CBPF-MO-001/87

CHIRAL SYMMETRY IN THE PATH-INTEGRAL APPROACH+

by

F.A. Schaposnik[1]*

Centro Brasileiro de Pesquisas Físicas - CNPq/CBPF
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

[1]Departamento de Física
Universidad Nacional de La Plata
C.C.67-1900 La Plata - Argentina

+ Lectures given at Centro Brasileiro de Pesquisas Físicas (CBPF)
Rio de Janeiro, Brasil - December, 1986.

* Member of CIC, Buenos Aires, Argentina.

TABLE OF CONTENTS

CBPF-MO-001/87

LECTURE I: CHIRAL ANOMALY IN THE PATH-INTEGRAL FRAMEWORK

§1.1 Introduction

The derivation of anomalous Ward-Takahashi identities related to chiral symmetries in the path-integral framework is rather recent. It was only 30 years after the discovery of anomalies[1] in the theoretical calculation of the $\pi^0 \to \gamma\gamma$ decay amplitudes that Fujikawa [2] showed that the path-integral fermionic measure cannot be defined respecting all clasical symmetries, this being at the origin of anomalies in the Ward-Takahashi identities obtained by differentiation of the generating functional.

This delay was fundamentally due to the fact people incorrectly assumed that the fermionic measure was invariant under chiral rotations (see for example Coleman's lectures on the uses of instantons [3]) and then quite artificial recipes to get the anomaly had to be invented.[4]

After Fujikawa's observation, the path-integral framework revealed to be very useful in the analysis of anomalous Ward-Takahashi identities. Not only it provided a non-perturbative approach to anomalies but also it proved to be a powerful tool in the complete solution of many interesting models. In the former aspect , gravitational and conformal anomalies were better understood by comparing the usual approach with the functional integral version[1],[5]-[6]. Concerning the exact solution of models like QED_2, QCD_2, Thirring and SU(N)-Thirring ones, etc. it provided a transparent derivation of the bosonization rules in the 2-dimensional world, not only in the well-understood Abelian case [7] but also in the non-Abelian case[9]-[12] in which the application of the path-integral technique extends today to the analysis of string theories[13]

More recently, the quantization of the so-called anomalous theories coupling gauge[14] and/or gravitational fields to Weyl fermions was finally understood by carefully applying the Faddeev-Popov technique The presence of a non-trivial Fujikawa Jacobian arising during the usual separation of the gauge group volume integration solved the anomaly problem showing, at the same time, the natural appearence of a Wess-Zumino term in the resulting effective Lagrangian[15]-[19]. It is important to stress that at the light of the results in [15] and [19] one can state that the symmetry group (the gauge group, the local Lorentz group, ...) becomes physical exactly as it happens with the conformal group in the quantization of the string theory when performed à la Polyakov[20].

I will discuss all this topics in 6 lectures, starting with the derivation of anomalous Ward-Takahashi identities (this lecture) then describing the solution of 2-dimensional models (lectures 2-4) in particular explaining how a Kac-Moody type current algebra can be obtained in the QCD_2 case (lecture 5) finally exposing the way gauge and gravitational theories with Weyl fermions have to be quantized. I will make special emphasis in the results derived in collaboration with many colleagues of the La Plata University (Quique Gamboa Saraví, Tato Solomin, Maria Amelia Muschietti, Carlos Naón, Virginia Manías, Cecilia von Reichenbach, Marta Trobo, Pipi Vucetich) Pittsburgh University (Ralph Roskies) Paris VI University (Olivier Babelon and Claude Viallet) Campinas University (Kyoko Furuya) and Manchester University (James Webb). It is thanks to their insights and patience (as well as those of Bruno Machet, Carlos Bollini, Carlos García Canal, Luis Epele and Huner Fanchiotti) that I learned how to handle symmetry transformations in the path-integral framework.

§1.2 Anomalous Ward-Takahashi identities

Let us discuss first how the non-conservation of the axial current at the quantum level can be derived in the path-integral framework (The case of any other anomaly, e.g. the conformal, Lorentz, Einstein ones, can be analysed exactly in the same form).

The fermionic (Euclidean) Lagrangian to be considered is:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} + e\not{A})\Psi = \bar{\Psi}\not{D}(A)\Psi \tag{1}$$

with Ψ a Dirac fermion in the fundamental representation of some symmetry group G, interacting with the gauge field A_μ taking values in the corresponding Lie algebra.

Consider some (continuous) symmetry leaving invariant the fermionic action:

$$\begin{gathered} \Psi \to \Psi' = \Psi + \delta_\alpha\Psi \ ; \quad \bar{\Psi}' \to \bar{\Psi} + \delta_\alpha\bar{\Psi} \\ S[A,\bar{\Psi},\Psi] = S[A,\bar{\Psi}',\Psi'] \end{gathered} \tag{2}$$

(we are considering a transformation depending on an infinitesimal parameter α. At the classical level, there is an associated conserved current (which can be derived via the Noether theorem) corresponding either to an internal or a space-time symmetry. Equivalently, there is a classically conserved quantity, the associated charge.

In the usual operator approach to the quantum theory, one studies, at this stage, the commutator algebra for currents and charges and determines if it corresponds to the one derived by naive manipulations of the canonical commutation relations obeyed by the fundamental fields. In this framework, it is sufficient to consider the transformation (2) as depending on an infinitesimal <u>global</u> parameter.

On the contrary, in the path-integral approach, one promotes (2) to a local transformation, writing $\alpha = \alpha(x)$. The action is in general non-invariant under this local transformation:

$$\Psi \to \Psi' = \Psi + \delta_{\alpha(x)}\Psi \quad ; \quad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi} + \delta_{\alpha(x)}\bar{\Psi} \tag{3}$$

$$S_F \to S_F' = S_F + \int dx\, \Lambda_\alpha(x) \tag{4}$$

Now, since the transformation (2) corresponds to a symmetry of the theory, Λ_α can be written as the divergence of some current. For example, in the case of a non-abelian symmetry transformation, from Lagrangian (1) and the transformation (3) one gets:

$$\Lambda_\alpha(x) = -\alpha^a D_\mu^{ab} (\bar{\Psi}\gamma^\mu t^b \Psi) \tag{5}$$

In order to study the possible quantum conservation of this current, that is, to derive the corresponding Ward-Takahashi identity, one just takes (3) as a change of variables to be performed in the generating functional:

$$Z_F = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{-S_F[A;\bar{\Psi},\Psi]} \equiv \det \not{D}(A) \tag{6}$$

It is in the integration measure where the quantum aspects are taken into account in this approach. Then, as Fujikawa stressed in his pioneering works [2] one should expect a non-trivial Jacobian associated with transformation (3) each time the operator approach to the quantum theory lead to an anomaly.

This Jacobian, whose evaluation will be discussed in details, can be also understood as the factor relating the fermionic determinants before and after the transformation (3):

$$\det \not{D}(A) = J(\delta\alpha) \det D_\alpha(A) \tag{7}$$

with $D_\alpha(A)$ defined from (4):

$$S' = \int \bar{\Psi}' D_\alpha(A) \Psi' \, dx \tag{8}$$

Now, both determinants appearing in (7) are ill-defined quantities since the product of eigenvalues of the Dirac operator increases without bound. Then, some regularization has to be adopted in order to make relation (7) (as well as the fermionic generating functional) meaningful. Consequently, the actual value of the Jacobian will depend on the selected regularization.

After the transformation, the generating functional Z can be written as:

$$Z = J(\delta\alpha) \int \mathcal{D}\bar{\Psi}' \mathcal{D}\Psi' \exp\left\{ -S'[A, \bar{\Psi}', \Psi'] - \int dx \, \lambda_\alpha \right\} \tag{9}$$

Since Z cannot depend on $\delta\alpha$,

$$\frac{1}{Z}\frac{\delta Z}{\delta\alpha} = 0 = \left\langle \frac{\delta}{\delta\alpha(x)} \int dy \, \lambda_\alpha(y) \right\rangle - \frac{\delta \log J(\delta\alpha)}{\delta\alpha(x)} \tag{10}$$

As a first example, suppose that (3) represents a gauge transformation:

$$\Psi \to \Psi' = \Psi + i t^a \delta\alpha^a \Psi \,, \qquad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi} - i \bar{\Psi} t^a \delta\alpha^a \tag{11}$$

with t^a the generators of the symmetry group.

If the regularization prescription used to define determinants and Jacobians respects gauge invariance (as it is the case, for example, of the ζ-function method, the Heat-kernel method - with $\not{D}$ as regulating operator, etc.) then J = 1 since, being $D_\alpha(A)$ the covariant transform of $\not{D}$, (7) becomes a trivial identity.

On the other hand, J_μ^a is given by (5) with

$$J_\mu^a = \bar{\Psi} \gamma_\mu t^a \Psi \tag{12}$$

and hence eq.(10) leads to the conservation of the vector current at the quantum level:

$$\langle D_\mu^{ab} J^{\mu b} \rangle = 0 \tag{13}$$

If instead of a gauge transformation, one considers a chiral (U(1) for simplicity) one:

$$\Psi \to \Psi' = \Psi + \gamma_5 \delta\alpha \Psi \quad ; \quad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi} + \bar{\Psi} \gamma_5 \delta\alpha \tag{14}$$

and one conserves the same regularization prescription leading to (13), one then has (see next section) a non-trivial Jacobian:

$$J(\delta\alpha) = \exp\left[\frac{1}{16\pi^2} \int \delta\alpha(x)\, \mathrm{tr}\, {}^*F_{\mu\nu} F_{\mu\nu}\, d^4x \right] \tag{15}$$

Concerning the change in the action, one easily finds:

$$\Lambda_{\alpha}(x) = -\alpha(x)\,\partial_{\mu}(\bar{\Psi}\gamma_{\mu}\gamma_{5}\Psi)$$

and hence, at the quantum level one gets the usual anomaly of the axial current:

$$\langle \partial_{\mu}\bar{\Psi}\gamma_{\mu}\gamma_{5}\Psi \rangle = \frac{1}{16\pi^{2}}\,\mathrm{tr}\,{}^{*}F_{\mu\nu}F_{\mu\nu}$$

Of course, one can consider also non-Abelian chiral rotations and study the corresponding Ward-Takahashi identities . Within the ζ-function approach one obtains in this case the so-called covariant anomally, which differs from the consistent (gauge-variant) one. This last can be obtained by a particular choice of the regulating operator in the Heat-kernel approach. This point will be discussed in more details in Lecture 6. Of course, one can derive one anomaly from the other by properly modifying the current definition.

§1.3 The Jacobian associated with an infinitesimal chiral rotation

We shall explicitly evaluate the Jacobian associated with an infinitesimal U(1) chiral rotation for an SU(N) gauge theory with Dirac fermions in the fundamental representation. We shall follow the original (i.e. Heat-kernel) Fujikawa's regularization scheme. The corresponding ζ-function calculation can be found in Ref.[9].

In the path-integral framework the measure is defined by expanding the (classical) fermion fields in terms of some complete set of functions $\{\varphi_n\}$:

$$\Psi(x) = \sum_n a_n \varphi_n(x)$$
$$\bar{\Psi}(x) = \sum_n \bar{a}_n \varphi_n^+(x) \tag{16}$$

where $a_n, \bar{a}_n$ are elements of an infinite Grassmann algebra. The fermionic measure is then defined as:

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \equiv \prod_n d\bar{a}_n da_n \tag{17}$$

In terms of the set $\{\bar{a}_n, a_n\}$ the transformation (14) can be written as

$$a' = C a \tag{18}$$

$$\bar{a}' = \bar{a} C \tag{19}$$

with

$$C_{mn} = \delta_{mn} + \int \varphi_m^+ \gamma_5 \varphi_n \delta\alpha \, dx \tag{20}$$

Hence, the Jacobian associated with the transformation (14) (or equivalently with (18)-(19)) is defined as:

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = J\mathcal{D}\bar{\Psi}'\mathcal{D}\Psi' = \det{}^{-2}C\,\mathcal{D}\bar{\Psi}'\mathcal{D}\Psi' \tag{21}$$

Using eqs. (20) and (21) we get:

$$J = \exp\left[-2\int dx\,\delta\alpha\,\mathcal{A}(x)\right. \tag{22}$$

where

$$\mathcal{A}(x) = \sum_n \varphi_n^+(x)\,\gamma_5\,\varphi_n(x) \tag{23}$$

It is important to note that the expression (23) is only formal and it has no meaning until a regularization prescription is introduced. Indeed, one can write $\mathcal{A}(x)$ in the form:

$$\mathcal{A}(x) = \operatorname{tr}\gamma_5 \sum_n \varphi_n(x)\otimes\varphi_n^+(x) = \operatorname{tr}\gamma_5 \times \delta^4(0) \quad !!! \tag{24}$$

which is obviously ill-defined.

