Entangling power of the baker's map: Role of symmetries

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The quantum baker map possesses two symmetries: a canonical "spatial" symmetry, and a time-reversal symmetry. We show that, even when these features are taken into account, the asymptotic entangling power of the baker's map does not always agree with the predictions of random matrix theory. We have verified that the dimension of the Hilbert space is the crucial parameter that determines whether the entangling properties of the baker map are universal or not. For power-of-2 dimensions, i.e., qubit systems, an anomalous entangling power is observed; otherwise the behavior of the baker map is consistent with random matrix theories. We also derive a general formula that relates the asymptotic entangling power of an arbitrary unitary with properties of its reduced eigenvectors.

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I. INTRODUCTION

The baker's map was quantized in 1987 by Balazs and Voros [1] and soon became a very useful toy model for investigating quantum-classical correspondence issues in closed chaotic systems, like the scarring phenomenon, Gutwiller trace formula, and long-time validity of semiclassical approximations (see, e.g., [2–7]).

Later on, the quantum baker map appeared in a variety of problems of quantum information, quantum computation, and quantum open systems. Schack noted that the quantum baker map could be efficiently realized in terms of quantum gates [8]. A three-qubit nuclear magnetic resonance experiment was proposed [9] and then implemented (with some simplifications) [10]. On the theoretical side, Schack and Caves [11] showed that the quantum baker map of Balazs and Voros' can be seen as a shift on a string of quantum bits—in full analogy with the classical case—and exhibited a family of alternative quantizations. This family of bakers maps was the subject of several studies [12–14]. Decoherent variants of the baker map have been constructed by including mechanisms of dissipation and/or diffusion [15–17].

The ability of the baker family to generate entanglement was studied by Scott and Caves [14]. They concluded that "the quantum baker's maps are, in general, good at creating multipartite entanglement amongst the qubits. It was found however, that some quantum baker's maps can, on average, entangle better than others, and that all quantum baker's maps fall somewhat short of generating the levels of entanglement expected in random states. This might be related to the fact that spatial symmetries in the baker's map allow deviations from the predictions of random matrix theory" [14].

The purpose of the present paper is to demonstrate that the spatial symmetry is not to blame for the reduced entangling power of the baker map. Two numerical complementary proofs will be presented. First we check that if the symmetry is removed from the baker map (by block diagonalization), the resulting desymmetrized baker maps produce the same levels of entanglement as the original one. Second we show that the entangling power of an ensemble of "spatially symmetric" unitary operators is not significantly different from that of the circular unitary ensemble of random operators (CUE) [18], i.e., imposing symmetry does not reduce the entangling power. We complete the analysis by verifying that the dimension of the Hilbert space is indeed the crucial parameter that determines whether the entangling properties of the baker map are universal or not. For qubit systems, i.e., power-of-2 dimensions, an anomalous entangling power is observed; otherwise the behavior of the baker map is consistent with random matrix theories.

The background of this contribution is the wider problem of understanding what makes a unitary operation a good entangler. There is not a definitive answer to this question yet, but some partial results have been obtained recently (see, e.g., [19–23]).

Section II presents the measure we use for quantifying the entangling power of a unitary operation. The quantum maps to be considered are introduced in Sec. III, and their symmetries analyzed. Sections IV and V contain the numerical analysis of the influence of the symmetries on the entangling power of the baker map. The general relation between asymptotic entangling power and eigenvectors is discussed in Sec. VI. Concluding remarks are presented in Sec. VII.

II. ENTANGLING POWER

We will only consider the case of bipartite entanglement of pure states.

