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# Inflationary nonsingular quantum cosmological model

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A stiff matter-dominated universe modeled by a free massless scalar field minimally coupled to gravity in a Friedmann-Lemaître-Robertson-Walker (FLRW) geometry is quantized. Generalized complex-width Gaussian superpositions of the solutions of the Wheeler-DeWitt equation are constructed and the Bohm-de Broglie interpretation of quantum cosmology is applied. A planar dynamical system is found in which a diversity of quantum Bohmian trajectories are obtained and discussed. One class of solutions represents nonsingular inflationary models starting at infinity past from flat space-time with Planckian size spacelike hypersurfaces, which inflates without inflaton but due to a quantum cosmological effect, until it makes an analytical graceful exit from this inflationary epoch to a decelerated classical stiff matter expansion phase.

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## I. INTRODUCTION

For more than 25 years, inflation [1] has been considered a paradigm to solve, at the same time, standard cosmological puzzles related to initial conditions like the flatness, horizon, and isotropy problems, and, as a bonus, astroparticle issues like the monopole excess. More important, it also predicts that primordial fluctuations, assumed to be of quantum origin, could be enhanced to the level required to trigger large scale structure formation, with an almost scale-invariant spectrum [2], which is confirmed by observations [3].

The inflation paradigm is also endowed with two specific problems, that may ultimately be related, namely, the meaning of the trans-Planckian [4] perturbations and the existence of a past singularity [5]. Concerning the latter, the existence of an initial singularity is one of the major drawbacks of classical cosmology. In spite of the fact that the standard cosmological model, based in the classical general relativity theory, has been successfully tested until the nucleosynthesis era (around $t \sim 1s$), the extrapolation of this model to higher energies leads to a breakdown of the geometry in a finite cosmic time. This breakdown of the geometry may indicate that the classical theory must be replaced by a quantum theory of gravitation: quantum effects may avoid the presence of the singularity, leading to a complete regular cosmological model.

Among the fundamental questions that come from the quantization of the Universe as a whole, one of the most important concerns the interpretation of the wave function of the Universe. In order to extract predictions from it, the Bohm-de Broglie (BdB) ontological interpretation of quantum mechanics [6,7] has been proposed [8–10], since it avoids many conceptual difficulties that follow from the application of the standard Copenhagen interpretation to a unique system that contains everything. In opposition to the latter one, the ontological interpretation does not need a classical domain outside the quantized system to generate the physical facts out of potentialities (the facts are there *ab initio* because the positions and trajectories of the particles (called bohmian trajectories) are considered to be part of objective reality, and hence it can be applied to the Universe as a whole. There are other alternative interpretations which can be used in quantum cosmology, like the many worlds interpretation of quantum mechanics [11], but we will not consider them here because they are probabilistic interpretations in essence. As we know [12], it is very difficult to obtain from the Wheeler-DeWitt equation, when applied to a closed universe, a probabilistic interpretation for their solutions because of its hyperbolic nature (see however further approaches [13]). In the case of the Bohm-de Broglie interpretation, probabilities are useful but not essential, as long as objective trajectories (universe histories) can be calculated and their properties studied. Probabilities can be recovered, as it has been suggested many times [14–17], at the semiclassical level, where a probability measure can be constructed with the quantum solutions. Hence, we can take the Wheeler-DeWitt equation as it is, without imposing any probabilistic interpretation at the most fundamental level, but still obtaining information using the Bohm-de Broglie interpretation, and then recover probabilities when we reach the semiclassical level. With this interpretation in hand, one can ask if the quantum scenario predicted by the wave function of the Universe is free of singularities, and which type of classical universe emerges from the quantum phase.

Quantum cosmology, in this framework, exhibits bouncing solutions [18] which can be interpreted as truly avoid-

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ing the singularity. Some of these bouncing models provide solutions to many cosmological puzzles, and may also yield a scale-invariant cosmological perturbation as in inflationary models [19]. Bouncing models are also obtained in loop quantum cosmology [20].

Some types of bouncing models were obtained in [21,22]. In these references, the matter content of the primordial model was considered to be stiff matter,[1] modeled by a free massless scalar field minimally coupled to gravity. The Wheeler-DeWitt equation turns out to be a two-dimensional Klein-Gordon equation, and Gaussian superpositions, with positive width, of negative and positive frequency modes solutions were considered. In the flat case, a two-dimensional dynamical system for the bohmian trajectories was obtained, yielding a variety of possibilities: big bang-big crunch models, oscillating universes, bouncing solutions, and ever expanding (contracting) big bang (big crunch) models with a period of acceleration in the middle of their evolution. No nonsingular purely expanding model with a primordial accelerated phase (a nonsingular inflationary model) was obtained (models of this type were obtained only with a nonminimal coupling between the scalar and gravitational fields [24]).

In the present paper we generalize the Gaussians superpositions of Refs. [21,22] to Gaussians with complex widths and non-negative real part. We obtain a richer two-dimensional dynamical system. When the real part of the width is made zero, we obtain nonsingular inflationary models which expand accelerately in the infinity past from flat universes with finite (not zero) size spatial sections, and is smoothly connected to a classical decelerated stiff matter expanding phase. It has features of the pre-big bang model [25] and the emergent model [26] without a graceful exit problem.

