and $J_{pn}^{(2)}$ by exchanging neutron and proton variables. B will be given by

$$B = \left[i L_u^{(1)} + L_u^{(2)} \right] \cos(\vec{\ell}_f, \vec{\Delta})(a_p - a_n) \tag{34}$$

The quantities b_o, b_1, b_2 are related to the P-wave phase shifts for the neutron-proton interaction in the triplet state by

$b_j = 2 \times 3(2\pi)^{-2}(M\ell_f)^{-1} \sin \delta_1^j \, e^{i\delta_1^j}$. So a_u is related to the triplet S-wave phase shift by $a_u = 2 \times (2\pi)^{-2} (M\ell_f)^{-1} \sin \delta_o^1 \, e^{i\delta_o^1}$. In the expression Eq. (20) for the cross section we have

$$E_f - E_i = E_{q_f} + \ell_f^2/M + (\vec{q}_o - \vec{q}_f)^2/(4M) + 2M - M_D - E_{q_o}. \qquad (35)$$

The integral over ℓ_f can be done at once, since $E_f - E_i$ does not depend on the direction of $\vec{\ell}_f$ but only on its modulus. The integral over q_f will be limited to the interval from $q_f = 0$ to a $q_{f\,max}$ which is the root of

$$E_{q_{f\,max}} + q_{f\,max}^2/(4M) - q_o \, q_{f\,max} \cos\theta/(4M) = E_{q_o} + M_D - 2M - q_o^2/(4M)$$

where θ is the angle between $\vec{q}_f$ and $\vec{q}_o$.

We can express $\psi_D(n_f)$ and $\psi_D(p_f)$ as functions of $\vec{\ell}_f$ and $\vec{\Delta}$ by writing

$$\psi_D(p_f) = (2\pi)^{-3/2} \int e^{-i\vec{\ell}_f \cdot \vec{r}} \, e^{i\frac{\vec{\Delta}}{2} \cdot \vec{r}} \, \psi_D(r) \, d_3 \vec{r} =$$

$$= 4\pi \, (2\pi)^{-3/2} \sum_\ell (2\ell + 1) \, P_\ell \left[\cos(\vec{\ell}_f, \vec{\Delta})\right] \Gamma_\ell \, (\ell_f, \Delta) \qquad (36)$$

and

$$\psi_D(n_f) = 4\pi \, (2\pi)^{-3/2} \sum_\ell (2\ell + 1)(-1)^\ell \, P_\ell \left[\cos(\vec{\ell}_f, \vec{\Delta})\right] \Gamma_\ell \, (\ell_f, \Delta) \qquad (37)$$

where we have called

$$\Gamma_\ell \, (\ell_f, \Delta) = \int_0^\infty j_\ell \, (\ell_f r) \, j_\ell\left(\frac{\Delta}{2} r\right) \psi_D(r) \, r^2 \, dr =$$

$$= N/(\ell_f, \Delta) \left[Q_\ell \left(\frac{\alpha^2 + \ell_f^2 + \frac{1}{4}\Delta^2}{\ell_f \Delta} \right) - Q_\ell \left(\frac{\beta^2 + \ell_f^2 + \frac{1}{4}\Delta^2}{\ell_f \Delta} \right) \right] \qquad (38)$$

where the Q_ℓ's are the Legendre polynomials of the second kind.

For simplicity let us neglect double scattering processes. We have seen that in the K^+-deuteron problem their effects are small due to the smallness of the K^+ nucleon scattering parameters. Then both quantities A and B that appear in Eq. (32) can be expressed in terms of Δ, ℓ_f and the angle between $\vec{\Delta}$ and $\vec{\ell}_f$. Thanks to the expansions in Eqs. (36) and (37), the integration over all directions of $\vec{\ell}_f$ can be done without need of the approximation, which was made in the previous papers dealing with the analysis of the meson-deuteron scattering, that the energy E_f in the final state, given by Eq. (35), does not depend on ℓ_f. We obtain

$$d\sigma = \frac{(2\pi)^4}{v} (M/2)\, \ell_f\, d_3 \vec{q}_f \Bigg\{ 8 \sum_{\ell=0}^{\infty} (2\ell+1)|a_p + (-1)^\ell a_n|^2 \Gamma_\ell^2(\ell_f,\Delta) -$$

$$- (4\pi)^2 (2\pi)^{-3/2} \Gamma_o(\ell_f,\Delta)|a_p + a_n|^2 a_u^* \left[-iI_{up}^{(1)} + I_{up}^{(2)}\right] -$$