Fujikawa's regularization choice is based in the introduction of a cut-off M^2 in the form:

$$\mathcal{A}_{reg}(x) = \lim_{M^2\to\infty} \sum_n \varphi_n^+(x)\,\gamma_5\, e^{-\frac{\lambda_n^2}{M^2}}\,\varphi_m(x) \tag{25}$$

where λ_n are the eigenvalues of the (hermitian) operator $\not{D}$ whose eigenfunctions are chosen as the basis set in expansions (15)-(16):

$$\not{D}\varphi_n = \lambda_n\varphi_n \tag{26}$$

Of course, the resulting Jacobian cannot depend on the basis choice since the fermionic measure must be defined independently of the complete set one selects. On the contrary, J depends on the regularization prescription. The choice of the $\not{D}$'s eigenvalues in the exponential in (25) ensures the gauge-invariance of the prescription (since $\{\lambda_n\}$ are gauge-invariant). It is for simplity in the explicit computations that the eigenfunctions of $\not{D}$ are chosen: in this case, (25) can be written as

$$\mathcal{A}_{reg}(x) = \lim_{M^2\to\infty} \sum_n \varphi_n^+(x)\gamma_5 e^{-\frac{\not{D}^2}{M^2}} \varphi_n(x) \tag{27}$$

a manifestly gauge invariant expression. Use of other eigenfunctions, as for example those of $i\not{\partial}$ as a basis set leads to the same result as first shown by Christos [22]. On the contrary, the use of a gauge-variant cut-off, like for example:

$$\text{“} 1 = \lim_{M^2\to\infty} e^{+\frac{\not{\partial}^2}{M^2}} \text{”} \tag{28}$$

leads to $\tilde{\mathcal{A}}_{reg} = 0$, i.e. a trivial Jacobian. This is the manifestation of the well-known fact that it is not possible to mantain both gauge and chiral invariances at the quantum level.

Coming back to (27), note that the regularized expression is obtained by first performing the sum and then taking $\lim M^2\to\infty$. This corresponds to make the sum disregarding "big eigenvalues" (those $|\lambda_n| > M$) For that reason, the resulting expression is finite.

Expression (27) can be rewritten in the form:

$$\mathcal{A}_{reg}(x) = \lim_{M^2\to\infty} \lim_{y\to x} \operatorname{tr} \gamma_5 e^{-\frac{\not{D}^2}{M^2}} \sum \varphi_n(x) \otimes \varphi_n^+(y) \tag{29}$$

$$\mathcal{A}_{reg} = \lim_{M^2\to\infty} \lim_{y\to x} \operatorname{tr} \gamma_5 e^{-\frac{\not{D}^2}{M^2}} \delta(x-y) \tag{30}$$

In this regularized form, one can think the basis $\{\varphi_n\}$ has been changed to a plane-wave basis, since $\mathcal{A}_{reg}$ can be written as:

$$\mathcal{A}_{reg}(x) = \lim_{M^2\to\infty} \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 e^{-\frac{\not{D}^2}{M^2}} e^{ikx} \tag{31}$$

$$\mathcal{A}_{reg}(x) = \lim_{M^2\to\infty} \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 \exp\left[-\frac{(ik_\mu + D_\mu)^2}{M^2} + \frac{i}{4M^2}[\gamma_\mu, \gamma_\nu] F_{\mu\nu}\right] \tag{32}$$

The trace in (31) runs over the Dirac an internal symmetry indices.

After rescaling $k_\mu \to M K_\mu$ one can perform the $1/M^2$ expansion in (31) . Taking the trace with γ_5 into account, one gets:

$$\mathcal{A}_{reg} = \lim_{M^2\to\infty} M^4 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 \exp\left[-k_\mu k^\mu + \frac{i}{4M^2} [\gamma_\mu, \gamma_\nu] F_{\mu\nu}\right] \tag{33}$$

$$\mathcal{A}_{reg} = \lim_{M^2\to\infty} \operatorname{tr} \gamma_5 \frac{1}{2!} \left(\frac{i}{4}[\gamma_\mu, \gamma_\nu] F_{\mu\nu}\right)^2 \int \frac{d^4k}{(2\pi)^4} e^{-k^2} = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}$$

Inserting (33) in (22) one gets the result announced in (15).

References

[1] S.Adler,Phys.Rev.177(1969),2426;J.S.Bell and R.Jackiw, N.C.60A(1969)47.

[2] K.Fujikawa, Phys.Rev.Lett.42(1979)1195; Phys.Rev.D21(1980)2848, ibid. D22(1980)1499 (E); ibid.D23(1981)2262; ibid.D25(1982)2584; ibid. D29(1984)285; Nucl.Phys. B226(1983)437.

[3] S.Coleman, The Uses of Instantons, 1977 Ettore Majorana Subnuclear Physics School, Ed. by A.Zichichi. 1977.

[4] D.Gross and R.Jackiw, Phys.Rev.D6(1972)477.

[5] L.Alvarez Gaumé,and E.Witten, Nucl.Phys.B234(1983)269.

[6] L.Alvarez Gaumé and P.Ginsparg, Nucl. Phys. B243(1984)449 and Ann. of Phys. (N.Y.)161(1984)423.

[7] R.Roskies and F.A.Schaposnik, Phys.Rev.D31(1981)558.

[8] R.E.Gamboa Saraví, F.A.Schaposnik and J.E.Solomin, Nucl.Phys. B185(1981)229.

[9] R.E.Gamboa Saraví, M.A.Muschietti, F.A:Schaposnik and J.E. Solomin, Ann. of Phys. (N.Y.) 157(1984)360.

[10] R.E.Gamboa Saraví, F.A.Schaposnik and J.E.Solomin, Phys.Rev. D 30(1984)1353.

[11] R.Roskies in Festchrift for Feza Gursey's 60th birthday. Eds. I.Bars, A.Chodos and C.Tze. Plenum, 1984.

[12] C.M.Naón, Phys.Rev. D31(1985)2035.

[13] A.Eastangh, L.Mezinescu, E.Sezguin, P.van Niewenhuizen, Phys.Rev. Lett.57(1986)29.

[14] L.Faddeev and S.L.Shatashvili, Phys.Lett 167B(1986)225.

[15] O.Babelon, F.A.Schaposnik and C.Viallet, Phys.Lett.177B(1986)385.

[16] A.V.Kulikov, IHEP preprint 86-83. Unpublished.

[17] K.Harada, I.Tsutsui, TIT-HEP preprint Nº94, 1986. Unpublished.

[18] M.Manías, M.C.von Reichenbach, F.A.Schaposnik and M.Trobo, La Plata preprint, 1986. Unpublished.

[19] F.A.Schaposnik and H.Vucetich, La Plata preprint, 1986. Unpublished.

[20] A.M.Polyakov, Phys. Lett.103B(1981)207.

LECTURE 2: SOLVING TWO-DIMENSIONAL MODELS: THE ABELIAN CASE

§2.1 Introduction

I shall describe in this and the next three lectures how very simple solutions of two-dimensional models, like the Schwinger and Thirring models, QCD_2, Gross Neveu model, chiral models, can be obtained within the path-integral framework.

The method I will explain parallels, in the path-integral framework, the operator fit first given by Lowenstein and Swieca[21] and then extended to many two-dimensional models. It is then a sort of path-integral version of the bosonization technique, particularly powerful in the non-Abelian case in which the usual (operator) bosonization scheme does not preserve the internal symmetry.

In the path-integral approach, the operator bosonization recipe is replaced by a chiral change in the fermion variables chosen so as to decouple fermions from other fields (gauge fields, auxiliary fields, etc.) at the classical level. At the quantum level this change leads to an effective Lagrangian which includes the contribution of the associated Fujikawa Jacobian. It is in this way that the Wess-Zumino term naturally arise in the non-Abelian case also for interacting theories (remember that Witten[22] bosonization scheme was originally applied to free fermions).

Since the "decoupling" change of fermionic variables corresponds to a finite chiral transformation, we shall need to develop a technique allowing for the computation of finite transformation chiral Jacobians. This will be done in the next section for the case of the Schwinger model. Then, in section 3, other Abelian models will be discussed.

§2.2 A simple example: the Schwinger model

The Schwinger Model (S.M.) is two-dimensional electrodynamics with massless fermions. The dynamics of the model is defined by the (Euclidean) Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\Psi}(i\not{\partial} + e\not{A})\Psi \tag{1}$$

Consider the field transformation:

$$\begin{aligned} \Psi &= \exp(\gamma_5\phi + i\eta)\,\chi \\ \bar{\Psi} &= \bar{\chi}\exp(\gamma_5\phi - i\eta) \end{aligned} \tag{2}$$

$$A_\mu = -\frac{1}{e}\varepsilon_{\mu\nu}\partial^\nu\phi + \frac{1}{e}\partial_\mu\eta \tag{3}$$

It is easy to see that after this change, the classical Lagrangian (1) becomes that of a massless free fermion decoupled from the massless gauge field * :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\chi}\, i\not{\partial}\,\chi \tag{4}$$

$$F_{\mu\nu} = \frac{1}{e}(\Box\phi)\varepsilon_{\mu\nu} \tag{5}$$

Evidently, the Schwinger mechanism (the "photon" getting a mass) cannot be discovered through this classical manipulations since it is a quantum effect related to the existence of an axial anomaly.

---

* My conventions for the (Euclidean) two-dimensional Dirac matrices are: $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$; $i\gamma_\mu\gamma_5 = \varepsilon_{\mu\nu}\gamma_\nu$; $\varepsilon_{01} = -\varepsilon_{10} = 1$.

In order to treat the problem at the quantum level one has to compute the Jacobian associated to the transformation (2) proceeding à la Fujikawa. Also the (Faddeev-Popov) Jacobian associated to (3) has to be evaluated but this is a trivial problem:

$$\mathcal{D}A_0\mathcal{D}A_1 = \Delta_{FP}\mathcal{D}\phi\mathcal{D}\eta \tag{6}$$

$$\Delta_{FP} = \det(-\nabla^2) \tag{7}$$

In fact, Δ_{FP} is the Faddeev-Popov Jacobian associated with the Lorentz ($\partial_\mu A^\mu = 0$) gauge. One has then to include a $\delta(\eta)$ term in the generating functional when fixing the gauge to the Lorentz condition and this eliminates the η -integration from Z .

Concerning the fermion Jacobian, the technique developed in the previous lecture has to be extended to the case of finite chiral transformations. In order to do that we shall apply the approach developed in Ref.[7]. Consider the following change of variables:

$$\begin{aligned} \Psi &= \exp[(\gamma_5\phi + i\eta)t]\,\Psi_t \\ \bar{\Psi} &= \bar{\Psi}_t\exp[(\gamma_5\phi - i\eta)t] \end{aligned} \tag{8}$$

depending on a parameter t, $0 \leq t \leq 1$ What we have in mind is to build up the finite transformation from infinitesimal ones by growing t from 0 to 1.