A full system, with Hilbert space of dimension $d=d_A \times d_B$, is partitioned into two subsystems A and B with dimensions d_A and d_B, respectively, and such that the full space $\mathcal{H}$ has the structure of a tensor product $\mathcal{H}=\mathcal{H}_A \otimes \mathcal{H}_B$. A usual definition of entangling power, $e_p(U)$, of a unitary operator U defined on $\mathcal{H}$ relies on a vector entanglement measure E, and on a suitable average over initial states [24]:

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$$e_p(U) = \langle E(U|\psi_A\rangle \otimes |\psi_B\rangle)\rangle_{|\psi_A\rangle,|\psi_B\rangle}. \tag{1}$$

Thus, $e_p(U)$ says how much entanglement U produces, on average, when acting on a set of nonentangled states. Here we take the measure E to be the linear entropy of the reduced density matrix. Let $|\psi\rangle$ be a (pure) separable state of the full system, corresponding to the density matrix $\rho=|\psi\rangle\langle\psi|$. In general, after application of U, the new density matrix $\rho' = U\rho U^\dagger$ will not correspond to a separable state any more. This will manifest in a positive linear entropy of the reduced density matrices,

$$S_L \equiv 1 - \mathrm{tr}(\rho_A')^2 = 1 - \mathrm{tr}(\rho_B')^2 > 0, \tag{2}$$

where $\rho_A' = \mathrm{tr}_B\rho'$ and $\rho_B' = \mathrm{tr}_A\rho'$ [25].

For the average over product states, indicated by $\langle\cdots\rangle_{|\psi_A\rangle,|\psi_B\rangle}$ in Eq. (1), we choose the unitarily invariant measures in both $\mathcal{H}_A$ and $\mathcal{H}_B$ [24,26]. That is, the components of $|\psi_A\rangle$ and $|\psi_B\rangle$ have the same distribution as the columns of CUE matrices of dimension $d_A \times d_A$ and $d_B \times d_B$, respectively [27].

Among various possible definitions of the entangling strength of a unitary [24,28–30], we chose Eq. (1) mainly for the purpose of comparing our results with those in Ref. [14], where that definition was adopted. The use of S_L instead of the more natural von Neumann entropy not only does not lead to qualitative differences [14] but has the essential advantage of allowing analytical calculations, which will be important for understanding and extending our results.

III. QUANTUM MAPS, SYMMETRIES

Following Schack and Caves [11], we write the unitary operator for the quantum baker map on N qubits, as ($d = 2^N$)

$$B_d = G_d(\mathbb{1}_2 \otimes G_{d/2}^{-1}), \tag{3}$$

where G_d is the antiperiodic quantum Fourier transform on N qubits, $\mathbb{1}_2$ is the unit operator for the first qubit, and $G_{d/2}^{-1}$ is the inverse (antiperiodic) Fourier transform on the remaining $N-1$ qubits. We use the standard ordering for the computational basis. If $|j\rangle$ is a tensor product of individual qubit basis states $|\epsilon_i\rangle$, with $\epsilon_i = 0,1$, i.e.,

$$|j\rangle = |\epsilon_1\rangle \otimes |\epsilon_2\rangle \otimes \cdots |\epsilon_N\rangle, \tag{4}$$

then j is given by the binary expansion

$$j = \sum_{i=1}^{N} \epsilon_i 2^{N-i} \equiv \epsilon_1 \cdots \epsilon_N, \tag{5}$$

$0 \leqslant j \leqslant d-1$

The definition (3) becomes equivalent to the baker map of Balazs-Voros and Saraceno when one identifies the computational basis states $|j\rangle$ with "position" eigenstates $|q_j\rangle$ having eigenvalues $q_j = 0.\epsilon_1 \ldots \epsilon_N 1$. In any case, the matrix representation of the baker map is

$$\|B_d\| = \|G_d\| \begin{pmatrix} \|G_{d/2}^{-1}\| & 0 \\ 0 & \|G_{d/2}^{-1}\| \end{pmatrix}, \tag{6}$$

where $\|G_d\|$ is the inverse Fourier *matrix* [2,3].

