

Octonionic M-theory and D=11 Generalized Conformal and Superconformal Algebras

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Abstract

Following [1] we further apply the octonionic structure to supersymmetric D=11 M -theory. We consider the octonionic $2^{n+1} \times 2^{n+1}$ Dirac matrices describing the sequence of Clifford algebras with signatures $(9 + n, n)$ ($n = 0, 1, 2, \dots$) and derive the identities following from the octonionic multiplication table. The case $n = 1$ (4×4 octonion-valued matrices) is used for the description of the $D = 11$ octonionic M -superalgebra with 52 real bosonic charges; the $n = 2$ case (8×8 octonion-valued matrices) for the D=11 conformal M -algebra with 232 real bosonic charges. The octonionic structure is described explicitly for $n = 1$ by the relations between the 512 Abelian $O(10, 1)$ tensorial charges $Z_\mu, Z_{\mu\nu}, Z_{\mu\dots\mu_5}$ of the M -superalgebra. For $n = 2$ we obtain 2048 real non-Abelian bosonic tensorial charges $Z_{\mu\nu}, Z_{\mu_1\mu_2\mu_3}, Z_{\mu_1\dots\mu_6}$ which, suitably constrained describe the generalized $D = 11$ octonionic conformal algebra. Further, we consider the supersymmetric extension of this octonionic conformal algebra which can be described as $D = 11$ octonionic superconformal algebra with a total number of 64 real fermionic and 239 real bosonic generators.

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1 Introduction

One of the consequences of the presence of membrane [2] and five-brane [3] solutions of the $D = 11$ supergravity is the appearance in the $D = 11$ superalgebra of 55 two-tensor and 462 five-tensor Abelian tensor charges, leading to the so-called M -superalgebra[§] [4–6] ($A, B = 1, \dots, 32$; $\mu, \nu = 0, 1, \dots, 10$)

$$\{Q_A, Q_B\} = Z_{AB} = (C\Gamma^\mu)_{AB} P_\mu + (C\Gamma^{\mu_1\mu_2})_{AB} Z_{[\mu_1\mu_2]} + (C\Gamma^{\mu_1\dots\mu_5})_{AB} Z_{[\mu_1\dots\mu_5]}. \quad (1.1)$$

The 528 real bosonic Abelian charges Z_{AB} ($Z_{AB} = Z_{BA}$) saturate the rhs of the anticommutator of 32 real supercharges Q_A , which are $D = (10, 1)$ as well as $D = (10, 2)$ Majorana spinors. In $D = 12$ the M -superalgebra (1.1) takes the form [7,8] ($A, B = 1, \dots, 32$; $\tilde{\mu}, \tilde{\nu} \dots = 0, 1, \dots, 11$)

$$\{Q_A, Q_B\} = Z_{AB} = (C\Gamma^{\tilde{\mu}_1\tilde{\mu}_2})_{AB} Z_{[\tilde{\mu}_1\tilde{\mu}_2]} + (C\Gamma^{\tilde{\mu}_1\dots\tilde{\mu}_6})_{AB} Z_{[\tilde{\mu}_1\dots\tilde{\mu}_6]}, \quad (1.2)$$

where $Z_{[\mu_1]} = P_\mu$ and the five-tensor charges in (1.1) are described by $D=12$ six-charges $Z_{[\tilde{\mu}_1\dots\tilde{\mu}_6]}$ satisfying the self-duality condition.

Let us recall that $O(10, 2)$ describes the standard $D = 10$ conformal algebra, with fundamental spinor ($D = 10$ twistor) built from a pair of 16-components $D = (9, 1)$ Majorana–Weyl spinors. It appears that these 16 components can be endowed with octonionic structure, and we can describe $D = 10$ spinors as a pair of octonions $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$.

Subsequently one can introduce the following relation of the octonionic matrices with $D=10$ Lorentz and conformal groups [9–12]

$$SL(2|\mathbf{O}) = \frac{SO(9, 1)}{G_2}, \quad U_\alpha(4|\mathbf{O}) = \frac{SO(10, 2)}{G_2}, \quad (1.3)$$

where the 14-generator G_2 algebra is the automorphism algebra of the multiplication table for the octonionic units t_k ($k = 1 \dots 7$)

$$t_k t_l = -\delta_{kl} + \frac{1}{2} f_{kl}^m t_m \quad (1.4)$$

and $U_\alpha(n|\mathbf{F})$ describes the antiunitary \mathbf{F} -valued group, with unitary norm and anti-symmetric invariant metric, where $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$. For $n = 4$ we obtain the sequence of $D = 3$ ($\mathbf{F} = \mathbf{R}$), $D = 4$ ($\mathbf{F} = \mathbf{C}$) and $D = 6$ ($\mathbf{F} = \mathbf{H}$) spinorial coverings of conformal groups; for $\mathbf{F} = \mathbf{O}$ we obtain the second formula (1.3). Subsequently we shall denote by $U_\alpha(n|\mathbf{F})$ also the corresponding algebras, with \mathbf{F} -Hermitian generators.