The fermion Lagrangian can be written in the form:

$$\mathcal{L}_F = \bar{\Psi}_t D_t \Psi_t \tag{9}$$

$$D_t = e^{(\gamma_5 \phi \quad i\eta)t} \not{D} e^{(\gamma_5 \phi + i\eta)t} \tag{10}$$

In this particular abelian case D_t has a very simple form:

$$D_t = i\not{\partial} + e\not{A} + it\gamma_\mu\gamma_5\partial_\mu\phi - \not{\partial}\eta \tag{11}$$

or

$$D_t = i\not{\partial} + e(1-t)\not{A} \tag{12}$$

Now, performing the change in the fermionic part of the generating functional

$$Z_F = \int J(t)\,\mathcal{D}\bar{\Psi}_t\,\mathcal{D}\Psi_t\, e^{-\int \bar{\Psi}_t D_t \Psi_t d^2x} \tag{13}$$

we have:

$$Z_F = J(t) \det D_t \tag{14}$$

Since Z_F cannot depend on t,

$$\frac{dZ_F}{dt} = 0 = \frac{dJ}{dt} \det D_t + J(t) \frac{d}{dt}(\det D_t) \tag{15}$$

Now, integrating (15) we get:

$$J \equiv J(1) = \exp\left[-\int_0^1 w'(t)\,dt\right] \tag{16}$$

where

$$\omega'(t) = \frac{d\omega}{dt} = \frac{d}{dt} \log \det D_t \tag{17}$$

It is precisely J(1) the Jacobian we were searching for. Now, the evaluation of $\omega'(t)$ is related to the infinitesimal calculation we performed in the previous lecture. Indeed, note that

$$D_{t+\Delta t} = D_t + \phi A(x) \Delta t + O(\Delta t^2) \tag{18}$$

with

$$\phi A(x) = \gamma_5 \phi D_t + D_t \gamma_5 \phi \tag{19}$$

Then

$$\omega'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\log \det D_{t+\Delta t} - \log \det D_t \right] \tag{20}$$

But:

$$\log\left[(\det D_{t+\Delta t}) / \det D_t\right] = \mathrm{tr} \log (D_{t+\Delta t}) D_t^{-1} \tag{21}$$

$$= \mathrm{tr} \log \left(1 + \phi A D_t^{-1} \Delta t + \cdots\right) \tag{22}$$

$$= \mathrm{tr}\, \phi A D_t^{-1} \Delta t = 2\, \mathrm{tr}\, \gamma_5 \phi \Delta t \tag{23}$$

and hence

$$w'(t) = \mathrm{tr}\, \phi(x) A(x) D_t^{-1} = 2\,\mathrm{tr}\, \gamma_5 \phi \tag{24}$$

Of course, all manipulations leading to (24) are merely formal since the determinants appearing in (20) (and consequently the trace in (24)) have to be regularized . For example, in the spirit of the previous lecture (the heat-kernel regularization scheme) the regularized form of $w'(t)$ is:

$$w'(t) = \lim_{M^2\to\infty} 2\,\mathrm{tr}\, \gamma_5 \phi\, e^{-D_t^2/M^2} \tag{25}$$

In the ζ-function approach, one has instead:

$$w'(t) = 2\,\mathrm{tr}\, D_t^{-s} \gamma_5 \phi \Big|_{s=0} \tag{26}$$

and in this case both expressions coincide.

One can evaluate $w'(t)$ easily. For example, from (25) one writes:

$$w'(t) = \lim_{M^2\to\infty} \mathrm{tr}\, \frac{2}{(2\pi)^2} \int d^2x\, d^2k\, e^{ikx}\, e^{-D_t^2/M^2} \gamma_5 \phi\, e^{-ikx} \tag{27}$$

$$w'(t) = \lim_{M^2\to\infty} \mathrm{tr}\, \frac{\gamma_5 \phi}{2\pi^2} \int d^2x\, d^2k\, \exp[-(D_t + k\!\!\!/)^2/M^2] \tag{28}$$

Expanding the exponential in (28) one gets:

$$w'(t) = \lim_{M^2\to\infty} \mathrm{tr}\, \frac{\gamma_5 \phi}{2\pi^2} \int d^2x\, d^2k\, e^{-k^2/M^2} \left(1 - D_t^2/M^2 + O\left(\frac{1}{M^4}\right)\right) \tag{29}$$

In two dimensions it is the $1/M^2$ term the one that contributes in the $M^2\to\infty$ limit . After rescaling variables we have:

$$w'(t) = \frac{1}{2\pi^2}\int d^2x \int d^2p\, e^{-p^2}\, \mathrm{tr}\, \gamma_5 \phi D_t^2 \tag{30}$$

or

$$w'(t) = \frac{1}{2\pi}\, \mathrm{tr}\, \gamma_5 \gamma_\mu \gamma_\nu \int d^2x\, D_{t\mu} D_{t\nu}\, \phi \tag{31}$$

Finally we get:

$$w'(t) = +\frac{e}{2\pi}(1-t)\int \phi\, \varepsilon_{\mu\nu} F_{\mu\nu}\, d^2x \tag{32}$$

and hence

$$\log J = \frac{e}{4\pi}\int d^2x\, \varepsilon_{\mu\nu} F_{\mu\nu}\phi = -\frac{e^2}{2\pi}\int A_\mu\left[\delta_{\mu\nu} - \partial_\mu \Box^{-1} \partial_\nu\right] A_\nu\, d^2x \tag{33}$$

Then the generating functional (including sources) reads after the decoupling:

$$Z(\eta,\bar{\eta}) = \int \mathcal{D}A_\mu\, \delta(\partial_\mu A^\mu)\, \mathcal{D}\bar{\chi}\mathcal{D}\chi\, e^{-\int\left[\mathcal{L}_{eff} + \mathcal{L}_{sources}\right] d^2x} \tag{34}$$

with

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{2\pi}A_\mu^2 + \bar{\chi}\, i\not{\partial}\chi \tag{35}$$

$$\mathcal{L}_{sources} = \bar{\chi}\, e^{\gamma_5\phi}\, \eta + \bar{\eta}\, e^{\gamma_5\phi}\, \chi$$

We have written Z in the Lorentz gauge. Note that it is the chiral Jacobian the responsible of the "photon mass ", absent from the classical decoupling.

Indeed, writing the effective Lagrangian in terms of ϕ , we have

$$\mathcal{L}_{eff} = -\frac{1}{2e^2}\Box\phi\Box\phi - \frac{1}{2\pi}\phi\Box\phi + \bar{\chi}i\not\partial\chi \tag{36}$$

and hence the ϕ-propagator reads:

$$\Delta_\phi(x) = + K_0\left(\frac{e}{\sqrt{\pi}}x\right) + \frac{1}{2\pi}\log\frac{e}{\sqrt{\pi}}x \tag{37}$$

That is, it corresponds to a massive scalar particle (with mass $\frac{e}{\sqrt{\pi}}$) and a massless gauge excitation, quantized with negative metric. Any Green Function can be obtained from the generating functional and the complete solution of the massless Schwinger model follows in a very economical way. For example, the two-point function reads:

$$\langle\Psi(x)\bar{\Psi}(0)\rangle = e^{-\Delta_\phi(x)}\langle\chi(x)\bar{\chi}(0)\rangle = cte\, e^{-\Delta_\phi(x)}\frac{x^\mu\gamma_\mu}{x^2} \tag{38}$$

§2.3 Other Abelian models

a) The Thirring model:

This is a purely fermionic model :

$$\mathcal{L}_{Th} = \bar{\Psi}i\not\partial\Psi - \frac{g^2}{2}(\bar{\Psi}\gamma_\mu\Psi)^2 \tag{39}$$

One can introduce an auxilliary vector field A_μ in the generating functional:

$$Z_{Th} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{-\int\mathcal{L}_{Th}d^2x} \tag{40}$$

through the identity:

$$\exp\left\{\frac{g^2}{2}\int(\bar{\Psi}\gamma_\mu\Psi)^2 d^2x\right\} = \int\mathcal{D}A_\mu \exp\left\{-\int\left(g\bar{\Psi}\not{A}\Psi + \frac{1}{2}A_\mu^2\right)\right\}d^2x \tag{41}$$

and hence one gets an effective vector theory which can be solved exactly as in the S.M. case. This was done in Ref.[23] by starting from the decoupling change of the fermionic variables (2)-(3). The resulting effective Lagrangian takes in this case the form:

$$\mathcal{L}_{eff} = \bar{\chi}i\not{\partial}\chi + \frac{1}{2g^2}\left(1+\frac{g^2}{\pi}\right)(\partial_\mu\phi)^2 + \frac{1}{2g^2}(\partial_\mu\eta)^2 \tag{42}$$

The two-point function reads:

$$\langle\Psi(x)\bar{\Psi}(0)\rangle = \exp\left\{-\frac{1}{2}\,\frac{g^2/\pi}{1+g^2/\pi}\,\log\mu|x|\right\}\langle\chi(x)\bar{\chi}(0)\rangle \tag{43}$$

It is important to stress that in this model A_μ is not a gauge field and hence, no necessity of using a gauge-invariant regularization prescription exists. In fact, use of a more general regulating operator than that employed in Ref.[23] leads to the well-known one-parameter family solutions of the Thirring model [24] as it was first shown in Ref.[25] This marks a limitation of the ζ-function method which is, per se, gauge-invariant and hence does not allow for this kind of generalizations. We shall return to this point in Lecture 6, when discussing chiral models.

b) CP^n model with fermions.

The Lagrangian for this model is:

$$\mathcal{L} = \left|\left(\partial_\mu - \frac{f}{n}\bar{z}\overleftrightarrow{\partial_\mu}z\right)z_i\right|^2 + \bar{\Psi}\left(i\not{\partial} - \frac{ief}{n}\bar{z}\overleftrightarrow{\not{\partial}}z\right)\Psi \tag{44}$$

with z an n-component complex field satisfying

$$|z|^2 = \bar{z}z = \frac{n}{2f} \tag{45}$$

Since $\bar{z}\overleftrightarrow{\partial}_\mu z$ acts like a gauge field, it can be eliminated from the covariant derivative by first introducing an auxilliary vector field and then decoupling it exactly as in the Schwinger and Thirring model cases. In the first step one gets the effective Lagrangian:

$$\tilde{\mathcal{L}}_{eff} = \overline{D_\mu z}\, D_\mu z + \bar{\Psi}\left(i\not{\partial} + \frac{e}{\sqrt{n}}\not{A}\right)\Psi + \frac{fe^2}{2n}\left(\bar{\Psi}\gamma_\mu\Psi\right)^2 \tag{46}$$

$$D_\mu = \partial_\mu + \frac{i}{\sqrt{n}} A_\mu$$

with the auxiliary field replacing the quartic term in (44). Then the usual chiral rotation decouples bosons from fermions. The resulting effective Lagrangian is

$$\mathcal{L}_{eff} = \overline{D_\mu z}\, D_\mu z + \frac{e^2}{2\pi} A_\mu A^\mu + \bar{\chi} i\not{\partial}\chi + \frac{e^2 f}{2n}\left(\bar{\chi}\gamma_\mu\chi\right)^2 \qquad [47]$$

As it is discussed in Ref.[26], the resulting theory corresponds to a Thirring-like fermionic part, decoupled from the purely massive bosonic sector which can be studied using the habitual-1/n expansion. The analysis of the spectrum is then straightforward and one discovers the factorization of a U(1) chiral factor in the Green functions responsible of a power-law behavior of the correlation functions (the almost long-range order predicted by Witten [27]).

c) Fermion number fractionization.

A curious behaviour of quantum systems, relevant in condensed matter physics (which is closer to real life than the physics we have been discussing in these lectures), can be understood by studying a two-dimensional fermionic model in the presence of a soliton background.

The corresponding Lagrangian is:

$$\mathcal{L} = \bar{\Psi}\left(i\not{\partial} + g e^{-\gamma_5 \phi}\right)\Psi \tag{48}$$

where ϕ is a pseudo-scalar soliton field which provides the background inducing unusual quantum numbers. In Refs.[28]-[30] the relation between this model and that describing for example a poliacetilene molecule, can be found. Supose we want to compute the conserved fermion number current in the presence of ϕ . We then define a generating functional with a source term s_μ:

$$Z[s] = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left(-\int \bar{\Psi}\left(i\not{\partial} + \not{s} + g e^{-\gamma_5\phi}\right)\Psi\, d^2x\right) \tag{49}$$

In terms of Z , the current reads:

$$J_\mu(x) \equiv \langle \bar{\Psi}\gamma_\mu\Psi \rangle = -\frac{1}{Z}\frac{\delta Z}{\delta s_\mu(x)}\bigg|_{s=0} \tag{50}$$

Performing the chiral rotation:

$$\begin{aligned} \Psi &= e^{\gamma_5\phi/2}\chi \\ \bar{\Psi} &= \bar{\chi}\, e^{\gamma_5\phi/2} \end{aligned} \tag{51}$$

the resulting classical Lagrangian becomes:

$$\mathcal{L}_D = \bar{\chi}\left(i\not\partial + \not{s} + \frac{1}{2}\gamma_\mu \varepsilon_{\mu\nu}\partial_\nu\phi\right)\chi \tag{52}$$

Concerning the Jacobian, it can be computed exactly as in the previous cases. The details are given in Ref.[31]. The answer is:

$$J(\phi) = \exp\left[\frac{1}{2\pi}\int d^2x\left[s_\mu\varepsilon_{\mu\nu}\partial_\nu\phi + \frac{1}{4}(\partial_\mu\phi)^2 - \frac{g^2}{2}(\cosh 2\phi - 1)\right]\right] \tag{53}$$

In terms of the decoupled variables the generating functional reads:

$$Z = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi\, e^{-\int \mathcal{L}_D d^2x}\, J(\phi) \tag{54}$$

and the fermion number current is:

$$J_\mu(x) = -\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\nu\phi + \frac{\delta}{\delta a_\mu}\log\det(i\not\partial + \not{a} + g)\Big|_{a_\mu = \frac{1}{2}\varepsilon_{\mu\nu}\partial_\nu\phi} \tag{55}$$

We shall see that the first term is responsible for the fermion number fractionization. One can expand the second one for slow varying ϕ fields getting

$$\varepsilon_{\mu\nu}\partial_\nu\left[\text{const.}\ \frac{1}{g^2}\Box\phi + \text{higher orders terms in } \Box\phi\right] \tag{55}$$

and hence to leading order in derivatives of ϕ we get:

$$J_\mu = -\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\nu\phi$$

Then, for a soliton with ϕ varying from 0 to π the fermion number is -1/2 ! :

$$N = \int_{-\infty}^{\infty} dx_1 J_0 = -\frac{1}{2\pi}\left[\phi(\infty) - \phi(-\infty)\right] = -1/2 \ . \tag{55}$$

this being the result first obtained in Refs.[28]-[30].

