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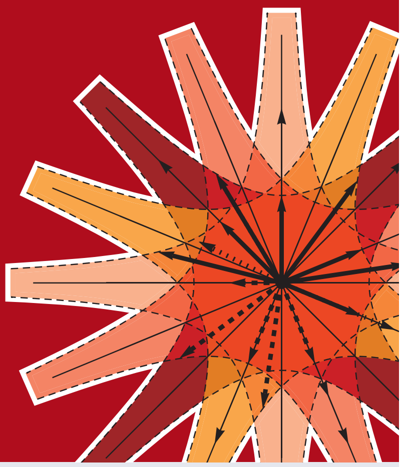

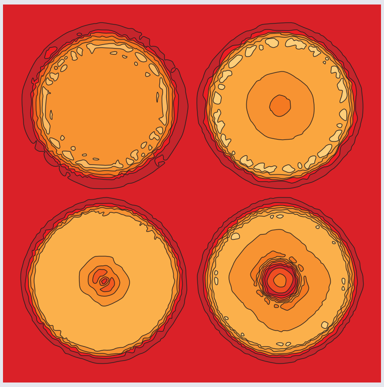

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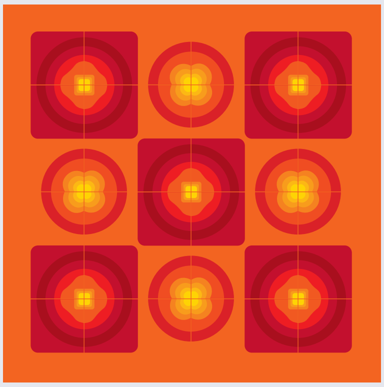

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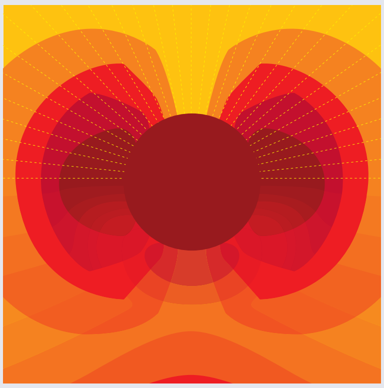

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Edge modes in the fractional quantum Hall effect without extra edge fermions

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Abstract – We show that the Chern-Simons-Landau-Ginsburg theory that describes the quantum Hall effect on a bounded sample is anomaly free and thus does not require the addition of extra chiral fermions on the boundary to restore local gauge invariance.

Introduction. – From a field-theoretical point of view, the fractional quantum Hall effect (FQHE) is well understood since 1989, when it was shown that it could be described by a Chern-Simons-Landau-Ginsburg (CSLG) effective theory [1]. Two years later, the effect of boundaries was analysed [2] and a breakdown of gauge invariance was found, due to a bounded Chern-Simons (CS) action. Noticing the fact that the microscopic theory *is* gauge invariant, several authors [2,3] concluded that an anomaly appeared, that would have to be cancelled in order to recover gauge invariance. This was done by the *addition of extra degrees of freedom*, in the form of chiral fermions circulating at the edge, whose well-known gauge anomaly restored gauge invariance of the complete theory. These extra degrees of freedom were, since then, usually identified with chiral edge modes, which are fundamental in the wave function approach to the FQHE [4].

In this letter, we briefly review the steps conducting from the microscopic Hamiltonian to the CSLG theory in the presence of boundaries. We show that, by carefully considering the influence of the boundary in the dynamics of the CS field, one finds no such anomaly and the resulting theory is *gauge invariant*. Thus, *there is no need to introduce extra degrees of freedom such as chiral edge fermions*. The edge modes appear naturally from the dynamics of the CS field, which is determined by its coupling to the Noether current of the CSLG theory. Thus, they have nothing to do with the chiral edge fermions, introduced in previous approaches.

Edge states were found experimentally [5] and appear as an essential ingredient in several recent works. We can quote some of them which deal with various subjects such as edge states in graphene [6,7], descriptions of chiral Luttinger liquids [8,9] and relations between edge electrons and Berry's phase [10]. Accordingly, the association of these modes with the original degrees of freedom of the CSLG effective theory is very important both for present and future applications.

