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THE QUANTUM MECHANICS SUSY ALGEBRA: AN INTRODUCTORY REVIEW

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Abstract

Starting with the Lagrangian formalism with $N = 2$ supersymmetry in terms of two Grassmann variables in Classical Mechanics, the Dirac canonical quantization method is implemented. The $N = 2$ supersymmetry algebra is associated to one-component and two-component eigenfunctions considered in the Schrödinger picture of Nonrelativistic Quantum Mechanics. Applications are contemplated.

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I. INTRODUCTION

We present a review work considering the Lagrangian formalism for the construction of one dimension supersymmetric (SUSY) quantum mechanics (QM) with $N = 2$ supersymmetry (SUSY) in a non-relativistic context. In this paper, the supersymmetry with two Grassmann variables ($N = 2$) in classical mechanics is used to implement the Dirac canonical quantization method and the main characteristics of the SUSY QM is considered in detail. A general review on the SUSY algebra in quantum mechanics and the procedure on like to build a SUSY Hamiltonian hierarchy in order of a complete spectral resolution it is explicitly applied for the Pöschl-Teller potential I. We will follow a more detailed discussion for the case of this problem presents unbroken SUSY and broken SUSY. We have include a large number of references where the SUSY QM works, with emphasis on the one-component eigenfunction under non-relativistic context. But we indicate some articles on the SUSY QM from Dirac equation of relativistic quantum mechanics. The aim of this paper is to stress the discussion how arise and to bring out the correspondence between SUSY and factorization method in quantum mechanics. A brief account of a new scenario on SUSY QM to two-component eigenfunctions, makes up the last part of this review work.

SUSY first appeared in field theories in terms of bosonic and fermionic fields[†], and the possibility was early observed that it can accommodate a Grand-Unified Theory (GUT) for the four basic interactions of Nature (strong, weak, electromagnetic and gravitational) [1]. The first work on the superalgebra in the space-time within the framework of the Poincaré algebra was investigated by Gol'fand and Likhtman [2]. On the other hand, Volkov-Akulov have considered a non-renormalizable realization of supersymmetry in field theory [3], and Wess-Zumino have presented a renormalizable supersymmetric field theory model [4].

Recently the SUSY QM has also been investigated with pedagogical purpose in some booktexts [5] on quantum mechanics giving its connections with the factorization method [7]. Starting from factorization method new class of one-parameter family of isospectral potential in one dimension has been constructed with the energy spectrum coincident with

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[†]A bosonic field (associated with particles of integral or null spin) is one particular case obeying the Bose-Einstein statistic and a fermionic field (associated to particles with semi-integral spin) is that obey the Fermi-Dirac statistic.

that of the harmonic oscillator by Mielnik [8]. In recent literature, there are some interesting books on supersymmetric classical and quantum mechanics emphasizing different approach and applications of the theory [6].

Fernandez *et al.* have considered the connection between factorization method and generation of solvable potentials [9]. The SUSY algebra in quantum mechanics initiated with the work of Nicolai [10] and elegantly formulated by Witten [11], has attracted interest and found many applications in order to construct the spectral resolution of solvable potentials in various fields of physics. However, in this work, SUSY $N = 2$ in classical mechanics [12–18] in a non-relativistic scenario is considered using the Grassmann variables [19]. Recently, we have shown that the $N = 1$ SUSY in classical mechanics depending on a single commuting supercoordinate exists only for the free case [20].

Nieto has shown that the generalized factorization observed by Mielnik [8] allow us to do the connection between SUSY QM and the inverse method [21,22]. The first technique that have been used to construct some families of isospectral order second differential operators is based on a theorem due to Darboux, in 1882 [23].

J. W van Holten *et al.* have written a number of papers dealing with SUSY mechanical systems [24–33]. The canonical quantization of $N = 2$ (at the time called $N = 1$) SUSY models on spheres and hyperboloids [25] and on arbitrary Riemannian manifolds have been considered in [26]; its $N = 4$ (at that time called $N = 2$) generalization is found in [27]; SUSY QM in Schwarzchild background was studied in [28]; New so-called Killing-Yano supersymmetries were found and studied in [29–31]; General multiplet calculus for locally supersymmetric point particle models was constructed in [32] and the relativistic and supersymmetric theory of fluid mechanics in 3+1 diemnsions has been investigated by Nyawelo-van Holten [33]. The vorticity in the hydrodynamics theory is generated by the fermion fields [34].

D'Hoker and Vinet have also written a number of papers dealing with classical and quantum mechanical supersymmetric Lagrangian mechanical systems. They have shown that a non-relativistic spin $\frac{1}{2}$ particle in the field of a Dirac magnetic monopole exhibits a large SUSY invariance [36]. Later, they have published some other interesting works on the construction of conformal superpotentials for a spin $\frac{1}{2}$ particle in the field of a Dyon and the magnetic monopole and $\frac{1}{r^2}-$potential for particles in a Coulomb potential [37]. However, the supersymmetrizatin of the action for the charge monopole system have been also developed by Balachandran *et al.* [38].

A new SUSY QM system given by a non-relativistic charged spin-$\frac{1}{2}$ particle in an extended external electromagnetic field was obtained by Dias-Helayel [39].

Using a general formalism for the non-linear quantum-mechanical σ model, a mechanism of spontaneous breaking of the supersymmetry at the quantum level related to the uncertainty of the operator ordering has been obtained by Akulov-Pashnev [40]. In [40] is noted the simplicity of the supersymmetric $O(3)$-or $O(2,1)$-ivariant Lagrangian deduced there when compared with the analogous obtained using real superfields [26]. The mechanism of spontaneous breaking of the supersymmetry in quantum mechanics has also been investigated by Fuchs [41].

Barcelos and others have implemented the Dirac quantization method in superspace and found the SUSY Hamiltonian operator [42]. Recently, Barcelos-Neto and Oliveira have investigated the transformations of second-class into first-class constraints in supersymmetric classical theories for the superpoint [43]. Junker-Matthiesen have also considered the Dirac's canonical quantization method for the non-relativistic superpaticle [230]. In the interest of setting an accurate historical record of the subject, we point out that, by using the Dirac's procedure for two-dimensional supersymmetric non-linear σ-model, Eq. (13) of the paper by Corrigan-Zachos [35] works certainly for a SUSY system in classical mechanics.

A generalized Berezin integral and fractional superspace measure arise as a deformed q-calculus is developed on the basis of an algebraic structure involving graded brackets. In such a construction of fractional supersymmetry the q-deformed bosons play a role exactly analogous to that of the fermions in the familiar supersymmetric case, so that the SUSY is identified as translational invariance along the braided line by Dunne *et al.* [45]. An explicit formula has been given in the case of real generalised Grassmann variable, $\Theta^n = 0$, for arbitrary integer $n = 2, 3, \cdots$ for the transformations that leave the theory invariant, and it is shown that these transformations possess interesting group properties by Azcárraga and Macfarlane [46]. Based on the idea of quantum groups [47] and paragrassmann variables in the q-superspace, where $\Theta^3 = 0$, a generalization of supersymmetric classical mechanics with a deformation parameter $q = \exp \frac{2\pi i}{k}$ dealing with the $k = 3$ case has been considered by Matheus-Valle and Colatto [48].

The reader can find a large number of studies of fractional supersymmetry in literature. For example, a new geometric interpretation of SUSY, which apllies equally in the fractional case. Indeed, by means of a chain rule expansion, the left and right derivatives are identified with the charge Q and covariant derivative D encountered in ordinary/fractional supersymmetry and this leads to new results for these operators [49].

Supersymmetric Quantum Mechanics is of intrinsic mathematical interest in its own as it connects otherwise apparently unrelated (Cooper and Freedman [50]) second-order differential equations.

For a class of the dynamically broken supersymmetric quantum-mechanical models proposed by Witten [11], various methods of estimating the ground-state energy, including the instanton developed by Salomonson-van Holten [13] have been examined by Abbott-Zakrzewski [51]. The factorization method [7] was generalized by Gendenstein [52] in context of SUSY QM in terms of the reparametrization of potential in which ensure us if the resolution spectral is achieved by an algebraic method. Such a reparametrization between the supersymmetric potential pair is called shape invariance condition.

In the Witten's model of SUSY QM the Hamiltonian of a certain quantum system is represented by a pair $H_{\pm}$, for which all energy levels except possibly the ground energy eigenvalue are doubly degenerate for both $H_{\pm}$. As an application of the simplest of the graded Lie algebras of the supersymmetric fields theories, the SUSY Quantum Mechanics embodies the essential features of a theory of supersymmetry, i.e., a symmetry that generates transformations between bosons and fermions or rather between bosonic and the fermionic sectors associated with a SUSY Hamiltonian. SUSY QM is defined (Crombrugghe and Rittenberg [53] and Lancaster [54]) by a graded Lie algebra satisfied by the charge operators $Q_i (i = 1, 2, \ldots, N)$ and the SUSY Hamiltonian H. The σ model and supersymmetric gauge theories have been investigated in the context of SUSY QM by Shifman *et al.* [55].

While in field theory one works with SUSY as being a symmetry associated with transformations between bosonic and fermionic particles. In this case one has transformations between the component fields whose intrinsic spin differ by $\frac{1}{2}\hbar$. The energy of potential models of the SUSY in field theory is always positive semi-defined [1,56,60–66]. Here is the main difference of SUSY between field theory and quantum mechanics. Indeed, due to the energy scale to be of arbitrary origin the energy in quantum mechanics is not always positive.

Using supergraph metods, Helayel-Neto *et al.* have derived the chiral and antichiral superpropagators [57]; have calculated the chiral and gauge anomalies for the supersymmetric Schwinger model [58]; under certain asumption on the torsion-like explicitly breaking term, one-loop finiteness without spoilong the Ricci-flatness of the target manifold [59].

After a considerable number of works investigating SUSY in Field Theory, confirmation of SUSY as high-energy unification theory is missing. Furthermore, there exist phenomenological applications of the $N = 2$ SUSY technique in quantum mechanics [67].

The SUSY hierarchical prescription [68] was utilized by Sukumar [69] to solve the energy spectrum of the Pöschl-Teller potential I (PTPI). We will use their notation.

The two first review work on SUSY QM with various applications were reported by Gendenshtein-Krive and Haymaker-Rau [70] but does they not consider the Sukumar's method [69]. In next year to the review work by Gendenshtein-Krive, Gozi implemented an approach on the nodal structure of supersymmetric wave functions [71] and Imbo-Sukhatme have investigated the conditions for nondegeneracy in supersymmetric quantum mechanics [72].

In the third in a series of papers dealing with families of isospectral Hamiltonians, Pursey has been used the theory of isometric operators to construct a unified treatment of three procedures existing in literature for generating one-parameter families of isospectral Hamiltonian [73]. In the same year, Castaños *et al.* have also shown that any n-dimensional scalar Hamiltonian possesses hidden supersymmetry provided its spectrum is bounded from below [74].

Lahiri *et al.* have investigated the transformation considered by Haymaker-Rau [70], viz., of the type $x = \ell n y$ so that the radial Schrödinger equation for the Coulomb potential becomes a unidimensional Morse-Schrödinger equation, and have stablished a procedure for constructing the SUSY transformations [75].

Cooper-Ginocchio have used the Sukumar's method [69] gave strong evidence that the more general Natanzon potential class not shape invariant and found the PTPI as particular case [76]. In the works of Gendenshtein [52] and Dutt *et al.* [77] only the energy spectrum of the PTPI was also obtained but not the excited state wave functions. The unsymmetric case has been treated algebraically by Barut, Inomata and Wilson [78]. However in these analysis only quantized values of the coupling constants of the PTPI have been obtained.

Roy-Roychoudhury have shown that the finite-temperature effect causes spontaneous breaking of SUSY QM, based on a superpotential with (non-singular) non-polynomial character, and Casahorran has investigated the superymmetric Bogomol'nyi bounds at finite temperature [79].

Jost functions are studied within framework of SUSY QM by Talukdar *et al.*, so that it is seen that some of the existing results follow from their work in a rather way [80].

Instanton-type quantum fluctuations in supersymmetric quantum mechanical systems with a double-well potential and a tripe-well potential have been discussed by Kaul-Mizrachi [81] and the ground state energy was found via a different method considered by Salomonson-van Holten [13] and Abbott-Zakrzewski [51].