$$- (4\pi)^2 (2\pi)^{-3/2} \Gamma_o(\ell_f,\Delta)|a_p+a_n|^2 a_u\left[iI_{up}^{(1)} + I_{up}^{(2)}\right] +$$

$$+ 4\pi|a_p + a_n|^2 a_u^2 \left[I_{up}^{(1)^2} + I_{up}^{(2)^2}\right] - \left[\frac{1}{9} (b_o + 3b_1 + 5b_2)\left(iL_u^{(1)} + L_u^{(2)}\right) +\right.$$

$$\left. + \frac{1}{9} (b_o^* + 3b_1^* + 5b_2^*)\left(-iL_u^{(1)} + L_u^{(2)}\right)\right](4\pi)^2(2\pi)^{-3/2}\Gamma_1(\ell_f,\Delta)|a_n - a_p|^2 +$$

$$+ \frac{4\pi}{9} \left[\frac{1}{2} |b_1 + b_2|^2 + \frac{1}{9} |b_o + 2b_2|^2\right]|a_n - a_p|^2 \left[L_u^{(1)^2} + L_u^{(2)^2}\right] \qquad (39)$$

The integrals Γ_ℓ in Eq. (38) decrease rapidly to zero as increases above a certain value. This can be seen in the following way. The function $r^2 \psi_D(r) = Nr(e^{-\alpha r} - e^{-\beta r})$ has the value zero at $r = 0$, increases to a maximum at $(\alpha r) \approx 1$, and then decreases rapidly: at $(\alpha r) \approx 5$ its value is 1/10 of the value at the maximum. The functions $j_\ell(ar)$ with $\ell \geqslant 1$ start from zero at $r = 0$, and remain very small up to $ar \approx \ell$, where a bump starts. So, if ℓ is so large that the bump in the function $j_\ell(ar)$ starts, say, after $ar = 5$, where $r^2 \psi_D(r)$ is very small, Γ_ℓ will be very small. So we expect that only few terms in the sum $\Sigma(2\ell+1)\Gamma_\ell^2$ that appears in the expression for the cross-section will be important. For the largest values of ℓ_f and $\Delta/2$ the integrals Γ_ℓ will have the largest values, because then the bumps in $j_\ell(\ell_f r)$ and $j_\ell(\frac{\Delta}{2} r)$ will occur for smaller values of r, and there will be stronger inersections with $r^2 \psi_D(r)$. We can adopt the following criterion to decide where to stop the sum. For a given incident momentum q_o we choose the highest values of ℓ_f and $\Delta/2$ that are compatible with energy conservation. Substituting these values in the explicit expressions for the Legendre polynomials Q_ℓ we can find the value of ℓ for which $(2\ell+1)\Gamma_\ell^2$ can be neglected. For K mesons incident with momentum $q_o = 200$ MeV/c for example, we find that cutting the series at $\ell = 4$ causes an error which is smaller than 3% at the extreme values $\ell_f = 150$ MeV/c and $\Delta/2 = 120$ MeV/c. For other values of ℓ_f and Δ the error is much smaller. For high incident energies we have to take more terms in the sum if we want to keep the error negligible.

If the scattered mesons momenta are not experimentally measured, it is interesting to integrate Eq. (39) over the spectrum of values of q_f for each scattering angle θ. The dependence of the neutron-proton scattering parameters a_u, b_o, b_1, b_2 on the relative momentum ℓ_f can easily be taken into account in the integration.

8. CHARGE EXCHANGE SCATTERING AND THE COULOMB INTERACTION IN THE FINAL STATE

If the incident meson exchanges charge with one of the nucleons of the deuteron, we may have two charged nucleons in the final state. An example is the process $K^+ + d \rightarrow K^o + p + p$. As the two nucleons will not have a very high relative energy, their Coulomb interaction can have a strong effect in the process.