[§]The superalgebra (1.1) is usually called M -algebra. We shall use the terminology “ M -superalgebra” in order to point out that it describes a supersymmetric theory.

Using the first relation in (1.3) it has been proposed [13] that the standard $D = 10$ Poincaré superalgebra can be described by a pair of octonionic supercharges $(\mathcal{Q}_1, \mathcal{Q}_2)$, with the following basic relations $(\mathcal{Q} = Q^{(0)} + t_r Q^{(r)} \rightarrow \mathcal{Q}^\dagger \equiv \overline{\mathcal{Q}} = Q^{(0)} - t_r Q^{(r)})$

$$\{\mathcal{Q}_a, \overline{\mathcal{Q}}_b\} = \mathcal{Z}_{ab} = \begin{pmatrix} P_0 + P_9 & P_8 + t_r P_r \\ P_8 - t_r P_r & P_0 - P_9 \end{pmatrix}, \quad (1.5)$$

where $Z_{ab} = Z_{ba}^+$ and $P_\mu = (P_0, P_1, \dots, P_9)$ describe $D = 10$ momentum generators.

Recently we proposed also to impose on the M -superalgebra generators (see formulae (1.1) and (1.2)) the octonionic structure [1]. The relations (1.1–1.2) we replaced by octonionic M -superalgebra ($r, s = 1, 2, 3, 4$)

$$\{\mathcal{Q}_r, \overline{\mathcal{Q}}_s\} = \mathcal{Z}_{rs}, \quad \mathcal{Z}_{rs} = \mathcal{Z}_{sr}^+, \quad (1.6)$$

where four real octonion-valued supercharges Q_r replace 32 real supercharges Q_A and in place of the 528 real Abelian charges present in the rhs of (1.1–1.2) we get only 52 independent real Abelian supercharges described by the real components $(Z_{rs}^{(0)}, Z_{rs}^{(k)})$ where

$$\mathcal{Z}_{rs} = Z_{rs}^{(0)} + t_k Z_{rs}^{(k)}, \quad (1.7)$$

$$Z_{rs}^{(0)} = Z_{sr}^{(0)}, \quad Z_{rs}^{(k)} = -Z_{sr}^{(k)}. \quad (1.8)$$

As a consequence, the octonionic M -algebra can be fully described by 11 four-momenta P_μ and only 41 generators $\mathcal{Z}_{\mu\nu}$ describing together the Abelian contractions of all the generators in the coset $\frac{O(10,2)}{G_2}$; another way of providing the bosonic generators of (1.6) is to introduce suitably constrained five-tensor charges $\mathcal{Z}_{[\mu_1 \dots \mu_5]}$.

In this paper we continue the considerations presented in [1]. In Sect. 2 we shall consider more in detail the properties of the octonionic M -superalgebra (1.6) and present it as a member of generalized octonionic supersymmetry algebras in $D = (9 + n, n)$ ($n = 0, 1, 2, \dots$). For $n = 0$ we obtain the generalized supersymmetry algebra for $D = 9$ Euclidean theory. If $n = 1$ we get the octonionic M -superalgebra considered in [1], and the case $n = 2$ provides the extension of M -superalgebra to $D = 13$ with signature (11, 2) and $D = 14$ with signature (11, 3), considered by Bars [14,15][¶]. It appears however that the relation (1.6) if $r, s, = 1, 2, \dots, 2^n$, can also be used for the supersymmetrization of the octonionic algebras $U_\alpha(2^n|\mathbf{O})$, which describe for $n = 1$ the octonionic $D = (9, 1)$ algebra with ten curved translations, for $n = 2$ the octonionic conformal algebra (see (1.3)) and for $n = 3$ the $D = 11$ octonionic generalized conformal algebra $U_\alpha(8|\mathbf{O})$ with 232 real charges. We shall argue that

[¶]Such a framework was proposed as a basis of the so-called S -theory unifying the algebraic description of all five superstring theories in $D = 10$. Similarly as the M -theory which can be described in the (10, 2)-covariant framework, the S -theory can be as well described by the $D = 14$ formalism, with (11, 3) signature [13].