We then see that all the results obtained previously with the bosonization recipe can be derived very simply within the path-integral approach. The case of massive theories can also be treated in this way. The equivalence between massive Thirring and sine-Gordon models can also be established [32] and the analysis of other massive fermionic models can be discussed. In particular we shall see how this technique can be straightforwardy extended to the case of non-Abelian models in the next lecture.

References

[21] J.Lowenstein and J.A.Swieca, Ann.of Phys (N.Y.) 68(1971)175.

[22] E.Witten, Comm.Math.Phys. 92 (1984) 455.

[23] K.Furuya, R.E.Gamboa Saraví and F.A.Schaposnik, Nucl.Phys. B208 (1982) 159.

[24] B.Klaiber in "Lectures in Theoretical Physics" Ed.A.O.Barut, Gordon and Beach, New York and references therein.

[25] M.Rubin, Journal of Physics A19(1986)2195

[26] F.A.Schaposnik, Phys.Lett. 126B(1983)351.

[27] E.Witten, Nuclear Phys. B145(1978)110.

[28] W.P.Su, J.R.Schrieffer, A.J.Heeger, Phys.Rev.Lett42(1979)1698; Phys.Rev. B22(1980)2099.

[29] J.Goldstone and F.Wilczek, Phys.Rev.Lett.47(1981)986.

[30] W.A.Bardeen, S.Elitzur, Y.Frishman and E.Rabinovici, Nucl. Phys. B218(1983)447.

[31] F.A.Schaposnik, Z.fur Phys.28C(1985)127.

[32] C.M.Naón, La Plata University Ph.D. Thesis, 1986 and Ref.[12].

LECTURE 3: SOLVING TWO-DIMENSIONAL MODELS: QCD_2

§3.1 Introduction

Quantum chromodynamics in two dimensions was popularized by 't Hooft as a laboratory for the study of 1/N expansions [33] but it was quickly captured by two-dimensional physicists anxious to extend the bosonization scheme to non-Abelian models. However, the attemps to determine the spectrum of QCD_2 in different regimes lead to contradictory results due to the fact the usual bosonization does not preserve internal symmetries (for a list of references on these first attemps, see Ref.[8]). Reliable results were at most conjectures on the existence of a non-Abelian extension of the Schwinger mechanism providing mass to the gluons. [34]

It was the path-integral framework that allowed to finally understand (at least qualitatively) the fundamental properties of QCD_2. At the same time, it gave an indication of how bosonization took place when non-Abelian symmetries were in the game. The sequence of advances in this approach can be summarized as follows:

i) The fermion determinant is computed exactly in Ref.[8] by extending the technique described in the previous lecture. In order to ensure the fermion decoupling, it was necessary to assume there exists some gauge condition in which the decoupling is automatic.

ii) In Ref.[11] Roskies prooved the existence of the gauge condition referred in (i).

iii) In Ref.[15] Polyakov and Wiegmann showed that in the light-cone gauge the SU(N) fermion determinant reduces to a non-linear sigma model plus a Wess-Zumino term.

iv) In Ref.[10] the connection between the results i) and iii) is established and the effective bosonic Lagrangian for QCD analysed. In Refs.[36]-[37] the current algebra for QCD_2 is presented.

v) In Ref.[38] Rothe extends the results in (i) to a general gauge.

Many other authors have simultaneously discovered or rediscovered these results. (See, for example, Refs. [39]-[41].However, the basic ingredients for treating QCD_2 in the path-integral framework are given in (i)-(v). Concerning the bosonization scheme in the operator approach, it was Witten [22] who suceed in finally giving the correct rules (at least for free fermions) making in this way contact with the results obtained in the path-integral approach for QCD_2 [8] and SU(N)-Thirring [35] models.

I shall describe in this lecture how the fermion determinant can be computed for fermions in the fundamental representation of SU(N) in the presence of an SU(N) gauge field acting as a background. I will follow the approach developed in the prevoius lecture. Then, I will discuss the principal features of QCD_2 emerging from the analysis of the effective Lagrangian obtained after the decoupling of fermions.

§3.2 The fermion determinant

In the Abelian case, the fermion determinant was exactly evaluated by performing a decoupling transformation where

$$\mathcal{L}_F = \bar{\Psi} \not{D}(A) \Psi \longrightarrow \bar{\Psi} i \not{\partial} \Psi \tag{1}$$

The computation of the Jacobian associated to the fermion transformation lead to the knowledge of the determinant since:

$$\det \not{D}(A) = J(A) \det i\not{\partial} \tag{2}$$

The QCD_2 determinant will then be solvable if there exists a transformation of the kind (1) and if the corresponding Jacobian is exactly computable. Both questions were answered afirmatively in Ref.[8]. Concerning (1), note that if one tries to decouple fermions in the fundamental representation of SU(N) by a chiral change:

$$\begin{aligned} \Psi &= e^{\gamma_5 \phi} \chi \\ \bar{\Psi} &= \bar{\chi} e^{\gamma_5 \phi} \end{aligned} \tag{3}$$

with $\phi = \phi^a t^a$ taking values in the Lie algebra of SU(N) and t_a the SU(N) generators, $\operatorname{tr} t_a t_b = \delta_{ab}$, this implies a restriction on the possible A_μ which can be decoupled:

$$\bar{\Psi} \not{D}(A) \Psi = \bar{\chi} \left(i\not{\partial} + e^{\gamma_5 \phi} (i\not{\partial} e^{\gamma_5 \phi}) + e\, e^{\gamma_5 \phi} \not{A} e^{\gamma_5 \phi} \right) \chi \tag{4}$$

In order (1) becomes an identity, one necessarily has:

$$\not{A} = -\frac{i}{e} \left(\not{\partial} e^{\gamma_5 \phi} \right) e^{-\gamma_5 \phi} \tag{5}$$

It can be proved [11] that any A_μ can be written in terms of some ϕ in a certain gauge, known as the decoupling gauge. Note that $\not{A}$ can be rewritten as:

$$\not{A} = -\frac{i}{e}\begin{pmatrix} 0 & (\partial_+ e^{\phi})e^{-\phi} \\ (\partial_- e^{-\phi})e^{\phi} & 0 \end{pmatrix} \tag{6}$$

Were we working in Minkowski space $e^{\phi_E} \to e^{i\phi_M}$ and then the resemblance between $\not{A}$ and the free-fermion bosonized currents introduced by Witten[22] becomes evident. Roskies has shown[11] that the decoupling gauge condition is a differentiable local one. It is not necessary to know its explicit form in order to compute the determinant in this gauge. Of course one can extend the transformation (3) in order to decouple a general $\not{A}$ just by also including a gauge rotation but for simplicity we shall work in the decoupling gauge. Exactly as we did in the previous lecture, we consider a transformation of the form:

$$\begin{aligned} \Psi &= e^{t\gamma_5\phi}\,\Psi_t \\ \bar{\Psi} &= \bar{\Psi}_t\, e^{t\gamma_5\phi} \end{aligned} \tag{7}$$

with $0 \leq t \leq 1$ and compute the Jacobian for the full transformation from eqs.(16) and (25) in the previous lecture:

$$\log J = -\int_0^1 \omega'(t)\,dt \tag{8}$$

$$\omega'(t) = \lim_{M^2\to\infty} 2\,\mathrm{Tr}\,\gamma_5\,\phi\, e^{-\not{D}_t^2/M^2} \tag{9}$$

with

$$D_t = e^{\gamma_5 \phi t} \not{D} e^{\gamma_5 \phi t} \tag{10}$$

and Tr now including a trace in the internal symmetry indices. Following identical steps as in the Abelian case, we get:

$$\log J = \frac{1}{2\pi}\int d^2x \int_0^1 dt \, \mathrm{Tr}\, \gamma_5 \phi D_t^2 \tag{11}$$

which can be written as

$$\log J = \frac{e^2}{2\pi}\int d^2x \int_0^1 dt \, \mathrm{Tr}\, \gamma_5 \phi \left(\not{\partial} \not{A}^t - ie \not{A}^t \not{A}^t\right) \tag{12}$$

with

$$\not{A}^t = -\frac{i}{e}\left(\not{\partial} e^{t\gamma_5\phi}\right) e^{-t\gamma_5\phi}$$

or (13)

$$\log J = -\frac{e^2}{2\pi}\int d^2x \, \mathrm{Tr}\left[\frac{1}{2}\not{A}\not{A} + \int_0^1 dt \, \not{A}^\delta \phi \not{A}^t dt\right]$$

This is the result first derived in [8]. The first term, where the t-integration was trivially performed corresponds to a gluon mass term: as in the Abelian case, the Schwinger mechanism takes place for gluons. The second term is related to the Wess-Zumino functional[42]. To see this, let us define:

$$\mathcal{U} = \mathcal{U}(\phi; t) = e^{2t\phi} \tag{14}$$

and write:

$$A_\mu^t = v_\mu^t - \varepsilon_{\mu\nu} a_\nu^t \tag{15}$$

with

$$\begin{aligned} v_\mu^t &= \frac{-i}{2e}\left[\partial_\mu U^{+1/2}, U^{-1/2}\right]_- \\ a_\mu^t &= \frac{1}{2e}\left[\partial_\mu U^{+1/2}, U^{-1/2}\right]_+ \end{aligned} \tag{16}$$

Eq.(15) corresponds, in the 2-dimensional world, to the decomposition of a gauge field into vector and axial parts since the relation $\gamma_\mu\gamma_5 = i\varepsilon_{\mu\nu}\gamma^\nu$ makes disappear the γ_5 matrix. The following useful identities hold:[43]

$$\begin{aligned} &\partial_\mu v_\nu^t - \partial_\nu v_\mu^t - ie\left[v_\mu^t, v_\nu^t\right] + ie\left[a_\mu^t, a_\nu^t\right] = 0 \\ &\partial_\mu \phi - ie\left[v_\mu^t, \phi\right] = e\frac{\partial a_\mu}{\partial t} \\ &\partial_\mu a_\nu^t - ie\left[v_\mu^t, \phi\right] = \partial_\nu a_\mu^t - ie\left[v_\nu^t, v_\mu^t\right] \end{aligned} \tag{17}$$

Using them, we can write, instead of (13)

$$\begin{aligned} \omega[U] = \log J = \frac{1}{8\pi}\int d^2x \, \mathrm{tr}\, \partial_\mu U^{-1}\partial_\mu U - \frac{i}{4\pi}\int_0^1 dt \int d^2x \times \\ \times \mathrm{tr}\left(\varepsilon_{\mu\nu} U^{-1}\partial_t U U^{-1}\partial_\mu U U^{-1}\partial_\nu U\right). \end{aligned} \tag{18}$$

and now the second term in (18), which we shall call ω_2, can be identified with a Wess-Zumino term. Indeed, consider for a moment the analytic continuation of U to an element U_c of SU(2) :

$$U_c(x,t) = \exp\left[2it\,\phi(x)\right] \tag{19}$$

Since we are taking space-time as a large sphere and $\pi_2(SU(2))=0$ we can think of U_c as a mapping from a solid ball B (whose boundary is S^2) into the SU(2) manifold. We shall take as coordinates in B the parameter t and the two-space-time coordinates (writing t=cosθ, we can think of B us the upper hemisphere of S^3)The analytically continued W_2 reads:

$$W_2^c = 4\pi i \Gamma \tag{20}$$

where Γ is the Wess-Zumino functional:

$$\Gamma = \frac{1}{24\pi^2}\int_B d^3x\, \varepsilon^{ijk}\, \mathrm{tr}\left[U_c^{-1}\partial_i U_c U_c^{-1}\partial_j U_c U_c^{-1}\partial_k U_c\right] \tag{21}$$

which has the very important property of being defined modulo 2π the ambiguity being related to the topologically inequivalent ways of extending a given mapping $U_c : S^2 \longrightarrow SU(2)$ into a mapping from B to SU(2). This topologically distinct possibilities are classified by $\pi_3(SU(2)) = Z$.