The terms in the expansion of the collision operator for charge-exchange meson-deuteron scattering which correspond to a single scattering (with charge exchange) of the meson by the neutron and to this scattering followed by a proton-proton interaction give (forgetting for the moment the antisymmetrization due to the presence of two identical particles in the final state)

$$\langle f | t_n^{ex} | i \rangle + \langle f | t_u (E-K+i\varepsilon)^{-1} t_n^{ex} | i \rangle =$$

$$= - \delta(\vec{P}_f - \vec{P}_i)\, a_{ex} \int \langle \psi^{(-)} | \vec{r} \rangle e^{\frac{i}{2} \vec{\Delta}.\vec{r}} \psi_D(r)\, d_3\vec{r} \qquad (40)$$

where the amplitude for meson-nucleon charge-exchange scattering has been taken as a constant a_{ex} over the range of integration. $\langle \psi^{(-)} | \vec{r} \rangle$ is the configuration space representation of the scattering state (with proper asymptotic behaviour) of the proton-proton system with nuclear and Coulomb interactions. Outside the range of the nuclear forces we have

$$\langle \psi^{(-)} | \vec{r} \rangle = (2\pi)^{-3/2} e^{-\frac{1}{2} n\pi} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1+in)}{(2\ell)!} (2i\ell_f r)^{\ell} e^{i\ell_f r} (-1)^{\ell} P_{\ell}\left[\cos(\vec{\ell}_f, \vec{r})\right] \times$$

$$\times \left\{ F(\ell+1+in,\ 2\ell+2,\ -2i\ell_f r) + (e^{i\delta_\ell} - 1)\, W_1(\ell+1+in,\ 2\ell+2,\ -2i\ell_f r) \right\} \quad (41)$$

where $n = \mu e^2/\ell_f$, with μ being the reduced mass of the proton-proton system, and δ_ℓ are the nuclear-phase shifts (spin effects are here ignored, for simplicity).

Our problem is to evaluate the integral in Eq. (40) using Eq. (41) for $\langle \psi^{(-)} | \vec{r} \rangle$. If we assume that there is no spin dependence in the charge-exchange scattering amplitude a_{ex}, the final state of the two nucleons will be a triplet state. Antisymmetrization will then require that only waves of odd values of ℓ will be present. This means that small values of the relative momenta of the two nucleons will not be often present, n being then small compared to unity. On the other hand, odd-wave phase shifts for the nucleon--nucleon system are small. All this indicates that the part of Eq. (41) containing the nuclear phase-shifts is negligible, or can be well represented by the pure-nuclear interaction already treated in section 6. This approximation corresponds to writing the odd-wave scattering amplitude for p-p scattering as a sum of the scattering amplitude for a pure Coulomb interaction with the amplitude for a purely nuclear interaction, namely $t_u = t_u^C + t_u^N$. Let us then consider the pure Coulomb part. Its contribution to Eq. (40) is (leaving out the factor $-\delta(\vec{P}_f - \vec{P}_i)$)

$$C = a_{ex}(2\pi)^{-3/2} e^{-\frac{1}{2}n\pi} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1+in)}{(2\ell)!} P_\ell\left[\cos(\vec{\ell}_f,\vec{\Delta})\right](4\pi)(-i)^\ell \times \tag{42}$$

$$\times \int_{r=0}^{\infty} (2i\ell_f r)\, e^{i\ell_f r} j_\ell\left(\frac{\Delta}{2} r\right) \psi_D(r)\, F(\ell+1+in,\ 2\ell+2,\ -2i\ell_f r) r^2 dr = \sum_\ell C_\ell$$

Now we substitute F by one of its integral representations (13), and integrate over r first. We obtain

$$C = -a_{ex}(2\pi)^{-3/2} e^{-\frac{1}{2}n\pi} 4\pi\ N \sum_\ell \frac{2^\ell}{(2\ell)!} (\ell_f \Delta)^\ell P_\ell\left[\cos(\vec{\ell}_f,\vec{\Delta})\right] \frac{\Gamma(2\ell+2)\Gamma(\ell+1)}{\Gamma(\ell+1-in)} \times$$

$$\times \int_0^1 du\ u^{(\ell+in)}(1-u)^{(\ell-in)} \left\{ \left(\frac{\Delta^2}{4} + \left[\alpha + i\ell_f(2u-1)\right]^2\right)^{-\ell-1} - \right.$$

$$\left. - \left(\frac{\Delta^2}{4} + \left[\beta + i\ell_f(2u-1)\right]^2\right)^{-\ell-1} \right\} \tag{43}$$