The paper is organized as follows: in the next section the classical model is presented. Section III is devoted to its quantization and the corresponding Wheeler-DeWitt equation is obtained. Generalized Gaussian superpositions of the quantum solutions, and their corresponding dynamical system are studied in Sec. IV. In Sec. V, the nonsingular inflationary model is studied and its properties are presented and discussed. In Sec. VI we present our conclusions.

## II. THE CLASSICAL MINISUPERSPACE MODEL

The model we take contains a massless free scalar field (stiff matter), and the total Lagrangian reads

$$L = \sqrt{-g}\left[\frac{R}{6l^2} - \frac{1}{2}\phi_{;\mu}\phi^{;\mu}\right], \tag{1}$$

where we are using natural units $\hbar = c = 1$, and $l^2 \equiv 8\pi G/3$, which is the Planck length squared in these units.

[1] A perfect fluid with a "super-rigid" equation of state $p = \rho$, that was proposed a long time ago in [23].

We will consider the spatially homogeneous and isotropic space-time line element,

$$ds^2 = -N^2 dt^2 + \frac{a(t)^2}{(1+\epsilon r^2/4)^2} \times [dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)], \tag{2}$$

where the spatial curvature $\epsilon$ takes the values 0, 1, − 1. Inserting this line element into the Lagrangian (1), and omitting a total time derivative, we obtain the following minisuperspace action:

$$S = \int\left(\frac{-\dot{a}^2 aV}{Nl^2} + \frac{N\epsilon aV}{l^2} + \frac{\dot{\phi}^2 a^3 V}{2N}\right)dt, \tag{3}$$

where $V$ is the total volume divided by $a^3$ of the spacelike hypersurfaces, which are supposed to be closed. $V$ depends on the value of $\epsilon$ and on the topology of the hypersurfaces. For $\epsilon = 0$, $V$ can have any value because the fundamental polyhedra of $\epsilon = 0$ hypersurfaces can have arbitrary size (see Ref. [27]). In the case of $\epsilon = 1$ and topology $S^3$, $V = 2\pi^2$.

Usually, the scale factor has dimensions of length because we use angular coordinates in closed spaces. Also, in natural units, the scalar field has dimensions of the inverse of a length. Hence we will define the dimensionless quantities $\bar{a} \equiv \sqrt{2V}a/l$, $\bar{\phi} \equiv l\phi/\sqrt{2}$. Calculating the Hamiltonian, and omitting the bars, yields

$$H = \frac{\sqrt{2V}N}{l}\left(-\frac{p_a^2}{2a} + \frac{p_\phi^2}{2a^3} - \frac{\epsilon a}{2}\right). \tag{4}$$

As $\sqrt{2V}/l$ appears as an overall multiplicative constant in the Hamiltonian, we can set it equal to one without any loss of generality, keeping in mind that the physical scale factor which appears in the metric is $la/\sqrt{2}$, not $a$. We can further simplify the Hamiltonian by defining $\alpha \equiv \ln(a)$ obtaining

$$H = \frac{N}{2\exp(3\alpha)}[-p_\alpha^2 + p_\phi^2 - \epsilon\exp(4\alpha)], \tag{5}$$

where

$$p_\alpha = -\frac{e^{3\alpha}\dot{\alpha}}{N}, \tag{6}$$

$$p_\phi = \frac{e^{3\alpha}\dot{\phi}}{N}. \tag{7}$$

The momentum $p_\phi$ is a constant of motion which we will call $\bar{k}$.

The classical solutions are, in the gauge $N = 1$ (cosmic time):

(1) *For* $\epsilon = 0$*:*

$$\phi = \pm\alpha + c_1, \tag{8}$$

where $c_1$ is an integration constant. In terms of cosmic time $\tau$ they read

$$a = e^{\alpha} = 3\bar{k}\tau^{1/3}, \tag{9}$$

$$\phi = \frac{\ln(\tau)}{3} + c_2. \tag{10}$$

The solutions contract or expand forever from a singularity, depending on the sign of $\bar{k}$, without any inflationary epoch.

(2) *For* $\epsilon = 1$*:*

$$a = e^{\alpha} = \frac{\bar{k}}{\cosh(2\phi - c_1)}, \tag{11}$$

where $c_1$ is an integration constant, and from the conservation of $p_\phi$ we get

$$\bar{k} = e^{3\alpha}\dot{\phi}. \tag{12}$$

The cosmic time dependence is complicated and we will not write it here. These solutions describe universes expanding from a singularity until a maximum size and contracting again to a big crunch. Near the singularity, these solutions behave as in the flat case. There is no inflation.

(3) *For* $\epsilon = -1$*:*

$$a = e^{\alpha} = \frac{\bar{k}}{|\sinh(2\phi - c_1)|}, \tag{13}$$

where $c_1$ is an integration constant, and again, from the conservation of $p_\phi$ we get

$$\bar{k} = e^{3\alpha}\dot{\phi}. \tag{14}$$

As before, the cosmic time dependence is complicated and we will not write it here. These solutions describe universes contracting forever to or expanding forever from a singularity. Near the singularity, these solutions behave as in the flat case. There is no inflation.