Stahlhofen showed that the shape invariance condition [52] for supersymmetric potentials and the factorization condition for Sturm-Liouville eigenvalue problems are equivalent [82]. Fred Cooper *et al.* starting from shape invariant potentials [52] applied an operator transformation for the Pöschl-Teller potential I and found that the Natanzon class of solvable potentials [83]. The supersymmetric potential partner pair through the Fokker-Planck superpotential has been used to deduce the computation of the activation rate in one-dimensional bistable potentials to a variational calculation for the ground state level of a non-stable quantum system [84].

A systematic procedure using SUSY QM has been presented for calculating the accurate energy eigenvalues of the Schrödinger equation that obviates the introduction of large-order determinants by Fernandez, Desmet and Tipping [85].

SUSY has also been applied for Quantum Optics. For instant, let us point out that the superalgebra of the Jaynes-Cummings model is described and the presence of a gap in the energy spectrum indicates a spontaneous SUSY breaking. If the gap tends to zero the SUSY is restored [86]. In another work, the Jaynes-Cummings model for a two-level atom interacting with an electromagnetic field is analyzed in terms of SUSY QM and their eigenfunctions are deduced [87]. Other applications on the SUSY QM to Quamtum Optics can be found in [88].

Mathur has shown that the symmetries of the Wess-Zumino model put severe constraints on the eigenstates of the SUSY Hamiltonian simplifying the solutions of the equation associated with the annihilation conditions for a particular superpotential [89]. In this interesting work, he has found the non-zero energy spectrum and all excited states are at least eightfold degenerate.

The connection of the PTPI with new isospectral potentials has been studied by Drigo Filho [90]. Some remarks on a new scenario of SUSY QM by imposing a structure on the raising and lowering operators have been found for the 1-component eigenfunctions [91]. The unidimensional SUSY oscillator has been used to construct the strong-coupling limit of the Jaynes-Cummings model exhibiting a noncompact ortosymplectic SUSY by Shimitt-Mufti [92].

The propagators for shape invariant potentials and certain recursion relations for them both in the operator formulation as well as in the path integrals were investigated with some examples by Das-Huang [93].

At third paper about a review on SUSY QM, the key ingredients on the quantization of the systems with anticommuting variables and supersymmetric Hamiltonian was constructed by emphasizing the role of partner potentials and the superpotentials have been discussed by Lahiri, Roy and Bagchi [94]. In which Sukumar's supersymmetric procedure was applied for the following potentials: unidimensional harmonic oscillator, Morse potential and $sech^2 x$ potential.

The formalism of SUSY QM has also been used to realize Wigner superoscillators in order to solve the Schrödinger equation for the isotonic oscillator (Calogero interaction) and radial oscillator [95].

Freedman-Mende considered the application in supersymmetric quantum mechanics for an exactly soluble N-particle system with the Calogero interaction [97]. The SUSY QM formalism associated with 1-component eigenfunctions was also applied to a planar physical system in the momentum representation via its connection with a PTPI system. There, such a system considered was a neutron in an external magnetic field [96].

A supersymmetric generalization of a known solvable quantum mechanical model of particles with Calogero interactions, with combined harmonic and repulsive forces have investigated by Freedman-Mende [97] and the explicit solution for such a supersymmetric Calogero were constructed by Brink *et al.* [98].

Dutt *et al.* have investigated the PTPI system with broken SUSY and new exactly solvable Hamiltonians via shape invariance procedure [99].

The formulation of higher-derivative supersymmetry and its connection with the Witten index has been proposed by Andrianov *et al.* [100] and Beckers-Debergh [101] have discussed a possible extension of the super-realization of the Wigner quantization procedure considered by Jayaraman-Rodrigues [95]. In [101] has been proposed a construction that was called of a parastatistical hydrogen atom which is a supersymmetric system but is not a Wigner system. Results of such investigations and also of the pursuit of the current encouraging indications to extend the present formalism for Calogero interactions will be reported separately.

In another work on SUSY in the non-relativistic hydrogen atom, Tangerman-Tjon have stressed the fact that no extra particles are needed to generate the supercharges of $N = 2$ SUSY algebra when we use the spin degrees of freedom of the electron [102]. Boya *et al.* have considered the SUSY QM approach from geometric motion on arbitrary rank-one Riemannian symmetric spaces via Jost functions and the Laplace-Beltrami operator [103].

In next year, Jayaraman-Rodrigues have also identified the free parameter of the Celka-Hussin's model with the Wigner parameter [95] of a related super-realized general 3D Wigner oscillator system satisfying a super generalized quantum commutation relation of the σ_3-deformed Heisenberg algebra [104]. In this same year, P. Roy has studied the possibility of contact interaction of anyons within the framework of two-particle SUSY QM model [105]; indeed, at other works the anyons have been studied within the framework of supersymmetry [106].

In stance in the literature, there exist four excellent review articles about SUSY in non-relativistic quantum mechanics [70,94,107]. Recently the standard SUSY formalism was also applied for a neutron in interaction with a static magnetic field in the coordinate representation [108] and the SUSY QM in higher dimensional was discussed by Das-Pernice [109].

Actually it is well known that the SUSY QM formalism is intrinsically bound with the theory of Riccati equation. Dutt *et al.* have ilusted the ideia of SUSY QM and shape invariance conditions can be used to obtain exact solutions of noncentral but separable potentials in an algebraic fashion [110]. A procedure for obtaining the complete energy spectrum from the Riccati equation has been illustrated by detailed analysis of several examples by Haley [111].

Including not only formal mathematical objects and schemes but also new physics, many different physical topics are considered by the SUSY technique (localization, mesoscopics, quantum chaos, quantum Hall effect, etc.) and each section begins with an extended introduction to the corresponding physics. Various aspects of SUSY may limit themselves to reading the chapter on supermathematics, in a book written by Efetov [112].

SUSY QM of higher order have been by Fernandez *et al.* [113]. Starting from SUSY QM, Junker-Roy [114], presented a rather general method for the construction of so-called conditionally exactly sovable potentials [115]. A new SUSY method for the generation of quasi-exactly solvable potentials with two known eigenstates has been proposed by Tkachuk [116].

Recently Rosas-Ortiz has shown a set of factorization energies generalizing the choice made for the Infeld-Hull [7] and Mielnik [8] factorizations of the hydrogen-like potentials [117]. The SUSY technique has also been used to generate families of isospectral potentials and isospectral effective-mass variations, which may be of interest, e.g., in the design of semiconductor quantum wells [118].

The soliton solutions have been investigated for field equations defined in a space-time of dimension equal to or higher than 1+1. The kink of a field theory is an example of a soliton in 1+1 dimensions [121–125]. In this work we consider the Bogomol'nyi [119] and Prasad-Sommerfield [120] (BPS) classical soliton (defect) solutions. Recently, from $N = 1$ supersymmetric solitons the connection between SUSY QM and the sphaleron and kinks has been established for relativistic systems of a real scalar field [126–132,134].

The shape-invariance conditions in SUSY [52] have been generalized for systems described by two-component wave functions [135], and a two-by-two matrix superpotential associated to the linear classical stability from the static solutions for a system of two coupled real scalar fields in (1+1)-dimensions have been found [136–138,141,142]. In Ref. [138] has been shown that the classical central charge, equal to the jump of the superpotential in two-dimensional models with minimal SUSY, is additionally modified by a quantum anomaly, which is an anomalous term proportional to the second derivative of the superpotential. Indeed, one can consider an analysis of the anomaly in supersymmetric theories with two coupled real scalar fields [140] as reported in the work of Shifman *et al.* [138]. Besides, the stability equation for a Q-ball in 1 dimension has also been related to the SUSY QM [139].

A systematic and critical examination, reveals that when carefully done, SUSY is manifest even for the singular quantum mechanical models when the regularization parameter is removed [143]. The Witten's SUSY formulation for Hamiltonian systems to also a system of annihilation operator eigenvalue equations associated with the SUSY singular oscillator, which, as was shown, define SUSY canonical supercoherent states containing mixtures of both pure bosonic and pure fermionic counterparts have been extended [144]. Also, Fernandez *et al.* have investigated the coherent states for SUSY partners of the oscillator [145], and Kinani-Daoud have built the coherent states for the Pöschl-Teller potential [146].

In the first work in Ref. [147], Plyushchay has used arguments of minimal bosonization of SUSY QM and R-deformed Heisenberg algebra in order to get in the second paper in the same Ref. a super-realization for the ladder operators of the Wigner oscillator [95]. While Jayaraman and Rodrigues, in Ref. [95], adopt a super-realization of the Wigner-Heisenberg algebra (σ_3-deformed Heisenberg algebra) as effective spectral resolution for the two-particle Calogero interaction or isotonic oscillator, in Ref. [147], using the same super-realization, Plyushchay showed how a simple modification of the classical model underlying Witten SUSY QM results in appearance of $N = 1$ holomorphic non-linear supersymmetry.

In the context of the symmetry of the fermion-monopole system [36], Pluyschay has shown that this system possesses $N = \frac{3}{2}$ nonlinear supersymmetry [148]. The spectral problem of the 2D system with the quadratic magnetic field is equivalent to that of the 1D quasi-exactly solvable systems with the sextic potential, and the relation of the 2D holomorphic n-supersymmetry to the non-holomorphic N-fold supersymmetry has been investigated [149].

In [150], it was shown that the problem of quantum anomaly can be resolved for some special class of exactly solvable and quasi-exactly solvable systems. So, in this paper it was discovered that the nonlinear supersymmetry is related with quasi-exact solvability. Besides, in this paper it was observed that the quantum anomaly happens also in the case of the linear quantum mechanics and that the usual holomorphic-like form of SUSYQM (in terms of the holomorphic-like operators $W(x) \pm i\frac{d}{dx}$) is special: it is anomaly free.

Macfarlane [151] and Azcárraga-Macfarlane [152] have investigated models with only fermionic dynamical variables. Azcárraga *et al.* generalises the use of totaly antisymmetric tensors of third rank in the definition of Killing-Yano tensors and in the construction of the supercharges of hidden supersymmetries that are at most third in fermionic variables [153].

The SUSY QM formulation has been applied for scattering states (continuum eigenvalue) in non-relativistic quantum mechanics [154,155]. However, a radically different theory for SUSY was recently putted forward, which is concerned with collision problems in SUSY QM by Shimbori-Kobayashi [156].

Zhang et al. have considered interesting applications of a semi-unitary formulation in SUSY QM [157]. Indded, in the papers of Ref. [157] a semi-unitary framework of SUSY QM was developed. This framework works well for multi-dimensional system. Besides Hamiltonian, it can simultaneously obtain superpartner of the angular momentum and other observables, though they are not the generators of the superalgebra in SUSY QM.

Recently, Mamedov *et al.* heve applied SUSY QM for the case of a Dirac particle moving in a constant chromomagnetic field [158].

The spectral resolution for the Pöschl-Teller potential I has been studied as shape-invariant potentials and their potential algebras [159]. For this problem we consider as complete spectral resolution the application of SUSY QM via Hamiltonian Hierarchy associated to the partner potential respective [160].

Rencently the group theoretical treatment of SUSY QM has also been investigated by Fernandez *et al.* [161]. The SUSY techniques has been applied to periodic potentials by Dunne-Feinberg [162], Sukhatme-khare [163] and by Fernandez *et al.* [164]. Rencently, the complex potentials with the so-colled PT symmetry in quantum mechanics [165] has also been investigated via SUSY QM [166].

This present work is organized in the following way. In Sec. II we start by summarizing the essential features of the formulation of one dimensional supersymmetric quantum mechanics. In Sec. III the factorization of the unidimensional Schrödinger equation and a SUSY Hamiltonian hierarchy considered by Andrionov *et al.* [68] and Sukumar [69] is presented. We consider in Sec. IV the close connection for SUSY method as an operator technique for spectral resolution of shape-invariant potentials. In Sec. V we present our own application of the SUSY hierarchical prescription for the first Pöschl-Teller potential. It is known that the SUSY algebraic method of resolution spectral via property of shape invariant which permits to work are unbroken SUSY. While the case of PTPI with broken SUSY in [99,159] has after suitable mapping procedures that becomes a new potential with unbroken SUSY, here, we show that the SUSY hierarchy method [69] can work for both cases. In Section VI, we present a new scenery on the SUSY when it is applied for a neutron in interaction with a static magnetic field of a straight current carrying wire, which is described by two-component wave functions [108,167].

Section VII contains the concluding remarks.