Making $t = u/(1-u)$, the integral in the expression above becomes:

$$\int_{t=0}^{\infty} dt \left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right]^{-\ell-1} \frac{t^{(\ell+in)}}{(t-t_1)^{\ell+1}(t-t_2)^{\ell+1}} \Bigg\} - \int_{t=0}^{\infty} dt \left\{\alpha \to \beta\right\}$$

with

$$t_1(\alpha) = \left[-\alpha^2 - \left(\frac{\Delta}{2} + \ell_f\right)^2\right] \Bigg/ \left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right]$$

$$t_2(\alpha) = \left[-\alpha^2 - \left(\frac{\Delta}{2} - \ell_f\right)^2\right] \Big/ \left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right] \tag{44}$$

and analogous expressions for $t_1(\beta)$, $t_2(\beta)$. The poles at $t = t_1$, $t = t_2$ are not in the path of integration. Integrating over the contour indicated in Fig. (3) we obtain

$$C = a_{ex}(2\pi)^{-3/2}\, e^{-\frac{1}{2}n\pi}\, 4\pi N(2\pi i)\left(1 - e^{-2\pi n}\right)^{-1} \cdot$$

$$\cdot \sum_\ell (2\ell_f\Delta)^\ell \frac{(2\ell+1)}{\Gamma(\ell+1-in)} P_\ell\left[\cos(\vec{\ell}_f, \vec{\Delta})\right] \times$$

$$\times \left[\left\{\left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right]^{-\ell-1} \left[\left(\frac{d^\ell}{dt^\ell} \frac{t^{\ell+in}}{(t-t_1)^{\ell+1}}\right)_{t=t_2(\alpha)} + \right.\right.\right.$$

$$\left.\left.\left. + \left(\frac{d^\ell}{dt^\ell} \frac{t^{\ell+in}}{(t-t_2)^{\ell+1}}\right)_{t=t_1(\alpha)}\right]\right\} - \left\{\alpha \rightarrow \beta\right\}\right] \tag{45}$$

The evaluation of the terms in this series ins straightforward.

Subtracting from Eq. (40) the single scattering term we obtain (without having yet antisymmetrized the final state)

$$\langle f | t_u (E-K+i\varepsilon)^{-1}\, t_n^{ex} | i \rangle = -\,\delta(\vec{P}_f - \vec{P}_i) \sum_\ell \left\{ C_\ell - \right.$$

$$\left. - a_{ex}(4\pi)(2\pi)^{-3/2}(2\ell+1)\, P_\ell\, |\cos(\vec{\ell}_f, \vec{\Delta})|\, \Gamma_\ell\,(\ell_f, \Delta) \right\} +$$

$$+ \langle f | t_u^N (E-K+i\varepsilon)^{-1}\, t_n^{ex} | i \rangle \tag{46}$$

Now, the wave scattered by the Coulomb field, which gives rise to all the part in the above expression which is inside the Σ symbol, will consist of only a few angular momentum waves, due to the fact that n is not very small. Thus, the series above will converge rapidly, only a few terms being necessary for our purposes. This has been confirmed by actual numerical computation.

For definiteness, let us consider the process $K^+d \rightarrow K^o pp$. Let us call a_{op} the amplitude for the $K^o p \rightarrow K^o p$ process. If a_{ex} does not depend on spin variables, antisymmetrization of the final state with respect to the coordinates of the two protons will elminate all even parity states. Squaring the property antisymmetrized amplitude, and summing over spin variables, we obtain for $\Sigma |(R_{inel})_{fi}|^2$ an expression of the form Eq. (32), with

$$A = \sqrt{2} \sum_{\ell \cdot odd} C_\ell - \frac{1}{\sqrt{2}} a_{ex} a_{op} \left[iJ^{(1)}_{p_2p_1} - iJ^{(1)}_{p_1p_2} + J^{(2)}_{p_2p_1} - J^{(2)}_{p_1p_2} \right] - \tag{47}$$

$$- \frac{1}{\sqrt{2}} a_{ex} a_p \left[iJ^{(1)}_{p_1p_2} - iJ^{(1)}_{p_2p_1} + J^{(2)}_{p_1p_2} - J^{(2)}_{p_2p_1} \right]$$

and

$$B = - \sqrt{2} a_{ex} \cos(\vec{\ell}_f, \vec{\Delta}) \left[iL^{(1)}_u + L^{(2)}_u \right] \tag{48}$$

where p_1, p_2 indicate the two protons in the final state and $\vec{p}_{1f}$, $\vec{p}_{2f}$ are their momenta. $\vec{\ell}_f$ is the vector $\frac{1}{2} (\vec{p}_{1f} - \vec{p}_{2f})$.

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* * *

APPENDIX - SPIN SUMS

The most general meson-proton interaction can be described by an operator $r_p = a_p + \vec{\sigma}_p \cdot \vec{b}_p$ where $\vec{\sigma}_p$ is the proton spin matrix and a_p and $\vec{b}_p$ do not depend on spin variables. For the meson-neutron interaction we have $r_n = a_n + \vec{\sigma}_n \cdot \vec{b}_n$. The general fermion-fermion interaction has a much more complicated dependence on spin variables. In our problems we shall be concerned with relative energies of the nucleon-nucleon system which are not very high, and for definiteness we here assume they interact in S and P waves only.