Hence, in all models there is at least one singularity and no acceleration phase, as it should be for a classical stiff matter fluid.

## III. QUANTIZATION AND THE BOHM-DE BROGLIE INTERPRETATION

Let us now quantize the model. The Wheeler-DeWitt equation is obtained through the Dirac quantization procedure, where the wave function must be annihilated by the operator version of the Hamiltonian constraint. For the case of homogeneous minisuperspace models, which have a finite number of degrees of freedom, the minisuperspace Wheeler-De Witt equation reads

$$\mathcal{H}(\hat{p}^{\mu}, \hat{q}_{\mu})\Psi(q) = 0. \tag{15}$$

The quantities $\hat{p}^{\mu}$, $\hat{q}_{\mu}$ are the phase space operators related to the homogeneous degrees of freedom of the model. Usually this equation can be written as

$$-\frac{1}{2}f_{\rho\sigma}(q_\mu)\frac{\partial\Psi(q)}{\partial q_\rho \partial q_\sigma} + U(q_\mu)\Psi(q) = 0, \tag{16}$$

where $f_{\rho\sigma}(q_\mu)$ is the minisuperspace DeWitt metric of the model, whose inverse is denoted by $f^{\rho\sigma}(q_\mu)$.

Writing $\Psi$ in polar form, $\Psi = R\exp(iS)$, and substituting it into (16), we obtain the following equations:

$$\frac{1}{2}f_{\rho\sigma}(q_\mu)\frac{\partial S}{\partial q_\rho}\frac{\partial S}{\partial q_\sigma} + U(q_\mu) + Q(q_\mu) = 0, \tag{17}$$

$$f_{\rho\sigma}(q_\mu)\frac{\partial}{\partial q_\rho}\left(R^2\frac{\partial S}{\partial q_\sigma}\right) = 0, \tag{18}$$

where

$$Q(q_\mu) \equiv -\frac{1}{2R}f_{\rho\sigma}\frac{\partial^2 R}{\partial q_\rho \partial q_\sigma} \tag{19}$$

is called the quantum potential.

The Bohm-de Broglie interpretation applied to quantum cosmology states that the trajectories $q_\mu(t)$ are real, independently of any observations. Equation (17) represents their Hamilton-Jacobi equation, which is the classical one added with a quantum potential term Eq. (19) responsible for the quantum effects. This suggests to define

$$p^{\rho} = \frac{\partial S}{\partial q_\rho}, \tag{20}$$

where the momenta are related to the velocities in the usual way:

$$p^{\rho} = f^{\rho\sigma}\frac{1}{N}\frac{\partial q_\sigma}{\partial t}. \tag{21}$$

To obtain the quantum trajectories we have to solve the following system of first order differential equations, called the guidance relations:

$$\frac{\partial S(q_\rho)}{\partial q_\rho} = f^{\rho\sigma}\frac{1}{N}\dot{q}_\sigma. \tag{22}$$

Equations (22) are invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same space-time geometry for a given nonclassical solution $q_\alpha(t)$.

There is no problem of time in the Bohm-de Broglie interpretation of minisuperspace quantum cosmology [28]. This is not the case, however, for the full superspace, see [8,29], although the theory remains consistent, see [9,29]).

Let us then apply this interpretation to our minisuperspace model. The operator version of Eq. (5), with the factor ordering which makes it covariant through field redefinitions, reads

$$\frac{1}{2e^{3\alpha}}\left(-\frac{\partial^2\Psi}{\partial\alpha^2} + \frac{\partial^2\Psi}{\partial\phi^2} + \epsilon e^{4\alpha}\Psi\right) = 0, \tag{23}$$

from where one can read that $f_{\rho\sigma} = \eta_{\rho\sigma}e^{-3\alpha}$, and $\eta_{\rho\sigma}$ is the Minkowski metric in two dimensions. Comparing Eq. (23) with Eqs. (16) and (17) yields for the quantum potential

$$Q(\alpha, \phi) \equiv -\frac{e^{3\alpha}}{R} f_{\rho\sigma} \frac{\partial^2 R}{\partial q_\rho \partial q_\sigma} = \frac{1}{R}\left[\frac{\partial^2 R}{\partial \alpha^2} - \frac{\partial^2 R}{\partial \phi^2}\right]. \quad (24)$$

The guidance relations (22) read

$$\frac{\partial S}{\partial \alpha} = -\frac{e^{3\alpha}\dot{\alpha}}{N}, \quad (25)$$

$$\frac{\partial S}{\partial \phi} = \frac{e^{3\alpha}\dot{\phi}}{N}. \quad (26)$$

We can write this equation in null coordinates,

$$u \equiv \tfrac{1}{\sqrt{2}}(\alpha + \phi), \qquad \alpha \equiv \tfrac{1}{\sqrt{2}}(u + v),$$
$$v \equiv \tfrac{1}{\sqrt{2}}(\alpha - \phi), \qquad \phi \equiv \tfrac{1}{\sqrt{2}}(u - v), \quad (27)$$

yielding

$$\left(-\frac{\partial^2}{\partial u \partial v} + \frac{\epsilon}{2} e^{2\sqrt{2}(u+v)}\right)\Psi(u, v) = 0. \quad (28)$$

The solutions are:

(1) *For* $\epsilon = 0$*:* In this case the general solution is

$$\Psi(u, v) = F(u) + G(v), \quad (29)$$

where $F$ and $G$ are arbitrary functions. Using a separation of variable method, one can write these solutions as Fourier transforms given by

$$\Psi(u, v) = \int dk U(k) \exp(iku) + \int dk V(k) \times \exp(ikv), \quad (30)$$

$U$ and $V$ also being arbitrary.

