
Gravitational Waves from First-Order Phase Transitions in the Early Universe

Master Thesis



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"GRAVITATIONAL WAVES FROM FIRST ORDER PHASE TRANSITIONS
IN THE EARLY UNIVERSE"

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Abstract

This thesis presents a theoretical and numerical study of stochastic gravitational-wave (GWs) backgrounds generated by first-order phase transitions (FOPTs) in the early Universe. In Chapters 2, 3 and 4, we develop a pedagogical construction of the physical chain that connects the finite-temperature effective potential, $V_{\text{eff}}(\phi, T)$, to the thermodynamic parameters of the transition and, finally, to the gravitational-wave spectrum $\Omega_{\text{GW}}(f)$. In Chapter 5, we apply this theoretical framework to the code developed throughout the Master's project, which implements a numerical pipeline capable of extracting the thermal history, the phase-transition parameters and the corresponding gravitational-wave spectrum.

We apply this pipeline to two classes of effective potentials associated with modifications of the scalar sector. The first contains a logarithmic term motivated by a correction to the functional measure of the path integral, following Ref. [1]. The second corresponds to the addition of higher-dimensional irrelevant operators, in particular ϕ^6 and ϕ^8 terms, to the scalar potential, following Ref. [2]. We also investigate the combination of these two deformations. With these examples, we analyze how the thermodynamic parameters of the transition vary according to the free parameters of theories beyond the Standard Model and, consequently, how these variations are reflected in the gravitational-wave spectrum. In addition, we study the impact of including thermal resummation corrections in the potentials and in the predicted gravitational signal.

The results show how gravitational waves can open a broad phenomenological window for testing high-energy physics. Different modifications beyond the Standard Model may, in principle, be confirmed, constrained or excluded through the future detection of stochastic backgrounds by space-based interferometers such as LISA, BBO and DECIGO. Thus, this dissertation reinforces the idea that gravitational waves are not only a new observational window into the early Universe, but also a tool complementary to terrestrial colliders in the search for new physics.

Keywords: gravitational waves; first-order phase transitions; effective potential; early Universe; scalar sector; electroweak transition; LISA; DECIGO; BBO.

Research is to see what everybody else has seen, and to think what nobody else has thought.
(Szent-Györgyi, Albert)

Resumo

Esta tese apresenta uma discussão teórica e numérica sobre a produção de um fundo estocástico de ondas gravitacionais (GWs) gerado por transições de fase de primeira ordem (FOPTs) no Universo primordial. Nos Capítulos 2, 3 e 4, construímos de forma instrutiva a física que conecta o potencial efetivo a temperatura finita, $V_{\text{eff}}(\phi, T)$, aos parâmetros termodinâmicos da transição e, finalmente, ao espectro de ondas gravitacionais $\Omega_{\text{GW}}(f)$. No Capítulo 5, aplicamos essa base teórica ao código desenvolvido ao longo do mestrado, que implementa um pipeline numérico capaz de extrair a história térmica, os parâmetros da transição de fase e o espectro de ondas gravitacionais correspondente.

Aplicamos esse pipeline a duas classes de potenciais efetivos associados a modificações do setor escalar. A primeira contém um termo logarítmico motivado por uma correção da medida funcional da integral de caminho, seguindo a referência [1]. A segunda corresponde à adição de operadores irrelevantes de ordem superior, em particular termos do tipo ϕ^6 e ϕ^8 , ao potencial escalar, seguindo a referência [2]. Também investigamos a combinação dessas duas deformações. Com esses exemplos, analisamos como os parâmetros termodinâmicos da transição variam de acordo com os parâmetros livres de teorias além do Modelo Padrão e, consequentemente, como essas variações se refletem no espectro de ondas gravitacionais. Além disso, estudamos o impacto da inclusão de correções de ressonância térmica sobre os potenciais e sobre o sinal gravitacional previsto.