FQHE on a bounded surface. – We briefly review the derivation of the CSLG model for the FQHE [1], modifying the procedure when necessary to take into account the finiteness of the surface. We start from the microscopic Hamiltonian of a two-dimensional system of polarized electrons interacting with an external electromagnetic field:

$$H_{\mathrm{F}} = \frac{1}{2m}\sum_r \left[\mathbf{p}_r - \frac{e}{c}\mathbf{A}(\mathbf{x}_r)\right]^2 + \sum_r eA_0(\mathbf{x}_r) + \sum_{r<s} V(\mathbf{x}_r - \mathbf{x}_s) + \sum_r V_c(\mathbf{x}_r). \tag{1}$$

In (1), $V_c(\mathbf{x}_r)$ is an electrostatic potential which is responsible for confining the electrons into the bounded region Γ. It is essentially zero in the bulk of Γ and very large as one approaches the boundary. As the electrons are polarized, the wave function is completely antisymmetric. It is possible to map this fermionic problem into a bosonic one. This can be done by means of an unitary transformation

$$U = \exp\left(-i\sum_{r<s}\frac{\theta}{\pi}\alpha_{rs}\right), \tag{2}$$

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where α_{rs} is the angle between $\mathbf{x}_r - \mathbf{x}_s$ with an arbitrary direction that may be chosen as the x-axis and $\theta = (2k+1)\pi$ with k being an integer. Under this choice, one can easily verify that an antisymmetric wave function ψ is mapped into a symmetric (bosonic) one $\phi \equiv U^{-1}\psi$. It is easy to check that

$$U^{-1}\left(\mathbf{p}_r - \frac{e}{c}\mathbf{A}(\mathbf{x}_r)\right)U = \mathbf{p}_r - \frac{e}{c}\mathbf{A}(\mathbf{x}_r) - \hbar\frac{\theta}{\pi}\sum_{r\neq s}\boldsymbol{\nabla}\alpha_{rs}. \tag{3}$$

The gradient of the angle between the vector $\mathbf{x}_r - \mathbf{x}_s$ and the x-axis is given by

$$\partial_i\alpha_{rs} \equiv \partial_i\alpha(\mathbf{x}_r - \mathbf{x}_s) = -\varepsilon_{ij}\frac{x_r^j - x_s^j}{|\mathbf{x}_r - \mathbf{x}_s|^2}. \tag{4}$$

Now, a statistical field is *defined* as

$$\mathbf{a}(\mathbf{x}_r) \equiv \frac{\phi_0}{2\pi}\frac{\theta}{\pi}\sum_{s\neq r}\boldsymbol{\nabla}\alpha_{rs}, \tag{5}$$

where $\phi_0 = hc/e$ is the quantum of flux. The bosonized Hamiltonian is $H_{\mathrm{B}} = U^{-1}H_{\mathrm{F}}U$ or

$$\begin{aligned} H_{\mathrm{B}} \;=\; & \frac{1}{2m}\sum_r\left(\mathbf{p}_r - \frac{e}{c}[\mathbf{A}(\mathbf{x}_r) + \mathbf{a}(\mathbf{x}_r)]\right)^2 \\ & + \sum_r eA_0(\mathbf{x}_r) + \sum_{r<s}V(\mathbf{x}_r - \mathbf{x}_s) + \sum_r V_c(\mathbf{x}_r). \end{aligned} \tag{6}$$

The second quantized Hamiltonian is obtained through the introduction of a bosonic field $\phi(\mathbf{x})$ ($\mathbf{x}$ denotes (x_1, x_2)) satisfying $[\phi(\mathbf{x}), \phi^\dagger(\mathbf{y})] = \delta^{(2)}(\mathbf{x}-\mathbf{y})$, and generalizing H_{B} to the (Hermitian) *matter Hamiltonian*

$$\begin{aligned} H_{\mathrm{M}} \;=\; & \int_\Gamma \mathrm{d}^2x\left\{\frac{\hbar^2}{2m}\left(\mathbf{D}\phi(\mathbf{x})\right)^\dagger\cdot\mathbf{D}\phi(\mathbf{x})\right\} \\ & + \int_\Gamma \mathrm{d}^2x\left\{eA_0(\mathbf{x})\,\phi^\dagger(\mathbf{x})\,\phi(\mathbf{x})\right\} \\ & + \frac{1}{2}\int_\Gamma \mathrm{d}^2x\,\mathrm{d}^2y\,\delta\rho(\mathbf{x})\,V(\mathbf{x}-\mathbf{y})\delta\rho(\mathbf{y}). \end{aligned} \tag{7}$$

Notice that we incorporated the effect of the confining potential V_c by restricting the integration domain over the region Γ. An alternative approach would be to explicitly take into account the confining potential by adding a Gaussian term $\int \mathrm{d}^2x\{V_c(\mathbf{x})\phi^\dagger(\mathbf{x})\phi(\mathbf{x})\}$ and not restricting integration. We will follow the first approach to make it easier to compare our results with the literature. In (7) the covariant derivative was defined as

$$D^k = \partial^k + \frac{ie}{\hbar c}(A^k(\mathbf{x}) + a^k(\mathbf{x})), \tag{8}$$