Writing the solution (29) in polar form,

$$\Psi(u, v) = R_+ e^{iS_+} + R_- e^{iS_-},$$

where

$$R_+ = R(u), \qquad S_+ = S(u),$$
$$R_- = R(v), \qquad S_- = S(v),$$

one obtains

$$R = \sqrt{R_+^2 + R_-^2 + 2R_+R_- \cos(S_+ - S_-)},$$
$$S = \arctan\left(\frac{R_+ \sin(S_+) + R_- \sin(S_-)}{R_+ \cos(S_+) + R_- \cos(S_-)}\right).$$

The derivative of $S$ with respect to some variable $x$ reads

$$\frac{\partial S}{\partial x} = \frac{R_+^2 \frac{\partial S_+}{\partial x} + R_-^2 \frac{\partial S_-}{\partial x} + (\frac{\partial S_+}{\partial x} + \frac{\partial S_-}{\partial x})R_+R_- \cos(S_+ - S_-) + (R_- \frac{\partial R_+}{\partial x} - R_+ \frac{\partial R_-}{\partial x}) \sin(S_+ - S_-)}{R_+^2 + R_-^2 + 2R_+R_- \cos(S_+ - S_-)}.$$

These equations will be used in the next section.

(2) *For* $\epsilon \neq 0$*:* In this case the general solution reads

$$\Psi(u, v) = \int dk U(k) \exp\left(\frac{k}{2\sqrt{2}} e^{2\sqrt{2}u} + \frac{\epsilon}{4\sqrt{2}k} e^{2\sqrt{2}v}\right) + \int dk V(k) \exp\left(\frac{k}{2\sqrt{2}} e^{2\sqrt{2}v} + \frac{\epsilon}{4\sqrt{2}k} e^{2\sqrt{2}u}\right), \quad (31)$$

where $f$ and $g$ are arbitrary functions. In Ref. [21] these solutions were expanded in terms of Bessel functions.

## IV. GENERALIZED GAUSSIAN SUPERPOSITIONS

In Refs. [21,22] we made Gaussian superpositions of the solutions with the choice $U(k) = V(\pm k) = A(k)$, with $A(k)$ given by

$$A(k) = \exp\left[-\frac{(k - \sqrt{2}d)^2}{\sigma^2}\right], \quad (32)$$

with $\sigma^2 > 0$, and the presence of $\sqrt{2}$ above is just for further convenience. Bouncing nonsingular solutions were obtained in [21], and expanding singular models with a period of acceleration between decelerated phases were proposed in [22].

In this paper we will consider the more general case where the parameter $\sigma^2$ in (32) is given by a complex number: $\sigma^2 + i4h$, where $h$ is an arbitrary real number. Under this assumption we have

$$A(k) = \exp\left[-\frac{(k - \sqrt{2}d)^2}{\sigma^2 + i4h}\right]. \quad (33)$$

From now on we will consider only flat spatial sections.

Integrating (30) with $U(k) = V(k) = A(k)$, we obtain the solution

$$\Psi(u, v) = \sqrt{\pi}\sqrt[4]{\sigma^4 + 16h^2} e^{i \arctan\sqrt{(\sqrt{\sigma^4+16h^2}-\sigma^2)/(\sqrt{\sigma^4+16h^2}+\sigma^2)}} \left\{\exp\left[-\frac{\sigma^2}{4}u^2 + i(-hu^2 + \sqrt{2}du)\right] + \exp\left[-\frac{\sigma^2}{4}v^2 + i(-hv^2 + \sqrt{2}dv)\right]\right\}. \tag{34}$$

To obtain the quantum trajectories, we have to calculate the phase $S$ of the above wave function and substitute it into the guidance equations. Computing the phase and recovering the original variables $\alpha$, $\phi$, we have

$$S = \arctan\left(\sqrt{\frac{\sqrt{\sigma^4 + 16h^2} - \sigma^2}{\sqrt{\sigma^4 + 16h^2} + \sigma^2}}\right) + d\alpha - \frac{h}{2}(\alpha^2 + \phi^2) + \arctan\left\{\tanh\left(\frac{\sigma^2\alpha\phi}{4}\right)\tan[\phi(h\alpha - d)]\right\}, \tag{35}$$

which, after substitution in Eqs. (25) and (26), yields a planar system given by

$$\dot{\alpha} = -\frac{N}{4e^{3\alpha}}\left\{4(d - h\alpha) + \frac{\sigma^2\phi\sin[2\phi(h\alpha - d)] + 4h\phi\sinh(\frac{\sigma^2\phi\alpha}{2})}{\cosh(\frac{\sigma^2\phi\alpha}{2}) + \cos[2\phi(h\alpha - d)]}\right\} =: f(\alpha, \phi), \tag{36}$$

and

$$\dot{\phi} = \frac{N}{4e^{3\alpha}}\left\{-4h\phi + \frac{\sigma^2\alpha\sin[2\phi(h\alpha - d)] + 4(h\alpha - d)\sinh(\frac{\sigma^2\phi\alpha}{2})}{\cosh(\frac{\sigma^2\phi\alpha}{2}) + \cos[2\phi(h\alpha - d)]}\right\} =: g(\alpha, \phi). \tag{37}$$