If the two nucleons interact in S-wave only, the most general form for r_u is $r_u = c_s P_s + c_t P_t$, where $P_s = (1 - \vec{\sigma}_p \cdot \vec{\sigma}_n) /4$ and $P_t = (3 + \vec{\sigma}_p \cdot \vec{\sigma}_n) /4$ are the projection operators for the singlet and triplet states of the nucleon-nucleon system. In the terms (up to second order processes) of the expansion of T_{inel} there appear products $r_u\, r_p$, $r_u\, r_n$, $r_p\, r_n$ besides the single scattering terms r_p and r_n. So, as far as spin variables are concerned, the most general form of matrix element which we have to consider (if the nucleon-nucleon interaction in final state is purely S-wave), is the matrix element of

$$R_1 = A+\vec{B}.\vec{\sigma}_p + \vec{C}.\vec{\sigma}_n + (\vec{D}.\vec{\sigma}_p)(\vec{E}.\vec{\sigma}_n) + P_s(F+\vec{H}.\vec{\sigma}_p+\vec{I}.\vec{\sigma}_n) + P_t(G+\vec{J}.\vec{\sigma}_p+\vec{K}.\vec{\sigma}_n) \tag{A.1}$$

calculated between the initial and final state of our system of one meson and two nucleons. The first three terms refer to single scattering of the meson by a nucleon, the fourth one to the double scattering processes, and the others to meson-nucleon collisions followed by nucleon-nucleon interactions in S-waves.

The initial state is a triplet (spin of deuteron is one). The correlation of neutron and proton spins in initial or final states can be taken into account by means of the appropriate projection operators. We get the following results.

i) Final state is a triplet state.

$$\frac{1}{3}\sum_{f,i}\langle i|R_1^+ P_t|f\rangle\langle f|R_1 P_t|i\rangle = \frac{1}{3}\Big\{3(A^*+G^*)(A+G) + (A^*+G^*)(\vec{D}.\vec{E}) + (A+G)(\vec{D}^*.\vec{E}^*) + 2(\vec{B}^*+\vec{J}^*+\vec{C}^*+\vec{K}^*).(\vec{B}+\vec{J}+\vec{C}+\vec{K}) + 2(\vec{D}^*.\vec{D})(\vec{E}^*.\vec{E}) - (\vec{D}^*.\vec{E}^*)(\vec{D}.\vec{E}) + 2(\vec{D}^*.\vec{E})(\vec{E}^*.\vec{D})\Big\} \tag{A.2}$$

ii) Final state is a singlet state.

$$\frac{1}{3}\sum_{f,i}\langle i|R_1^+ P_s|f\rangle\langle f|R_1 P_t|i\rangle = \frac{1}{3}\Big\{(\vec{H}^*-\vec{I}^*-\vec{C}^*+\vec{B}^*).(\vec{H}-\vec{I}-\vec{C}+\vec{B}) - (\vec{D}^*.\vec{E})(\vec{E}^*.\vec{D}) + (\vec{D}^*.\vec{D})(\vec{E}^*.\vec{E}) + i\Big[(\vec{B}^*+\vec{H}^*-\vec{C}^*-\vec{I}^*).(\vec{D}\wedge\vec{E}) - (\vec{B}+\vec{H}-\vec{C}-\vec{I}).(\vec{D}^*\wedge\vec{E}^*)\Big]\Big\} \tag{A.3}$$

Now let us take the two nucleons interacting in P-waves. In this case, if the total spin of the two nucleons is $S = 0$, only one value of the total angular momentum is possible, namely $J = L = 1$. There is no spin dependence of the nucleon-nucleon interaction in this particular state, and its contribution can be absorbed in Eq. (A.1) by suitable modifying the quantities F, $\vec{H}$, $\vec{I}$ of the S-wave case.