II. N=2 SUSY IN CLASSICAL MECHANICS

Recently, we present a review work on Supersymmetric Classical Mechanics in the context of a Lagrangian formalism, with $N = 1-$supersymmetry. We have shown that the $N = 1$ SUSY does not allow the introduction of a potential energy term depending on a single commuting supercoordinate, $\phi(t;\Theta)$ [20].

In the construction of a SUSY theory with $N > 1$, referred to as extended SUSY, for each space commuting coordinate, representing the degrees of freedom of the system, we associate one anticommuting variable, which are known that Grassmannian variables. However, we consider only the $N = 2$ SUSY for a non-relativistic point particle, which is described by the introduction of two real Grassmannian variables Θ_1 and Θ_2, in the configuration space, but all the dynamics are putted in the time t [13,18,42,43,230,50,94].

SUSY in classical mechanics is generated by a translation transformation in the superspace, viz.,

$$\Theta_1 \rightarrow \Theta_1' = \Theta_1 + \epsilon_1, \quad \Theta_2 \rightarrow \Theta_2' = \Theta_2 + \epsilon_2, \quad t \rightarrow t' = t + i\epsilon_1\Theta_1 + i\epsilon_2\Theta_2, \tag{1}$$

whose are implemented for maintain the line element invariant

$$dt - i\Theta_1 d\Theta_1 - i\Theta_2 d\Theta_2 = \text{ invariant}, (\text{Jacobian} = 1), \tag{2}$$

where Θ_1, Θ_2 and ϵ_1 and ϵ_2 are real Grassmannian paramenters. We insert the $i = \sqrt{-1}$ in (1) and (2) to obtain the real character of time.

The real Grassmannian variables satisfy the following algebra:

$$[\Theta_i, \Theta_j]_+ = \Theta_i\Theta_j + \Theta_j\Theta_i = 0 \Rightarrow (\Theta_1)^2 = 0 = (\Theta_2)^2. \tag{3}$$

They also satisfy the Berezin integral rule [19]

$$\int d\Theta_i\Theta_j = \delta_{ij} \Rightarrow \sum_{i=1}^{2}\int d\Theta_i\Theta_i = 2, \quad \int d\Theta_i = 0 = \partial_{\Theta_i}1, \quad \int d\Theta_i\Theta_j = \delta_{ij} = \partial_{\Theta_i}\Theta_j, \tag{4}$$

where $\partial_{\Theta_i} = \frac{\partial}{\partial\Theta_1}$ so that

$$[\partial_{\Theta_i}, \Theta_j]_+ = \partial_{\Theta_i}\Theta_j + \Theta_j\partial_{\Theta_i} = \delta_{ij}, \quad \partial_{\Theta_i}(\Theta_k\Theta_j) = \delta_{ik}\Theta_j - \delta_{ij}\Theta_k, \tag{5}$$

with $i = j \Rightarrow \delta_{ii} = 1$; and if $i \neq j \Rightarrow \delta_{ij} = 0, (i, j = 1, 2)$.

Now, we need to define the derivative rule with respect to one Grassmannian variable. Here, we use the right derivative rule i.e. considering $f(\Theta_1, \Theta_2)$ a function of two anticommuting variables, the right derivative rule is the following:

$$\begin{aligned} f(\Theta_\alpha) &= f_0 + \sum_{\alpha=1}^{2} f_\alpha\Theta_\alpha + f_3\Theta_1\Theta_2 \\ \delta f &= \sum_{\alpha=1}^{2} \frac{\partial f}{\partial\Theta_\alpha}\delta\Theta_\alpha. \end{aligned} \tag{6}$$

where $\delta\Theta_1$ and $\delta\Theta_2$ appear on the right side of the partial derivatives.

Defining Θ and $\bar{\Theta}$ (Hermitian conjugate of Θ) in terms of $\Theta_i (i = 1, 2)$ and Grassmannian parameters ϵ_i,

$$\begin{aligned} \Theta &= \frac{1}{\sqrt{2}}(\Theta_1 - i\Theta_2), \\ \bar{\Theta} &= \frac{1}{\sqrt{2}}(\Theta_1 + i\Theta_2), \\ \epsilon &= \frac{1}{\sqrt{2}}(\epsilon_1 - i\epsilon_2), \\ \bar{\epsilon} &= \frac{1}{\sqrt{2}}(\epsilon_1 + i\epsilon_2), \end{aligned} \tag{7}$$

the supertranslations become:

$$\Theta \rightarrow \Theta' = \Theta + \epsilon, \qquad \bar{\Theta} \rightarrow \bar{\Theta}' = \bar{\Theta} + \bar{\epsilon}, \qquad t \rightarrow t' = t - i(\bar{\Theta}\epsilon - \bar{\epsilon}\Theta). \tag{8}$$

In this case, we obtain

$$[\partial_\Theta, \Theta]_+ = 1, \quad [\partial_{\bar{\Theta}}, \bar{\Theta}]_+ = 1, \quad \Theta^2 = 0. \tag{9}$$

The Taylor expansion for the real scalar supercoordinate is given by

$$\phi(t;\Theta,\bar{\Theta}) = q(t) + i\bar{\Theta}\psi(t) + i\Theta\bar{\psi}(t) + \Theta\bar{\Theta}A(t), \tag{10}$$

which under infinitesimal SUSY transformation law provides

$$\begin{aligned}\delta\phi &= \phi(t';\Theta',\bar{\Theta}') - \phi(t;\Theta,\bar{\Theta}) \\ &= \partial_t\phi\delta t + \partial_\Theta\phi\delta\Theta + \partial_{\bar{\Theta}}\phi\delta\bar{\Theta} \\ &= (\bar{\epsilon}Q + \bar{Q}\epsilon)\phi, \end{aligned}\tag{11}$$

where $\partial_t = \frac{\partial}{\partial t}$ and the two SUSY generators

$$Q \equiv \partial_{\bar{\Theta}} - i\Theta\partial_t, \quad \bar{Q} \equiv -\partial_\Theta + i\bar{\Theta}\partial_t. \tag{12}$$

Note that the supercharge $\bar{Q}$ is not the hermitian conjugate of the supercharge Q. In terms of $(q(t); A)$ bosonic (even) components and $(\psi(t), \bar{\psi}(t))$ fermionic (odd) components we get:

$$\delta q(t) = i\{\epsilon\bar{\psi}(t) + \bar{\epsilon}\psi(t)\}, \qquad \delta A = \epsilon\dot{\bar{\psi}}(t) - \bar{\epsilon}\dot{\psi}(t) = \frac{d}{dt}\{\epsilon\bar{\psi} - \bar{\epsilon}\psi), \tag{13}$$

$$\delta\psi(t) = -\epsilon\{\dot{q}(t) - iA\}, \qquad \delta\bar{\psi}(t) = -\bar{\epsilon}\{\dot{q}(t) + iA\}. \tag{14}$$

Therefore making a variation in the even components we obtain the odd components and vice-versa i.e. SUSY mixes the even and odd coordinates.

A super-action for the superpoint particle with N=2 SUSY can be written as the following tripe integral‡

$$S[\phi] = \int\int\int dt d\bar{\Theta} d\Theta\{\frac{1}{2}(D\phi)(\bar{D}\phi) - U(\phi)\}, \qquad \bar{D} \equiv \partial_\Theta + i\bar{\Theta}\partial_t, \tag{15}$$

where D is the covariant derivative ($D = -\partial_{\bar{\Theta}} - i\Theta\partial_t$), $\bar{\partial}_{\bar{\Theta}} = -\partial_\Theta$ and $\partial_\Theta = \frac{\partial}{\partial\Theta}$, built so that $[D, Q]_+ = 0 = [\bar{D}, \bar{Q}]_+$ and $U(\phi)$ is a polynomial function of the supercoordinate.

The covariant derivatives of the supercoordinate $\phi = \phi(\Theta, \bar{\Theta}; t)$ become

$$\begin{aligned}\bar{D}\phi &= (\partial_\Theta + i\bar{\Theta}\partial_t)\phi = -i\bar{\psi} - \bar{\Theta}A + i\bar{\Theta}\partial_t q + \Theta\bar{\Theta}\dot{\bar{\psi}}, \\ D\phi &= (-\partial_{\bar{\Theta}} - i\Theta\partial_t)\phi = i\psi - \Theta A - i\Theta\dot{q} + \Theta\bar{\Theta}\dot{\psi} \\ (D\phi)(\bar{D}\phi) &= \psi\bar{\psi} - \bar{\Theta}(\psi\dot{q} - iA\psi) + \Theta(iA\bar{\psi} + \bar{\psi}\dot{q}) \\ &\quad + \Theta\bar{\Theta}\left(\dot{q}^2 + A^2 + i\psi\dot{\bar{\psi}} + i\dot{\psi}\bar{\psi}\right). \end{aligned}\tag{16}$$

‡In this section about supersymmetry we use the unit system in which $m = 1 = \omega$, where m is the particle mass and ω is the angular frequency.

Expanding in series of Taylor the $U(\phi)$ superpotential and maintaining $\Theta\bar{\Theta}$ we obtain:

$$\begin{aligned} U(\phi) &= \phi U'(\phi) + \frac{\phi^2}{2}U''(\phi) + \cdots \\ &= A\Theta\bar{\Theta}U'(\phi) + \frac{1}{2}\psi\bar{\psi}\bar{\Theta}\Theta U'' + \frac{1}{2}\bar{\psi}\psi\Theta\bar{\Theta}U'' + \cdots \\ &= \Theta\bar{\Theta}\{AU' + \bar{\psi}\psi U''\} + \cdots, \end{aligned} \tag{17}$$

where the derivatives (U' and U'') are such that $\Theta = 0 = \bar{\Theta}$, whose are functions only the $q(t)$ even coordinate. After the integrations on Grassmannian variables the super-action becomes

$$S[q;\psi,\bar{\psi}] = \frac{1}{2}\int\left\{\dot{q}^2 + A^2 - i\dot{\psi}\bar{\psi} + i\psi\dot{\bar{\psi}} - 2AU'(q) - 2\bar{\psi}\psi U''(q)\right\}dt \equiv \int L dt. \tag{18}$$

Using the Euler-Lagrange equation to A, we obtain:

$$\frac{d}{dt}\frac{\partial L}{\partial\partial_t A} - \frac{\partial L}{\partial A} = A - U'(q) = 0 \Rightarrow A = U'(q). \tag{19}$$

Substituting Eq. (19) in Eq. (18), we then get the following Lagrangian for the superpoint particle:

$$L = \frac{1}{2}\left\{\dot{q}^2 - i(\dot{\psi}\bar{\psi} + \psi\dot{\bar{\psi}}) - 2\left(U'(q)\right)^2 - 2U''(q)\bar{\psi}\psi\right\}, \tag{20}$$

where the first term is the kinetic energy associated with the even coordinate in which the mass of the particle is unity. The second term in bracket is a kinetic energy piece associated with the odd coordinate (particle's Grassmannian degree of freedom) dictated by SUSY and is new for a particle with a potential energy. The Lagrangian is not invariant because its variation result in a total derivative and consequently is not zero, however, the super-action is invariant, $\delta S = 0$, which can be obtained from $D\mid_{\Theta=0} = -Q\mid_{\Theta=0}$ and $\bar{D}\mid_{\bar{\Theta}=0} = -\bar{Q}\mid_{\bar{\Theta}=0}$.

The canonical Hamiltonian for the $N = 2$ SUSY is given by:

$$H_c = \dot{q}\frac{\partial L}{\partial\dot{q}} + \frac{\partial L}{\partial(\partial_t\psi)}\dot{\psi} + \frac{\partial L}{\partial(\partial_t\bar{\psi})}\dot{\bar{\psi}} - L = \frac{1}{2}\left\{p^2 + \Big(U'(q)\Big)^2 + U''(q)[\bar{\psi},\psi]_-\right\}, \tag{21}$$

which provides a mixed potential term. Putting $U'(x) = -\omega x$ Eq. (18) describes the super-action for the superymmetric oscillator, where ω is the angular frequency.

A. CANONICAL QUATIZATION IN SUPERSPACE

The supersymmetry in quantum mechanics, first formulated by Witten [11], can be deduced via first canonical quantization or Dirac quantization of above SUSY Hamiltonian which inherently contain constraints. The first work on the constraint systems without SUSY was implemented by Dirac in 1950. The nature of such a constraint is different from the one encountered in ordinary classical mechanics.