$U_\alpha(8|\mathbf{O})$ can be obtained from the generalized $D = 11$ conformal algebra $Sp(64|\mathbf{R})$ [16,17] by imposing constraints describing the octonionic structure. It is known [17] that the $Sp(64|\mathbf{R})$ generators can be described by $O(11, 2)$ two-tensors, three-tensors and six-tensors. We shall show that the octonionic conformal M -algebra $U_\alpha(8|\mathbf{O})$ will be completely described only in terms of the two-tensor generators from the coset $\frac{O(11,2)}{G_2}$ supplemented by a suitably restricted set of three-tensor generators. It appears that $U_\alpha(8|\mathbf{O})$ supplemented with $S^7 \simeq U(1|\mathbf{O})$ generators describe the bosonic sector of the supersymmetric extension $U_\alpha(8; 1|\mathbf{O})$ of the octonionic conformal M -algebra.

The conformal algebras in $D = 10$, since fundamental $D=10$ octonionic conformal spinors have four components, belong to the framework of conformal Jordan algebras [9–11,18,12]). Beyond $D = 10$ and in particular for the eleven dimensional conformal M -algebra $U_\alpha(8; \mathbf{O})$, we are outside of the framework of conformal algebras associated with Jordan algebras. The construction of the conformal algebra can be however linked with the group of invariance of the metric for “doubled Lorentz spinors”. This procedure we propose to apply to octonionic conformal spinors in dimensions $D = (9 + n, n)$, providing $D = 11, 13$ etc. We consider in Sect. 3 the generalized octonionic conformal algebras and superalgebras in the form of octonionic-valued (super-)matrix realizations which admit closed algebraic relations with the (super-)matricial (anti-)commutator structure. For $D=11$, considering 8×8 octonionic matrices $U_\alpha(8|\mathbf{O}) \simeq Sp(8|\mathbf{O})$, one can show that it contains the generators parametrizing the coset $\frac{O(11,2)}{G_2}$ (64 real generators), but with additional 168 real generators not closing to any subset of Lie algebra generators. Such algebra is also not of Malcev type [20]; it is an interesting task to elucidate its algebraic characterization.

Further, in Sect. 4, we provide the table listing the number of independent n -fold antisymmetric products of octonionic Dirac matrices in odd dimensions $D = 7 + 2k$ ($k = 0, 1, 2, 3$) and we interpret them as the relation between the p -brane degrees of freedom in corresponding supersymmetric D -dimensional theory with the most general set of central charges.

2 Octonionic-Valued Clifford Algebras and Corresponding Space–Time Superalgebras

a) D=(0,7)

Let us observe that (0,7) spinors are real eight–dimensional and $C = C^{T\parallel}$. Introducing seven 8×8 real Γ_i matrices, $i = 1, \dots, 7$,

$$\{\Gamma_i, \Gamma_j\} = -2\delta_{ij}, \quad (2.1)$$

one obtains the relations $(\Gamma_{i_1 \dots i_n} = \frac{1}{n} \sum_{\text{perm}} (-1)^{\text{perm}} \Gamma_{i_{k_1}} \Gamma_{i_{k_2}} \dots \Gamma_{i_{k_n}})$

$$C\Gamma_i = -(C\Gamma_i)^T, \quad C\Gamma_{ij} = -(C\Gamma_{ij})^T, \quad C\Gamma_{ijk} = -(C\Gamma_{ijk})^T. \quad (2.2)$$

If we use 8 real supercharges one obtains the D=7 generalized Euclidean superalgebra ($\alpha = 0, 1, 2, \dots, 7$)

$$\{Q_\alpha, Q_\beta\} = Z \cdot (C)_{\alpha\beta} + Z^{ijk} (\Gamma_{ijk})_{\alpha\beta}, \quad (2.3)$$

which unfortunately does not contain the 7-momentum sector which should be linear in Γ_i .

One obtains the octonionic $C_{\mathbf{O}}(0, 7)$ Clifford algebra with generators $\Gamma_i^{(7)}$ satisfying the relation (2.1) by assuming

$$\Gamma_i^{(7)} = t_i, \quad C = 1, \quad (2.4)$$

where t_i are seven octonionic units with the multiplication table (1.4). The Hermitean $N = 1$ octonionic superalgebra generated by the supercharge $\mathcal{Q} = Q_0 + t_a Q_a$

$$\{\bar{\mathcal{Q}}, \mathcal{Q}\} = \mathcal{Z}, \quad (2.5)$$

describes the octonionic $N = 8$ supersymmetric mechanics, with the real generator \mathcal{Z} playing the role of the Hamiltonian [18].

b) D= (1,8),(1,9) and D=(9,0),(9,1).