Coming back to our actual problem, the extension from S^2 to B arose naturally when we constructed the finite chiral transformation $e^{\gamma_5\phi}$ from $e^{t\gamma_5\phi}$. Any other extension than the one defined by (7) would have yielded the same W_2 since there are no ambiguities when there is a γ_5 in the exponential since in this case the group is not SU(2) but SL(2,C) which is homotopically equivalent to R^3 and hence $\pi_3(R^3)=0$. We then conclude that any other extension different from that of eq.(7) would lead to the same result.

The presence of Γ in our approach has a transparent origin: the non-Abelian chiral anomaly, responsible for the Jacobian. As we stated above, the evaluation of J can be performed in an arbitrary gauge by noting that the substitution

$$e^{\gamma_5 \phi} \rightarrow e^{i\eta} e^{\gamma_5 \phi} \tag{22}$$

changes A_μ from the decoupling gauge to a general one:

$$A[\phi] \rightarrow A[\eta,\phi] = e^{i\eta} A[\phi] e^{-i\eta} + \frac{i}{e} e^{i\eta} \not{\partial} e^{-i\eta} \tag{23}$$

This was done by Rothe in Ref.[38] where the QCD_2 determinant in an arbitrary gauge can be found. We shall end this lecture by quoting an important property of the functional (18):

$$W[UV] = W[U] + W[V] - \frac{1}{16\pi} \operatorname{tr} \int U^{-1} \partial_+ U \, V \partial_- V^{-1} d^2x \tag{24}$$

This allows the obtention of the determinant in a general gauge by noting that (23) can be written as:

$$A[\eta,\phi] = -\frac{i}{e} [\not{\partial}(VU)] (VU)^{-1} \tag{25}$$

with

$$V = e^{i\eta} \tag{26}$$

The answer conincides with the result of Polyakov and Wiegmann[35].

§3.3 QCD_2 properties

Once the decoupling change in the fermionic variables has been done, one can write the complete generating functional for QCD_2 in terms of the decoupled fermion fields and $\mathcal{U} = e^{2t\phi}$ (we are still working in the decoupling gauge):

$$Z = \int \mathcal{D}\mathcal{U}\, \Delta_{FP}[\mathcal{U}]\, \mathcal{D}\bar{\chi}\mathcal{D}\chi\, e^{-S_{eff}} \tag{27}$$

with S_{eff} the effective action including the Jacobian contribution:

$$S_{eff} = -\frac{1}{4} \mathrm{tr} \int F^2_{\mu\nu}[\mathcal{U}]\, d^2x + W[\mathcal{U}] + \bar{\chi} i \not{\partial} \chi \tag{28}$$

and Δ_{FP} the Faddeev-Popov determinant,

$$\Delta_{FP} = \det \frac{\delta A}{\delta \mathcal{U}}$$

which, as Polyakov and Wiegmann [43] explained for the case of the axial gauge, can be shown to lead also to a Wess-Zumino term which just changes the factor multiplying the effective action.

It is instructive to investigate (28) by performing a perturbative expansion[10]:

$$\mathcal{U} = 1 + \phi^a t^a + O(\phi^2) \tag{29}$$

leading to an effective Lagrangian of the form:

$$\mathcal{L}_{eff} = \frac{1}{e^2} \mathrm{tr} \left\{ \phi \left(\nabla^2 \nabla^2 + \frac{e^2}{2\pi} \nabla^2 \right) \phi + \frac{e^2}{6\pi} \phi \partial_\mu \phi \partial_\nu \phi \, \epsilon_{\mu\nu} + 2\phi \partial_\mu \nabla^2 \phi \partial_\nu \phi \epsilon_{\mu\nu} \right\} \tag{30}$$

As usual in the path-integral approach to bosonization, one gets an effective Lagrangian with high-order derivative terms. It corresponds to N^2-1 massive scalars (with mass $m=e/\sqrt{2\pi}$) and N^2-1 massless excitations since the propagator associated to this Lagrangian is again:

$$\Delta = \Delta_F(m,x) - \Delta_F(0,x) \; ; \quad \Delta_F(mx) = -K_0(mx)$$

In summary, we have computed the fermion determinant for QCD_2 and stressed the appearence of a Wess-Zumino term analogous to that present in other two-dimensional models. We have shown that the fermions completely decouple from bosons, these last being massive self-interacting scalars. Concerning fermionic Green functions, the decoupling implies, as in the Schwinger model, that at short distances fermions are free. It is interesting to note that the presence of the $F_{\mu\nu}^2$ term in QCD_2 is responsible for the mass of the bosons, in contrast with the case of purely fermionic models, as we shall see in next lecture.

References

[33] G.'t Hooft, Nucl.Phys.B75(1974)461.

[34] I.V.Belvedere, J.A.Swieca, K.D.Rothe and B.Schroer, Nucl.Phys. B153(1979)112.

[35] A.M.Polyakov and P.B.Wiegmann, Phys.Lett.131B(1983)121.

[36] R.E.Gamboa Saraví, C.M.Naón and F.A.Schaposnik, Phys.Lett.153B (1985)97.

[37] R.E.Gamboa Saraví, C.M.Naón and F.A.Schaposnik, Phys.Lett.163B (1985)

[38] K.D.Rothe, Nucl.Phys.B269(1986)269.

[39] O.Alvarez, Nucl.Phys.B238(1984)61.

[40] P.di Vechia, B.Durhus and J.Petersen, Phys.Lett. 144B(1984)245.

[41] D.González and A.Redlich, Phys.Lett.147B(1984)150.

[42] J.Wess and B.Zumino, Phys.Lett.37B(1971)95.

[43] A.M.Polyakov and P.B.Wiegmann, Phys.Lett.141B(1984)223.

LECTURE 4: PURELY FERMIONIC MODELS: THE CHIRAL GROSS-NEVEU MODEL.

§4.1 The model

The chiral Gross-Neveu model [44] (CGN) dynamics is defined by the Lagrangian:

$$\mathcal{L} = -i\bar{\Psi}\not{\partial}\Psi + \mathcal{L}_{int} \tag{1}$$

$$\mathcal{L}_{int} = -\frac{g^2}{4N}\left[(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2\right] \tag{2}$$

with the fermions in the fundamental representation of SU(N). It is invariant under global U(1)xU(1) and global SU(N)xSU(N) transformations:

$$\Psi \to \Omega\Psi \ , \quad \Omega \in SU(N) \tag{3}$$

A Fierz-type transformation can be used to write $\mathcal{L}_{int}$ in terms of U(N) generators in the form:

$$\mathcal{L}_{int} = -\frac{g^2}{2N}\left(\bar{\Psi}\gamma_\mu\lambda^a\Psi\right)^2 \qquad a = 0, 1, 2 \ldots, N^2-1 \tag{4}$$

with

$$\begin{aligned} \lambda^0 &= I/2 \\ \lambda^a &= t^a \ , \quad a = 1, 2, \ldots, N^2-1 \end{aligned} \tag{5}$$

Exactly as in the abelian case, we can introduce auxiliary vector fields in order to eliminate quartic interactions. To this end, we use the identity:

$$e^{\frac{g^2}{2N}\int(\bar{\Psi}\gamma_\mu\lambda^a\Psi)^2 d^2x} = \int \mathcal{D}A^0_\mu \mathcal{D}A_\mu \, e^{-\int\left[-\frac{g}{\sqrt{N}}\bar{\Psi}\left(\not{A}t + \not{A}^0\frac{I}{2}\right)\Psi + \frac{A_\mu^2}{2} + \frac{A_\mu^{0\,2}}{2}\right] d^2x} \qquad (6)$$

At this point it is interesting to note that keeping the CGN Lagrangian in the form (1)-(2), the auxiliary fields that eliminates the quartic interaction have to be scalars. As originally shown by Gross and Neveu [44] these fields are helpful to explore global aspects of the theory and to perform 1/N expansions; in this way, relevant aspects of the theory, like dynamical mass generation and asymptotic freedom were discovered. However, as we shall see below, some features where the symmetry plays an important rôle (for example the realization of the chiral symmetry [27]) can be more clearly discussed in a framework where Lagrangians whith one component (where the introduction of scalars is evident) and N-component fermions (where the SU(N) symmetry affects neatly the properties of the system) are not treated on the same footing.

The generating functional for the CGN model then reads:

$$Z = \mathcal{N}\int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}A^a_\mu \, e^{-\int\left[\mathcal{L}'(A^a_\mu, \bar{\Psi}, \Psi) + \mathcal{L}_{sources}\right] d^2x} \qquad (7)$$

$$\mathcal{L}_{sources} = \int(\bar{\Theta}\Psi + \bar{\Psi}\Theta)\, d^2x \qquad (8)$$

$$\mathcal{L}' = -\bar{\Psi}\not{D}\Psi + \frac{1}{2}(A_\mu^2 + A_\mu^{0\,2})$$

(9)

Here A_μ^0 is an abelian vector field while A_μ is a $2(N^2-1)$ component vector. The covariant derivative is defined as:

$$D_\mu = i\partial_\mu + \frac{g}{\sqrt{N}}\left(A_\mu^0 \frac{I}{2} + A_\mu^a t^a\right) \tag{10}$$

Following Ref.[45] we first extract the U(1) part of the fermionic variables by considering an Abelian chiral transformation, already familiar from Lecture 2. Namely, we perform the change of variables:

$$\begin{aligned} \Psi &= e^{i\eta_0 + \gamma_5 \phi_0}\chi = \mathcal{U}_0 \chi \\ \bar{\Psi} &= \mathcal{U}_0^* \bar{\chi} \end{aligned} \tag{11}$$

$$A_\mu^0 = -\frac{\sqrt{N}}{g}\left(\varepsilon_{\mu\nu}\partial_\nu \phi_0 - \partial_\mu \eta_0\right) \tag{12}$$

In terms of the new variables, Z reads:

$$\begin{aligned} Z = \mathcal{N} \int & \mathcal{D}\bar{\chi}\mathcal{D}\chi\mathcal{D}A_\mu^a \mathcal{D}\phi_0 \mathcal{D}\eta_0 \, J_{A^0} J^F_{U(1)} \times \\ & \times \exp\left\{-\int\left[\bar{\chi} D' \chi + \frac{A_\mu^2}{2} - \frac{N^2}{2e^2}(\partial_\mu \phi_0)^2 - \frac{N^2}{2e^2}(\partial_\mu \eta_0)^2\right] d^2x\right\} \\ & \times \exp\left\{\int\left[\bar{\theta}\, e^{(\gamma_5 \phi_0 + i\eta_0)}\chi + \bar{\chi}\, e^{\gamma_5 \phi_0 - i\eta_0}\theta\right] d^2x\right\} \end{aligned} \tag{13}$$

where

$$D'_\mu = i\partial_\mu + \frac{g}{\sqrt{N}} A_\mu \tag{14}$$

The Jacobian $\mathcal{J}_{A^0}$ is the trivial one given by eq.(7) in Lecture 2 while the fermionic one $\mathcal{J}^F_{U(1)}$ is:

$$\log \mathcal{J}^F_{U(1)} = -\frac{N}{2\pi} \int d^2x \left(\partial_\mu \phi_0\right)^2 \tag{15}$$

The generating functional takes then the form:

$$\begin{aligned} Z = \mathcal{N} \int & \mathcal{D}\bar{\chi}\,\mathcal{D}\chi\,\mathcal{D}A_\mu\,\mathcal{D}\phi_0\,\mathcal{D}\eta_0 \;\times \\ & \times \exp\left\{ \int d^2x \left[-\bar{\chi}\,\partial\!\!\!/\,'\chi + \frac{A_\mu^2}{2} + \frac{N}{2g^2}\left(1+\frac{g^2}{\pi}\right)\left(\partial_\mu \phi_0\right)^2 + \frac{N^2}{2g^2}\left(\partial_\mu \eta_0\right)^2 \right. \right. \\ & \qquad \left. \left. + \text{source terms} \right] \right\} \end{aligned} \tag{16}$$