Salomonson *et al.* [13], F. Cooper *et al.*, Ravndal [50] do not consider such constraints. However, they have maked an adequate choice for the fermionic operator representations corresponding to the odd coordinates $\bar{\psi}$ and ψ. The question of the constraints in SUSY classical mechanics model have been implemented via Dirac method by Barcelos-Neto and Das [42,43], and by Junker [230]. According the Dirac method the Poisson brackets $\{A, B\}$ must be substituted by the modified Posion bracket (called Dirac brackets) $\{A, B\}_D$, which between two dynamic variables A and B is given by:

$$\{A, B\}_D = \{A, B\} - \{A, \Gamma_i\} C^{-1}_{ij} \{\Gamma_j, B\} \tag{22}$$

where Γ_i are the second-class constraints. These constraints define the C matrix

$$C_{ij} \simeq \{\Gamma_i, \Gamma_j\}, \tag{23}$$

which Dirac show to be antisymmetric and nonsingular. The fundamental canonical Dirac brackets associated with even and odd coordinates become:

$$\{q, \dot{q}\}_D = 1, \qquad \{\psi, \bar{\psi}\}_D = i \qquad \text{and} \qquad \{A, \dot{q}\}_D = \frac{\partial^2 U(q)}{\partial q^2}. \tag{24}$$

All Dirac brackets vanish. It is worth stress that we use the right derivative rule while Barcelos-Neto and Das in the Ref. [42] have used the left derivative rule for the odd coordinates. Hence unlike of second Eq. (24), for odd coordinate there appears the negative sign in the corresponding Dirac brackets, i.e., $\{\psi, \bar{\psi}\}_D = -i$.

Now in order to implement the first canonical quantization so that according with the spin-statistic theorem the commutation $[A, B]_- \equiv AB - BA$ and anti-commutation $[A, B]_+ \equiv AB + BA$ relations of quantum mechanics are given by

$$\begin{aligned} \{q, \dot{q}\}_D = 1 \rightarrow \frac{1}{i}[\hat{q}, \dot{\hat{q}}]_- = 1 \quad &\Rightarrow [\hat{q}, \dot{\hat{q}}]_- = \hat{q}\dot{\hat{q}} - \dot{\hat{q}}\hat{q} = i, \\ \{\psi, \bar{\psi}\}_D = i \rightarrow \frac{1}{-i}[\hat{\psi}, \hat{\bar{\psi}}]_+ = i &\Rightarrow [\hat{\psi}, \hat{\bar{\psi}}]_+ = \hat{\psi}\hat{\bar{\psi}} + \hat{\bar{\psi}}\hat{\psi} = 1. \end{aligned} \tag{25}$$

Now we will consider the effect of the constraints on the canonical Hamiltonian in the quantized version. The fundamental representation of the odd coordinates, in $D = 1 = (0 + 1)$ is given by:

$$
\begin{aligned}
\hat{\psi} = \sigma_+ &= \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv b^+ \\
\hat{\bar{\psi}} = \sigma_- &= \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv b^- \\
[\hat{\psi}, \hat{\bar{\psi}}]_+ &= 1_{2\times 2}, \qquad [\hat{\bar{\psi}}, \hat{\psi}]_- = \sigma_3,
\end{aligned}
\tag{26}
$$

where σ_3 is the Pauli diagonal matrix, σ_1 and σ_2 are off-diagonal Pauli matrices. On the other hand, in coordinate representation, it is well known that the position and momentum operators satisfy the canonical commutation relation ($[\hat{x}, \hat{p}_x]_- = i$) with the following representations:

$$
\hat{x} \equiv \hat{q}(t) = x(t), \qquad \hat{p}_x = m\dot{x}(t) = -i\hbar\frac{d}{dx} = -i\frac{d}{dx}, \quad \hbar = 1. \tag{27}
$$

In next section we present the various aspects of the SUSY QM and the connection between Dirac quantization of the SUSY classical mechanics and the Witten's model of SUSY QM.

III. THE FORMULATION OF SUSY QM

The graded Lie algebra satisfied by the odd SUSY charge operators $Q_i (i = 1, 2, \ldots, N)$ and the even SUSY Hamiltonian H is given by following anti-commutation and commutation relations:

$$
[Q_i, Q_j]_+ = 2\delta_{ij}H, \quad (i, j = 1, 2, \ldots, N), \tag{28a}
$$

$$
[Q_i, H]_- = 0. \tag{28b}
$$

In these equations, H and Q_i are functions of a number of bosonic and fermionic lowering and raising operators respectively denoted by $a_i, a_i^\dagger (i = 1, 2, \ldots, N_b)$ and $b_i, b_i^\dagger (i = 1, 2, \ldots, N_f)$, that obey the canonical (anti-)commutation relations:

$$
[a_i, a_j^\dagger]_- = \delta_{ij}, \tag{29a}
$$

$$
[b_i, b_j^\dagger]_+ = \delta_{ij}, \tag{29b}
$$

all other (anti-)commutators vanish and the bosonic operators always commute with the fermionic ones.

If we call the generators with these properties "even" and "odd", respectively, then the SUSY algebra has the general structure

$$[even, even]_- = even$$

$$[odd, odd]_+ = even$$

$$[even, odd]_- = odd$$

which is called a graded Lie algebra or Lie superalgebra by mathematicians. The case of interest for us is the one with $N_b = N_f = 1$ so that $N = N_b + N_f = 2$, which corresponds to the description of the motion of a spin $\frac{1}{2}$ particle on the real line [11].

Furthermore, if we define the mutually adjoint non-Hermitian charge operators

$$Q_\pm = \frac{1}{\sqrt{2}}(Q_1 \pm iQ_2), \tag{30}$$

in terms of which the Quantum Mechanical SUSY algebraic relations, get recast respectively into the following equivalent forms:

$$Q_+^2 = Q_-^2 = 0, \qquad [Q_+, Q_-]_+ = H \tag{31}$$

$$[Q_\pm, H]_- = 0. \tag{32}$$

In (31)), the nilpotent SUSY charge operators $Q_\pm$ and SUSY Hamiltonian H are now functions of a^-, a^+ and b^-, b^+. Just as $[Q_i, H] = 0$ is a trivial consequence of $[Q_i, Q_j] = \delta_{ij}H$, so also (32) is a direct consequence of (31) and expresses the invariance of H under SUSY transformations.

We illustrate the same below with the model example of a simple SUSY harmonic oscillator (Ravndal [50] and Gendenshtein [70]). For the usual bosonic oscillator with the Hamiltonian[§]

$$H_b = \frac{1}{2}\left(p_x^2 + \omega_b^2 x^2\right) = \frac{\omega_b}{2}[a^+, a^-]_+ = \omega_b\left(N_b + \frac{1}{2}\right), \quad N_b = a^+a^-, \tag{33}$$

$$a^\pm = \frac{1}{\sqrt{2\omega_b}}\left(\pm ip_x - \omega_b x\right) = \left(a^\mp\right)^\dagger, \tag{34}$$

$$[a^-, a^+]_- = 1, \quad [H_b, a^\pm]_- = \pm\omega_b a^\pm, \tag{35}$$

[§]NOTATION: Throughout this section, we use the systems of units such that $c = \hbar = m = 1$.

one obtains the energy eigenvalues

$$E_b = \omega_b \left(n_b + \frac{1}{2} \right), \quad = 0, 1, 2, \ldots, \tag{36}$$

where n_b are the eigenvalues of the number operator indicated here also by N_b.

For the corresponding fermionic harmonic oscillator with the Hamiltonian

$$H_f = \frac{\omega_f}{2}[b^+, b^-]_- = \omega_f \left(N_f - \frac{1}{2} \right), \quad N_f = b^+ b^-, \quad (b^+)^\dagger = b^-, \tag{37}$$

$$[b^-, b^+]_+ = 1 \quad , (b^-)^2 = 0 = (b^+)^2 \quad , [H_f, b^\pm]_- = \pm \omega_f b^\pm, \tag{38}$$

we obtain the fermionic energy eigenvalues

$$E_f = \omega_f \left(\eta_f - \frac{1}{2} \right) \quad , \eta_f = 0, 1, \tag{39}$$

where the eigenvalues $\eta_f = 0, 1$ of the fermionic number operator N_f follow from ${N_f}^2 = N_f$.

Considering now the Hamiltonian for the combined system of a bosonic and a fermionic oscillator with $\omega_b = \omega_f = \omega$, we get:

$$H = H_b + H_f = \omega \left(N_b + \frac{1}{2} + N_f - \frac{1}{2} \right) = \omega (N_b + N_f) \tag{40}$$

and the energy eigenvalues E of this system are given by the sum $E_b + E_f$, i.e., by

$$E = \omega(n_b + n_f) = \omega n, \qquad (n_f = 0, 1; \quad n_b = 0, 1, 2, \ldots; \quad n = 0, 1, 2, \ldots). \tag{41}$$

Thus the ground state energy $E^{(0)} = 0$ in (41) corresponds to the only non-degenerate case with $n_b = n_f = 0$, while all the excited state energies $E^{(n)}(n \geq 1)$ are doubly degenerate with $(n_b, n_f) = (n, 0)$ or $(n-1, 1)$, leading to the same energy $E^{(n)} = n\omega$ for $n \geq 1$.

The extra symmetry of the Hamiltonian (40) that leads to the above of double degeneracy (except for the singlet ground state) is in fact a supersymmetry, i.e., one associated with the simultaneous destruction of one bosonic quantum $n_b \to n_b - 1$ and creation of one fermionic quantum $n_f \to n_f + 1$ or vice-versa, with the corresponding symmetry generators behaving like $a^- b^+$ and $a^+ b^-$. In fact, defining,

$$Q_+ = \sqrt{\omega} a^+ b^-, \qquad Q_- = (Q_+)^\dagger = \sqrt{\omega} a^- b^+, \tag{42}$$

it can be directly verified that these charge operators satisfy the SUSY algebra given by Eqs. (31) and (32).

Representing the fermionic operators by Pauli matrices as given by Eq. (26), it follows that

$$N_f = b^+ b^- = \sigma_- \sigma_+ = \frac{1}{2}(1 - \sigma_3), \tag{43}$$

so that the Hamiltonian (12) for the SUSY harmonic oscillator takes the following form:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\sigma_3 \omega, \tag{44}$$

which resembles the one for a spin $\frac{1}{2}$ one dimensional harmonic oscillator subjected to a constant magnetic field. Explicitly,

$$\begin{aligned} H = & \begin{pmatrix} \frac{1}{2}p_x^2 + \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega & 0 \\ 0 & \frac{1}{2}p_x^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega \end{pmatrix} \\ = & \begin{pmatrix} \omega a^+ a^- & 0 \\ 0 & \omega a^- a^+ \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \end{aligned} \tag{45}$$

where, from Eqs. (26) and (42), we get

$$Q_+ = \sqrt{\omega} \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix}, \qquad Q_- = \sqrt{\omega} \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix}. \tag{46}$$

The eigenstates of N_f with the fermion number $n_f = 0$ is called bosonic states and is given by

$$\chi^- = \chi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{47}$$

Similarly, the eigenstates of N_f with the fermion number $n_f = 1$ is called fermionic state and is given by

$$\chi^+ = \chi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{48}$$

The subscripts $-(+)$ in $\chi_-(\chi_+)$ qualify their non-trivial association with $H_-(H_+)$ of H in (45). Accordingly, H_- in (45) is said to refer to the bosonic sector of the SUSY Hamiltonian H while H_+, the fermionic sector of H. (Of course this qualification is only conventional as it depends on the mapping adopted in (26) of $b^\mp$ onto $\sigma^\pm$, as the reverse mapping is easily seen to reverse the above mentioned qualification.)