Os resultados mostram como ondas gravitacionais podem abrir um espaço fenomenológico amplo para testar física de altas energias. Diferentes modificações além do Modelo Padrão podem, em princípio, ser confirmadas, restringidas ou excluídas por meio da futura detecção de fundos estocásticos por interferômetros espaciais como LISA, BBO e DECIGO. Assim, esta dissertação reforça a ideia de que ondas gravitacionais não são apenas uma nova janela observacional para o Universo primordial, mas também uma ferramenta complementar aos colisores terrestres na busca por nova física.

Palavras-chave: ondas gravitacionais; transições de fase; potencial efetivo; Universo primordial; setor escalar; transição eletrofraca; LISA; DECIGO; BBO.

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For more thoughts, see Appendix F.

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1 Chapter One

Introduction

Why are first-order phase transitions (First-Order Phase Transitions, FOPTs) in the early Universe such privileged sources of gravitational waves (Gravitational Waves, GWs) – and, more than that, why is it worth studying gravitational waves *now*? In other words: what makes this topic genuinely relevant in the current context of physics, and not merely a technical curiosity?

These are, for me, the first natural questions when facing such a specific subject. The goal of this introductory part is exactly to answer them: to motivate why GWs have become a central observable in modern cosmology and why FOPTs occupy a special role within this story. The main idea is simple: GWs can carry information from epochs of the Universe that are inaccessible to photon-based astronomy, opening a window into high-energy physics and, in particular, into scenarios beyond the Standard Model.

1.1 Motivations

Before going deeper into FOPTs, it is worth motivating why studying GWs themselves matters. To understand the importance of this “new messenger”, it is useful to take a step back and look at common ground in observational cosmology: the cosmic microwave background radiation (Cosmic Microwave Background, CMB).

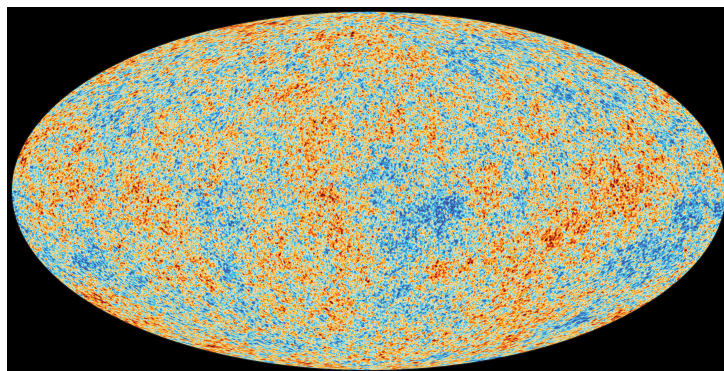


Figure 1: Map of CMB temperature anisotropies measured by the *Planck* satellite.

The CMB is, essentially, a “thermal fossil” of the primordial plasma: a background of photons that decoupled from matter when the Universe cooled enough for electrons and protons to combine into neutral atoms (recombination) [3] . From that moment – which we associate

with the last scattering surface – photons stopped interacting strongly with baryons and began to travel freely through the expanding space, making the Universe transparent to electromagnetic radiation – the earlier period is often described as opaque to photons. Today, we observe this radiation as an almost perfect blackbody spectrum with mean temperature $T \simeq 2.725\text{K}$, containing small-amplitude anisotropies that encode deep information about the content and geometry of the Universe. [3]

Historically, these photons have dominated observational cosmology: the CMB allows us to reconstruct, with remarkable precision, the cosmic composition (baryonic matter, dark matter, dark energy), the physics of acoustic oscillations in the primordial plasma, and fundamental cosmological parameters – including the value of H_0 , whose determination lies at the center of current observational tensions. In summary, studying the CMB is an extremely efficient way to access a well-defined epoch in the thermal history of the Universe and extract from it a huge amount of physics.