The baker map has a time-reversal symmetry, corresponding to the Fourier transform followed by complex conjugation in the computational basis:

$$\|G_d^{-1} B_d G_d\|^* = \|B_d^{-1}\|. \tag{7}$$

The use of the antiperiodic Fourier transform makes the baker map also reflection symmetric, in agreement with its classical counterpart [3]. That is, if we define the reflection operator

$$R_d|j\rangle = |d-1-j\rangle, \tag{8}$$

then

$$B_d R_d = R_d B_d. \tag{9}$$

For a qubit system the reflection operator can be factored into a tensor product of N single-qubit reflections

$$R_d|j\rangle = R_2|\epsilon_1\rangle \otimes R_2|\epsilon_2\rangle \otimes \cdots \otimes R_2|\epsilon_N\rangle, \tag{10}$$

where R_2 is just the negation operator (Pauli-X gate). The reflection symmetry of the baker map, Eq. (9), can be easily proved using the reflection symmetry of the antiperiodic Fourier transform $R_d G_d = G_d R_d$, and the factorization property $R_d = R_2 \otimes R_{d/2}$.

Given that the quantum baker map is a unitary operator with a chaotic classical limit, one may expect that the iterative application of the baker map to random nonentangled states could produce states with levels of entanglement typical of random states. However, Scott and Caves verified that, in spite of being a good entangler, the baker map generates states that are somewhat less entangled than random states. They suggested that this deviation might be due to the spatial symmetry R_d, which, as any unitary symmetry, is known to produce deviations from standard random matrix behavior.

In the following we implement a simple test to decide if the spatial symmetry does really play a significant role in the reduction of the entangling power of the baker map. Due to the symmetry R_d, with eigenvalues ± 1, the baker map can be cast into block-diagonal form:

$$\Lambda^\dagger B_d \Lambda = |0\rangle\langle 0| \otimes B_{d/2}^{(-)} + |1\rangle\langle 1| \otimes B_{d/2}^{(+)}, \tag{11}$$

where $B_{d/2}^{(\pm)}$ are the symmetry-reduced baker maps, and $\Lambda^\dagger$ is a unitary mapping of the computational basis to an R_d-symmetrical basis:

$$\Lambda = \frac{1}{\sqrt{2}}(\mathbb{1}_d + iY \otimes R_{d/2}), \tag{12}$$

with Y the second Pauli matrix.

The maps $B_{d/2}^{(\pm)}$ have well-known classical limits: They correspond to conservative, piecewise linear versions of the Smale horseshoe [31,32]. Instead of working with the exact $B_{d/2}^{(\pm)}$, we prefer the simpler approximate expressions

$$B_d^{(-)} \cong G_d(|0\rangle\langle 0| \otimes G_{d/2}^{-1} + |1\rangle\langle 1| \otimes G_{d/2}) \equiv D_d, \tag{13}$$

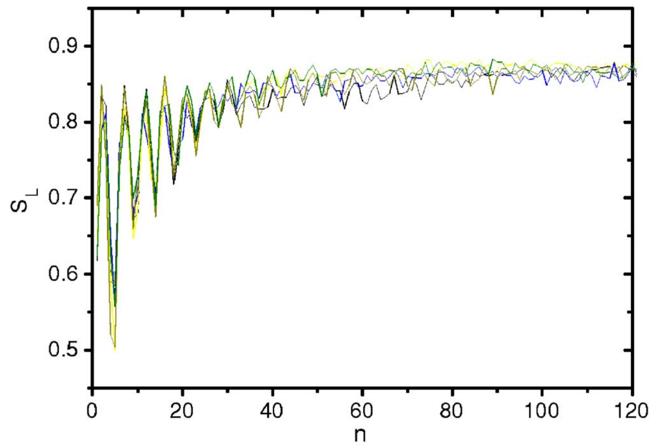

FIG. 1. (Color online) Five initial pure product states were chosen randomly according to the CUE measure, and then evolved by applying the D map n times. At each time the linear entropy $S_L(n)$ is calculated. The system consists of eight qubits divided into two subgroups of the most (least) significant four.