A. WITTEN'S QUANTIZATION WITH SUSY

Witten's model [11] of the one dimensional SUSY quantum system is a generalization of the above construction of a SUSY simple harmonic oscillator with $\sqrt{\omega}a^- \to A^-$ and $\sqrt{\omega}a^+ \to A^+$, where

$$A^{\mp} = \frac{1}{\sqrt{2}}(\mp ip_x - W(x)) = (A^{\pm})^{\dagger}, \tag{49}$$

where, $W = W(x)$, called the superpotential, is an arbitrary function of the position coordinate. The position x and its canonically conjugate momentum $p_x = -i\frac{d}{dx}$ are related to a^- and a^+ by (34), but with $\omega_b = 1$:

$$a^{\mp} = \frac{1}{\sqrt{2}}(\mp ip_x - x) = (a^{\pm})^{\dagger}. \tag{50}$$

The mutually adjoint non-Hermitian supercharge operators for Witten's model [11,107] are given by

$$Q_+ = A^+\sigma_- = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \quad Q_- = A^-\sigma_+ \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \tag{51}$$

so that the SUSY Hamiltonian H takes the form

$$\begin{aligned} H = [Q_+, Q_-]_+ &= \frac{1}{2}\left(p_x^2 + W^2(x) - \sigma_3 \frac{d}{dx}W(x)\right) \\ &= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} \end{aligned} \tag{52}$$

where σ_3 is the Pauli diagonal matrix and, explicitly,

$$\begin{aligned} H_- = A^+A^- &= \frac{1}{2}\left(p_x^2 + W^2(x) - \frac{d}{dx}W(x)\right) \\ H_+ = A^-A^+ &= \frac{1}{2}\left(p_x^2 + W^2(x) + \frac{d}{dx}W(x)\right). \end{aligned} \tag{53}$$

In this stage we present the connection between the Dirac quantization and above SUSY Hamiltonian. Indeed, from Eq. (26) and (21), and defining

$$W(x) \equiv U'(x) \equiv \frac{dU}{dx}, \tag{54}$$

the SUSY Hamiltonian given by Eq. (52) is reobtained.

Note that for the choice of $W(x) = \omega x$ one reobtains the unidimensional SUSY oscillator (44) and (45) for which

$$A^- = a^- = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} - \omega x\right) = \psi_-^{(0)}\left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\frac{1}{\psi_-^{(0)}}$$
$$A^+ = a^+ = (A^-)^\dagger = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} - \omega x\right) = \frac{1}{\psi_-^{(0)}}\left(\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\psi_-^{(0)}, \tag{55}$$

where

$$\psi_-^{(0)} \propto exp\left(-\frac{1}{2}\omega x^2\right) \tag{56}$$

is the normalizable ground state wave function of the bosonic sector Hamiltonian H_-.

In an analogous manner, for the SUSY Hamiltonian (52), the operators $A^\pm$ of (49) can be written in the form

$$\begin{aligned} A^- &= \psi_-^{(0)}\left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\frac{1}{\psi_-^{(0)}} \\ &= \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \frac{1}{\psi_-^{(0)}}\frac{d\psi_-^{(0)}}{dx}\right) \end{aligned} \tag{57}$$

$$\begin{aligned} A^+ &= (A^-)^\dagger = \frac{1}{\psi_-^{(0)}}\left(\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\psi_-^{(0)} \\ &= \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \frac{1}{\psi_-^{(0)}}\frac{d\psi_-^{(0)}}{dx}\right), \end{aligned} \tag{58}$$

where

$$\psi_-^{(0)} \propto exp\left(-\int^x W(q)dq\right) \tag{59}$$

and

$$\psi_+^{(0)} \propto exp\left(\int^x W(q)dq\right) \Rightarrow \psi_+^{(0)} \propto \frac{1}{\psi_-^{(0)}} \tag{60}$$

are symbolically the ground states of H_- and H_+, respectively. Furthermore, we may readily write the following annihilation conditions for the operators $A^\pm$:

$$A^-\psi_-^{(0)} = 0, \qquad A^+\psi_+^{(0)} = 0. \tag{61}$$

Whatever be the functional form of $W(x)$, we have, by virtue of Eqs. (47), (48), (51), (59), (60) and (61),

$$Q_-\psi_-^{(0)}\chi_- = 0, \qquad |\phi_- >\equiv \psi_-^{(0)}\chi_- \propto exp\left(-\int^x W(q)dq\right)\begin{pmatrix}1\\0\end{pmatrix} \tag{62}$$

and

$$Q_+\psi_+^{(0)}\chi_+ = 0, \qquad |\phi_+ >\equiv \psi_+^{(0)}\chi_+ \propto \exp\left(\int^x W(q)dq\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{63}$$

so that the eigensolution $|\phi_- >$ and $|\phi_+ >$ of (62) and (63) are both annihilated by the SUSY Hamiltonian (52), with $Q_-\chi_+ = 0$ and $Q_+\chi_- = 0$, trivially holding good. If only one of these eigensolution, $|\phi_- >$ or $|\phi_+ >$, are normalizable, it then becomes the unique eigenfunction of the SUSY Hamiltonian (52) corresponding to the zero energy of the ground state. In this situation, SUSY is said to be unbroken. In the case when neither $|\phi_- >$, Eq. (62), nor $|\phi_+ >$, Eq. (63), are normalizable, then no normalizable zero energy state exists and SUSY is said to be broken. It is readily seen from (62) and (63) that if $W(x) \to \infty(-\infty)$, as $x \to \pm\infty$, then $|\phi_- >$ ($|\phi_+ >$) alone is normalizable with unbroken SUSY while for $W(x) \to -\infty$ or $+\infty$, for $x \to \pm\infty$ neither $|\phi_- >$ nor $|\phi_+ >$ are normalizable and one has broken SUSY dynamically [11,51,71,107]. In this case, there are no zero energy for the ground state and so far the spectra to $H_\pm$ are identical.

Note from the form of the SUSY Hamiltonian H of (52), that the two second-order differential equations corresponding to the eigenvalue equations of H_- and H_+ of Eq. (53), by themselves apparently unconnected, are indeed related by SUSY transformations by $Q_\pm$, Eq. (51), on H, which operations get translated in terms of the operators $A^\pm$ in $Q_\pm$ as discussed below.

1. FACTORIZATION OF THE SCHRÖRDINGER AND A SUSY HAMILTONIAN

Considering the case of unbroken SUSY and observing that the SUSY Hamiltonian (52) is invariant under $x \to -x$ and $W(x) \to -W(x)$ there is no loss of generality involved in assuming that $|\phi_- >$ of (62) is the normalizable ground state wave function of H so that $\psi_-^{(0)}$ is the ground state wave function of H_-. Thus, from (52), (53), (57) and (62), it follows that

$$H_-\psi_-^{(0)} = \frac{1}{2}\left(p_x^2 + W^2(x) - W'(x)\right)\psi_-^{(0)} = A^+A^-\psi_-^{(0)} = 0, \tag{64}$$

$$E_-^{(0)} = 0, \quad V_-(x) = \frac{1}{2}W^2(x) - \frac{1}{2}W'(x), \quad W'(x) = \frac{d}{dx}W(x). \tag{65}$$

Them from (52) and (55),

$$H_+ = A^-A^+ = A^+A^- - [A^+, A^-]_- = H_- - \frac{d^2}{dx^2}\ell n\psi_-^{(0)}, \tag{66}$$

$$V_+(x) = V_-(x) - \frac{d^2}{dx^2}\ell n\psi_-^{(0)} = \frac{1}{2}W^2(x) + \frac{1}{2}W'(x). \tag{67}$$

From (64) and (65) it is clear that any Schrödinger equation with potential $V_-(x)$, that can support at least one bound state and for which the ground state wave function $\psi_-^{(0)}$ is known, can be factorized in the form (64) with V_- duly readjusted to give $E_-^{(0)} = 0$ (Andrianov *et al.* [68] and Sukumar [69]). Given any such readjusted potential $V_-(x)$ of (65), that supports a finite number, M, of bound states, SUSY enables us to construct the SUSY partner potential $V_+(x)$ of (67). The two Hamiltonians H_- and H_+ of (52), (64) and (66) are said to be SUSY partner Hamiltonians. Their spectra and eigenfunctions are simply related because of SUSY invariance of H, i.e., $[Q_\pm, H]_- = 0$.

Denoting the eigenfunctions of H_- and H_+ respectively by $\psi_-^{(n)}$ and $\psi_+^{(n)}$, the integer $n = 0, 1, 2\ldots$ indicating the number of nodes in the wave function, we show now that H_- and H_+ possess the same energy spectrum, except that the ground state energy $E_-^{(0)}$ of V_- has no corresponding level for V_+.

Starting with

$$H_-\psi_-^{(n)} = E_-^{(n)}\psi_-^{(n)} \Longrightarrow A^+A^-\psi_-^{(n)} = E_-^{(n)}\psi_-^{(n)} \tag{68}$$

and multiplying (68) from the left by A^- we obtain

$$A^-A^+(A^-\psi_-^{(n)}) = E_-^{(n)}(A^-\psi_-^{(n)}) \Rightarrow H_+(A^-\psi_-^{(n)}) = E_-^{(n)}(A^-\psi_-^{(n)}). \tag{69}$$

Since $A^-\psi_-^{(0)} = 0$ [see Eq. (61)], comparison of (69) with

$$H_+\psi_+^{(n)} = A^-A^+\psi_+^{(n)} = E_+^{(n)}\psi_+^{(n)}, \tag{70}$$

leads to the immediate mapping:

$$E_+^{(n)} = E_+^{(n+1)}, \quad \psi_+^{(n)} \propto A^-\psi_-^{(n+1)}, \quad n = 0, 1, 2, \ldots. \tag{71}$$

Repeating the procedure but starting with (70) and multiplying the same from the left by A^+ leads to

$$A^+A^-(A^+\psi_+^{(n)}) = E_+^{(n)}(A^+\psi_+^{(n)}), \tag{72}$$

so that it follows from (68), (71) and (72) that

$$\psi_-^{(n+1)} \propto A^+\psi_+^{(n)}, \quad n = 0, 1, 2, \ldots. \tag{73}$$

The intertwining operator $A^-(A^+)$ converts an eigenfunction of $H_-(H_+)$ into an eigenfunction of $H_+(H_-)$ with the same energy and simultaneously destroys (creates) a node of $\psi_-^{(n+1)}\left(\psi_+^{(n)}\right)$. These operations just express the content of the SUSY operations effected by Q_+ and Q_- of (51) connecting the bosonic and fermionic sectors of the SUSY Hamiltonian (52).

The SUSY analysis presented above in fact enables the generation of a hierarchy of Hamiltonians with the eigenvalues and the eigenfunctions of the different members of the hierarchy in a simple manner (Sukumar [69]). Calling H_- as H_1 and H_+ as H_2, and suitably changing the subscript qualifications, we have

$$H_1 = A_1^+ A_1^- + E_1^{(0)}, \quad A_1^{(-)} = \psi_1^{(0)}\left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\frac{1}{\psi_1^{(0)}} = (A_1^+)^\dagger, \quad E_1^{(0)} = 0, \tag{74}$$

with supersymmetric partner given by

$$H_2 = A_1^- A_1^+ + E_1^{(0)}, \qquad V_2(x) = V_1(x) - \frac{d^2}{dx^2}\ell n\psi_1^{(0)}. \tag{75}$$

The spectra of H_1 and H_2 satisfy [see (71)]

$$E_2^{(n)} = E_1^{(n+1)}, \qquad n = 0, 1, 2, \ldots, \tag{76}$$

with their eigenfunctions related by [see (73)]

$$\psi_1^{(n+1)}\alpha A_1^+\psi_2^{(n)}, \qquad n = 0, 1, 2, \ldots. \tag{77}$$

Now factoring H_2 in terms of its ground state wave function $\psi_2^{(0)}$ we have

$$H_2 = -\frac{1}{2}\frac{d^2}{dx^2} + V_2(x) = A_2^+ A_2^- + E_2^{(0)}, \quad A_2^- = \psi_2^{(0)}\left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\frac{1}{\psi_2^{(0)}}, \tag{78}$$

and the SUSY partner of H_2 is given by

$$H_3 = A_2^- A_2^+ + E_2^{(0)}, \quad V_3(x) = V_2(x) - \frac{d^2}{dx^2}\ell n\psi_1^{(0)}. \tag{79}$$

The spectra of H_2 and H_3 satisfy the condition

$$E_3^{(n)} = E_2^{(n+1)}, \quad n = 0, 1, 2, \ldots, \tag{80}$$

with their eigenfunctions related by

$$\psi_2^{(n+1)}\alpha A_2^+\psi_3^{(n)}, n = 0, 1, 2, \ldots. \tag{81}$$

Repetition of the above procedure for a finite number, M, of bound states leads to the generation of a hierarchy of Hamiltonians given by

$$H_n = -\frac{1}{2}\frac{d^2}{dx^2} + V_n(x) = A_n^+ A_n^- + E_n^{(0)} = A_{n-1}^- A_{n-1}^+ + E_{n-1}^{(0)}, \tag{82}$$

where

$$\begin{aligned} A_n^- &= \psi_n^{(0)}\left(-\frac{1}{\sqrt{2}}\frac{d}{dx}\right)\frac{1}{\psi_n^{(0)}} = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} - W_n(x)\right), \\ W_n(x) &= -\frac{d}{dx}\ell n(\psi_n^{(0)}), \quad A_n^+ = \left(A_n^-\right)^\dagger, \end{aligned} \tag{83}$$

and

$$\begin{aligned} V_n(x) &= V_{n-1}(x) - \frac{d^2}{dx^2}\ell n(\psi_{n-1}^{(0)}) \\ &= V_1(x) - \frac{d^2}{dx^2}\ell n(\psi_1^{(0)}\psi_2^{(0)}\ldots\psi_{n-1}^{(0)}), \quad n = 2, 3, \ldots, M, \end{aligned} \tag{84}$$

whose spectra satisfy the conditions

$$E_1^{n-1} = E_2^{n-2} = \ldots = E_n^{(0)}, \qquad n = 2, 3, \ldots, M, \tag{85}$$

$$\psi_1^{n-1} \propto A_1^+ A_2^+ \ldots A_{n-1}^+ \psi_n^{(0)}. \tag{86}$$

Note that the nth-member of the hierarchy has the same eigenvalue spectrum as the first member H_1 except for the missing of the first $(n-1)$ eigenvalues of H_1. The energy eigenvalue of the (n-1)th-excited state of H_1 is degenerate with the ground state of H_n and can be constructed with the use of (86) that involves the knowledge of $A_i (i = 1, 2, \ldots, n-1)$ and $\psi_n^{(0)}$.