One can introduce the following 2×2 matrix realizations of the octonionic–valued Clifford algebras:

b1) The $C_{\mathbf{O}}(1, 8)$ Clifford algebra ($C^T = -C$)

$$\Gamma_i^{(9)} = \left(\begin{array}{c|c} 0 & t_i \\ \hline t_i & 0 \end{array} \right), \quad \Gamma_8^{(9)} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \Gamma_0^{(9)} = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad (2.6)$$

^{\parallel}In all the examples below the C matrix is given by the product of the time-like Clifford's Gamma matrices.

where $C = \Gamma_0$. The generalized octonionic D=9 Poincaré algebra takes the form ($r, s = 1, 2; \mu = 0, 1 \dots 8$)

$$\{\mathcal{Q}_r, \overline{\mathcal{Q}}_s\} = C_{rs}Z + (C\Gamma_\mu)_{rs}P^\mu. \quad (2.7)$$

We see that the generator Z in D=9 is the central charge. Due however to the first relation (1.3) the superalgebra (2.7) can be promoted to D=(1,9) superPoincaré algebra [10], where now the $D = 9$ central charge Z is the component P^9 of the ten-dimensional momenta. One can also assume that the generators on the rhs of (2.7) are nonAbelian and form the algebra $U_\alpha(2|\mathbf{O})$ containing only ten curved (1.9) translations, which is an octonionic counterpart of the $D = 10$ AdS algebra.

b2) *The $C_{\mathbf{O}}(9, 0)$ Clifford algebra ($C^T = C$)*

$$\tilde{\Gamma}_i^{(9)} = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}, \quad \tilde{\Gamma}_i^{(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\Gamma}_i^{(8)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.8)$$

where $C = \mathbb{I}_2$. We obtain the general octonionic $D = 9$ Euclidean superalgebra with $D = 9$ Euclidean momenta \tilde{P}_μ ($\mu = 1 \dots 9$) and central charge \tilde{Z} ($r, s = 1, 2$)

$$\{\mathcal{Q}_r, \overline{\mathcal{Q}}_s\} = \delta_{rs}\tilde{Z} + (\tilde{\Gamma}_\mu)_{rs}\tilde{P}^\mu. \quad (2.9)$$

Again, the superalgebra (2.9) can be promoted to $D = (9, 1)$ Poincaré algebra if we choose $\tilde{Z} = P_0$ (the $D = 10$ energy operator).

c) **D = (10, 1), (10, 2)**

c1) *The $C_{\mathbf{O}}(10, 1)$ Clifford algebra and D=11 M-superalgebra.*

Such an algebra can be represented by the following 4×4 octonionic matrices** ($a = b + 1 = 1, \dots, 9$)

$$\Gamma_a^{(11)} = \begin{pmatrix} 0 & \Gamma_b^{(9)} \\ -\Gamma_b^{(9)} & 0 \end{pmatrix}, \quad \Gamma_0^{(11)} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \Gamma_{10}^{(11)} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \quad (2.11)$$

where

$$C^{(11)} = \begin{pmatrix} 0 & \Gamma_8^{(9)} \\ -\Gamma_8^{(9)} & 0 \end{pmatrix}. \quad (2.12)$$

**One can represent $C_{\mathbf{O}}(10, 1)$ in two ways. The second choice is obtained when the following $D = 11$ octonionic 4×4 Dirac matrices are taken (now $C = \tilde{\Gamma}_0^{(11)}$)

$$\tilde{\Gamma}_a^{(11)} = \begin{pmatrix} 0 & \tilde{\Gamma}_b^{(9)} \\ \tilde{\Gamma}_b^{(9)} & 0 \end{pmatrix}, \quad \tilde{\Gamma}_0^{(11)} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \quad \tilde{\Gamma}_{10}^{(11)} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}. \quad (2.10)$$