$$R_d B_d^{(+)} R_d \cong G_d(|0\rangle\langle 0| \otimes G_{d/2}^{-1} - |1\rangle\langle 1| \otimes G_{d/2}) \equiv D_d'. \tag{14}$$

The operators on the right-hand sides in the equations above, D_d and D_d', are both unitary, time-reversal symmetric, and do not have spatial symmetries. They share the same classical limit with the reduced baker maps, but correspond to a slightly different quantization of the classical baker map [31]. Instead of (11), D and D' satisfy

$$\Lambda^\dagger \bar{B}_d \Lambda = |0\rangle\langle 0| D_{d/2} + |1\rangle\langle 1| R_d D_{d/2}' R_d, \tag{15}$$

where $\bar{B}_d$ coincides with Schack and Caves' baker $\hat{B}_2$ [11].

IV. COMPARING B_d WITH D_d

In order to assess the effect of symmetries in the production of entanglement, it suffices to consider the nonsymmetric operators D_d or D_d', and compare with B_d.

The iterative application of B_d to an initial product state typically makes the entropy grow quickly from zero to some "equilibrium" value, which depends on the initial state. After that, the entropy keeps fluctuating with small amplitude around the equilibrium. This is the behavior observed by Scott and Caves for a variety of ways of partitioning the qubits [14]. Our analysis of the map D_d [Eq. (13)] verify the same qualitative features. As an example, we show in Fig. 1 some plots of entropy vs time for a system of eight qubits split into two groups: the four most significant qubits on one side, the remaining least significant on the other.

From now on we focus on the asymptotic regime of large times, when the system has already relaxed to equilibrium. In this regime we expect the statistical properties of entangled states to be described by some random matrix model. The simplest ansatz associates evolved pure states with random states of the same Hilbert space, chosen according to the CUE measure. The average linear entropy of these states is given by

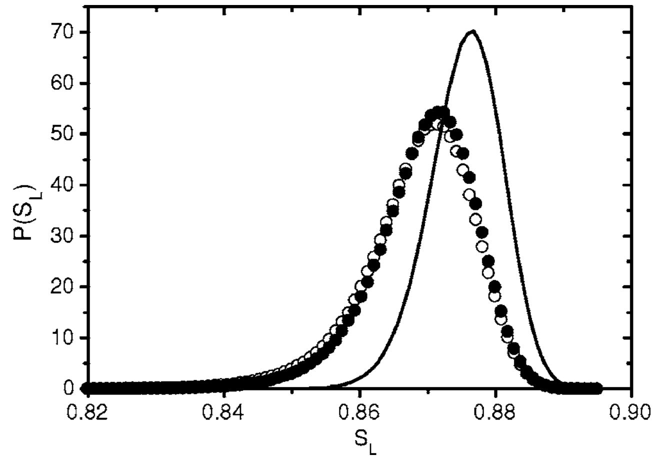

FIG. 2. Histograms of asymptotic linear entropies generated by the baker map (open circles) and by the map D_d (full circles). The line corresponds to the distribution of entropies for a large set of CUE random states calculated numerically. In all cases the system consists of eight qubits divided into two subgroups of the most (least) significant four.

$$\langle S_L \rangle_{\mathrm{CUE}} = \frac{(d_A - 1)(d_B - 1)}{d_A d_B + 1} \tag{16}$$

analytical expressions for the second and third cumulants are also known (see [14] and references therein).

For a quantitative comparison between D and baker maps, we considered the same system as before, but this time we generated a set of 2×10^6 data by gathering values of $S_L(n)$ for $513 \leqslant n \leqslant 2512$ and 10^3 random initial states. These data, properly binned, are displayed in Fig. 2 together with analogous data for the baker map. It can be immediately seen that both maps, baker and D_d, produce very similar distributions of entropies, both shifted to values lower than those of random states. The conclusion of this comparative simulation is that the reflection symmetry is not the cause for the states generated by the baker map being less entangled, in average, that random states (because the D map is nonsymmetric and also shows a reduced entangling power). However, absence of symmetry may be the explanation for a very small, though perceptible, increase of entangling power of the D map as compared with the baker (see Fig. 2). (A similar influence of symmetry in coupled tops was reported by Bandyopadhyay and Lakshminarayan [33].)