Below, I present a schematic figure of the cosmological timeline, highlighting the position of the CMB within this panorama:

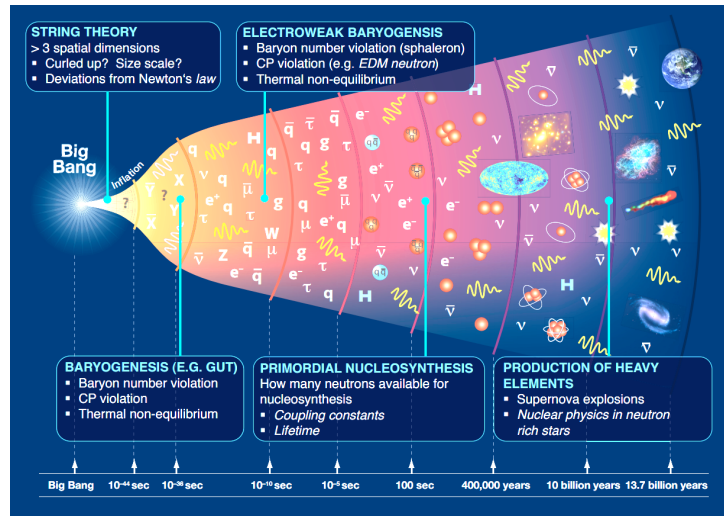


Figure 2: Schematic timeline of the Universe, indicating the CMB and different cosmological epochs. (Source [arXiv:1205.2451](https://arxiv.org/abs/1205.2451).)

For a long time, it seemed almost unthinkable to “see” epochs earlier than the CMB using photons: before decoupling, the Universe was opaque to electromagnetic radiation, and light simply could not travel freely over long distances without being scattered and thermalized. This naturally raises the central question: is there any other messenger capable of crossing the primordial Universe and reaching us carrying information from older times?

The answer is yes. In principle, neutrinos also carry very primordial information (although their direct cosmological detection is extremely challenging). But this is where gravitational waves enter as a particularly promising observable: because they interact extremely weakly with the medium, they can propagate practically decoupled from the plasma after their production, without being absorbed or scattered, suffering mainly cosmological redshift, and preserving memory of the physical conditions of the primordial Universe where they were generated.

With technological progress, this messenger stopped being only a theoretical prediction of relativistic equations and became part of our observational toolkit: gravitational waves were directly detected for the first time in 2015 by the LIGO collaborations [4], inaugurating a new era in astronomy and cosmology. Since then, GW sources can be divided, schematically, into two broad categories:

- **Astrophysical sources:** compact binary systems (black holes and neutron stars), mergers, pulsars, possible emissions associated with supernovae, among others.
- **Cosmological sources:** mechanisms from the primordial Universe, such as inflation, cosmic strings and, of particular interest in this thesis, **first-order phase transitions**.

The conceptual difference is important. Astrophysical sources typically generate “localized” signals (events) associated with massive and compact objects. Cosmological sources, on the other hand, may produce a *stochastic background* of gravitational waves: a diffuse and stochastic signal, analogous in spirit to the CMB itself, but originating in much earlier epochs – closer to the Big Bang and, therefore, at much higher energy scales, a background not of light, but of gravitational waves. [5, 6]

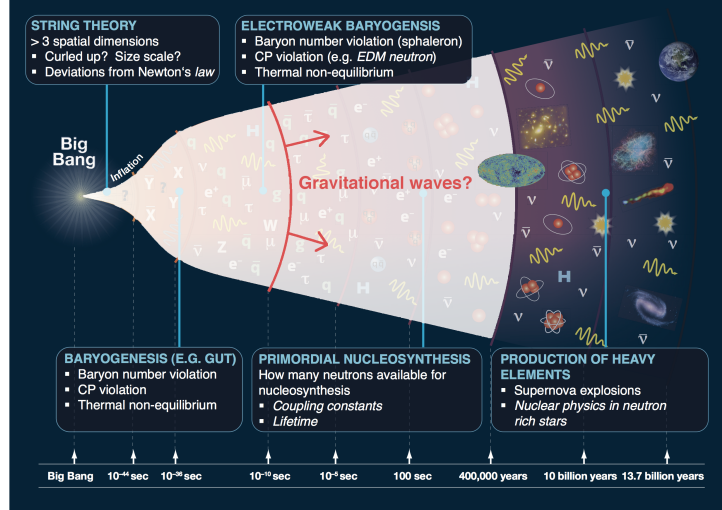


Figure 3: Cosmological timeline emphasizing that, unlike photons, GWs can carry information from epochs earlier than CMB decoupling.