If $S = 1$, three values of J are possible, namely 0, 1, 2 with three corresponding independent scattering amplitudes b_o, b_1, b_2. We shall then have to calculate the matrix element of [12]

$$P_t \left\{ \frac{1}{12} (b_o - b_2)(\vec{\sigma}_p - \vec{\sigma}_n) \cdot \frac{\vec{k}_i}{k_i} (\vec{\sigma}_p - \vec{\sigma}_n) \cdot \frac{\vec{k}_f}{k_f} + b_2 \cos\theta + \right.$$

$$\left. + \frac{1}{8} (b_1 - b_2)(\vec{\sigma}_p + \vec{\sigma}_n) \cdot \frac{\vec{k}_i}{k_i} (\vec{\sigma}_p + \vec{\sigma}_n) \cdot \frac{\vec{k}_f}{k_f} \right\} (Q + \vec{S}\cdot\vec{\sigma}_p + \vec{T}\cdot\vec{\sigma}_n)$$

Here $\vec{k}_i$ and $\vec{k}_f$ are the relative momenta in the centre-of-mass system of the nucleons before and after the collision. The term $b_2 \cos\theta\,(Q + \vec{S}\cdot\vec{\sigma}_p + \vec{T}\cdot\vec{\sigma}_n)$ can be considered as absorbed in the equivalent (with respect to spin variables) terms of Eq. (A.1). We can write the expression above in a more compact form, which maintains all the necessary spin dependence,

$$R_2 = P_t\left[(\vec{L}\cdot\vec{\sigma}_p)(\vec{N}\cdot\vec{\sigma}_p) + (\vec{L}\cdot\vec{\sigma}_n)(\vec{N}\cdot\vec{\sigma}_n) + M(\vec{L}\cdot\vec{\sigma}_p)(\vec{N}\cdot\vec{\sigma}_n) + M(\vec{L}\cdot\vec{\sigma}_n)(\vec{N}\cdot\vec{\sigma}_p)\right]\cdot$$
$$(Q + \vec{S}\cdot\vec{\sigma}_p + \vec{T}\cdot\vec{\sigma}_n) \qquad \text{(A.4)}$$

We have to square the matrix element of $R = R_1$ and sum over the possible polarization of the two nucleons. R_2 will not contribute to final singlet states, and the transition probability to singlet states will be given by Eq. (A.3). For final triplet states we have

$$\frac{1}{3}\sum_{f,i}\langle i|R^+P_t|f\rangle\langle f|R\,P_t|i\rangle = \frac{1}{3}\,Tr_pTr_n(R_1^+P_tR_1P_t)+$$

$$+\frac{1}{3}\,Tr_pTr_n(R_1^+P_tR_2P_t+R_2^+P_tR_1P_t)+\frac{1}{3}\,Tr_p\;Tr_n\;(R_2^+P_tR_2P_t) \qquad \text{(A.5)}$$

The first term of the right hand side is given by (A.2). For the second term we obtain

$$\frac{1}{3}\,Tr_pTr_n\left[R_1^+P_tR_2P_t+R_2^+P_tR_1P_t\right]=\Big\{\frac{2}{3}(A^*+G^*)\left[(3+M)(\vec{L}.\vec{N})Q+2i(\vec{T}+\vec{S}).(\vec{L}\wedge\vec{N})\right]+$$

$$+\frac{2}{3}\,(\vec{B}^*+\vec{J}^*+\vec{C}^*+\vec{K}^*).\Big[(\vec{L}\wedge\vec{N})\;2i\;Q+2\;(\vec{S}+\vec{T})(\vec{L}.\vec{N})+$$

$$+\vec{L}\,\vec{N}.\;(\vec{S}+\vec{T})(M+1)+\vec{N}\,\vec{L}.\;(\vec{S}+\vec{T})(M-1)\Big]+\frac{2}{3}\,Q\Big[2M(\vec{D}^*.\vec{L})(\vec{E}^*.\vec{N})+$$

$$+2M(\vec{D}^*.\vec{N})(\vec{E}^*.\vec{L})+(-M+1)(\vec{D}^*.\vec{E}^*)(\vec{L}.\vec{N})\Big]+\frac{2}{3}\,i(\vec{S}+\vec{T}).\Big[\vec{D}^*\vec{E}^*.(\vec{L}\wedge\vec{N})+$$

$$+\vec{E}^*\vec{D}^*.(\vec{L}\wedge\vec{N})+M(\vec{E}^*\wedge\vec{N})(\vec{D}^*.\vec{L})+M(\vec{D}^*\wedge\vec{N})(\vec{E}^*.\vec{L})+M(\vec{E}^*\wedge\vec{L})(\vec{D}^*.\vec{N})+$$

$$+M\;(\vec{D}^*\wedge\vec{L})(\vec{E}^*.\vec{N})\Big]\Big\}+\text{complexe conjugate}$$