Introducing 4 octonionic supercharges we obtain the octonionic M -superalgebra (1.6) considered in our previous paper [1]. The superalgebra (1.6) has 52 independent real bosonic charges, which can be equivalently expressed in two ways ($r, s = 1, \dots, 4$)

$$\mathcal{Z}_{rs} = P^\mu (C\Gamma_\mu^{(11)})_{rs} + Z_{\mathbf{O}}^{\mu\nu} (C\Gamma_{\mu\nu}^{(11)})_{rs} = Z_{\mathbf{O}}^{[\mu_1 \dots \mu_5]} (C\Gamma_{\mu_1 \dots \mu_5}^{(11)})_{rs}, \quad (2.13)$$

where all 11 components of P^k are independent, while the generators $Z_{\mathbf{O}}^{\mu\nu}$ describe the coset $\frac{O(10,1)}{G_2}$ (i.e. there are 14 relations between the 55 generators $Z_{\mathbf{O}}^{\mu\nu}$), and out of the 462 components of $Z_{\mathbf{O}}^{[\mu_1 \dots \mu_5]}$ there are only 52 independent ones.

Similar constraints are satisfied by the coordinates of the generalized octonionic $D = 11$ space-time

$$\mathcal{X}_{rs} = X^\mu (C\Gamma_\mu^{(11)})_{rs} + X_{\mathbf{O}}^{\mu\nu} (C\Gamma_{\mu\nu}^{(11)})_{rs} = X_{\mathbf{O}}^{\mu_1 \dots \mu_5} (C\Gamma_{\mu_1 \dots \mu_5}^{(11)})_{rs}. \quad (2.14)$$

We see that in the $D = 11$ supersymmetric theory with basic superalgebra (2.13) the two-brane and five-brane tensorial central charges are strongly constrained, and the theory can be described completely

- either by one-brane and two-brane sectors ($(P^\mu, Z_{\mathbf{O}}^{\mu\nu})$ in generalized momentum space or, using the dual picture, by $(X^\mu, X_{\mathbf{O}}^{\mu\nu})$ in generalized space-time,
- or by the constrained five-brane sector ($(Z_{\mathbf{O}}^{\mu_1 \dots \mu_5}$ or $X_{\mathbf{O}}^{\mu_1 \dots \mu_5}$).

c2) The $D = 10$ octonionic conformal superalgebra $U_\alpha(4|\mathbf{O})$

The form (1.6) of the octonionic superalgebra is also obtained for the generalized octonionic $D = 10$ conformal superalgebras. For that purpose one can write the basic relation in $O(10, 2)$ - covariant form ($r, s = 1, \dots, 4; \tilde{\mu}_1, \tilde{\mu}_2 = 0, 1, \dots, 11$)

$$\{\tilde{\mathcal{Q}}_r, \overline{\tilde{\mathcal{Q}}}_s\} = \mathcal{M}_{rs} = (C\Gamma_{\tilde{\mu}_1 \tilde{\mu}_2}^{(12)})_{rs} M^{[\tilde{\mu}_1 \tilde{\mu}_2]}, \quad (2.15)$$

where ($\mu_1, \mu_2 = 0, 1, \dots, 10$)

$$\Gamma_{\mu_1 \mu_2}^{(12)} = \Gamma_{\mu_1 \mu_2}^{(11)} \equiv [\Gamma_{\mu_1}^{(11)}, \Gamma_{\mu_2}^{(12)}], \quad \Gamma_{11\mu}^{(12)} = \Gamma_\mu^{(11)}, \quad (2.16)$$

with the 66 generators $M_{[\tilde{\mu}_1, \tilde{\mu}_2]}$ describing 52 independent generators of $\frac{O(10,2)}{G_2}$ (see (1.3b)). In the octonionic framework the remaining six-tensor generators $M_{[\tilde{\mu}_1 \dots \tilde{\mu}_6]}$ (compare with (1.2)) are not independent. The octonionic superalgebra (2.15) can be also treated as describing $D=11$ octonionic AdS algebra [10], with $M^{[\mu_1 \mu_2]}$ describing $\frac{O(10,2)}{G_2}$ and $M^{12\mu}$ the curved $D = 11$ AdS translations. In particular if we introduce the rescaling of the generators $M^{12\mu} = R \cdot \mathcal{P}^\mu$ by performing the limit $R \rightarrow \infty$ one can obtain from the relations (2.15) the $D=11$ octonionic M -algebra, given by (1.6).