2. *SUSY METHOD AND SHAPE-INVARIANT POTENTIALS*

It is particularly simple to apply (86) for shape-invariant potentials (Gendenshtein [52], Cooper *et al.* [76], Dutt *et al.* [77] and in review article [107]) as their SUSY partners are similar in shape and differ only in the parameters that appear in them. More specifically, if $V_-(x; a_1)$ is any potential, adjusted to have zero ground state energy $E_-^{(0)} = 0$, its SUSY partner $V_+(x; a_1)$ must satisfy the requirement

$$V_+(x; a_1) = V_-(x; a_2) + R(a_2), \qquad a_2 = f(a_1), \tag{87}$$

where a_1 is a set of parameters, a_2 a function of the parameters a_1 and $R(a_2)$ is a remainder independent of x. Then, starting with $V_1 = V_-(x; a_2)$ and $V_2 = V_+(x; a_1) = V_1(x; a_2) + R(a_2)$ in (87), one constructs a hierarchy of Hamiltonians

$$H_n = -\frac{1}{2}\frac{d^2}{dx^2} + V_-(x; a_n) + \Sigma_{s=2}^{n} R(a_s), \tag{88}$$

where $a_s = f^s(a_1)$, i.e., the function f applied s times. In view of Eqs. (87) and (88), we have

$$H_{n+1} = -\frac{1}{2}\frac{d^2}{dx^2} + V_-(x; a_{n+1}) + \Sigma_{s=2}^{n+1} R(a_s) \tag{89}$$

$$= -\frac{1}{2}\frac{d^2}{dx^2} + V_+(x; a_n) + \Sigma_{s=2}^{n} R(a_s). \tag{90}$$

Comparing (88), (89) and (90), we immediately note that H_n and H_{n+1} are SUSY partner Hamiltonians with identical energy spectra except for the ground state level

$$E_n^{(0)} = \Sigma_{s=2}^{n} R(a_s) \tag{91}$$

of H_n, which follows from Eq. (88) and the normalization that for any $V_-(x; a)$, $E_-^{(0)} = 0$. Thus Eqs. (85) and (86) get translate simply, letting $n \to n + 1$, to

$$E_1^n = E_2^{n-1} = \ldots = E_{n+1}^{(0)} = \sum_{s=2}^{n+1} R(a_s), \quad n = 1, 2, \ldots \tag{92}$$

and

$$\psi_1^{(n)} \propto A_1^+(x; a_1) A_2^+(x; a_2) \ldots A_n^+(x; a_n) \psi_{n+1}^{(0)}(x; a_{n+1}). \tag{93}$$

Equations (92) and (93), succinctly express the SUSY algebraic generalization, for various shape-invariant potentials of physical interest [52,69,77], of the method of constructing energy eigenfunctions ($\psi_{osc}^{(n)}$) for the usual ID oscillator problem. Indeed, when $a_1 = a_2 = \ldots = a_n = a_{n+1}$, we obtain $\psi_{osc}^{(n)} \propto (a^+)^n \psi_1^{(0)}$, $A_n^+ = a^+$, $\psi_{osc}^{(0)} = \psi_{n+1}^{(0)} = \psi_1^{(0)} \propto e^{-\frac{\omega x^2}{2}}$, where ω is the angular frequency.

The shape invariance has an underlying algebraic structure and may be associated with Lie algebra [168]. In next Section of this work, we present our own application of the Sukumar's SUSY method outlined above for the first Pöschl-Teller potential with unquantized coupling constants, while in the earlier SUSY algebraic treatment by Sukumar [69] only the restricted symmetric case of this potential with quantized coupling constants was considered.

IV. THE FIRST PÖSCHL-TELLER POTENTIAL VIA SUSY QM

We would like to stress the interesting approaches for the Pöschl-Teller I potential. Utilizing the SUSY connection between the particle in a box with perfectly rigid walls and the symmetric first Pöschl-Teller potential, the SUSY hierarchical prescription (outlined in Section III) was utilized by Sukumar [69] to solve the energy spectrum of this potential. The unsymmetric case has recently been treated algebraically by Barut, Inomata and Wilson [78]. However in these analysis only quantized values of the coupling constants of the Pöschl-Teller potential have been obtained. In the works of Gendenshtein [52] and Dutt *et al.* [77] treating the unsymmetric case of this potential with unquantized coupling constants by the SUSY method for shape-invariant potentials, only the energy spectrum was obtained but not the excited state wave functions. Below we present our own application of the Sukumar's SUSY method obtaining not only the energy spectrum but also the complete excited state energy eigenfunctions.

It is well known that usual shape invariance procedure [52] is not applicable for computation energy spectrum of a potential without zero energy eigenvalue. Recently, an approach was implemented with a two-step shape invariant in order to connect broken and unbroken SUSY QM potentials [99,159]. In this references it is considered the Pöschl-Teller I potential, showing the types of shape invariance it possesses. In Ref. [159], the PTPI and the three-dimensional harmonic oscillator both with broken SUSY have been investigated, for the first time, in terms of a novel two-step shape invariance approach via a group theoretic potential algebra approach [168]. In the work present is the first spectral resolution, to our knowledge, via SUSY hierarchy in order to construct explicitly the energy eigenvalue and eigenfunctions of the Pöschl-Teller I potential.

Starting with the first Pöschl-Teller Hamiltonian [169]

$$H_{PT} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\alpha^2\left\{\frac{k(k-1)}{sin^2\alpha x} + \frac{\lambda(\lambda-1)}{cos^2\alpha x}\right\}, \tag{94}$$

where $0 \leq \alpha x \leq \pi/2, k > 1, \lambda > 1; \alpha =$ real constant. The substitution $\Theta = 2\alpha x, 0 \leq \Theta \leq \pi$, in (94) leads to

$$H_{PT} = 2\alpha^2 H_1 \tag{95}$$

where

$$\begin{aligned} H_1 &= -\frac{d^2}{d\Theta^2} + V_1(\Theta) \\ V_1(\Theta) &= \frac{1}{4}\left[k(k-1)sec^2(\Theta/2) + \lambda(\lambda-1)cossec^2(\Theta/2)\right]. \end{aligned} \tag{96}$$

Defining

$$A_1^{\pm} = \pm\frac{d}{d\Theta} - W_1(\Theta) \tag{97}$$

and

$$\begin{aligned} H_1 &= A_1^+ A_1^- + E_1^{(0)} \\ &= -\frac{d^2}{d\Theta^2} + W_1^2(\Theta) - W_1'(\Theta) + E_1^{(0)} \end{aligned} \tag{98}$$

where the prime means a first derivative with respect to Θ variable. From both above definitions of H_1 we obtain the following non-linear first order differential equation

$$W_1^2(\theta) - W_1'(\Theta) = \frac{1}{4}\left\{\frac{k(k-1)}{sin^2(\Theta/2)} + \frac{\lambda(\lambda-1)}{cos^2(\Theta/2)}\right\} - E_1^{(0)}, \tag{99}$$

which is exactly a Riccati equation.

Let be superpotential Ansatz

$$W_1(\Theta) = -\frac{k}{2}cot(\Theta/2) + \frac{\lambda}{2}tan(\Theta/2), \quad E_1^{(0)} = \frac{1}{4}(k+\lambda)^2. \tag{100}$$

According to Sec. III, the energy eigenfunction associated to the ground state of PTI potential becomes

$$\psi_1^{(0)} = exp\left\{-\int W_1(\theta)d\Theta\right\} \propto sin^k(\Theta/2)cos^\lambda(\Theta/2). \tag{101}$$

In this case the first order intertwining operators become

$$A_1^- = -\frac{d}{d\Theta} + \frac{k}{2}cot(\Theta/2) - \frac{\lambda}{2}tan(\Theta/2) = \psi_1^{(0)}\left(-\frac{d}{d\Theta}\right)\frac{1}{\psi_1^{(0)}} \tag{102}$$

and

$$\begin{aligned} A_1^+ &= (A_1^-)^\dagger = \frac{1}{\psi_1^{(0)}}\left(\frac{d}{d\Theta}\right)\psi_1^{(0)} \\ &= \frac{d}{d\Theta} + \frac{k}{2}cot(\Theta/2) - \frac{\lambda}{2}tan(\Theta/2). \end{aligned} \tag{103}$$

In Eqs. (102)) and (103), $\psi_1^{(0)}$ is the ground state wave function of H_1.

The SUSY partner of H_1 is H_2, given by

$$\begin{aligned} H_2 &= A_1^- A_1^+ + E_1^{(0)} = H_1 - [A_1^+, A_1^-]_- \\ V_2(\Theta) &= V_1(\Theta) - 2\frac{d^2}{d\Theta^2}\ell n\psi_1^{(0)} \\ &= V_1(\Theta) - 2\frac{d^2}{d\Theta^2}\ell n\left[sin^k(\Theta/2)cos^\lambda(\Theta/2)\right] \\ &= \frac{1}{4}\left(\frac{k(k+1)}{sin^2(\Theta/2)} + \frac{\lambda(\lambda+1)}{cos^2(\Theta/2)}\right). \end{aligned} \tag{104}$$

Let us now consider a refactorization of H_2 in its ground state

$$H_2 = A_2^+ A_2^- + E_2^{(0)}, \quad A_2^- = -\frac{d}{d\Theta} - W_2(\Theta). \tag{105}$$

In this case we find the following Riccati equation

$$W_2^2(\theta) - W_2'(\Theta) = \frac{1}{4}\left\{\frac{k(k+1)}{sin^2(\Theta/2)} + \frac{\lambda(\lambda+1)}{cos^2(\Theta/2)}\right\} - E_2^{(0)}, \tag{106}$$

which provides a new superpotential and the ground state energy of H_2

$$W_2(\Theta) = -\frac{(k+2)}{2}cot(\Theta/2) + \frac{(\lambda+2)}{2}tan(\Theta/2), \quad E_2^{(0)} = \frac{1}{4}(k+\lambda+2)^2. \tag{107}$$

Thus the eigenfunction associated to the ground state of H_2 is given by

$$\psi_2^{(0)} = exp\left\{-\int W_2(\Theta)d\Theta\right\} \propto sin^{k+1}(\Theta/2)cos^{\lambda+1}(\Theta/2). \tag{108}$$

Hence in analogy with (102) and (103) the new intertwining operators are given by

$$\begin{aligned} A_2^\pm &= \pm\frac{d}{d\Theta} - W_2(\Theta) \\ &= \pm\frac{d}{d\Theta} + \frac{(k+1)}{2}cot(\Theta/2) - \frac{(\lambda+1)}{2}tan(\Theta/2) \\ A_2^- &= \psi_2^{(0)}\left(-\frac{d}{d\Theta}\right)\frac{1}{\psi_2^{(0)}}, \quad A_2^-\psi_2^{(0)} = 0. \end{aligned} \tag{109}$$