$\rho(\mathbf{x}) = \phi^\dagger(\mathbf{x})\phi(\mathbf{x})$ and $\delta\rho(\mathbf{x}) = \rho(\mathbf{x}) - \bar{\rho}$ (the average density $\bar{\rho}$ is included here to avoid problems in the case of a long-range potential [1]). Taking all operators in the Heisenberg picture, they become functions of time. The action below generates the correct Heisenberg equations

$$\begin{aligned} S_{\mathrm{M}} \;=\; & \int \mathrm{d}^3x\left\{\frac{i\hbar c}{2}\Theta(\mathbf{x})\,\phi^\dagger(x)\,D_0\phi(x)\right\} \\ & + \int \mathrm{d}^3x\left\{-\frac{i\hbar c}{2}\Theta(\mathbf{x})\left(D_0\phi(x)\right)^\dagger\phi(x)\right\} \\ & + \int \mathrm{d}^3x\left\{-\frac{\hbar^2}{2m}\Theta(\mathbf{x})\left(\mathbf{D}\phi(x)\right)^\dagger\cdot\mathbf{D}\phi(x)\right\} \\ & - \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\Theta(\mathbf{x})\,\Theta(\mathbf{y})\,\delta\rho(x)\,V(\mathbf{x}-\mathbf{y})\,\delta\rho(y), \end{aligned} \tag{9}$$

with

$$\begin{aligned} D_0 \;=\; & \frac{1}{c}\partial_t + \frac{ie}{\hbar c}A_0(\mathbf{x}) \\ \equiv\; & \partial_0 + \frac{ie}{\hbar c}A_0(\mathbf{x}), \end{aligned} \tag{10}$$

and the integration being effectively over the surface of the sample, which is obtained by the use of a step function Θ defined as

$$\Theta(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x}\in\Gamma, \\ 0, & \text{if } \mathbf{x}\notin\Gamma. \end{cases} \tag{11}$$

Global phase invariance of the action (9) under the transformations $\phi'(x) = e^{i\alpha}\phi(x)$ implies the continuity equation

$$\partial_\mu j^\mu_{\mathrm{M},\Gamma} = \partial_0 j^0_{\mathrm{M},\Gamma} + \partial_i j^i_{\mathrm{M},\Gamma}(x) = 0, \tag{12}$$

where the components of the matter current are given by

$$\begin{aligned} & j^0_{\mathrm{M},\Gamma} = \Theta(\mathbf{x})\,\phi^\dagger(x)\,\phi(x) = \Theta(\mathbf{x})\,\rho(x), \\ & \mathbf{j}_{\mathrm{M},\Gamma} = \Theta(\mathbf{x})\,\mathbf{j}_{\mathrm{M}}(x), \end{aligned} \tag{13}$$

with $\mathbf{j}_{\mathrm{M}}$ given by

$$\mathbf{j}_{\mathrm{M}}(x) = \frac{i\hbar}{2m}\{\phi^\dagger(x)\,\mathbf{D}\phi(x) - \left(\mathbf{D}\phi(x)\right)^\dagger\phi(x)\}. \tag{14}$$

The field $\mathbf{a}(x)$ is completely determined in terms of the density operator $\rho(x)$, and is given by the second quantized version of eq. (5),

$$\begin{aligned} a_i(x) \;=\; & -\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\varepsilon_{ij}\int_\Gamma \mathrm{d}^2y\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}\rho(y) \\ =\; & -\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\varepsilon_{ij}\int \mathrm{d}^2y\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}\Theta(\mathbf{y})\,\rho(y). \end{aligned} \tag{15}$$

The field $\mathbf{a}(x)$ shown above can be seen as an auxiliary field. It is the solution of the following pair of equations:

$$\begin{aligned} & \varepsilon_{ij}\partial_i a_j(x) = \phi_0\frac{\theta}{\pi}\Theta(\mathbf{x})\,\rho(x), \\ & \partial_i a_i = 0. \end{aligned} \tag{16}$$

Using the continuity equation (13) we can derive a third equation for the field $a_i(x)$ involving a time derivative

$$\varepsilon_{ij}\partial_0 a_j(x) = -\phi_0\frac{\theta}{\pi}\Theta(\mathbf{x})\,j^i_{\mathrm{M}}. \tag{17}$$