For the third term in (A.5) we obtain

$$\frac{1}{3}\,Tr_pTr_n\left[R_2^+P_tR_2P_t\right]=\frac{4}{3}\,Q^*Q\Big\{(\vec{N}^*.\vec{L}^*)(\vec{N}.\vec{L})(3+M+M^*-M^*M)+2(\vec{N}^*.\vec{N})(\vec{L}^*.\vec{L})(1+M^*M)+$$

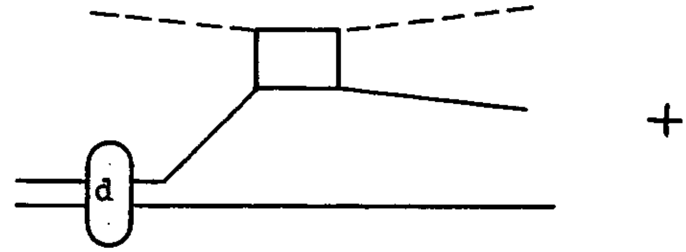
$$+2(\vec{N}^*.\vec{L})(\vec{L}^*.\vec{N})(-1+M^*M)\Big\}+\frac{4}{3}i\left[Q^*(\vec{S}+\vec{T})+Q(\vec{S}^*+\vec{T}^*)\right].\Big\{2(\vec{N}^*\wedge\vec{L}^*)(\vec{L}.\vec{N})+$$

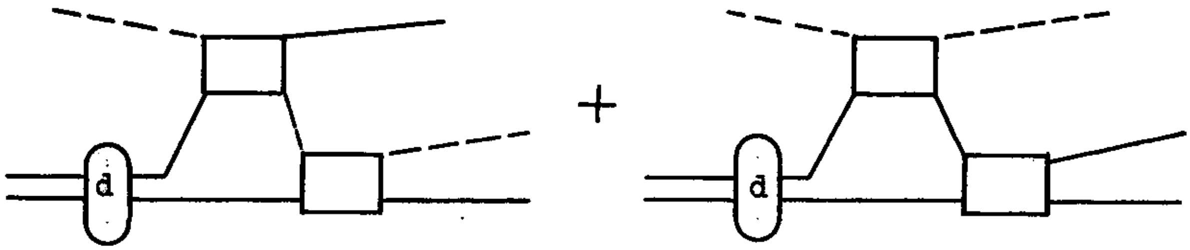
$$+2(\vec{L}\wedge\vec{N})(\vec{L}^*.\vec{N}^*)+(\vec{L}\wedge\vec{N})(\vec{N}^*\wedge\vec{L}^*)+\vec{L}\;M(\vec{N}^*\wedge\vec{L}^*).\vec{N}-\vec{L}\;M^*(\vec{N}\wedge\vec{L}).\;\vec{N}^*+$$