Note that the $V_2(\Theta)$ partner potential has a symmetry, viz.,

$$V_2(\Theta) = \frac{1}{4}\left(\frac{k(k+1)}{sin^2(\Theta/2)} + \frac{\lambda(\lambda+1)}{cos^2(\Theta/2)}\right) = V_1(k \to k+1, \lambda \to \lambda+1) \tag{110}$$

which is leads to the shape-invariance property (outlined in subsection III.2) for the first unbroken SUSY potential pair

$$\begin{aligned} V_{1-} &= \frac{1}{4}\left(\frac{k(k-1)}{sin^2(\Theta/2)} + \frac{\lambda(\lambda-1)}{cos^2(\Theta/2)}\right) - \frac{1}{4}(k+\lambda)^2 \\ V_{1+} &= \frac{1}{4}\left(\frac{k(k+1)}{sin^2(\Theta/2)} + \frac{\lambda(\lambda+1)}{cos^2(\Theta/2)}\right) - \frac{1}{4}(k+\lambda)^2 \\ &= V_{1-}(k \to k+1, \lambda \to \lambda+1) + (\lambda+k+1). \end{aligned} \tag{111}$$

In this case, one can obtain energy eigenvalues and eigenfunctions by means of the shape-invariance condition. However, we have derived the excited state algebraically, by exploiting the Sukumar's method for the construction of SUSY hierarchy [69]. Furthermore, note that $\psi_{1-}^{(0)} = \psi_1^{(0)}$ is normalizable with zero energy for the ground state of bosonic sector

Hamiltonian $H_{1-} = H_1 - E_1^{(0)}$ and the energy eigenvalue for the ground state of fermionic sector Hamiltonian $H_{1+} = H_2 - E_1^{(0)}$ is exactly the first excited state of H_{1-}, but the eigenfunction $\frac{1}{\psi_1^{(0)}}$ is not the ground state of H_{1+}, for $k > 0$ and $\lambda > 0$.

Let us again consider the Sukumar's method in order to find the partner potential of $V_2(\Theta)$ is

$$\begin{aligned} V_3(\Theta) &= V_2(\Theta) - 2\frac{d^2}{d\Theta^2}\ell n\psi_2^{(0)} = V_2(\Theta) + \frac{1}{2}\left(\frac{(2k+1)}{sin^2(\Theta/2)} + \frac{(2\lambda+1)}{cos^2(\Theta/2)}\right) \\ &= \frac{1}{4}\left(\frac{(k+2)(k+1)}{sin^2(\Theta/2)} + \frac{(\lambda+2)(\lambda+1)}{cos^2(\Theta/2)}\right) = V_1(k\rightarrow k+2, \lambda\rightarrow\lambda+2). \end{aligned} \tag{112}$$

Now one is able to implement the generalization for nth-member of the hierarchy, i.e. the general potential may be written for all integer values of n, viz.,

$$\begin{aligned} V_n(\Theta) &= V_1(\Theta) + \frac{1}{4}(n-1)\left\{\frac{2k+n-2}{sin^2(\Theta/2)} + \frac{2\lambda+n-2}{cos^2(\Theta/2)}\right\} \\ &= \frac{k^2 - k + k(n-1) + (n-1)(n-2)}{4sin^2(\Theta/2)} \\ &+ \frac{\lambda^2 - \lambda + \lambda(n-1) + (n-1)(n-2)}{4cos^2(\Theta/2)} \\ &= \frac{1}{4}\left\{\frac{(k+n-1)(k+n-2)}{sin^2(\Theta/2)} + \frac{(\lambda+n-1)(\lambda+n-2)}{cos^2(\Theta/2)}\right\}. \end{aligned} \tag{113}$$

Note that $V_n(\Theta) = V_1(\Theta; k\rightarrow k+n-1, \lambda\rightarrow\lambda+n-1)$ so that the (n+1)th-member of the hierarchy is given by

$$H_{n+1} = A^+_{n+1}A^-_{n+1} + E^{(0)}_{n+1}, \quad E^{(0)}_{n+1} = \frac{1}{4}(k+\lambda+2n)^2 \tag{114}$$

where

$$\begin{aligned} A^-_{n+1} &= \psi^{(0)}_{n+1}\left(-\frac{d}{d\Theta}\right)\frac{1}{\psi^{(0)}_{n+1}} = \left(A^+_{n+1}\right)^\dagger \\ \psi^{(0)}_{n+1} &\propto sin^{k+n}(\Theta/2)cos^{\lambda+n}(\Theta/2). \end{aligned} \tag{115}$$

Applying the SUSY hierarchy method (92), one gets the nth-excited state of H_1 from the ground state of H_{n+1}, as given by

$$\begin{aligned} \psi_1^{(n)} &\propto A_1^+A_2^+\ldots A_n^+ sin^{k+n}(\Theta/2)cos^{\lambda+n}(\Theta/2) \\ &= \prod_{s=0}^{n-1}\left[\frac{1}{sin^{k+s}(\Theta/2)cos^{\lambda+s}(\Theta/2)}\left(\frac{d}{d\Theta}\right)\right. \\ &\left. sin^{k+s}(\Theta/2)cos^{\lambda+s}(\Theta/2)\right] sin^{k+n}(\Theta/2)cos^{\lambda+n}(\Theta/2) \end{aligned}$$

$$\propto \frac{1}{sen^{k-1}(\Theta/2)cos^{\lambda-1}(\Theta/2)}\left(\frac{1}{sin(\Theta/2)}\frac{d}{d\Theta}\right)^{n} sin^{2(k+n)-1}(\Theta/2)cos^{2(\lambda+n)-1}(\Theta/2)$$

$$\propto (1-u)^{\frac{k}{2}}(1+u)^{\frac{\lambda}{2}}\left[(1-u)^{-k+\frac{1}{2}}(1+u)^{-\lambda+\frac{1}{2}}\left(\frac{d^n}{du^n}\right)(1-u)^{k-\frac{1}{2}+n}(1+u)^{\lambda-\frac{1}{2}+n}\right], \quad (116)$$

where ($u = cos\Theta$), so that the ground state of nth-member of hierarchy is given by

$$\psi_n^{(0)} \propto sin^{k+n-1}(\Theta/2)cos^{\lambda+n-1}(\Theta/2) \quad (117)$$

and the nth-excited state of PTI potential become

$$\psi_1^{(n)}(\Theta) \propto sin^{k}(\Theta/2)cos^{\lambda}(\Theta/2)F\left(-n, n+k+\lambda; k+\frac{1}{2}; sin^2(\Theta/2)\right), \quad (118)$$

which follows on identification of the square bracketed quantity in (116) with the Jacobi polynomials (Gradshteyn and Ryzhik [170])

$$J_n^{\left(k-\frac{1}{2},\lambda-\frac{1}{2}\right)} \propto F\left(-n, n+k+\lambda; k+\frac{1}{2}; \frac{1-u}{2}\right).$$

Here F are known as the confluent hypergeometric functions which clearly they are in the region of convergency and defined by [70]

$$F(a,b;c;x) = 1 + \frac{ab}{c}x + \frac{a(a+1)(b(b+1)}{1.2.c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.c(c+1)(c+2)}x^3 + \cdots$$

and its derivative with respect to x becomes

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x).$$

The excited state eigenfunctions (118) here obtained by the SUSY algebraic method agree with those given in Flügge [169] using non-algebraic method. Note that the coupling constants k and λ in above analysis are unquantized. Besides from Sec. III, Eq. (114) and Eq. (95) we readily find the following energy eigenvalues for the PTI potential

$$E_1^{(n)} = E_2^{(n-1)} = \cdots = 2\alpha^2 E_{n+1}^{(0)} = \frac{\alpha^2}{4}(k+\lambda+2n)^2,$$

$$E_{PT}^{(n)} = 2\alpha^2 E_1^{(n)} = \frac{\alpha^2}{4}(k+\lambda+2n)^2, \quad n = 0,1,2,\cdots. \quad (119)$$

Let us now point out the existence of various possibilities to supersymmetrize the PTI Hamiltonian with broken SUSY, for a finite interval $[0,\pi]$. with unbroken and broken SUSY. Indeed both ground states do not have zero energy, so that when k and λ are in a particular interval one can have a broken SUSY, because there such eigenstates are also not normalizable. In these cases, the shape invariant procedure is not valid but the Sukumar's SUSY hierarchy procedure [69] can be applied. Therefore, we see that other combinations of the k and λ parameters are also possible to provide distinct superpotentials.

V. NEW SCENARIO OF SUSY QM

In this Section we apply the SUSY QM for a neutron in interaction with a static magnetic field of a straight current carrying wire. This system is described by two-component wave functions, so that the development considered so far for SUSY QM must be adapted.

The essential reason for the necessity of modification is due to the Riccati equation may be reduced to a set of first-order coupled differential equations. In this case the superpotential is not defined as $W(x) = -\frac{d}{dx}\ell n\left(\psi_0(x)\right)$, where $\psi_0(x)$ is the two-component eigenfunction of the ground state. Only in the case of 1-component wave functions one may write the superpotential in this form. Recently two superpotentials, energy eigenvalue and the two-component eigenfunction of the ground state have been found [108,167].

In this Subection we investigate a symmetry between the supersymmetric Hamiltonian pair $\mathbf{H}_\pm$ for a neutron in an external magnetic field. After some transformations on the original problem which corresponds to a one-dimensional Schrödinger-like equation associated with the two-component wavefunctions in cylindrical coordinates, satisfying the following eigenvalue equation

$$\mathbf{H}_\pm \mathbf{\Phi}_\pm^{(n_\rho,m)} = E_\pm^{(n_\rho,m)} \mathbf{\Phi}_\pm^{(n_\rho,m)}, \quad n_\rho = 0, 1, 2, \cdots, \tag{120}$$

where n_ρ is the radial quantum number and m is the orbital angular momentum eigenvalue in the z-direction. The two-component energy eigenfunctions are given by

$$\mathbf{\Phi}_\pm^{(n_\rho,m)} = \mathbf{\Phi}_\pm^{(n_\rho,m)}(\rho,k) = \begin{pmatrix} \phi_{1\pm}^{(n_\rho,m)}(\rho,k) \\ \phi_{2\pm}^{(n_\rho,m)}(\rho,k) \end{pmatrix} \tag{121}$$

and the supersymmetric Hamiltonian pair

$$\begin{aligned} \mathbf{H}_- &\equiv \mathbf{A}^+\mathbf{A}^- = -\mathbf{I}\frac{d^2}{d\rho^2} + \begin{pmatrix} \frac{m^2-\frac{1}{4}}{\rho^2} + \frac{1}{8(m+1)^2} & \frac{1}{\rho} \\ \frac{1}{\rho} & \frac{(m+1)^2-\frac{1}{4}}{\rho^2} + \frac{1}{8(m+1)^2} \end{pmatrix} \\ \mathbf{H}_+ &\equiv \mathbf{A}^-\mathbf{A}^+ = \mathbf{H}_- - [\mathbf{A}^+, \mathbf{A}^-]_- \\ &= -\mathbf{I}\frac{d^2}{d\rho^2} + \mathbf{V}_+, \end{aligned} \tag{122}$$

where $\mathbf{I}$ is the 2x2 unit matrix and

$$\mathbf{A}^\pm = \pm\frac{d}{d\rho} + \mathbf{W}(\rho). \tag{123}$$

In this Section we are using the notation of Ref. [167]. Thus in this case the Riccati equation in matrix form, becomes

$$\mathbf{W}'(\rho) + \mathbf{W}^2(\rho) = \frac{(m+\frac{1}{2})(m+\frac{1}{2}-\sigma_3)}{\rho^2} + \frac{\sigma_1}{\rho} + \frac{\mathbf{I}}{8(m+1)^2}, \tag{124}$$

where the two-by-two hermitian superpotential matrix recently calculated in [167], which is given by

$$\mathbf{W}(\rho;m) = \mathbf{W}^\dagger = (m+\frac{1}{2})(\mathbf{I}+\sigma_3)\frac{1}{2\rho} + (m+\frac{3}{2})(\mathbf{I}-\sigma_3)\frac{1}{2\rho} + \frac{\sigma_1}{2m+2}, \tag{125}$$

where σ_1, σ_3 are the well known Pauli matrices.