Thus, a natural way to investigate the early Universe is to look at GWs. They can, in principle, reveal information about epochs inaccessible to light, when the Universe was hotter and the fundamental interactions operated in high-energy regimes. This means that we can complement the results obtained by terrestrial colliders with GWs: GWs provide an indirect, cosmological and global probe, capable of testing classes of particle-physics models, at high energies, that leave specific signatures in the spectrum $\Omega_{\text{GW}}(f)$. In this way, we can test different particle-physics models in an extremely high-temperature regime through this observable and, therefore, complement the results of current colliders and even generate new information about energy scales still inaccessible on Earth.

There is, however, an essential detail: different primordial production mechanisms appear today in different frequency ranges after cosmological redshift. Ground-based detectors such as LIGO/Virgo/KAGRA typically observe frequencies in the range of tens to thousands of Hz, while cosmological signals associated with phase transitions at electroweak/TeV scales are often expected in the mHz range or below – exactly where space-based missions such as LISA, BBO and DECIGO are more sensitive [7–10]. This makes the theoretical and phenomenological study of these signals urgent (and unavoidable): the relevant observational window is consolidating, and we need to know what to look for, how to parametrize it, and how to interpret a possible detection.

This is where first-order phase transitions enter. We know that the Universe went through (and may have gone through) important thermal transitions along its history (e.g., the electroweak transition, the QCD transition). If any of them was first order, bubbles of the true phase

nucleate, expand, collide and stir the relativistic plasma, producing GWs with a characteristic spectrum [11–14]. More than that: FOPTs are one of the most direct paths to connect gravitational waves to new physics beyond the Standard Model, since different extensions of the scalar sector and of the particle content can modify the order and strength of the transition, changing parameters such as α and β/H_* and, consequently, the observable signal, as we will see throughout this study. In addition, an extra motivation for focusing on FOPTs is that one of the basic requirements for explaining the matter-antimatter asymmetry in the Universe is a departure from thermal equilibrium, which is naturally provided by a first-order phase transition, making it an even richer physical scenario to explore.

In short, gravitational waves are a new messenger that we have only recently begun to explore concretely, driven by technological advances in instrumentation and data analysis. They offer the possibility of accessing more primitive epochs of the Universe and, with that, of investigating physics at energy scales that cannot be directly reproduced in the laboratory. And even in the scenario where no cosmological signature is detected, the result is still informative: strong experimental limits allow us to constrain entire classes of particle-physics models and to strengthen (or put pressure on) what we understand as fundamental physics and the Standard Model.

With this motivation established, the next step is to define the specific objectives of this thesis: which models and potentials will be considered, which physical quantities will be extracted from the transitions, and how these results translate into observational predictions for gravitational-wave detectors.

Until recently, we could only “see” the Universe through photons; now we can also “hear” it through GWs, and we should use this new sense to advance our physical understanding of the Universe.

1.2 Objectives

In the previous section, we discussed why gravitational waves have become a central observable in present-day cosmology and why first-order phase transitions (FOPTs) are an especially interesting mechanism: besides producing a potentially detectable stochastic background, they directly connect the thermal history of the primordial Universe with the microphysics of the particle sector. In other words, a detection (or even an experimental limit) in $\Omega_{\text{GW}}(f)$ can be translated into quantitative constraints on high-energy models.

With this in mind, the natural question that remains is: *which transition are we talking about, and which effective potential will be used to describe it?*

A particularly well-defined starting point is the electroweak transition associated with the Higgs field. We know experimentally, since the detection of the Higgs boson, that the mechanism of spontaneous electroweak symmetry breaking occurred in the history of the Universe. However, for the measured Higgs mass, the Standard Model predicts that this process occurs smoothly (as a *crossover*) , that is, without a first-order phase transition with bubble nucleation. Consequently, no strong cosmological gravitational-wave signal is expected from the electroweak transition *within* the Standard Model. [15, 16]

This is exactly where the problem (and the motivation) of this thesis becomes clear: if we want to associate a GW signal with a transition close to the electroweak scale, we need to consider extensions of the Standard Model or effective modifications of the scalar sector that make the transition sufficiently strong and first order [2, 17, 18] . These modifications may arise from different physical origins (new fields, new couplings, effective operators, or corrections associated with the quantum and thermal description of the potential), and each scenario leaves its own signature in the parameters that control the gravitational signal. Each model may therefore leave distinguishable imprints in such a background.