Another simple complementary test, which also shows that the influence of the symmetries is very limited, consists of calculating the entangling power of a random map having the same symmetries as the baker map, i.e., time-reversal and reflection symmetries.

The problem of introducing symmetries in a random matrix model is well known, e.g., in the scattering approach to electronic transport through mesoscopic cavities [34–37]. The unitary (scattering matrix) is brought to block diagonal form by choosing a basis with well-defined symmetry, and each block is modeled by a circular ensemble. Using this recipe for the baker map we arrive at an ensemble of random matrices $\mathcal{B}$ with the structure

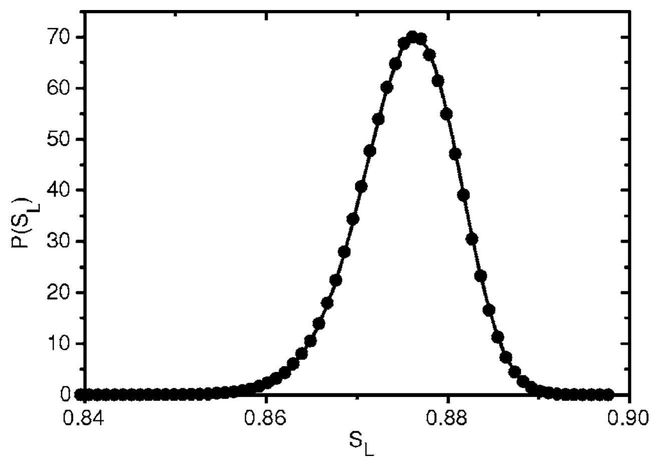

FIG. 3. Histograms of linear entropies generated by one application of random maps belonging either to the symmetric ensemble of Eq. (17) (circles), or to the CUE ensemble (line). In the first case we applied 1000 symmetric maps to 1000 random nonentangled states. The CUE data set is the same as in Fig. 2, and so is the system of qubits.

$$\mathcal{B} = \Lambda(|0\rangle\langle 0| \otimes W^{(1)} + |1\rangle\langle 1| \otimes W^{(2)})\Lambda^{\dagger}, \tag{17}$$

where $W^{(1)}$ and $W^{(2)}$ are drawn independently from the circular orthogonal ensemble (COE), appropriate for unitary maps with time-reversal symmetry [18], and Λ is defined in Eq. (12).

In Fig. 3 we calculate the entangling power of the symmetric random maps defined above and compare with random maps having no symmetries at all, i.e., the CUE ensemble. The figure shows that differences between the two ensembles are not significant. (CUE matrices were generated using the Hurwitz parametrization [38,39]. COE matrices were obtained simply by forming the products VV^T, with V belonging to CUE [18].)

V. FROM QUBITS TO ARBITRARY DIMENSIONS

In the search for an alternative explanation for the anomalies observed, we recall that baker maps in spaces of power-of-2 dimensionality are known to exhibit peculiar properties. For instance, Balazs-Voros and Sano observed that baker maps of dimension 256 and 1024, respectively, display spectral statistics quite far from universal behavior [2,40]. Though asymptotic entangling power is not a property determined by the eigenvalues, but by the eigenvectors (see below), the latter may also be anomalous for the qubit case. Thus, we now proceed to check if the precise value of the Hilbert space dimension has a definite influence in the entangling properties of the baker map.