Equations (16) and (17) may be viewed as the equations of motion of a new dynamical field, if we make the substitution $D_0 = \partial_0 + \frac{ie}{\hbar c} A_0(\mathbf{x}) \to \partial_0 + \frac{ie}{\hbar c}(A_0(\mathbf{x}) + a_0(x))$ (which means that now $S_{\rm M} = S_{\rm M}(a_0)$), and replace $S_{\rm M}$ by the action

$$S = S_{\rm M}(a_0) + S_{\rm CS}, \tag{18}$$

with

$$S_{\rm CS} \equiv \frac{\sigma_{xy}}{2} \int {\rm d}^3x\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \tag{19}$$

where $S_{\rm CS}$ is known as the Chern-Simons (CS) action and we define the Hall conductivity as

$$\sigma_{xy} \equiv \frac{\pi}{\theta}\frac{1}{\phi_0}. \tag{20}$$

The additional field $a_0(x)$ introduced in (19) and in $S_{\rm M}$ can be eliminated by requiring the condition $a_0(x) = 0$, what is legitimate in the context of a gauge field theory. *Thus, gauge invariance is crucial for the correct introduction of the statistical CS field* $a_\mu(x)$. Without it, the equations of motion obeyed by the field are not enough to eliminate this extra component and the identification of the dynamics of $a_\mu(x)$ with that of a CS field cannot be made.

The action S is gauge invariant, because the CS part $S_{\rm CS}$ (eq. (19)) is not being integrated over a finite surface. Then, under a gauge transformation,

$$\begin{aligned} S_{\rm CS}[a_\mu + \partial_\mu \alpha] &= S_{\rm CS}[a_\mu] + \frac{\sigma_{xy}}{2}\int {\rm d}^3x\, \varepsilon^{\mu\nu\rho}(\partial_\mu\alpha)\partial_\nu a_\rho \\ &= S_{\rm CS}[a_\mu] - \frac{\sigma_{xy}}{2}\int {\rm d}^3x\, \partial_\mu(\varepsilon^{\mu\nu\rho}\alpha\partial_\nu a_\rho) \\ &= S_{\rm CS}[a_\mu]. \end{aligned} \tag{21}$$

The restriction of the domain of integration to the area of the sample is the origin of the destruction of gauge invariance [2,3]. This problem is absent here. Thus, gauge invariance is a consequence of unbounded integration in eq. (19).

If one insisted to consider the CS action on a surface with a boundary,

$$S^b_{\rm CS} \equiv \frac{\sigma_{xy}}{2}\int {\rm d}^3x\, \Theta(\mathbf{x})\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \tag{22}$$

the resulting equations of motion for the CS field would be (setting $a_0 = 0$ artificially, as this is only allowed if the theory is gauge invariant)

$$\varepsilon_{ij}\partial_i a_j(x) = \phi_0 \frac{\theta}{\pi}\rho(x), \tag{23}$$

$$\varepsilon_{ij}\partial_0 a_j(x) = -\phi_0 \frac{\theta}{\pi} j^i, \tag{24}$$

and these equations would result in the solution

$$a_i(x) = -\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\varepsilon_{ij}\int {\rm d}^2y \frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}\rho(y), \tag{25}$$

which does not contain the restriction of the integration to the surface of the sample and, so, does not coincide with (15). This would result in a second quantized action which is not equivalent to the original problem. Therefore another reason why the CS action *must* be unbounded is to provide the correct solution for the statistical field.

Although the Chern-Simons part of the action makes no reference to the boundary of the sample, we have to remember that the CS field is minimally coupled to the matter fields. This coupling involves the Noether current and forces the equations of motion for the CS field to depend on the boundary, as can be explicitly seen in (16) and (17). This has to be so, if one wants to recover (15).

Compatibility with known results. – We can follow [1] and seek for a mean-field solution in the presence of a magnetic field $\Theta(\mathbf{x})B = -\varepsilon_{ij}\partial_i A_j$. Again the proposed form is

$$\phi(x) = \sqrt{\bar\rho}, \qquad \mathbf{a} = -\mathbf{A}, \qquad a_0(x) = 0, \tag{26}$$

where $\bar\rho$ is the average particle density. The current takes the form $j^\mu = (\Theta(\mathbf{x})\bar\rho, \mathbf{0})$. The equations of motion in the CS sector are

$$\begin{aligned} &\varepsilon_{ij}\partial_0 a_j(x) = 0 \ \text{(identically satisfied)}, \\ &\varepsilon_{ij}\partial_i a_j(x) = \Theta(\mathbf{x})B = \Theta(\mathbf{x})\phi_0\frac{\theta}{\pi}\bar\rho. \end{aligned} \tag{27}$$

Identifying the density of magnetic flux as $\rho_\Gamma = B/\phi_0$, one obtains the condition for the validity of the mean-field approximation

$$\nu = \frac{\bar\rho}{\rho_\Gamma} = \frac{\pi}{\theta} = \frac{1}{2k+1}. \tag{28}$$

This is exactly the same result obtained in [1] for the filling factors.