In this way, the central objective of this thesis is twofold:

1. **General objective (didactic):** to explain and build the theoretical basis needed to construct gravitational-waves predictions from a given effective potential in a general way.
2. **Specific objective (physical case study):** to apply this framework to a concrete modification of the scalar sector associated with the Higgs, in which the electroweak transition may become first order, and to investigate how the shape of the potential and its parameters are reflected in the predicted spectrum, in the detectability by future observatories, and in what conclusions can be extracted from this spectrum.

The importance of this approach is that it does not depend on a single model: the method built here is designed to be modular and generalizable, allowing us to explore different potentials that generate FOPTs and to compare, in observational language, what each scenario “says” to the GW spectrum.

In the end, therefore, what we seek is not merely a set of curves, but a clear physical interpretation: which parameter regimes produce potentially detectable signals, which remain

below sensitivity, and what types of conclusions can be extracted from a future detection (or from upper limits). At a time when missions such as LISA are on the observational horizon, this kind of prior mapping is essential: when the data arrive, we need to be ready to translate them into information about fundamental physics.

In the next subsection, I present the road map of this thesis and how each chapter contributes to building this bridge between effective potential, phase transition and observational prediction of gravitational waves.

1.3 Road Map

In the previous section, we established the objective of this research. Now we need to make explicit *how* this investigation will be conducted and which didactic sequence will be followed throughout the chapters.

Chapter 2, *Effective Potential*, builds the central object of the whole thesis: the finite-temperature effective potential $V_{\text{eff}}(\phi, T)$. There we discuss (i) what characterizes a first-order phase transition, in contrast with a smooth transition (*crossover*); (ii) how the structure of minima and barriers in the potential emerges with temperature; and (iii) which modifications of the scalar sector – in particular in the context associated with the Higgs – can make the electroweak transition first order, as well as their physical motivations.

Once the potential and its phase history are defined, Chapter 3, *Thermodynamics of the Transition*, shows how to quantitatively extract the parameters that control the transition. In practical terms, this chapter builds the bridge between $V_{\text{eff}}(\phi, T)$ and quantities such as the critical and nucleation temperatures (T_c , T_n), as well as the parameters that summarize the strength and effective duration of the transition (for example α and β/H_*), always making clear the approximations and physical assumptions involved.

With these ingredients in hand, Chapter 4, *Gravitational Waves*, discusses how the transition parameters are translated into an observable prediction: the stochastic gravitational-wave spectrum $\Omega_{\text{GW}}(f)$. The objective here is to make explicit the chain connecting microphysics (the potential and the transition dynamics), production mechanisms (bubbles/plasma) and the spectral shape that would be measured by detectors.

Finally, in Chapter 5, *Results and Discussion*, we return to the particular case of the chosen potentials and present the numerical results obtained: thermal history, values of T_n , α , β/H_* and the corresponding spectrum $\Omega_{\text{GW}}(f)$. These results are then compared with sensi-

tivity curves of future detectors, allowing us to discuss concretely what could (or could not) be observed, which parameter regions are favored and which scenarios may be constrained. We also investigate the effect of adding or removing thermal resummation in the potential and its impact on the observed spectrum.

It is important to emphasize that a significant part of the calculations in this thesis requires a dedicated numerical implementation. To ensure reproducibility and facilitate extensions to other models, I developed a code in `Python` that automates this pipeline. The repository is public and available at:

github.com/IgorStellet/GW_FOPT_Simulation.

Figure 4 summarizes, schematically, this logical sequence: from the effective potential to the thermal transition, and from there to the observable gravitational-wave spectrum. With this map established, in the next chapter we begin the discussion of the effective potential and of the physical conditions that make a first-order phase transition possible.