Note that Eq. (6) defines a quantum baker map for any even dimension d. We shall consider, for instance, two systems with dimensions $d=238$ and 162, partitioned as $238 = 14\times 17$ and $162 = 9\times 18$. The analyses of these cases is presented in Fig. 4. The histograms clearly show that, by avoiding power-of-2 dimensions, we recover universal behavior. These results give additional support to the belief that qubit baker maps (and D maps) possess hidden symmetries,

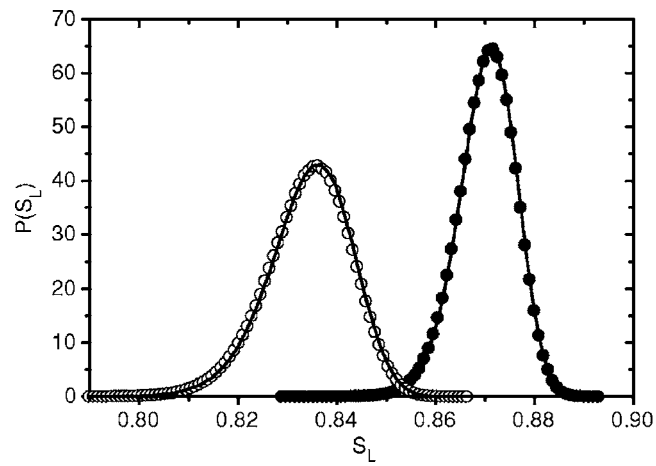

FIG. 4. Histograms of asymptotic linear entropies generated by two nonqubit baker maps with (full circles) $d=238$, $d_A=14$, and $d_B=17$; (open circles) $d=162$, $d_A=9$, and $d_B=18$. Data were collected by the same procedure used for Fig. 2. Full lines are the corresponding CUE predictions.

as happens with cat maps and with certain triangles of the hyperbolic plane [41].

The case $d=162=2\times 3^4$ was chosen to demonstrate that a power-of-3 factor in Hilbert space dimension is not a source of anomaly for the baker map. Presumably, power-of-3 dimensions, i.e., qutrit systems, are anomalous for *ternary* baker maps, like those considered by Nonnenmacher and Zworsky [42,43].

VI. RELATION TO EIGENVECTORS

We have characterized the entangling abilities of a unitary operator by the distribution of entropies it produces when acting iteratively on a set of nonentangled states. But such an entropy distribution depends (weakly) on the number of iterations, even in the long-time regime. For this reason, in the histograms of Figs. 2 and 4 we also included data corresponding to different times.

Then it is natural to consider an average of the entropy over both initial states *and* time. In this way we arrive at a quantity that characterizes unambiguously the *asymptotic entangling power* of a unitary $e_p^{\infty}(U)$, defined by

$$e_p^{\infty}(U) \equiv \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} e_p(U^k), \tag{18}$$

i.e., the time average of the entangling power of Eq. (1).

In the case of the baker and D maps considered in the previous sections e_p^{∞} can be estimated as the mean value of the data set used to generate each histogram in Figs. 2 and 4. However, extracting $e_p^{\infty}(U)$ from a finite data set introduces undesirable statistical errors. This could be avoided by implementing the averages in (18) analytically, followed by numerical evaluation of the resulting expression. We shall see in the following that this procedure leads to a relation between the asymptotic entangling power of an arbitrary unitary U and its reduced eigenvectors. Such a relation is not only very interesting by itself, but it will give us the possibility of checking the numerical simulations of previous sections.

We start by noting that if the average over initial states is removed from Eq. (18) one obtains the *asymptotic entropy* considered by Demkowicz-Dobrzanski and Kus [20]:

$$S_L^{\infty}(|\psi\rangle) = 1 - \sum_i |\langle e_i|\psi\rangle|^4 \mathrm{tr}_A(\rho_A^{ii})^2 - \sum_{i\neq j} |\langle e_i|\psi\rangle|^2 |\langle e_j|\psi\rangle|^2 [\mathrm{tr}_A(\rho_A^{ii}\rho_A^{jj}) + \mathrm{tr}_A(\rho_A^{ij}\rho_A^{ji})], \tag{19}$$

which depends on the initial state $|\psi\rangle$. In the expression above $|e_i\rangle$ stands for an eigenvector of U, and ρ_A^{ij} is the reduced operator

$$\rho_A^{ij} \equiv \mathrm{tr}_B |e_i\rangle\langle e_j|. \tag{20}$$

Of course, ρ_A^{ii} is the reduced density matrix obtained from the ith eigenvector, to be denoted simply by ρ_A^i.

