Division Algebras and Physics

Francesco Toppan

CBPF - CCP
Rua Dr. Xavier Sigaud 150, cep 22290-180
Rio de Janeiro (RJ), Brazil

Abstract

A quick and condensed review of the basic properties of the division algebras is presented. Some applications to physics and to physically motivated mathematical problems are discussed.

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E-mail: toppan@cbpf.br

Introduction: what is a division algebra?

The most familiar division algebras are $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$ (i.e. the algebra of the quaternions, spanned by the identity and the three Pauli matrices).

They all admit the unit element and an anti-involution (the conjugation) s.t.

$$(a^*)^* = a, \qquad\qquad (ab)^* = b^* a^*$$

(the conjugation corresponds to the transposition in the matrix representation).

Definition of a division algebra over $\mathbf{R}$ (see **[Por]**).

It is a finite-dimensional real linear space X with a bilinear product $X^2 \rightarrow X$ s.t.

$$ab = 0 \qquad iff \qquad a = 0 \vee b = 0.$$

A division algebra is normed if $\exists N : X \rightarrow \mathbf{R}^+$ s.t.

$$N : a \mapsto a^* a \in \mathbf{R}^+, \qquad N(ab) = N(a)N(b)$$

N is the norm, while $\mathbf{R}^+$ denotes the non-negative real numbers.

A division algebra is said alternative if $\forall a, b \in X$ both properties below hold

$$a(ab) = a^2 b, \qquad (ab)b = ab^2.$$

Comment: the alternativity is a weakening of the notion of associativity.

The Cayley algebra, also called algebra of octaves or octonions and denoted by $\mathbf{O}$, is an alternative division algebra of dimension 8.

The 14-dimensional exceptional Lie group G_2 is its group of automorphisms.

Some results on division algebras.

i) Frobenius' theorem: any associative division algebra over $\mathbf{R}$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$.

ii) Hurwitz' theorem: any normed division algebra over $\mathbf{R}$ with unit element is isomorphic to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$.

iii) any alternative division algebra over $\mathbf{R}$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or $\mathbf{O}$.

iv) any division algebra over $\mathbf{R}$ has dimension 1, 2, 4 or 8.

Explicit construction of division algebras through a procedure, see [Pos], known as **"doubling of an algebra"**.

$$\mathbf{R} \to \mathbf{C} \to \mathbf{H} \to \mathbf{O} \to \dots$$

Comment: the simplest example of doubling corresponds to the geometrical identification of complex numbers with points in the real plane.

Let $a, b, c, d \in X$, then $(a, b), (c, d) \in X^2$.
In X^2 we define the product through

$$(a, b)(c, d) \; = \; (ad - c^* b, bc^* + da).$$

The identity $\mathbf{1}_{X^2}$ in X^2 is $\mathbf{1}_{X^2} = (\mathbf{1}_X, 0)$.
By setting $\mathbf{I}_{X^2} = (0, \mathbf{1}_X)$, we can write
$(a, b) = a\mathbf{1}_{X^2} + b\mathbf{I}_{X^2}$.

The conjugation in X^2 is introduced through

$$(a\mathbf{1}_{X^2} + b\mathbf{I}_{X^2})^* \; = \; a^*\mathbf{1}_{X^2} - b\mathbf{I}_{X^2}.$$

Comment: the "doubling of an algebra" can be performed *ad libitum*. However, each time that a doubling is performed some properties are lost.

In the passage from complex numbers to quaternions the commutativity is lost.

In the passage from quaternions to octonions the associativity is lost while the weaker property of alternativity is preserved.

The doubling of the octonions produces an algebra which is no longer normed.

Applications of the division algebras

$*$) Octonions and the seven-sphere S^7.

S^7 is a compact, simply connected, parallelizable manifold (the last property means $\exists$ d nowhere vanishing, linearly independent vector fields, with d the dimensionality of the manifold). It can be expressed as [CP]

$$S^7 \;=\; \{x \in \mathbf{O} | x^* x = \mathbf{1}\}.$$

The remaining compact, simply connected parallelizable manifolds are S^1 and S^3, associated to complex numbers and quaternions respectively. They are both group-manifolds

$$S^1 \sim U(1), \qquad\qquad S^3 \sim SO(3).$$

Comment: S^7 is not a group-manifold due to the non-associativity character of the octonions.