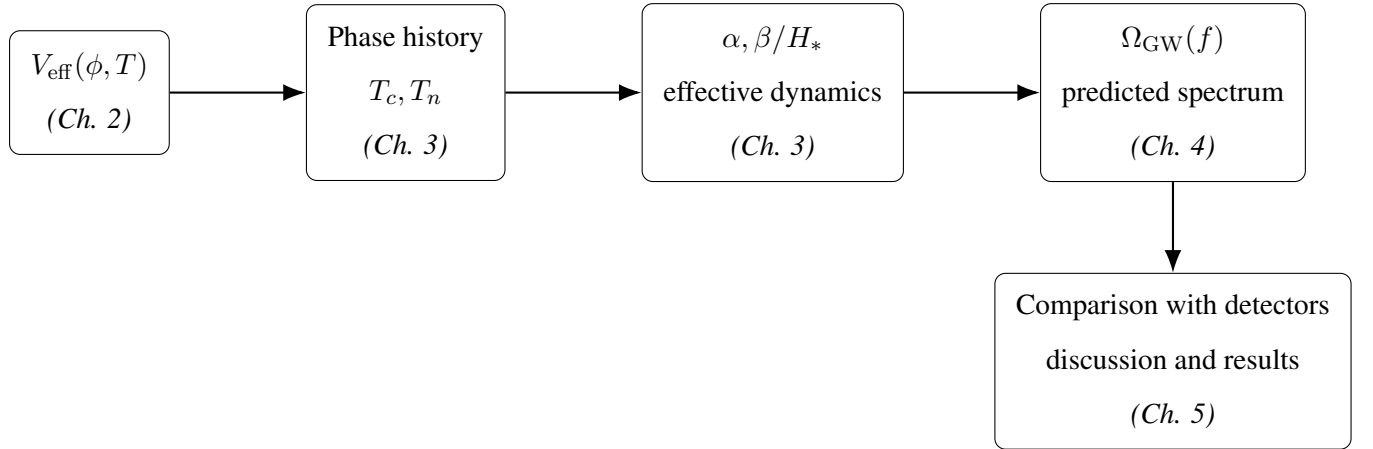


Figure 4: Logical map of the thesis: from the finite-temperature effective potential to the prediction and observational interpretation of the gravitational-wave spectrum.

2 Chapter Two

The effective potential

In the previous chapter, we saw that GWs have become a new gateway into cosmological and astrophysical physics. Unlike photons, which stop informing us directly about epochs prior to the CMB, gravitational waves can carry information from much earlier periods in the history of the Universe. In addition, we discussed that one of the possible cosmological sources of this new messenger are FOPTs that may have occurred when the Universe was still extremely hot and dense.

Thus, before computing any gravitational-wave spectrum, we need to answer more fundamental questions: what, in fact, is a phase transition? What is the physical object that undergoes the transition? What does it mean to say that a transition is first order? What is the difference between a smooth transition, a second-order transition and a first-order transition? And, finally, how does all of this connect with the electroweak transition, with the Higgs field and with possible modifications of the Standard Model?

As we will see throughout this chapter, these questions are deeply connected to thermodynamics, to Landau theory for phase transitions, to quantum field theory (QFT) at finite temperature and to particle physics.

The role of this chapter is, therefore, to establish the theoretical base of the pyramid that will lead us, at the top, to gravitational waves. At its base lies the microscopic physics of the scalar sector, and this is what we will study next.

2.1 What is a phase transition?

When we think about everyday situations, we do not always notice how phase transitions are present all around us. The expression “phase transition” may seem, at first contact, something distant or overly technical, especially for those who are not familiar with the language of physics. However, the idea appears in very simple phenomena.

The most immediate example is water. When we heat liquid water up to a certain temperature, it can undergo a transition to the gaseous state. When we cool it down, we can take it to the solid state. This change between liquid, vapor and ice is the first intuitive contact we usually have with a phase transition: the same physical system, made of the same molecules,

begins to display completely different macroscopic properties.

This example already contains an essential part of the idea. To trigger the change of phase, we vary an external parameter, in this case the temperature. By varying this parameter, we change the thermodynamic condition of the system, and the preferred state of matter can change. Liquid water, vapor and ice correspond to different ways of organizing the water molecules. The phase transition occurs when, by changing an external condition, the system reorganizes its degrees of freedom and moves from one phase to another.