We add here that by doubling the realization of $C_{\mathbf{O}}(10, 1)$ given in footnote ⁴⁾ one obtains the realizations of the Clifford algebras $C_{\mathbf{O}}(2, 9)$ and $C_{\mathbf{O}}(2, 10)$.

d) D=(11,2)

In D=13 we shall consider only one choice of signature (11,2)^{††}. The corresponding representation of $C_{\mathbf{O}}(11,2)$ in terms of 8×8 octonionic matrices is explicitly given, for $(a = b + 1 = 1, \dots, 11)$, as follows:

$$\Gamma_a^{(13)} = \begin{pmatrix} 0 & \Gamma_b^{(11)} \\ \Gamma_b^{(11)} & 0 \end{pmatrix}, \quad \Gamma_0^{(13)} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \quad \Gamma_{12}^{(13)} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \quad (2.17)$$

with $C^{(13)} = \begin{pmatrix} C^{(11)} & 0 \\ 0 & -C^{(11)} \end{pmatrix}$, while $\Gamma_b^{(11)}$ and $C^{(11)}$ are given by (2.11) and (2.12) respectively.

The basis of the 232 octonionic hermitian generators is given by the 64 antisymmetric two-tensors $C\Gamma_{[\mu_1\mu_2]}$ and the 168 antisymmetric three tensors $C\Gamma_{[\mu_1\mu_2\mu_3]}$ or, equivalently, by the 232 antisymmetric six-tensors $C\Gamma_{[\mu_1\dots\mu_6]}$.

The D=(11,2) generalized octonionic supersymmetry algebra (called also S -algebra [14,15]) takes the form $(r, s = 1, 2 \dots 8; M, N = 0, 1, 2, \dots 12)$

$$\{Q_r, Q_s\} = \left(C^{(13)} \Gamma_{MN}^{(13)} \right)_{rs} Z_0^{MN} + \left(C^{(13)} \Gamma_{M_1M_2M_3} \right)_{rs} Z_0^{M_1M_2M_3}, \quad (2.18)$$

with 232 real bosonic charges, which is an octonionic counterpart of the real generalized D=(11,2) superalgebra with 64 real Majorana supercharges $(A, B = 1 \dots 64)$ and 2048 bosonic charges.

$$\begin{aligned} \{Q_A, Q_B\} &= \left(C^{(13)} \Gamma_{MN}^{(13)} \right)_{AB} Z^{MN} + \left(C^{(13)} \Gamma_{M_1M_2M_3} \right)_{AB} Z^{M_1M_2M_3} \\ &\quad - \left(C^{(13)} \Gamma_{M_1\dots M_6} \right)_{AB} Z^{M_1\dots M_6}, \end{aligned} \quad (2.19)$$

where now $(\Gamma_M^{(13)})_{AB}$ are the 64×64 real Majorana representations of $C(11,2)$. From the formula (2.20) follows that

i) The octonionic six-charges are not needed in the relation (2.18) because they can be expressed by two-charges and three-charges.

ii) The relations (2.19) with non Abelian charges $Z^{MN}, Z^{M_1M_2M_3}, Z^{M_1\dots M_6}$ describe the superalgebra $OSp(1|64; R)$, which was proposed as the generalized D=11 conformal superalgebra [17,19]. In such a case the generators Z^{MN} describe the $O(11,2)$ algebra. The octonionic counterpart of $Sp(64; R)$ is provided by $U_\alpha(8|\mathbf{O})$, which is supersymmetrized by relation (2.18). The generators Z_0^{MN} describe in such a case the coset $\frac{O(11,2)}{G_2}$.

^{††}The other possible choice of octonionic 8×8 Clifford algebra representations can be considered for signatures (3, 10) and (3, 11).

3 D=11 Octonionic Generalized (Super)conformal Transformations as Automorphisms

It is known that the conformal algebra can be introduced as the algebra of transformations leaving invariant the inner product of fundamental conformal spinors called also twistors. We shall apply this method to derivation of octonionic conformal algebra from octonionic spinors with inner product. In $D = (10, 1)$ such inner product is given by $\psi^\dagger C \eta$, where ψ, η are eightdimensional octonionic conformal $O(11, 2)$ spinors described by pairs of octonionic $O(10, 1)$ Lorentz spinors and the matrix C given by the product of the two space-like Clifford's Gamma matrices $\Gamma_0^{(13)}, \Gamma_{12}^{(13)}$ (see (2.17)) is real-valued and totally antisymmetric. Therefore, the conformal transformations are realized by the octonion-valued 8-dimensional matrices \mathcal{M} leaving C invariant, i.e. satisfying

$$\mathcal{M}^\dagger C + C \mathcal{M} = 0. \quad (3.1)$$

This allows identifying the octonionic conformal transformations with the octonionic unitary-symplectic transformations $U_\alpha(8|\mathbf{O})$.