$$
\begin{aligned}
&+ \vec{N} M (\vec{N^*} \wedge \vec{L^*}).\vec{L} - \vec{N^*} M^*(\vec{N} \wedge \vec{L}).\, \vec{L^*} + (\vec{L^*} \wedge \vec{L}) M^* M (\vec{N^*}.\, \vec{N}) + \\
&+ (\vec{N^*} \wedge \vec{N}) M^* M (\vec{L^*}.\vec{L}) + (\vec{L^*} \wedge \vec{N}) M^* M (\vec{N^*}.\vec{L}) - (\vec{L} \wedge \vec{N^*}) M^* M (\vec{N}.\vec{L^*}) \Big\} + \\
&+ \frac{8}{3} (\vec{S^*} + \vec{T^*}).(\vec{S} + \vec{T}) \Big\{ (\vec{L}.\vec{N})(\vec{L^*}.\vec{N^*})(1 - M^* M) + (\vec{N^*}.\vec{L})(\vec{L^*}.\vec{N})(-1 + M^* M) + \\
&+ (\vec{N^*}.\vec{N})(\vec{L^*}.\vec{L})(1 + M^* M) \Big\} + \frac{4}{3}\Big[(\vec{S^*} + \vec{T^*}) \wedge (\vec{S} + \vec{T})\Big].\Big\{(\vec{L}\ \ \vec{N})(\vec{L^*}.\vec{N^*})(1 + M^*) + \\
&+ (\vec{N^*} \wedge \vec{L^*})(\vec{N}.\vec{L})(1 + M) + \frac{4}{3}\Big[\vec{N^*}.(\vec{S} + \vec{T})\Big] (\vec{L^*}.\vec{N})\Big[\vec{L}.(\vec{S^*} + \vec{T^*})\Big](1 + M - M^* - M^* M) + \\
&+ (\vec{L^*}.\vec{L})\Big[\vec{N}.(\vec{S^*}.\vec{T^*})\Big](-1 + M + M^* - M^* M)\Big\} + \frac{4}{3}\Big[\vec{L^*}.(\vec{S} + \vec{T})\Big] \Big\{ (\vec{N^*}.\vec{L})\Big[\vec{N}.(\vec{S^*} + \vec{T^*})\Big](1 + \\
&+ M^* - M - M^* M) + (\vec{N^*}.\, \vec{N}) \Big[\vec{L}.\, (\vec{S^*} + \vec{T^*})\Big](-1 - M - M^* - M^* M) \Big\} + \\
&+ \frac{4}{3} (M + M^* M)(\vec{N^*}.\vec{L^*})\Big\{\Big[\vec{N}.(\vec{S^*} + \vec{T^*})\Big]\Big[\vec{L}.(\vec{S} + \vec{T})\Big] + \Big[\vec{L}.(\vec{S^*} + \vec{T^*})\Big]\vec{N}.(\vec{S} + \vec{T})\Big] \Big\} + \\
&+ \frac{4}{3} (M^* + M^* M)(\vec{N}.\vec{L})\Big\{\Big[\vec{N^*}.(\vec{S} + \vec{T})\Big]\Big[\vec{L^*}.(\vec{S^*} + \vec{T^*})\Big] + \Big[\vec{L^*}.(\vec{S} + \vec{T})\Big]\Big[\vec{N^*}.(\vec{S^*} + \vec{T^*})\Big]\Big\}
\end{aligned}
\tag{A.7}
$$

* * *

$\langle \phi_f | T_{inel} | \psi_i \rangle =$

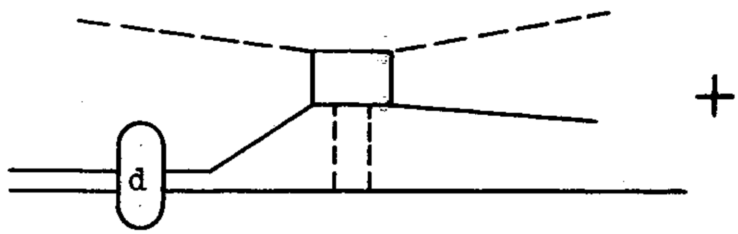




+ remainder

Fig. 1 - The main terms of the expansion of the matrix element of the scattering operator for meson-deuteron inelastic scattering in terms of two-particle processes.

remainder =



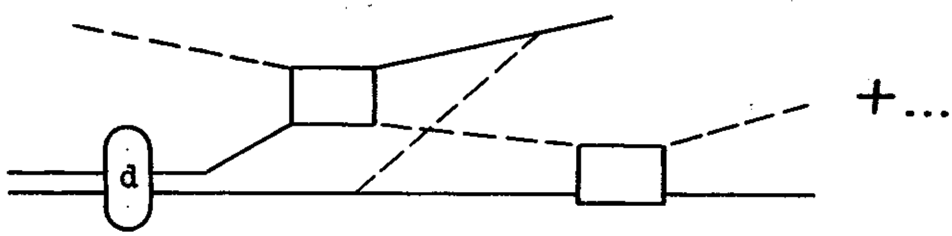


Fig. 2 - Examples of contributions to the residual terms in the expansion represented in Fig. 1. These are essentially three-body effects which cannot be represented by two-particle scattering operators only.

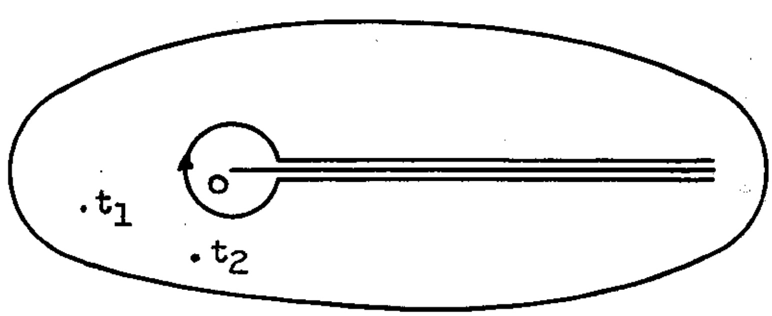


Fig. 3 - The path of integration in the complexe plane to evaluate Eq. (44).

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