Next, we shall consider a non-Abelian transformation involving the generators of the SU(N) group:

$$\begin{aligned} \chi &= \mathcal{U}^{-1}(x)\,\mathcal{U}_5(x)\,\xi \\ \bar{\chi} &= \bar{\xi}\,\mathcal{U}_5(x)\,\mathcal{U}(x) \end{aligned} \tag{17}$$

$$\gamma^\mu A_\mu = \mathcal{U}^{-1}\left(i\frac{\sqrt{N}}{g}\,\partial\!\!\!/\,\mathcal{U}_5\right)\mathcal{U}_5^{-1}\,\mathcal{U} - \mathcal{U}^{-1}\, i\frac{\sqrt{N}}{g}\,\partial\!\!\!/\,\mathcal{U} \tag{18}$$

where

$$\begin{aligned} \mathcal{U} &= e^{i\eta} \\ \mathcal{U}_5 &= e^{\gamma_5 \phi} \end{aligned} \tag{19}$$

Under this change of variables the effective lagrangian becomes now:

$$\mathcal{L}_0'' = \bar{\varsigma}\left(-i\not{\partial}\varsigma\right) + \frac{N}{2g^2}\left(1+\frac{g^2}{\pi}\right)\left(\partial_\mu\phi_0\right)^2 + \frac{N}{2g^2}\left(\partial_\mu\eta\right)^2$$
$$+ \mathrm{tr}\left[u^{-1}\frac{i\sqrt{N}}{g}(\not{\partial}U_5)U_5^{-1}u - i\frac{\sqrt{N}}{g}u^{-1}\not{\partial}u\right]^2 \qquad (20)$$

or

$$\mathcal{L}'' = \bar{\varsigma}\, i\not{\partial}\varsigma + \frac{N}{2g^2}\left(1+\frac{g^2}{\pi}\right)\left(\partial_\mu\phi_0\right)^2 + \frac{N}{2g^2}\left(\partial_\mu\eta_0\right)^2 + \mathrm{tr}\frac{A_\mu^2}{2} \qquad (21)$$

Hence, the generating functional in terms of the new variables reads:

$$Z = N\int \mathcal{D}\bar{\varsigma}\mathcal{D}\varsigma\, e^{-\int\bar{\varsigma}i\not{\partial}\varsigma\, d^2x}\, \mathcal{D}\eta^0\mathcal{D}\phi^0\mathcal{D}\eta^a\mathcal{D}\phi^a$$
$$\times \mathcal{J}_F^{SU(N)}\mathcal{J}_A\, e^{-\int\left[\frac{N}{2g^2}\left(1+\frac{g^2}{\pi}\right)\left(\partial_\mu\phi_0\right)^2 + \frac{N}{2g^2}\left(\partial_\mu\eta_0\right)^2\right]d^2x} \qquad (22)$$
$$\times e^{-\mathrm{tr}\int A_\mu^2[\phi_a,\eta_a]\,d^2x} \times e^{-\int\mathcal{L}_{\text{sources}}\,d^2x}$$

Here $\mathcal{J}_F^{SU(N)}$ represents the Jacobian of the U_5 change in the fermionic variables. We can use Rothe result [38] for it [beeing A_μ a true vector field, one does not have to fix the gauge and hence one has necessarily to work in a general gauge].Concerning the change in the vector field,

$$\mathcal{D}A_0^a \mathcal{D}A_1^a = J_A \, \mathcal{D}\phi^a \mathcal{D}\eta^a \qquad a = 1, 2, \dots, N^2-1 \tag{23}$$

it will now depend on ϕ^a and η^a (remember that for QCD_2 in the decoupling gauge, it just contributed with a Wess-Zumino type Lagrangian).

Even without computing $J_F^{SU(N)}$ and J_A one can already see from (22) a remarkable property of the model: First, the massless excitations coming from the U(1) charge decouple from the rest as it was conjectured by Witten [27] and confirmed at the spectral level by Andrei and Lowenstein[46]. Moreover, in agreement with the work of Rothe and Swieca [47] we can show that this part of the Green functions, that indeed factorizes, coincides with the Thirring model U(1) part, computed in Lecture 2. To see all these properties, let us compute the two-point correlation function from the generating functional (22):

$$\langle \Psi_\ell(x) \bar{\Psi}_{\ell'}(0) \rangle = \frac{1}{Z} \frac{\delta^2 Z}{\delta \bar{\theta}_\ell \delta \theta_{\ell'}} \bigg|_{\theta=\theta'=0} = \tag{24}$$

$$= \frac{1}{Z} \int \mathcal{D}\phi^0 \mathcal{D}\eta^0 \, \mathcal{U}_0 \mathcal{U}_0^* \times$$

$$\times \exp\left\{ -\int \left[\frac{N}{2\beta} (\partial_\mu \phi_0)^2 + \frac{N}{2\alpha} (\partial_\mu \eta_0)^2 \right] d^2x \right.$$

$$\times \int \mathcal{D}\bar{\zeta} \mathcal{D}\zeta \; \zeta_k(0) \bar{\zeta}_{k'}(x) \; e^{-\int \bar{\zeta} i\partial\!\!\!/ \zeta \, d^2x}$$

$$\times \int \mathcal{D}\phi^a \mathcal{D}\eta^a \, J_F^{SU(N)} J_A \times \left(\mathcal{U}^{-1}(x) \mathcal{U}_5(x) \right)_{k\ell} \left(\mathcal{U}(0) \mathcal{U}_5^{(0)} \right)_{k'\ell'}$$

$$\times \; e^{-\frac{1}{4} \int A_\mu^2 \, d^2x}$$

or, shortly,

$$\langle \psi_\ell(x)\bar{\psi}_{\ell'}(0)\rangle = \langle u_0(x)u_0^*(0)\rangle_{u(1)}\langle g_k(x)\bar{g}_{k'}(0)\rangle_0 \times \langle (u^{-1}(x)\, u_5(x))_{k\ell}(u(0)v_5(0))_{k'\ell}\rangle \tag{25}$$

The first factor corresponds to the U(1) part of the Green function and is similar to the one appearing in the Thirring model:

$$\langle u_0(x)u_0^*(0)\rangle = \mu|x|^{-\chi(g)/N} \tag{26}$$

with

$$\chi(g) = \frac{g^2}{2\pi}\left(1 - \frac{1}{1+g^2/\pi}\right) \geq 0 \tag{27}$$

We can then infer that the "almost long range order" of the Kosterlitz-Thouless type occurs[27] in the infrared region (dominated by the massless particles). On the other hand, for short distances the coupling constant goes to zero (due to asymptotic freedom) the dynamically generated mass remains and the chirality carrying term (26) goes to 1 ensuring that in the asymptotically free region the massless part is not important. A more detailed discussion of all these facts can be found in Ref.[45]

References

[44] D.Gross and A.Neveu, Phys.Rev. D10(1974)3235.

[45] K.Furuya, R.E.Gamboa Saraví and F.A.Schaposnik, Nucl.Phys.B208(1982)159.

[46] N.Andrei and J.H.Lowenstein, Phys.Rev.Lett.43(1979)135; Phys.Lett.90B (1980)106, 91B(1980)401.

J.H.Lowenstein in Surveys in High energy physics 2(1981)207.

[47] K.D.Rothe and J.A.Swieca, Nucl.Phys.B168(1980)454.

LECTURE 5: THE "WORLD" OF CURRENTS (IN TWO-DIMENSIONS)

§5.1 Introduction

Since the basic ideas of current algebra were introduced by Gell-Mann[48] in 1961, the subject found an increasing number of applications, both from the theoretical and the experimental sides[49]. An important result already obtained in 1969 [50] is the (unexpected) dependence of current commutators on the interaction, due to the existence of anomalies (related in this framework to the singularity structure of operator products for small separations).

Since operator products can be defined only as a limit, in order to specify a theory completely, one has to specify the nature of the limit. Two-dimensional models were then used in this context in order to investigate the detailed structure of singularities. In this way, the so-called Sugawara construction[51] was established for different fermion models in two-dimensions [52]-[53] showing the possibility of providing a complete formulation of an interacting theory in terms of current dynamics (in the same way equal-time (e.t.) commutation algebra of fields does in conventional canonical field theory).

The interest in constructing the commutator algebra of currents and energy momentum tensor revived when it was observed that this construction lead to very simple realizations of Kac-Moody and Virasoro algebras which can then be used to analyse the spectrum of candidate string theories [54]-[57]. This fact was crucial in the derivation by Witten[22] of the non-abelian bosonization rules. Of course, these investigation were originally restricted to conformally invariant field theories but the understanding of current algebra in other models (like QCD_2, chiral Schwinger model, etc) may have interesting applications in physics and, in particular, in string theories [13],[58].

We shall discuss in this lecture a systematic way of getting current algebra results while working in the path-integral framework. In particular, we shall apply our approach to the well-known Schwinger-model current algebra and then present the more interesting non-Abelian case where a Kac-Moody type algebra is gotten for the QCD_2 model.

§5.2 Current commutators and the path-integral

Once the fermion generating functional is properly defined (i.e. the fermion determinant is adequately regularized) any correlation function can be obtained by functional differentiation with no need of new regularizations. In particular, vacuum expectation values of fermionic currents for the abelian model

$$\langle J_\mu \rangle = e \langle \bar{\Psi}(x) \gamma_\mu \Psi(x) \rangle \tag{1}$$

or the non-Abelian one,

$$\langle J_\mu^a \rangle = e \langle \bar{\Psi}(x) \gamma_\mu t^a \Psi(x) \rangle \tag{2}$$

can be obtained from the fermionic generating functional

$$Z_F = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, e^{-\int \bar{\Psi}(i\not\partial + e\not{A})\Psi dx} = \det \not{D}(A) \tag{3}$$

by using a gauge-invariant regularization prescription for the fermion determinant definition (as for example the ζ-function one [59]).

$$\langle J_\mu(x)\rangle = -\frac{1}{Z_F}\frac{\delta Z_F}{\delta A_\mu(x)} = -\frac{\delta}{\delta A_\mu(x)}\log\det \not{D}(A) \tag{4}$$

$$\langle J^a_\mu(x)\rangle = -\frac{\delta}{\delta A^a_\mu(x)}\log\det\not{D}(A)$$

Other possible approach starts by the observation that one can formally differentiate an unregularized generating functional (i.e. an unregularized determinant),

$$-\frac{\delta}{\delta A_\mu}\log\det\not{D} = -\frac{\delta}{\delta A_\mu}\operatorname{tr}\log\not{D} = -\operatorname{tr}\gamma_\mu G(x,x) \tag{5}$$

$$\not{D}(A)\,G(x,y) = \delta(x-y) \tag{6}$$

and then define the current v.e.v. by gauge-invariantly regularizing (5), for example by the point-splitting method, as first explained by Schwinger [60]:

$$\langle \tilde{J}_\mu(x)\rangle = -\lim_{y\to x}\operatorname{tr}\gamma_\mu G(x,y)\,e^{-ie\int_x^y A_\mu dz^\mu} \tag{7}$$

Here the phase-factor ensures the gauge invariance of the result. Of course both definitions yield to the same result. From the v.e.v. of currents and product of currents one can compute current commutators using the Bjorken-Jhonson-Low (BJL) method [61][62]. Indeed, consider the current-current correlation function:

$$G_{\mu\nu}(x,y) \equiv \langle J_\mu(x)\,J_\nu(y)\rangle = \frac{1}{Z_F}\frac{\delta^2 Z_F}{\delta A_\mu(x)\,\delta A_\nu(y)} \tag{8}$$

Noting that the l.h.s. in (8) is a time-ordered v.e.v., one has

$$\langle [J_\mu(x), J_\nu(y)]_{e.t} \rangle = \lim_{\varepsilon \to 0} \left[G_{\mu\nu}(\vec{x}, t+\varepsilon; \vec{y}, t) - G_{\mu\nu}(\vec{x}, t-\varepsilon; \vec{y}, t) \right] \tag{9}$$

and hence, from the knowledge of $G_{\mu\nu}$ one can derive the current commutators.

§5.3 The abelian case

Let us consider as a first simple example the Schwinger Model. We have, from eq.(33) in Lecture 2

$$\det \not{D}(A) = J \det i\not{\partial} = \mathcal{N} e^{-(e^2/2\pi)\int A_\mu [\delta_{\mu\nu} - \partial_\mu \Box^{-1} \partial_\nu] A_\nu d^2x} \tag{10}$$

and hence using eq.(3):

$$\langle J_\mu(x) \rangle = \frac{e^2}{\pi} \left(A_\mu - \partial_\mu \Box^{-1} \partial_\nu A_\nu \right) \tag{11}$$

In order to compute $G_{\mu\nu}$ we write, instead of (8),

$$G_{\mu\nu}(x,y) = -\frac{\delta \langle J_\mu(x) \rangle}{\delta A_\nu(y)} + \langle J_\mu(x) \rangle \langle J_\nu(y) \rangle \tag{12}$$

It is the first term the one which contributes to the commutator of currents since, using (9), the second one cancels within the BJL limit .