Formula (19) is not of completely general validity. However, its derivation makes only the weak assumption that eigenvalues $\exp(i\phi_k)$ do not satisfy the commensurability relation

$$\phi_k - \phi_l + \phi_m - \phi_n = 0, \tag{21}$$

except for the trivial cases $k=l$ and $m=n$, or $k=n$ and $l=m$ (this condition is more general than merely requiring absence of degeneracies). We have verified numerically that the eigenvalues of the maps studied in this paper, even if anomalous in another sense, are not commensurable. So we can safely use (19).

Averaging (19) over initial random product states we arrive at a formula for $e_p^{\infty}(U)$ in terms of the eigenvectors of U. The derivation is somewhat lengthy but simple. It requires averaging products of the type

$$c_\alpha c_\beta c_\gamma^* c_\delta^*, \tag{22}$$

where c_α are the coefficients that arise from expanding the state $|\psi\rangle$ in the eigenbasis of U. The coefficients c_α are distributed like the elements in a column of a CUE matrix. The average above is one among others calculated by Mello some time ago [44]. We omit the details and just show the final result:

$$e_p^{\infty}(U) = \frac{d+1}{d'} - \frac{2}{dd'}\sum_i [\mathrm{tr}_A(\rho_A^i)^2]^2 - \frac{1}{dd'}\sum_{i\neq j} [\mathrm{tr}_A(\rho_A^i\rho_A^j) + \mathrm{tr}_B(\rho_B^i\rho_B^j)]^2. \tag{23}$$

Here we used the abbreviations $d=d_A d_B$ and $d'=(d_A+1)(d_B+1)$, together with the property

$$\mathrm{tr}_A(\rho_A^{ij}\rho_A^{ji}) = \mathrm{tr}_B(\rho_B^i\rho_B^j), \tag{24}$$

showing that Eqs. (23) and (19) are indeed invariant with respect to the exchange of subsystems A and B.

Equation (23) is a useful formula which has absorbed the averages analytically; it expresses the asymptotic entangling power of a unitary as a function of a special combination of pairs of eigenvectors,

$$\mathrm{tr}_A(\rho_A^i\rho_A^j) + \mathrm{tr}_B(\rho_B^i\rho_B^j), \tag{25}$$

i.e., the symmetrized Hilbert-Schmidt scalar product of the reduced density matrices. For the maps considered here, we have checked that the calculation of e_p^{∞} either using Eq. (23) or by straightforward time and ensemble averages leads to consistent results; thus we verified the correctness of both procedures.

We remark that e_p^{∞} is not directly related to the eigenvector entropies

$$1 - \mathrm{tr}_A(\rho_A^i)^2, \tag{26}$$

even though bounds relating asymptotic entangling power and eigenvector average entropies can be obtained by the use of the Cauchy-Schwartz inequality [20]. (One must remember that in some cases eigenvector entanglement may give a wrong estimation of e_p^{∞} [20,45].)

VII. CONCLUDING REMARKS

We demonstrated that the deviations from universal behavior reported by Scott and Caves [14] are not due to spatial or time-reversal symmetries. Instead, the anomalous entangling power of the qubit baker map originates from specificities associated with the dimension of the Hilbert space being a power of 2. When other dimensions are considered, a behavior consistent with random matrix theory is recovered. Presumably qubit baker maps possess symmetries of number theoretic origin, i.e., with no classical analogs ("pseudosymmetries" [41]). All the members of the Schack-Caves family [11] (of which the baker map considered here is a special case) suffer, to different extents, from a reduction of the entangling power [14]. It is tempting to speculate that those differences may be related to each member having a different number of pseudosymmetries.

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