The most general octonionic-valued matrix leaving invariant Ω can be expressed as follows

$$\mathbf{M} = \begin{pmatrix} D & B \\ C & -D^\dagger \end{pmatrix}, \quad (3.2)$$

where the 4×4 octonionic matrices B, C are hermitian

$$B = B^\dagger, \quad C = C^\dagger. \quad (3.3)$$

It is easily seen that the total number of independent components in (3.2) is precisely 232, as we expected from the previous considerations.

It should be noticed that the set of octonionic matrices \mathbf{M} of (3.2) type forms a closed algebraic structure under the usual matrix commutation. Indeed one gets $[\mathbf{M}, \mathbf{M}] \subset \mathbf{M}$, endowing the structure of $U_\alpha(8|\mathbf{O})$ to \mathbf{M} . As recalled in the introduction, $U_\alpha(2n; \mathbf{O})$ for $n > 3$ is no longer a conformal algebra associated with a Jordan-algebra (see e.g [18]), nevertheless it admits the Lie-algebraic commutation relations which, however, do not satisfy the Jacobi identities.

In the procedure of supersymmetric extension to the superconformal algebra we have to accommodate the components of 8 octonionic spinors of $(11, 2)$ into a supermatrix enlarging $U_\alpha(8|\mathbf{O})$. This can be achieved as follows. The two 4-column octonionic spinors α and β can be accommodated into a supermatrix of the form

$$\mathcal{M}^{(1)} = \left(\begin{array}{c|cc} 0 & -\beta^\dagger & \alpha^\dagger \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{array} \right). \quad (3.4)$$

Under anticommutation, the lower bosonic diagonal block reduces to $U_\alpha(8|\mathbf{O})$. On the other hand, extra real seven generators, associated to the 1-dimensional antihermitian matrix A

$$A^\dagger = -A, \quad (3.5)$$

i.e. described by seven imaginary octonions, are obtained in the upper bosonic diagonal block. Therefore, the generic bosonic element is of the form

$$\mathcal{M}^{(0)} = \left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & D & B \\ 0 & C & -D^\dagger \end{array} \right), \quad (3.6)$$

with A , B and C satisfying (3.5) and (3.3).

It can be shown that in analogy to the relation (3.1) one can derive (3.6) from the invariance of inner product for octonion-valued supertwistor (ψ, ξ) , where $\xi = \xi^{(0)} + \xi^{(i)}t_i$ ($\{\xi^{(a)}, \xi^{(b)}\} = 0; a, b = 0, 1, \dots, 7$ and $\xi^{(a)}$ real):

$$\mathcal{M}^\# \mathbb{C} = -\mathbb{C} \mathcal{M}^\#, \quad \mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(1)}, \quad (3.7)$$

where $\mathbb{C} = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$ is the $OSp(1; 8)$ metric and $\#$ describe graded-Hermitian conjugation. The closed superalgebraic structure, with (3.4) as generic fermionic element and (3.6) as generic bosonic element, we denote as $OSp(1, 8|\mathbf{O})$. It can be considered as the superconformal algebra of the octonionic M -theory or generalized octonionic AdS algebra in $D = (11, 1)$ and admits a total number of 239 real bosonic generators.

4 Octonionic Structure and p -Superbranes

We have shown that the 52 independent components of the Hermitian-octonionic Z_{ab} matrix can be represented either as the 11 + 41 bosonic generators entering

$$\mathcal{Z}_{ab} = P^\mu (C\Gamma_\mu)_{ab} + Z_{\mathbf{O}}^{\mu\nu} (C\Gamma_{\mu\nu})_{ab}, \quad (4.1)$$

or as the 52 bosonic generators entering

$$\mathcal{Z}_{ab} = Z_{\mathbf{O}}^{[\mu_1 \dots \mu_5]} (C\Gamma_{\mu_1 \dots \mu_5})_{ab}. \quad (4.2)$$