Now,

$$\frac{\delta \langle J_\mu(x) \rangle}{\delta A_\nu(y)} = \frac{e^2}{\pi} \left[\delta_{\mu\nu} \delta(x-y) + \partial^x_\mu \partial^z_\nu \Box^{-1}_{xz} \, \delta(z-y) \right) \tag{13}$$

or

$$\frac{\delta \langle J_\mu(x)\rangle}{\delta A_\nu(y)} = \frac{e^2}{\pi}\left[\delta_{\mu\nu}\delta(x-y) - \partial_\mu^x \partial_\nu^x \Box_{xy}^{-1}\right] \tag{14}$$

Then, we have for $G_{\mu\nu}$

$$G_{\mu\nu} = \langle J_\mu(x) J_\nu(y)\rangle - \frac{e^2}{\pi}\delta_{\mu\nu}\delta(x-y) - \frac{e^2}{2\pi^2}\partial_\mu^x \partial_\nu^y \log\mu|x-y| \tag{15}$$

where we have used:

$$\Box_{xy}^{-1} = \frac{1}{2\pi}\log\mu|x-y| \tag{16}$$

Now, we can construct $[J_0, J_1]_{e.t}$ from (9):

$$\langle [J_0, J_1]_{e.t}\rangle = \lim_{\varepsilon\to 0}\frac{2e^2}{\pi^2}\frac{\varepsilon(x_1-y_1)}{[(x_1-y_1)^2+\varepsilon^2]^2} = \lim_{\varepsilon\to 0} -\frac{e^2}{\pi^2}\partial_1 \frac{\varepsilon}{\varepsilon^2+(x_1-y_1)^2} \tag{17}$$

and hence

$$[J_0(x_1,t), J_1(y_1,t)] = -\frac{e^2}{\pi}\delta'(x_1-y_1) \tag{18}$$

which gives the usual (finite in two dimensions) Schwinger term . The other commutators are of course zero.

§5.4 The non-abelian case: QCD_2

Instead of differentiating the fermion determinant (eq.(18)

in Lecture 3) it is easier to use eq.(7) in order to construct current commutators for QCD_2.

First, note that in the decoupling gauge one knows exactly the fermion Green function:

$$G(x,y) = e^{\gamma_5 \phi(x)} G_0(x,y) e^{\gamma_5 \phi(y)} \tag{19}$$

with G_0 the free fermion Green function,

$$G_0(z) = \frac{i}{2\pi} \frac{\not{z}}{z^2} \tag{20}$$

Calling $\varepsilon_\mu = x_\mu - y_\mu$ we have,

$$\langle J_\mu^a(x) \rangle = -\mathrm{tr}\, \gamma_\mu t^a e G(x,x)\big|_{Reg} =$$

$$\langle J_\mu^a(x) \rangle = \lim_{\varepsilon \to 0} -\frac{iee}{2\pi}^{\gamma_5 \phi(x)} \frac{\not{\varepsilon}}{\varepsilon^2} e^{\gamma_5 \phi(x+\varepsilon)} e^{ieA_\sigma \varepsilon^\sigma} \tag{21}$$

One has to expand all exponentials and use:

$$\not{A} = -\frac{i}{e} \left(\not{\partial} e^{\gamma_5 \phi} \right) e^{-\gamma_5 \phi} \tag{22}$$

Then, taking the symmetric $\varepsilon \to 0$ limit,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^\mu \varepsilon^\nu}{\varepsilon^2} = \frac{1}{2} \delta^{\mu\nu} \tag{23}$$

we have:

$$\langle J_\mu^a(x)\rangle = -\frac{ie^2}{2\pi}\, \mathrm{tr}\, t^a \gamma_\nu \gamma_5 a_\nu \gamma_\mu \tag{24}$$

$$a_\nu = \frac{1}{2e}\left[e^{-\phi}, \partial_\nu e^{\phi}\right]$$

or

$$\langle J_\mu^a(x)\rangle = -\frac{e^2}{2\pi}\, a_\nu^a \, \varepsilon_{\mu\nu} \tag{25}$$

a result first given in Ref.[36]. It is interesting to note that, defining

$$J_\pm = \langle J_0 \rangle \pm i \langle J_1 \rangle \tag{26}$$

one has:

$$J_+ = -\frac{e}{4\pi}\, \mathcal{U} D_+ \mathcal{U}^{-1} \tag{27}$$

$$J_- = -\frac{e}{4\pi}\, \mathcal{U}^{-1} D_- \mathcal{U}$$

with $\mathcal{U} = e^{2\phi}$

These are the currents for QCD_2 (The gauge field taken as a background). They differ from the free ones, first written by Witten [22] in the bosonized form, in the fact they contain a covariant (and not ordinary) derivative. One can however define new currents:

$$\overset{\circ}{J}_{\pm} = \mathcal{U}^{1/2} J_{\pm} \mathcal{U}^{-1/2} \tag{28}$$

which do coincide with the free bosonized fermion currents.

In order to compute current commutators we again evaluate $G_{\mu\nu}(x|y)$ taking care this time of the point-splitting. The answer is[37]:

$$\left[J_{\pm}^{a}(x), J_{\pm}^{b}(y) \right]_{e.t} = f^{abc} J_{\pm}^{c}(x) \delta(x_1 - y_1) + \frac{i}{2\pi} D_{\pm}^{ab} \delta(x_1 - y_1) \qquad [29]$$

Again, the current algebra for QCD_2 is similar to that arising in the case of free fermion models except that, instead of a normal Schwinger term, we have the covariant derivative of the δ-function, due to the fact we are considering an interacting model. However, the algebra of $\overset{\circ}{J}_{\pm}$ defined in (28) is an ordinary Kac-Moody one.

It is interesting to note that the energy momentum tensor algebra can be also determined in the path-integral framework by functional differentiating the fermion determinant. First, one computes:

$$\langle T_{\mu\nu}(x) \rangle = 2 \frac{\delta \log \det D}{\delta g^{\mu\nu}(x)} \tag{30}$$

and then one uses the BJL method to determine the commutator algebra [63]. For example we have:

$$\langle T_{++} \rangle = 2\pi \, \mathrm{tr} \langle J_{+} \rangle^{2} \quad , \quad \langle T_{--} \rangle = 2\pi \, \mathrm{tr} \langle J_{-} \rangle^{2} \tag{31}$$

with

$$T_{++} = T_{00} + T_{01}$$
$$T_{--} = T_{00} - T_{01} \qquad (32)$$

and:

$$\left[T_{++}(x), J_+^b(y)\right]_{e.t.} = i J_+^a(x) D_+^{ab} \delta(x_1 - y_1) \qquad (33)$$

Since

$$\mathrm{tr}\langle J_+\rangle^2 = \mathrm{tr}\,\mathring{j}_+^2 \qquad (34)$$

the $\left[T_{\pm\pm}, T_{\pm\pm}\right]$ commutators coincide with those of a free theory.

References

[48] M.Gell-Mann, Phys.Rev.125(1962)1067; Physics 1(1964)63 and references therein.

[49] S.Treiman, R.Jackiw and D.Gross, Lectures on Current Algebra and its applications, Priceton Univ.Press, 1972.

[50] S.Adler, Phys.Rev.177(1969)2426

J.S.Bell and R.Jackiw, Nuovo Cim 60A(1969)67.

[51] H.Sugawara, Phys.rev.170(1968)1659.

[52] S.Coleman, D.Gross and R.Jackiw, Phys.Rev.180(1969)1359.

[53] C.G.Callan,R.F.Dashen and D.H.Sharp, Phys.rev.165(1968)1883.

[54] K.Bardakci and M.Halpern, Phys.Rev.D3(1971)2493.

[55] D.Nemechansky and S.Yankielowicz, Phys.Rev.Lett.54(1985)502.

[56] S.Antoniadis and C.Bachas, SLAC preprint 3625, 1985.

[57] S.Jain, R.Shankar and S.Wadia, Phys.Rev.D32(1985)2713.

[58] D.Chang,A.Kumar, R.Mohapatra, Maryland preprint 86 (unpublished)

[59] R.E.Gamboa Saraví, F.A.Schaposnik and J.E.Solomin, Journal of Math. Phys. 26(1985)2045.

[60] J.Schwinger, Phys.rev.82(1951)664 ; Phs.Rev.Lett.3(1959)296.

[61] J.D.Bjorken, Phys.rev.148(1966)1467.

[62] K.Johnson and F.E.Low, Progr.Theor.Phys.(Kyoto)Supp.37-38(1966)74.

[63] R.E.Gamboa Saraví, F.A.Schaposnik and J.E.Solomin, Phys.Rev.D33 (1986)3762.

LECTURE 6: QUANTIZATION OF THEORIES WITH WEYL FERMIONS

§6.1 Introduction

After Polyakov's beautiful paper on quantization of strings[64] it has become clear that theories with some symmetry (like conformal symmetry in the case of strings, gauge symmetry in the case of Yang-Mills theory, Einstein and Lorentz symmetry in the case of gravitation, ...) may have a peculiar behavior: even if the degrees of freedom associated with that symmetry are decoupled at the classical level (i.e. they do not appear in the classical equations of motion) they can reappear playing a physical rôle at the quantum level.

This idea was extended by Faddeev and Shatashvili [65] in their proposal of quantizing gauge theories with Weyl fermions (a potentially anomalous case) by adding a physical chiral field which cancelled out the anomaly. However, in this approach, this field was added *manu militari* while Polyakov's idea was that the model itself already contains (hidden at the classical level) new physical degree of freedom charged of absorbing the anomaly.

Very recently it was proved that this is indeed what happens for gauge theories [66]-[68] and gravitational ones [69] and to this point it is addressed this last lecture.

§6.2 Gauge theories with Weyl fermions

Consider a gauge theory coupled with Weyl fermions. The generating functional reads:

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{-S[A,\bar{\Psi},\Psi]} \tag{1}$$

The action is invariant under the gauge transformation:

$$A_\mu \to A_\mu^g = g A_\mu g^{-1} + i g \partial_\mu g^{-1} \tag{2}$$

$$\Psi \to \Psi^g = g\Psi \; ; \quad \bar{\Psi} \to \bar{\Psi}^g = \bar{\Psi} g^{-1} \tag{3}$$

For definiteness, we shall consider left-handed fields,

$$\Psi = -\gamma_5 \Psi \tag{4}$$

Now, as it is well-known, Z has to be considered as an integral over the whole space of connections rather than as an integral over the orbit space. (See Ref.[70] for a discussion). Were the fermions Dirac fermions, the distinction makes no difference: in the fixing gauge procedure one passes from the integration on the whole A_μ-space to that over the orbit space by factorizing a trivial integration over the gauge-group volume. The presence of Weyl fermions radically changes this situation.

The group space factorization is habitually obtained by following the original Faddeev-Popov procedure[71]: one writes a " resolution of the identity " in the form

$$1 = \Delta_{FP}[A] \int \mathcal{D}g \, \delta(F[A^g]) \tag{5}$$

and then inserts it in eq.(1). Here $F[A] = 0$ is the gauge condition choosing one representative of A_μ on each orbit. Using (5) in (1) as well as the trivial identities:

$$\mathcal{D}A_\mu = \mathcal{D}A_\mu^g \tag{6}$$

$$\Delta_{FP}[A] = \Delta_{FP}[A^g] \tag{7}$$

In pure gauge theories or theories with Dirac fermions (in which the fermion integration measure satisfies an identity of the kind (6)) the group integration factorizes trivially and the generating functional is defined in a fixed gauge (a certain section).

In the case of Weyl fermions one cannot define a gauge-invariant measure (we shall explain the reason below). Hence, instead of a relation like (6) one has for the fermionic measure:

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = \mathcal{J}(g,A)\,\mathcal{D}\bar{\Psi}^g\mathcal{D}\Psi^g \tag{8}$$

where J is the Fujikawa Jacobian:

$$\mathcal{J}(g,A) = \det D(A) / \det D(A^g) \tag{9}$$

whose non-triviallity signals the existence of an anomaly.

It is for that reason that the group integration does not factorize: J depends not only on g but also on A_μ (through the regularization prescription). Instead of the habitual result one has:

$$Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}A_\mu\,\delta(F[A])\,\mathcal{J}(g,A^{g^{-1}})\,e^{-S[A,\bar{\Psi},\Psi]}\,\Delta_{FP}[A] \tag{9}$$

This expression coincides with the one proposed by Faddeev-Shatashvilli[65] except that in the present approach g has not been added by hand as a new physical degree of freedom. It naturally appears in the fixing gauge procedure.