$*$) Lorentz groups and division algebras [KT].
The universal covering groups of some Lorentz-groups are isomorphic to division algebra-valued $Sl(2)$ groups. The following isomorphisms hold

$$
\begin{aligned}
\overline{S0(2,1)} &\sim Sl(2,\mathbf{R}),\\
\overline{S0(3,1)} &\sim Sl(2,\mathbf{C}), \qquad (\flat)\\
\overline{S0(5,1)} &\sim Sl(2,\mathbf{H}). \qquad (\sharp)
\end{aligned}
$$

$(\flat)$ provides the 2-component spinor decomposition in the standard Minkowski space.

$(\sharp)$ provides a 2-component spinor decomposition in a 6-dimensional Minkowski space.

10-dimensional supersymmetric theories admit an octonionic description based on the Jordan algebra realization of $\overline{SO(9,1)}$ in terms of 2×2 hermitian matrices over $\mathbf{O}$.

Further associations (see [GK]):

$SO(8)$ admits a realization through the G_2 automorphisms group of the octonions plus left and right multiplications by unit octonions x_L, x_R.

$SO(7)$ admits a realization through the G_2 automorphisms group of the octonions plus left and right multiplications by conjugated unit octonions x_L, $x_R = {x_L}^*$.

Consistency checks:

$$\frac{1}{2}(8 \cdot 7) = 28 \iff 14 + 7 + 7 = 28,$$

$$\frac{1}{2}(7 \cdot 6) = 21 \iff 14 + 7 = 21.$$

$*$) Octonions and Clifford Γ-matrices [Oku].

The seven-dimensional Euclidean real Clifford algebra $C(0,7)$ is expressed by seven real anti-symmetric Γ matrices of size 8×8 which can be constructed in terms of the octonionic structure constants.

$*$)Division algebras and extended supersymmetries [Top].

The simplest association of division algebras and extended supersymmetries is for $1D$ N-extended SUSY Quantum Mechanical Systems.
For $N = 1, 2, 4, 8$ the supersymmetric algebra

$$\{Q_i, Q_j\} = \delta_{ij} H, \quad i, j = 1, \ldots, N,$$

can be realized through

$$Q = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$$

and

$$Q_a = \tau_a D,$$

where τ_a denotes the generators of the corresponding division algebra (besides the identity) and D is the supersymmetric derivative

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}.$$

$*$) Γ-matrices and extended supersymmetries [PT].

The irreducible multiplets of representation of the $1D$ N-extended SUSY algebra are in one-to-one correspondence with the real-valued Clifford Γ-matrices of Weyl type (i.e. decomposable in antidiagonal block form).

The dimensionality D of the Clifford algebra corresponds to N, the number of extended supersymmetries.

The size d of the Clifford Γ-matrices corresponds to n, the number of states in the irreducible multiplet.

Comment: Just for $N = 8$ there are two realizations of the $N = 8$ supersymmetry, namely

i) a matrix realization, corresponding to the $N = 8$ case of the above classification, which further implies $n = 16$,

ii) the octonionic realization discussed above. This is of course *not* a matrix realization since the octonions are non-associative.

Open problem: find a connection between the two realizations of the $N = 8$ supersymmetry.

$*$) String theories and division algebras [Omn].

According to the theory of geometrical embeddings, the motion of an Euclidean string in a D-dimensional Euclidean target manifold is reduced, for $D = 3, 4, 6, 10$, to the Liouville theory of a single field Φ taking values in a division algebra, according to the table

$$
\begin{aligned}
D = 3 &\iff \mathbf{R}, \\
D = 4 &\iff \mathbf{C}, \\
D = 6 &\iff \mathbf{H}, \\
D = 10 &\iff \mathbf{O} \quad (?).
\end{aligned}
$$

Comment: (?) is conjectured.

Notice that the dimension of the corresponding division algebra is $D - 2$, the dimension of the transverse coordinates space.

$*$) **Further applications:**

- The supersymmetric affinization $\hat{\mathbf{0}}$ of the octonions. An $N = 8$ supersymmetric algebra of Malcev type **[CRT]**.

- Division algebras and $N = 2, 4, 8$ extended supersymmetrizations of the KdV hierarchy **[CRT2]**.

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