The hermiticity condition on the superpotential matrix allows us to construct the following supersymmetric potential partner

$$\begin{aligned}\mathbf{V}_+(\rho;m) &= \mathbf{V}_- - 2\mathbf{W}'(\rho)\\ &= \mathbf{W}^2(\rho) - \mathbf{W}'(\rho)\\ &= \begin{pmatrix} \frac{(m+\frac{1}{2})(m+\frac{3}{2})}{\rho^2} & \frac{1}{\rho} \\ \frac{1}{\rho} & \frac{(m+\frac{1}{2})(m+\frac{7}{2})+2}{\rho^2} \end{pmatrix} + \frac{\mathbf{I}}{8(m+1)^2}.\end{aligned} \tag{126}$$

Note that in this case we have unbroken SUSY because the ground state has zero energy, viz., $E_-(0) = 0$, with the annihilation conditions

$$A^-\Phi_-^{(0)} = 0 \tag{127}$$

and

$$A^+\Phi_+^{(0)} = 0. \tag{128}$$

Due to the fact these eigenfunctions to be of two components one is not able to write the superpotential in terms of them in a similar way of that one-component eigenfunction belonging to the respective ground state.

Furthermore, we have a symmetry between $\mathbf{V}_\pm(\rho;m)$. Indeed, it is easy to see that

$$\begin{aligned}\mathbf{V}_+(\rho;m) &= \frac{(m+1)^2-\frac{1}{4}}{\rho^2}\mathbf{I} + (2m+3)\frac{(\mathbf{I}-\sigma_3)}{2\rho^2} + \frac{\sigma_1}{\rho} + \frac{\mathbf{I}}{8(m+1)^2}\\ &= \mathbf{V}_-(\rho; m \to m+1) + \mathbf{R}_m,\end{aligned} \tag{129}$$

where $\mathbf{R}_m = -\frac{\mathbf{I}}{8}(2m+3)(m+1)^{-2}(m+2)^{-2}$. Therefore, we can find the energy eigenvalue and eigenfunction of the ground state of $\mathbf{H}_+$ from those of $\mathbf{H}_-$ and the resolution spectral of the hierarchy of matrix Hamiltonians can be achieved in an elegant way via the shape invariance procedure.

VI. CONCLUSIONS

We start considering the Lagrangian formalism for the construction of one dimension supersymmetric quantum mechanics with N=2 SUSY in a non-relativistic context, viz., two Grassmann variables in classical mechanics and the Dirac canonical quantization method was considered.

This paper also relies on known connections between the theory of Darboux operators [23] in factorizable essentially isospectral partner Hamiltonians (often called as supersymmetry in quantum mechanics "SUSY QM"). The structure of the Lie superalgebra, that incorporates commutation and anticommutation relations in fact characterizes a new type of a dynamical symmetry which is SUSY, i.e., a symmetry that converts bosonic state into fermionic state and vice-versa with the Hamiltonian, one of the generators of this superalgebra, remaining invariant under such transformations [5,11]. This aspect of it as well reflects in its tremendous physical content in Quantum Mechanics as it connects different quantum systems which are otherwise seemingly unrelated.

A general review on the SUSY algebra in quantum mechanics and the procedure on like to build a SUSY Hamiltonian hierarchy in order of a complete spectral resolution it was explicitly applied for the Pöschl-Teller potential I. We will now do a more detail discussion for the case of this problem presents broken SUSY.

It is well known that usual shape invariant procedure [52,200] is not applicable for computation energy spectrum of a potential without zero energy eigenvalue. Recently, the approach implemented with a two-step shape invariance in order to connect broken and unbroken [99] is considered in connections [159]. In these references it is considered the Pöschl-Teller potential I (PTPI), showing the types of shape invariance it possesses. In this work we consider superpotential continuous and differentiable that provided us the PTPI SUSY partner with the nonzero energy eigenvalue for the ground state, a broken SUSY system, or containing a zero energy for the ground state with unbroken SUSY. We have presented our own application of the SUSY hierarchy method, which can also be applied for broken SUSY [69]. The potential algebras for shape invariance potentials have been considered in the references [159,168].

We have also applied the SUSY QM formalism for a neutron in interaction with a static magnetic field of a straight current carrying wire, which is described by two-component wave functions, and presented a new scenario in the coordinate representation. Parts of such an application have also been considered in [108,167].

Furthermore, we stress that defining $k = -2(m + \frac{1}{2})$ and $\lambda = 2(m + \frac{1}{2})$, where m is angular moment along z axis, in the PTPI it is possible to obtain the energy eigenvalue and eigenfunctions for such a planar physical system as an example of the 2-dimensional supersymmetic problem in the momentum representation [96]. We see from distinct superpotentials may be considered distinct supersymmetrizations of the PTPI potential [160].

In this article some applications of SUSY QM were not commented. As examples, the connection between SUSY and the variational and the WKB methods. In [94,107] the reader can find various references about useful SUSY QM and in improving the old WKB and variational method. However, the WKB approximations provide us good results for higher states than for lower ones. Hence if we apply the WKB method in order to calculate the ground state one obtain a very poor approximation. The $N = 2$ SUSY algebra and many applications including its connection with the variational method and supersymmetric WKB have been recently studied in the literature [171,172]. Indeed, have been suggested that supersymmetric WKB method may be useful in studying the deviation of the energy levels of a quantum system due to the presence of spherically confining boundary [172]. There they have observed that the confining geometry removes the angular-momentum degeneracy of the electronic energy levels of a free atom. Khare has investigated the supersymmetric WKB quantization approximation [173], and Khare-Yarshi have studied the bound state spectrum of two classes of exactly solvable non-shape-invariant potentials in the SWKB approximation and shown that it is not exact [174]. A method to obtain wave function in a uniform semiclassical approximation to SUSY QM has been applied for the Morse, Rosen-Morse, and anharmonic oscillator potentials [175]. Inomata *et al.* have applied the WKB quantization rule for the isotropic harmonic oscillator in three dimensions, quadratic potential and the PTPI [176].

In literature, the SUSY algebra has also been applied to invetigate a variety of one-parameter families of isospectral SUSY partner potentials [21,22,177] in non-relativistic quantum mechanics which are phase-equivalent [178]. By phase-equivalent potentials it is understood that the potentials relate all Hamiltonians which have the same phase shifts and essentially the same bound-state spectrum. Lévai-Baye-Sparenberg have obtained potentials which are phase-equivalent with the generalised Ginochio potential [179]. Nag-Roychoudhury show that the repeated application of Darboux's theorem [23] for an isospectral Hamiltonian provides a new potential which can be phase equivalent. However, such a similar procedure is inequivalent to the usual approach on Darboux' theorem [180].

Another important approach is the connection between SUSY QM and the Dirac equation, so that many authors have considered in their works. For example, Ui [181] has shown that a Dirac particle coupled to a Gauge field in three spacetime dimensions possesses a SUSY analogous to Witten's model [11] and Gamboa-Zanelli [182] have discussed the extension to include non-Abelian Gauge fields, based on the ground-state wavefunction representation for SUSY QM. The SUSY QM has also been applied for the Dirac equation of the electron in an attractive central Coulomb field by Sukumar [183], to a massless Dirac particle in a magnetic field by Huges-Kostelecty-Nieto [184], and to second-order relativistic equations, based on the algebra of SUSY by Jarvis-Stedman [185] and for a neutral particle with an anomalous magnetic moment in a central electrostatic field by Fred *et al.* [186] and Semenov [187]. Beckers *et al.* have shown that $2n$ fermionic variables of the spin-orbit coupling procedure may generate a grading leading to a unitary Lie superalgebra [188]. Using the intertwining of exactly solvable Dirac equations with one-dimensional potentials, Anderson has shown that a class of exactly solvable potentials corresponds to solitons of the modified Korteweg de Vries equation [189]. Njock *et al.* [190] have investigated the Dirac equation in the central approximation with the Coulomb potential, so that they have derived the SUSY-based quantum defect wave functions from an effective Dirac equation for a valence that is solvable in the limit of exact quantum-defect theory. Dahl-Jorgensen have investigated the relativistic Kepler prblem with emphasis on SUSY QM via Jonson-Lippman operator [191]. The energy eigenvalues of a Dirac electron in a uniform magnetic field has been analyzed via SUSY QM by Lee [192]. The relation between superconductivity and Dirac SUSY has been generalized to a multicomponent fermionic system by Moreno *et al.* [193].

An interesting quantum system is the so-called Dirac oscillator, first introduced by Moshinsky-Szczepaniak [194]; its spectral resolution has been investigated with the help of techniques of SUSY QM [195]. The Dirac oscillator with a generalized interaction has been treated by Castaños *et al.* [196]. In another work, Dixit *et al.* [197] have considered the Dirac oscillator with a scalar coupling whose non-relativistic limit leads to a harmonic oscillator Hamiltonian plus a $\vec{S}\cdot\vec{r}$ coupling term. The wave equation is not invariant under parity. They have worked out a parity-invariant Dirac oscillator with scalar coupling by doubling the number of components of the wave function and using the Clifford algebra $C\ell_7$. These works motivate the construction of a new linear Hamiltonian in terms of the momentum, position and mass coordinates, through a set of seven mutually anticommuting 8x8-matrices yielding a representation of the Clifford algebra $C\ell_7$. The seven elements of the Clifford algebra $C\ell_7$ generate the three linear momentum components, the three position coordinates components and the mass, and their squares are the 8x8-identity matrix $\mathbf{I}_{8X8}$.

Recently, the Dirac oscillator have been approached in terms of a system of two particles [198] and to Dirac-Mörsen problem [199] and relativistic extensions of shape invariant potential classes [200].

Results of our analysis on the SUSY QM and Dirac equation for a linear potential [199,201] and for the Dirac oscillator via R-deformed Heisenberg algebra [95,147], and the new Dirac oscillator via Clifford algebra $C\ell_7$ are in preparation.

Crater and Alstine have applied the constraint formalism for two-body Dirac equations in the case of general covariant interactions [202,203], which has its origin in the work of Galvão-Teitelboim [12]. This issue and the discussion on the role SUSY plays to justify the origin of the constraint have recently been reviewed by Crater and Alstine [204].

In [205], Robnit purposes to generate the superpotential in terms of arbitrary higher excited eigenstates, but there the whole formalism of the 1-component SUSY QM is needed of an accurate analysis due to the nodes from some excited eigenstates. Results of such investigations will be reported separately.

Now let us point out various other interesting applications of the superymmetric quantum mechanics, for example, the extension dynamical algebra of the n-dimensional harmonic oscillator with one second-order parafermionic degree of freedom by Durand-Vinet [206]. Indeed, these authors have shown that the parasupersymmetry in non-relativistic quantum mechanics generalize the standard SUSY transformations [207]. The parasupersymmetry has also been analyzed in the following references [208–213].

Other applications of SUSY QM may be found in [214–255]. All realizations of SUSY QM in these works is based in the Witten's model [11]. However, another approach on the SUSY has been implemented in classical and quantum mechanics. Indeed, a N=4 SUSY representation in terms of three bosonic and four fermionic variables transforming as a vector and complex spinor of rotation group O(3) has been proposed, based on the supercoordinate construction of the action [256]. However, the superfield SUSY QM with 3 bosonic and 4 fermionic fields was first described by Ivanov-Smilga [257].

It is well known that N=4 SUSY is the largest number of extended SUSY for which a superfield (supercoordinate) formalism is known. However, using components fields and computations the $N > 4$ classification of N-extended SUSY QM models have been implemented via irreducible multiplets of their representation by Gates *et al.* [258]. Recently, Pashnev-Toppan have also shown that all irreducible multiplets of representation of N extended SUSY are associated to fundamental short multiplets in which all bosons and all fermions are accommodated into just two spin states [259].

$N = 4$ supersymmetric quantum mechanics many-body systems in terms of Calogero models and $N = 4$ superfield formulations have been investigated by Wyllard [260]. This Ref. and the N=4 superfield formalism used there are actually based on the paper [261].

SUSY N=4 in terms of the dynamics of a spinning particle in a curved background has also been described using the superfield formalism [262]. There are a few more of works where SUSY QM in higher dimensions is investigated [263,265,266]. However, the Ref. [263] is a further extension of the results obtained earlier in the basic paper [264].

The paper of Claudson-Halpern, [267], was the first to give the $N = 4$ and $N = 16$ SUSY gauge quantum mechanics, the latter now called M(atrix) theory [268].

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