But the physical concept of a phase transition goes far beyond changes of state of matter. A particularly important example for physics is ferromagnetism. In a ferromagnetic material, we can imagine the microscopic spins as tiny magnetic moments. At very high temperatures, thermal agitation tends to disorder these spins, so that there is no preferred macroscopic magnetization. However, below a critical temperature, called the Curie temperature, the spins can align spontaneously and the material begins to exhibit magnetization even in the absence of an external magnetic field. This alignment can be in any direction, so we say that there is a spontaneous symmetry breaking in this transition, due to the presence of more than one possible equilibrium state at low temperatures.

This example is extremely instructive because it shows that a phase transition does not need to be associated only with “solid, liquid and gas”. It can be associated with the emergence of order in a system. In the case of the ferromagnet, what changes is not the substance that composes the material, but the collective organization of its microscopic degrees of freedom. Above the critical temperature, the spins are disordered; below it, a preferred direction emerges and the system moves to an ordered phase.

This observation leads us to a central question: what, exactly, characterizes a phase?

The modern answer is that a phase can be characterized by macroscopic properties that distinguish one equilibrium state from another. Often, this distinction is made through a quantity called the **order parameter**.

Another essential ingredient is the free energy. At finite temperature, the system does not simply choose the state of lowest internal energy, but the state that minimizes the appropriate free energy. A phase transition occurs when the structure of the equilibrium states of the system reorganizes as we vary some external parameter, such as temperature, pressure, density or an external field. For this reason, the language of phase transitions will be, throughout the rest of this chapter, a language of minima.

Thus, to define a phase transition in a general way, we need to understand three elements: first, what is the physical system we are studying; second, what is the external parameter that is being varied; and third, what is the quantity that characterizes the change of phase.

In the early Universe, the physical system is the set of quantum fields that make up the fundamental theory in a finite-temperature situation. As the Universe expands, its temperature decreases. This drop in temperature can modify the shape of the free energy associated with the fields, changing which configurations are stable, metastable or unstable. In a field theory, this free energy will be described by the finite-temperature effective potential, $V_{\text{eff}}(\phi, T)$. [18, 19]

Instead of water molecules reorganizing, or spins aligning in a material, we will study scalar fields whose equilibrium values change with temperature. The phase of the system will be identified with a minimum of the effective potential. If the minimum is at $\phi = 0$, we may be in a symmetric phase. If the minimum is at $\phi \neq 0$, we may be in a symmetry-broken phase, especially when different minima related by a symmetry of the theory become physically equivalent, that is, when the minimum of the broken phase is degenerate. As the temperature changes, these minima can appear, disappear, exchange stability or be separated by a barrier.

Different languages for the same idea: a phase changes when the preferred state changes

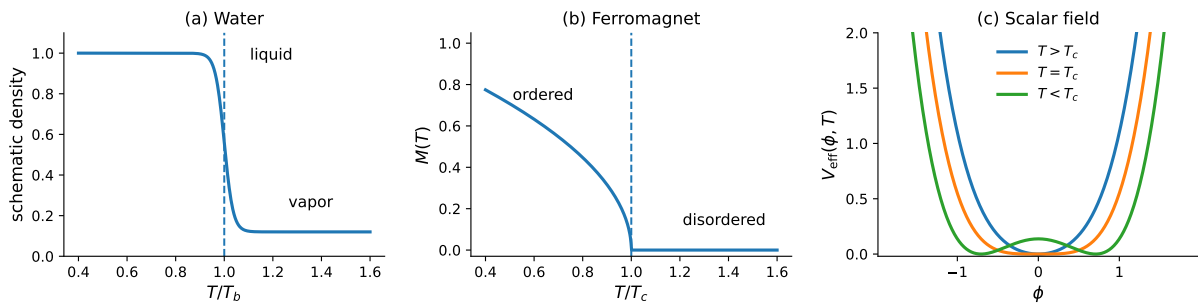


Figure 5: Schematic comparison between different physical realizations of a phase transition. In ordinary matter, a phase transition may be described by a change in density, as in the liquid–vapor transition of water. In a ferromagnet, the transition is characterized by the appearance of a spontaneous magnetization below the critical temperature. In a scalar field theory, the same idea is translated into the displacement of the minimum of the finite-temperature effective potential.