It is easy to show that

$$\mathcal{J}(g,A) = \mathcal{J}(h,A)\,\mathcal{J}(h^{-1}g, A^{h}) \qquad (11)$$

(a so-called one-cocycle condition, as can be seen by writing $\mathcal{J} = e^{2\pi i w_1}$)

Using this property it is easy to show that the resulting theory is gauge invariant. Indeed, consider the effective action defined after integration over fermions and the g-field:

$$e^{-S_{eff}[A]} \equiv \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}g\, e^{-S[A,\bar{\Psi},\Psi]}\,\mathcal{J}(g, A^{g^{-1}}) \qquad (12)$$

For a gauge transformed A_μ^h, one has:

$$\begin{aligned} e^{-S_{eff}[A^h]} &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}g\, e^{-S[A^h,\bar{\Psi},\Psi]}\,\mathcal{J}(g, A^{hg^{-1}}) \\ &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}g\, e^{-S[A,\bar{\Psi}^{h'},\Psi^{h'}]}\,\mathcal{J}(g, A^{hg^{-1}}) \end{aligned} \qquad (13)$$

Making the change of variables:

$$\begin{aligned} \Psi &\to h^{-1}\Psi = \Psi' \\ \bar{\Psi} &\to \bar{\Psi}h = \bar{\Psi}' \end{aligned} \qquad (14)$$

with the Jacobian:

$$\mathcal{D}\bar{\Psi}'\mathcal{D}\Psi' = \mathcal{J}(h,A)\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \tag{15}$$

we have:

$$e^{-S_{eff}[A^h]} = \int \mathcal{D}\bar{\Psi}'\mathcal{D}\Psi'\mathcal{D}g\, e^{-S[A,\bar{\Psi}',\Psi']} \mathcal{J}(g,A^{hg^{-1}})\mathcal{J}^{-1}(h,A) \tag{16}$$

But, from eq.(11)

$$\mathcal{J}(g,A^{hg^{-1}}) = \mathcal{J}(h,A)\,\mathcal{J}(gh^{-1},A^{hg^{-1}}) \tag{17}$$

and hence, using

$$\mathcal{D}g = \mathcal{D}(gh^{-1}) \tag{18}$$

we finally have:

$$e^{-S_{eff}[A^h]} = e^{-S_{eff}[A]} \tag{19}$$

From this fact we can conclude that the fermion gauge-current:

$$\langle J^a_\mu(x)\rangle = \frac{\delta S_{eff}}{\delta A^a_\mu(x)} \tag{20}$$

is conserved:

$$\langle D_\mu J^\mu \rangle = 0 \tag{21}$$

Let us now understand why the Jacobian relating the Weyl fermion measures before and after a gauge transformation is non-trivial. The point is that the Dirac operator appearing in the generating functional (1) maps negative chirality spinors into positive chirality ones and consequently it does not have a well-defined eigenvalue problem. This is the reason why the definition of Weyl-fermion determinants is problematic [2],[72]-[73]. To handle this problem, one can define an operator $\hat{D}$ acting on Dirac fermions,

$$\hat{D}(A) = \not{D}(A)\frac{(1-\gamma_5)}{2} + i\not{\partial}\frac{(1+\gamma_5)}{2} = i\not{\partial} + e\not{A}\frac{(1-\gamma_5)}{2} \tag{22}$$

which then leads to a well-defined eigenvalue problem. The doubling in the number of degrees of freedom implied by (22) affects only the normalization factor since the positive chirality pieces do not couple to the gauge field. One then defines:

$$\det D(A)\Big|_{\text{Weyl}} \equiv \det \hat{D}(A)\Big|_{\text{Dirac}} \tag{23}$$

with the r.h.s. in (23) appropriately regularized since the product of eigenvalues of the Dirac operator (22) grows without bound. The crucial point in this scheme is that $\hat{D}(A)$'s eigenvalues are not gauge invariant, this being the origin of the non-triviality of J.

It is important to note that J can be closely computed in any number of dimensions. Indeed, for an infinitesimal transformation:

$$g = 1 + i\delta\theta \tag{24}$$

the corresponding Jacobian is, as we have seen:

$$\mathcal{J}(\delta\theta) = e^{i\,\mathrm{tr}\int \mathcal{A}(A)\,\delta\theta\,dx} \tag{25}$$

with $\mathcal{A}$ the anomaly,

$$\langle D_\mu J_\mu \rangle_{\text{Fermions}} = \mathcal{A}(A) = -\frac{\delta \log \mathcal{J}}{\delta\theta} \tag{26}$$

The finite transformation Jacobian is gotten just by iteration of infinitesimal transformations. The simplest way is to build up this finite rotation by introducing a parameter t, $0 \leq t \leq 1$. The answer is, as we have seen:

$$\mathcal{J}(g, A) = \frac{\det D(A)}{\det D(A^g)} = e^{-\mathrm{tr}\int \mathcal{A}(A^{g(t)})\,\theta\,dt} \tag{27}$$

where:

$$g(t) = e^{it\theta} \qquad\qquad g(1) = g \tag{28}$$

From the effective action (12) one can compute v.e.v.'s of product of currents. For example the current-current correlation function is given by:

$$\langle J_\mu(x) J_\nu(y) \rangle = \frac{\delta^2 S_{eff}}{\delta J_\mu(x)\,\delta J_\nu(y)} - \frac{\delta S_{eff}}{\delta J_\mu(x)}\frac{\delta S_{eff}}{\delta J_\nu(y)}$$

Then, current commutators can be evaluated as we did in the previous lecture. In Ref.[74] we have shown, in this way, that the current commutators for the Scwhinger model coupled to Weyl fermions (chiral Schwinger model) are those expected in a non-anomalous theory:

$$\begin{aligned} [j_0, j_0]_{e.t} &= 0 \\ [j_0, j_1]_{e.t} &= -\frac{e^2}{4\pi}\frac{a^2}{a-1}\delta'(x-y) \end{aligned} \tag{29}$$

It is interesting to discuss the presence of the parameter a. Once the definition (22) is accepted, there is no gauge-principle to invoke justifying the choice of a particular regularization for the fermion determinants since, as we stated before, $\hat{D}(A)$ is not gauge covariant. In two dimensions, a general operator which can be a candidate in a heat-kernel regularization scheme is:

$$D_{Reg} = \hat{D}(A) + a \not{A} \frac{(1+\gamma_5)}{2} \tag{30}$$

Here a is an arbitrary parameter. Now, from (29) and the usual arguments about the definiteness of the Schwinger term sign, one necessarily has $a > 1$. This was also discovered in the original investigations on the chiral Schwinger model[75].

Concerning more realistic models, we can conclude that the correct treatment of the gauge degrees of freedom ensures that the resulting quantum theory is non-anomalous. The study of the effective Lagrangian, which contains a Wess-Zumino term comming from the chiral Jacobian, will decide if the theory is consistent (in the sense the gauge group acts as a physical field and there are no tachyons).

§6.3 Gravitation with Weyl fermions

It is always possible to define a Weyl fermion measure invariant under general coordinate transformations (Einstein transformations) [76] but, in spaces of dimension 2 [mod 4], it is not possible simultaneously to mantain local Lorentz invariance. This is the manifestation, in the path-integral framework, of the gravitational anomalies.

Exactly as it hapens in gauge theories, this fact makes the quantization of gravitation with Weyl fermions a delicate problem. It has been shown [69] that a careful treatment of the symmetries at the quantum level, makes the Lorentz group to acquire the status of a physical field and the resulting quantum theory non-anomalous.

The generating functional is, in this case, of the form:

$$Z = \int \mathcal{D}\mu(g)\,\mathcal{D}(g^{1/4}\bar{\psi})\,\mathcal{D}(g^{1/4}\psi)\,e^{-S[h,\bar{\psi},\psi]} \tag{31}$$

where $\mathcal{D}\mu(g)$ is the integration measure over the metric $g_{\mu\nu}$; h are the vierbeins

$$h_{\nu}{}^{a} h_{\mu a} = g_{\mu\nu} \quad ; \quad g = \det g_{\mu\nu} \geq 0 \tag{32}$$

in Euclidean space and the fermionic measure is defined in a coordinate invariant way [77]. The action contains a gravitational part (whose explicit form is unimportant for the present discussion) and a fermionic part:

$$S_F = \int \sqrt{g}\, dx\, \bar{\Psi} \gamma^c h_c^{\ \mu} \left(\overleftrightarrow{\partial}_\mu - \frac{i}{4} \sigma_{ab} \omega_\mu^{ab} \right) \left(\frac{1-\gamma_5}{2} \right) \Psi \qquad (33)$$

The invariance of S under Einstein and Lorentz transformations makes necessary two "gauge conditions":

$$F_1 [g_{\mu\nu}] = 0 \quad ; \quad F_2 [h_\mu^{\ a}] = 0 \qquad (34)$$

Again, one inserts the "resolution of the identity" but, when changing variables, the Weyl-fermionic measure is not invariant:

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = J(\lambda, h)\, \mathcal{D}\bar{\Psi}^\lambda \mathcal{D}\Psi^\lambda \qquad (35)$$

with λ a Lorentz transformation:

$$h_a^{\ \mu,\lambda} = h_a^{\ \mu} - \lambda_a^{\ b} h^\mu_{\ b} \qquad (36)$$

Hence, the λ-integration does not factorize and the λ-field becomes a physical field:

Again the effective action, defined through the identity:

$$e^{-S_{eff}[h]} = \int \mathcal{D}\lambda \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, J(\lambda, h^{\lambda^{-1}}) \, e^{-S(h,\bar{\Psi},\Psi)} \tag{37}$$

satisfies the identity:

$$S_{eff}[h] = S_{eff}[h^{\lambda}] \tag{38}$$

and hence the theory is non-anomalous.

We have studied the theory in a simple two-dimensional case[69]. We have shown that the λ-field is indeed physical (a massless scalar field for a particular family of regularization prescriptions) but the analysis of more realistic models remains to be performed.

It is interesting to note that in the case of supersymmetric gauge models coupled to (super)gravity (as it is the case of the low-energy limit of certain superstring modles) one may have two non-trivial Jacobians, one associated with the gauge symmetry and the other with the local Lorentz symmetry. The Green-Schwartz[78] cancellation mechanism for certain particular symmetry groups will correspond, in the present path-integral approach, to the cancellation of one Jacobian with the other in such a way both group-integration then trivially factorize.

Acknowledgements

I wish to thank the Centro Brasileiro de Pesquisas Físicas (CBPF/CNPq) for the hospitality extended during the stay at Rio and specially to Profs. J.J. Giambiagi, J.A. Mignaco and F. Simão for the warm welcome and interest along this series of lectures.

References

[64] A.M.Polyakov, Phys.Lett.103B(1981)207.

[65] L.D.Faddeev and S.L.Shatashvili, Phys.Lett.167B(1986)225.

[66] O.Babelon, F.A.Schaposnik and C-M.Viallet, Phys.Lett.177B(1986)385.

[67] A.V.Kulikov, IHEP preprint 86-83 (unpublished)

[68] K.Harada and I.Tsutsui, TIT-HEP preprint, 86 (unpublished)

[69] F.A.Schaposnik and H.Vucetich, La Plata preprint, 86 (unpublished)

[70] O.Babelon and C-M.Viallet, Phys.Lett.85B(1979)246.

[71] L.D.Faddeev and N.V.Popov, Phys.Lett.25B(1967)29.

[72] M.P.Balachandran, G.Marmo, V.P.Nair and C.G.Trahern, Phys.Rev.D25 (1982)2718.

M.B.Einhorn and D.R.T.Jones, Phys.rev.D29(1984)331.

S.K.Hu, B.L.Young and D.W.McKay, Phys.Rev.D30(1984)836.

A.Andrianov and L.Bonora, Nucl.Phys.233(1984)232.

R.E.Gamboa Saraví, F.A.Schaposnik and J.E.Solomin, Phys.Rev.D28 (1983)856.

[73] L.Alvarez Gaumé and P.Ginsparg, Nucl.Phys.B243(1984)449.

[74] M.V.Manías, M.C.vonReichenbach, F.A.Schaposnik and M.Trobo, La Plata preprint 86 (unpublished).

[75] R.Jackiw and R.Rajaraman, Phys.Rev.Lett.54(1985)1219.

[76] W.A.Bardeen and B .Zumino, Nucl.Phys.B244(1984)421.

[77] K.Fujikawa and O.Yamada, Nucl.Phys.B245(1984)436.

[78] M.Green and J.Schwartz, Phys.Lett. 149B(1984)117.