The norm of the solution (37) is given by

$$R = \sqrt{2\pi}\sqrt[4]{\sigma^4 + 16h^2} e^{-(\sigma^2/8)(\alpha^2+\phi^2)} \times \sqrt{\cosh\left(\frac{\sigma^2\phi\alpha}{2}\right) + \cos[2\phi(h\alpha - d)]}. \tag{38}$$

Equations (36) and (37) give the directions of the geometrical tangents to the trajectories which solves this planar system. We shall work in the gauge $N = e^{3\alpha}$. By plotting the tangent direction field, it is possible to obtain the trajectories. Because of the symmetries $f(\alpha, \phi; h, d) = f(-\alpha, -\phi; -h, d)$, $g(\alpha, -\phi; h, d) = g(-\alpha, -\phi; -h, d)$, and $f(\alpha, \phi; h, d) = -f(-\alpha, \phi; h, -d)$, $g(\alpha, \phi; h, d) = g(-\alpha, \phi; h, -d)$, one concludes that a change in sign of $h$ corresponds to an inversion around the origin with time reversion, and a change in sign of $d$ corresponds to a reflexion in the $\phi$ axis. Hence, one can make definite choices of sign for these parameters without loss of generality.

Field plots of this planar system are shown in Figs. 1–4, for the choice of parameters $\{\sigma^2 = 2, h = 1/8\}$, $\{\sigma^2 = 2, h = 0.5\}$, and $\{\sigma^2 = 2, h = 5\}$ (two portraits), respectively, all with $d = -1$.

The line $\phi = 0$ divides configuration space in two symmetric regions as long as $f(\alpha, \phi) = f(\alpha, -\phi)$ in Eq. (36), and $g(\alpha, \phi) = -g(\alpha, -\phi)$ in Eq. (37). This can be seen in Fig. 1.

The line $\alpha = 0$ contains all the nodes of this system, as is shown in all figures. They appear when the denominator of the above equations, which is proportional to the norm of the wave function (see Eq. (38)), is zero. No trajectory can pass through these points. They happen when $\alpha = 0$ and $\cos(2d\phi) = -1$, or $\phi = (2n + 1)\pi/(2d)$, $n$ an integer, with separation $\pi/d$.

The center points appear when the numerators are zero, their locations depend on the values of $h$, $\sigma^2$, and $d$, and they are not on the line $\alpha = 0$, unless $h = 0$ (case of Refs. [21,22]). Note in Fig. 1, where $h$ is relatively much smaller than $\sigma^2$, that the centers are close but not on the line $\alpha = 0$.

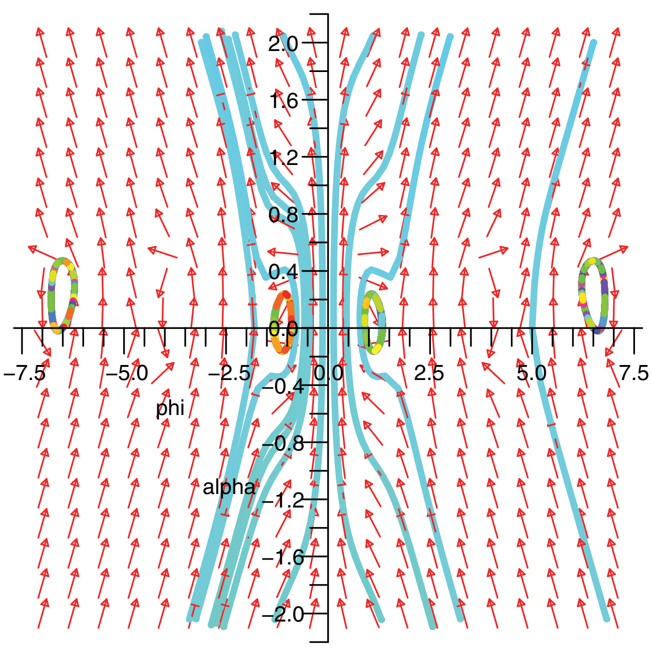


FIG. 1 (color online). Field plot for the system (36) and (37), giving the direction of the geometrical tangent to the trajectories, for the values $\sigma^2 = 2$, $h = 1/8$, and $d = -1$.