The reason for that lies in the fact that, unlike in the real case, the sectors individuated by (4.1) and (4.2) are not independent. This is a consequence of the multiplication table of the octonions. Indeed, when we multiply antisymmetric products of k octonionic-valued Gamma matrices, a certain number of them are redundant. For $k = 2$, due to the G_2 automorphisms, 14 such products have to be erased. In

the general case [22] a table can be produced, which we write down (see Table 1) for $7 \leq D \leq 13$ odd-dimensional spacetime corresponding to octonionic realizations of Clifford algebras considered in Sect. 2. The Table 1 was constructed from the $D = 7$ results (which can be easily computed), by taking into account that out of the D Gamma matrices, 7 of them are octonion-valued, while the remaining $D - 7$ are purely real. The columns in Table 1 are labeled by k , the number of antisymmetrized Gamma matrices.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$D = 7$	<u>1</u>	7	7	<u>1</u>	<u>1</u>	7	7	<u>1</u>						
$D = 9$	<u>1</u>	<u>9</u>	22	22	<u>10</u>	<u>10</u>	22	22	<u>9</u>	<u>1</u>				
$D = 11$	1	<u>11</u>	<u>41</u>	75	<u>76</u>	<u>52</u>	<u>52</u>	76	<u>75</u>	<u>41</u>	<u>11</u>	1		
$D = 13$	1	13	<u>64</u>	<u>168</u>	267	279	<u>232</u>	<u>232</u>	279	267	<u>168</u>	<u>64</u>	13	1

Table 1: Number of independent octonionic tensorial charges with underlined octonionic-Hermitian matrices. The signatures entering the tables are respectively given by $(0, 7)$, $(9, 0)$ or $(1, 8)$, $(10, 1)$ or $(2, 9)$, $(11, 2)$ or $(3, 10)$.

The Table 1 is valid for octonionic generalized Poincaré superalgebras, with Abelian generators, as well as for their nonAbelian counterparts $U_\alpha(k, \mathbf{O})$ ($k = 2$ for $D = 9$, $k = 4$ for $D = 11$, $k = 8$ for $D = 13$) describing octonionic D -dimensional AdS superbranes. The octonionic equivalence of different sectors, via generalized Poincaré or generalized AdS supersymmetry algebras interpreted as branes sectors, can be symbolically expressed in different odd space-time dimensions according to the Table 2.

$D = 7$	$M0 \equiv M3$
$D = 9$	$M1 + M2 \equiv M4$
$D = 11$	$M1 + M2 \equiv M5$
$D = 13$	$M2 + M3 \equiv M6$

Table 2. The relation between octonionic super- p -branes M_p .

In $D = 11$ dimensions the relation between $M1 + M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into the 4 real indices, labeled by a, b, c, \dots and the 7 octonionic indices labeled by i, j, k, \dots . We get:

$M1 + M2$	$M5$
4 $M1_a$	7 $M5_{[abcdi]} \equiv M5_i$
7 $M1_i$	$4 \times 7 = 28$ $M5_{[abcij]} \equiv M5_{[ai]}$
6 $M1_{[ab]}$	6 $M5_{[abijk]} \equiv M5_{[ab]}$
$4 \times 7 = 28$ $M2_{[ai]}$	4 $M5_{[aijkl]} \equiv M5_a$
7 $M2_{[ij]} \equiv M2_i$	7 $M5_{[ijklm]} \equiv \widetilde{M5}_i$

which shows the equivalence of the two sectors, as far as the octonionic content and tensorial properties are concerned. Please notice that the correct total number of 52 independent components is recovered

$$52 = 2 \times 7 + 28 + 6 + 4.$$

It would be very interesting to find a dynamical realization of presented above octonionic super- p -branes framework. Similarly one can reproduce the count of independent degrees of freedom for octonionic $M2, M3, M6$ in $D = 13$.

5 Outlook

The octonions are the basic ingredients of many exceptional structures in mathematics. It is very well known, that the octonions provide the algebraic and geometric framework for the exceptional Lie algebras. Indeed, G_2 is the automorphism group of the octonions, while F_4 is the automorphism group of the 3×3 octonionic-valued hermitian matrices realizing the exceptional $J_3(\mathbf{O})$ Jordan algebra. F_4 and the remaining exceptional Lie algebras (E_6, E_7, E_8) are recovered from the so-called “magic square Tits’ construction” which associates a Lie algebra to any given pair of division algebras, if one of these algebras coincide with the octonionic algebra [12]. We would like also to point out here that exceptional Lie algebras have numerous applications in elementary particle physics (see e.g. [21])

We have applied the octonionic structure to the description of a new version of M -theory. The main outcome of our considerations, which is symbolically represented in Table 2, implies that in such a framework the different brane sectors are no longer independent. We would like also to point out (see formula (2.14)) that octonionic structure imposes in extended space-times [23,24,19] additional constraints on central charge tensor coordinates, without restricting however $D=11$ spacetime.

Our considerations here are purely algebraic - the step which would be desirable is to provide some corresponding geometrical notions. It should be pointed out, however, that for nonassociative algebras the distinction between algebraic and geometric considerations rather disappears.

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