Concerning the expression of the current in terms of the CS field, we can see, using the CS equations of motion,

$$\frac{\delta S}{\delta a_\mu} = \frac{\delta}{\delta a_\mu}(S_{\rm M} + S_{\rm CS}) = j^\mu_{\rm M,\Gamma} + \frac{\delta S_{\rm CS}}{\delta a_\mu} = 0. \tag{29}$$

So, $j^\mu_{\rm M,\Gamma} = -\delta S_{\rm CS}/\delta a_\mu$. Computing this last quantity, we obtain

$$\frac{\delta S_{\rm CS}}{\delta a_\mu(x)} \equiv e j^\mu_{\rm CS} = -\sigma_{xy}\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho(x). \tag{30}$$

This is precisely the final current obtained in [3], after the effect of the chiral edge fermions has been taken into account. In this paper, the author integrates over chiral $(1+1)$-dimensional edge fermions to obtain an effective action which, considered along with a bounded Chern-Simons action (not gauge invariant), results in a gauge-invariant theory. The anomalous term in the current is cancelled and it remains only the contribution equal to that of a non-bounded Chern-Simons action (which is gauge invariant). Our approach leads directly to this result.

It must also be emphasised that the result (30), contained in [2,3], means that the edge modes cannot be identified with the extra edge fermions. This is explicitly said by the author of ref. [3], which follows the approach proposed in ref. [2]. Our approach gives the same current without the need of introduction of any extra degrees of freedom.

All that remains is to see what happens at the edge using the equation $\partial_\mu j^\mu_{\mathrm{M},\Gamma} = 0$. This equation gives

$$\partial_\mu \left(\Theta\left(\mathbf{x}\right) j^\mu_{\mathrm{M}}\right) = \Theta\left(\mathbf{x}\right) \partial_\mu j^\mu_{\mathrm{M}} + \left(\partial_\mu \Theta\left(\mathbf{x}\right)\right) j^\mu_{\mathrm{M}} = 0. \qquad (31)$$

Equation (31) implies two separated equations: one for the bulk and other for the edge of the surface of the sample. We obtain,

$$\partial_\mu j^\mu_{\mathrm{M}} = 0, \text{ in the bulk,} \qquad (32)$$

and

$$j_{\mathrm{M},n} = n_i j^i_{\mathrm{M}} = 0, \text{ in the edge,} \qquad (33)$$

where n_i is a vector field that is normal to the boundary. Condition (33) says that the current at the edge is only tangential, and this completes the identification of the edge modes relevant to the FQHE with the matter current, as expected.

Conclusion. – Close inspection on the calculation done in refs. [1–3] shows that one obtains, starting from S, given by eq. (18) and gauge invariant, the same value for the Hall conductance and edge current that were obtained previously, starting with a theory which was not gauge invariant and adding extra degrees of freedom in the form of chiral edge fermions. So, there is no reason for the introduction of extra one-dimensional chiral fermions circulating on the boundary.

* * *

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REFERENCES

[1] Zhang S. C., Hansson T. H. and Kivelson S., *Phys. Rev. Lett.*, **62** (1989) 82; Zhang S. C., *Int. J. Mod. Phys. B*, **6** (1992) 25.
[2] Wen X. G., *Phys. Rev. B*, **43** (1991) 11025.
[3] Maeda N., *Phys. Lett. B*, **376** (1996) 142.
[4] Girvin S. M., *The Quantum Hall Effect: Novel Excitations and Broken Symmetries,* in *Les Houches Lecture Notes: Topological Aspects of Low Dimensional Systems*, edited by Comtet A., Jolicoeur T., Ouvry S. and David F. (Springer-Verlag, Berlin) 2000.
[5] Wang J. K. and Goldman V. J., *Phys. Rev. Lett.*, **67** (1991) 749.
[6] Yao W., Yang S. A. and Niu Q., *Phys. Rev. Lett.*, **102** (2009) 096801.
[7] Hatsugai Y., *Solid State Commun.*, **149** (2009) 1061.
[8] Boyarsky A., Cheianov V. V. and Ruchayskiy O., *Phys. Rev. B*, **70** (2004) 235309.
[9] Chang A. M., *Rev. Mod. Phys.*, **75** (2003) 1449.
[10] Basu B. and Bandyopadhyay P., *Phys. Scr.*, **73** (2006) 332.