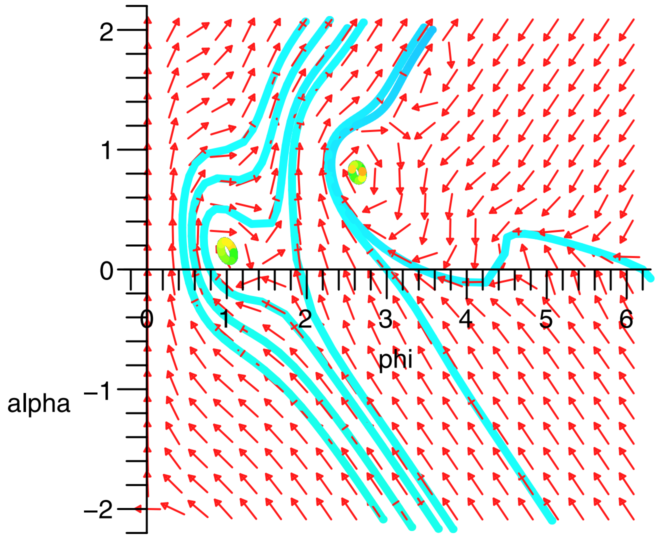


FIG. 2 (color online). Field plot for the system (36) and (37), giving the direction of the geometrical tangent to the trajectories, for $\sigma^2 = 2$, $h = 0.5$, and $d = -1$. For a bigger value of $h$ the center points are farther from the axis $\alpha = 0$. Note the change in the sign of $\dot{\alpha}$ when $\phi$ is big enough.

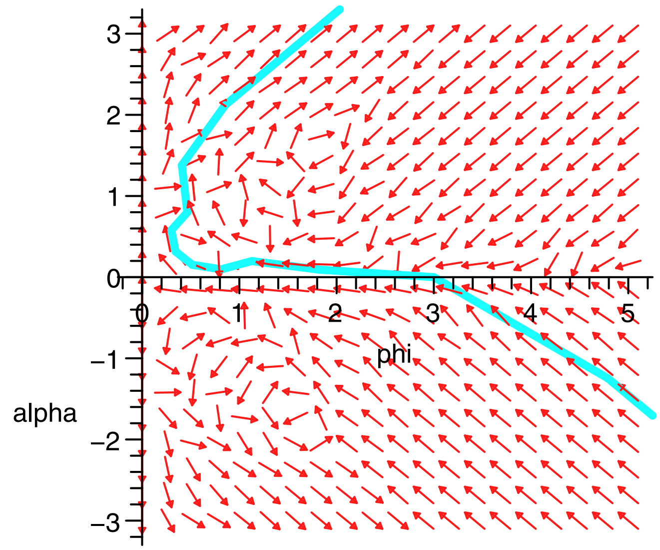


FIG. 4 (color online). Field plot for the planar system (36) and (37) giving the direction of the geometrical tangent to the trajectories for $\sigma^2 = 2$, $h = 5$, and $d = -1$. The trajectory for a universe coming from a singularity, experiencing a long static phase and finally expanding, is depicted.

Apart from the changes in the sign of $\dot{\alpha}$, which happen around the center points, there are regions with different signs of $\dot{\alpha}$ when $|\phi| \gg 0$, $|\alpha| \gg 0$ where the hyperbolic functions dominate over the trigonometric. For instance, considering $\phi > 0$, if $d < 0$, $h > 0$, and $\alpha > 0$ for $\alpha$ fixed, Eq. (36) shows that $\dot{\alpha} > 0$ for small $\phi$ when the cosh term dominates, but it changes signs when $\phi$ is big enough in order for the sinh term to be greater than the cosh term in this equation, changing the sign of $\dot{\alpha}$. This situation is depicted in Figs. 2–4.

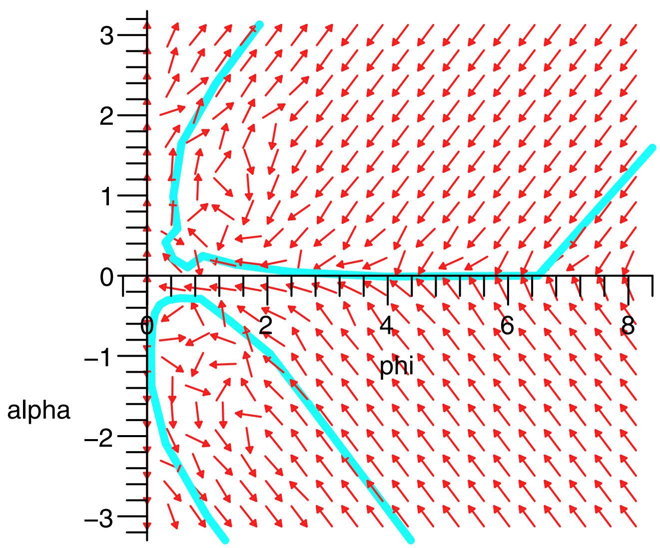


FIG. 3 (color online). Field plot for the planar system (36) and (37) for $\sigma^2 = 2$, $h = 5$, and $d = -1$. Two trajectories are depicted: one representing a bouncing universe spending a long time on the bounce and the other which corresponds to a universe which begins and ends in singular states (''big bang-big crunch'' universe). Note the change in the sign of $\dot{\alpha}$ when $\phi$ is big enough.

Note finally that the classical solutions for $\alpha(\tau)$ ($a(\tau) \propto \tau^{1/3}$) and $\phi(\tau)$ are recovered when $|\alpha| \longmapsto \infty$ or $|\phi| \longmapsto \infty$, and none of them are null.

We can see plenty of different trajectories, depending on the initial conditions and parameter values, in Figs. 1–4. Near the center points we can have oscillating universes without singularities. One may have universes expanding classically from a singularity, experiencing quantum effects in the middle of their expansion, with possible accelerating and/or static phases, and recovering their classical expansion behavior for large values of $\alpha$.

There are big bang-big crunch models and bouncing models, where the bounce may take long and connects two asymptotically classical contracting and expanding phases (see Fig. 3).

Note that, if we choose $V(k) = A(-k)$ (see Eqs. (30) and (32)), one obtains the same field plots as above with the axis $\alpha$ and $\phi$ interchanged, obtaining more possibilities for bouncing models.

Finally, there is the special situation of $\sigma^2 = 0$, which leads to qualitative different solutions, as we will see in the next section.

## V. NONSINGULAR INFLATIONARY BOHMIAN TRAJECTORIES

There is an interesting case which is obtained when $\sigma^2 = 0$ in the Gaussian (33).

Hence, $A(k)$ reads

$$A(k) = \exp\left[i\frac{(k-\sqrt{2}d)^2}{4h}\right]. \tag{39}$$

Then the wave function (34) reduces to

$$\Psi(u,v) = 2\sqrt{\pi|h|}\left[\exp i\left(-hu^2 + \sqrt{2}du + \frac{\pi}{4}\right) + \exp i\left(-hv^2 + \sqrt{2}dv + \frac{\pi}{4}\right)\right]. \tag{40}$$

Its norm is given by $R = 4\sqrt{\pi|h|}\cos[\phi(h\alpha - d)]$, yielding the quantum potential

$$Q = (h\alpha - d)^2 - h^2\phi^2. \tag{41}$$

The guidance relations given by (36) and (37) now reduce to

$$\dot{\alpha} = h\alpha - d, \tag{42}$$

$$\dot{\phi} = -h\phi. \tag{43}$$

The Eqs. (42) and (43) represent a linear dynamical system. The only critical point ($\phi = 0$, $\alpha = \frac{d}{h}$) is a saddle point and, as is well known, it represents an unstable equilibrium.

The field plot of these solutions is depicted in Fig. 5 for $d = -1$, $h = 0.5$. Note that there are two regions of different signs of $\dot{\alpha}$ separated by the line $\alpha = d/h$.

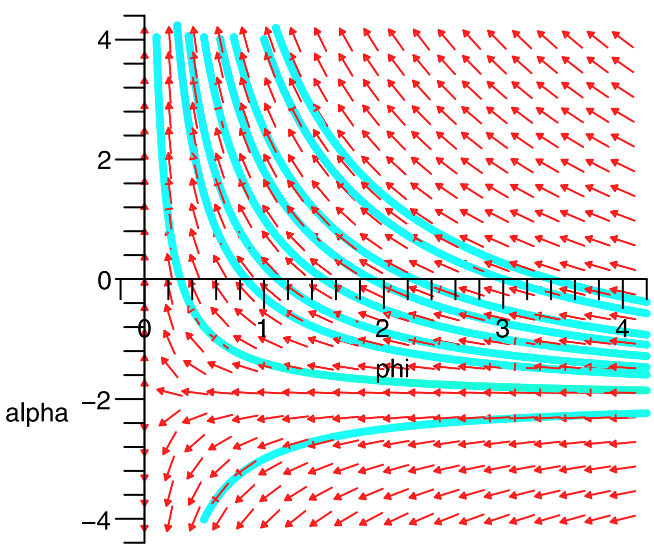


FIG. 5 (color online). Field plot for the linear planar system (42) and (43) for $\sigma^2 = 0$, $h = 0.5$, and $d = -1$. A class of trajectories represent a universe that begins to inflate quantum mechanically from a Planckian size closed flat space-time in the infinite past, and when it becomes large enough it undergoes an analytical graceful exit to a decelerated classical stiff matter expansion phase.

In this case there are analytical solutions, which read

$$a = e^{\alpha} = e^{d/h}\exp(\alpha_0 e^{ht}) \quad \text{and} \quad \phi = \phi_0 e^{-ht}, \tag{44}$$

where $\alpha_0$ and $\phi_0$ are integration constants, and remembering that the time parameter $t$ is related to cosmic time $\tau$ through $\tau = \int dt e^{3\alpha(t)} \Rightarrow \tau - \tau_0 = \mathrm{Ei}(3\alpha_0 e^{ht})/h$, where $\mathrm{Ei}(x)$ is the exponential-integral function.

These solutions represent ever expanding or contracting nonsingular models, depending on the sign of $h$. Let us consider the physical situation of expanding solutions with $h > 0$. The Hubble and deceleration parameters $a'/a$ and $a''/a$ read (a prime denotes a derivative in cosmic time)

$$\frac{a'}{a} = \frac{\alpha_0 h e^{ht}}{a^3}, \tag{45}$$

and

$$\frac{a''}{a} = \frac{\alpha_0 h^2}{a^6} e^{ht}(1 - 2\alpha_0 e^{ht}), \tag{46}$$

and the scalar curvature is

$$\mathcal{R} = -\frac{6\alpha_0 h^2}{a^6} e^{ht}(1 - \alpha_0 e^{ht}). \tag{47}$$

There are three important phases in this model. For $t \ll 0$ the universe expands accelerately from its minimum size $a_0 = e^{d/h}$ (remember that for the physical scale factor one has $a_0^{\mathrm{phys}} = le^{d/h}/\sqrt{2V}$), which occurs in the infinity past $t \to -\infty$ when the curvature is null but increasing while the scale factor grows. The scalar field is very large in that phase. For $t \gg 0$ the universe expands decelerately, the scale factor is immensely big, the scalar field becomes negligible, and the curvature approaches zero again. The transition occurs when $ht_{\mathrm{tran}} = -\ln(2\alpha_0)$.

Around $ht = 0$ one has

$$a \approx e^{\alpha_0 + d/h}[1 + \alpha_0 ht + (\alpha_0 h^2 + \alpha_0^2 h^2)t^2/2! + \ldots]. \tag{48}$$

If $\alpha_0 \gg 1$ (and hence $t > t_{\mathrm{tran}}$, which means in the deceleration phase), one can write $a \approx e^{\alpha_0 + d/h}\exp(\alpha_0 ht)$. In that case, from $\tau = \int dt a^3(t)$, one obtains that $a \propto (\tau - \tau_0)^{1/3}$ and $\phi' \propto 1/\tau \propto 1/a^3$, as in the classical regime.

Collecting all these phases, and considering $\alpha_0 \gg 1$, one has a nonsingular ever expanding universe, starting with a constant and finite size (which may be of the order of the Planck length if $d = 0$, a Planckian flat space) in the infinity past, which inflates afterwards until it attains an almost[2] classical decelerating expanding regime, with a size $e^{\alpha_0}$ times bigger than it was initially. At this time,

[2]We would like to say that it is not necessary that classical behavior appears when the universe is large, see [30]. In some circumstances, a quantum behavior should be desirable, as in [22] to produce late acceleration in the universe, or to avoid a big rip, see [31].

radiation may start to dominate over the scalar field since it becomes much smaller than in the infinity past, and its energy density goes like $1/a^6$. There are no event nor particle horizons. Hence we have an inflationary nonsingular model which can be smoothly joined to the standard model.

## VI. CONCLUSION

In the present paper we generalized the Gaussians superpositions of Refs. [21,22] to Gaussians with complex widths with non-negative real part. We obtained a richer two-dimensional dynamical system, with oscillating universes without singularities, universes expanding classically from a singularity, experiencing quantum effects in the middle of their expansion with possible accelerating and/or static phases whose duration depend on the three free parameters of the Gaussian (its complex width and the location of its center) and two initial conditions, which recover their classical expansion behavior for large scale factors. There are also big bang-big crunch models, and bouncing models, where the bounce may take long, and connects two asymptotically classical contracting and expanding phases.

The most interesting solutions occur when the real part of the width is made zero. Then there were obtained analytical nonsingular inflationary models which expand accelerately in the infinity past from flat universes with finite (not zero) size spatial sections, and are smoothly connected to a classical decelerated stiff matter expanding phase. It is like the pre-big bang model [25], with a minimum volume spatial section in the infinity past (which can be of the order of the Plank volume), or as the emergent model [26] for flat spatial sections, without any graceful exit problem. Sufficient inflation can be obtained with reasonable choices of initial conditions ($\alpha_0 > 70$). The picture is then of a universe that begins to inflate quantum mechanically from a Planckian size closed flat space-time in the infinity past, and when it becomes large enough it makes a smooth transition to a decelerated classical stiff matter expansion, which will eventually be dominated by radiation before nucleosynthesis. Some reheating process must then also be presented in our model, and this could happen through interactions of the type $\sqrt{-g}\upsilon\phi\Psi\bar{\Psi}$, where $\Psi$ represents some fermion field, $\upsilon$ is a coupling constant, and the scalar field decays into these fermions. Such fermions would not be present in the beginning but they would have been produced when the universe got bigger and evolved into the decelerated phase. Such an extra term would not be relevant in the Hamiltonian which describes the model when the universe was small (it is proportional to $a^3$) and empty of these fermions, justifying our quantum solutions in this regime. However, when the fermions begin to be produced afterwards, our quantum solutions could not be any more reliable. At this stage, when the model becomes decelerated and gets closer to classical behavior, this solution must be connected to the classical radiation plus stiff matter solution. The nonclassical behavior of our bohmian quantum solution for $t > 0$ should then be of course discarded, they are out of the range of the validity of our assumptions. Therefore, our nonsingular inflationary solution should be utilized only as a zeroth-order approximation to a realistic cosmological inflationary model, in the same way as the de Sitter model or power law solutions are used for usual inflation [32]. The improvement of the model, including reheating and a radiation field deserves further research as there are many possibilities to be investigated. Another important step forward is to evaluate the evolution of cosmological perturbations using the formalism developed in Ref. [19], and confront it with observations [3]. This will be the subject of our future publications.

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