It is precisely this geometry of the potential that will determine the type of transition. In a smooth transition, the relevant minimum shifts continuously and there is no bubble nucleation. In a first-order transition, on the other hand, two distinct phases coexist over some temperature interval: a false phase, initially metastable, and a true phase, energetically favored. The passage from one to the other does not occur homogeneously throughout space, but through the nucleation of bubbles of the true phase inside the false phase.

In the context of this thesis, this is the case of greatest interest. First-order phase transitions in the early Universe can produce violent disturbances in the plasma: bubbles nucleate, expand, interact with the surrounding fluid and eventually collide. This out-of-equilibrium dynamics can generate a stochastic background of gravitational waves that, after propagating through cosmic history, could reach us as a signature of high-energy physics.

Even if these concepts may seem abstract at first, the physical logic is the same as in the more familiar examples: an external parameter changes, the equilibrium state reorganizes and the system moves from one phase to another. The difference is that, now, the object that undergoes the transition is not a common substance, but the very thermal state of the fundamental fields of the Universe. As distant as it may seem, first-order transitions are also in our everyday life, such as the freezing of water.

2.2 Free energy, order parameters and Landau theory

In the previous section, we discussed in a qualitative way the important variables that define a phase transition. In this section, therefore, our task is to formalize these concepts in a mathematical language. The idea is to turn the intuition built from everyday examples – water, magnets, ordered and disordered phases – into quantitative objects that can later be used in field theory. In particular, we want to understand the role of free energy, the order parameter and Landau theory as a first phenomenological description of phase transitions.

To describe a phase transition, we need to associate to the system a parameter capable of characterizing and distinguishing its phases. This quantity, as we have already discussed, is the **order parameter**. In general, the order parameter is a macroscopic quantity that takes different values in different phases of the system. In water, for example, we can associate the difference between liquid and vapor to quantities such as density, specific volume or volume per molecule. In a ferromagnet, the natural order parameter is the magnetization. In the disordered phase, at sufficiently high temperatures, the spontaneous magnetization is zero; in the ordered phase, below the critical temperature, it becomes nonzero.

Let us represent this order parameter generically by M . We want to use M as a general notation for any macroscopic quantity that distinguishes one phase from another. In many problems it is convenient to normalize this order parameter so that it varies between 0 and 1, but this is not a physical necessity. What matters is that M carries the information about the phase of the system. In a symmetric or disordered phase, we may have $M = 0$; in an ordered

or symmetry-broken phase, we may have $M \neq 0$.

Since M is a macroscopic quantity, it is naturally tied to a statistical description of the system. A phase transition is not a phenomenon of a single molecule or of a single isolated spin, but rather a collective property of many degrees of freedom. This is why the language of statistical ensembles becomes so important: the phase of a system is an emergent property that appears when we consider a large number of microscopic constituents and study the equilibrium state they assume under certain external conditions.

In the context of a scalar theory, which will be the central case of this thesis, the order parameter will be associated with the mean value of the scalar field,

$$M \sim \langle \phi \rangle. \quad (1)$$

This mean value is often called the VEV, from *vacuum expectation value*. At finite temperature, it is more precise to think of this object as the thermal equilibrium value of the field, that is, the value of ϕ that minimizes the free energy of the theory for a given temperature. Thus, when we say that a phase is at $\phi = 0$ or at $\phi \neq 0$, we are saying that the relevant minimum of the effective potential occurs, respectively, at the origin or away from it.

This brings us to the second fundamental ingredient: free energy. At finite temperature, the system does not simply choose the state of lowest internal energy. Thermodynamic equilibrium is determined by the minimization of the available free energy. This distinction is important because temperature introduces a competition between internal energy and entropy. A state may have higher internal energy, but be thermodynamically favored for having higher entropy. Therefore, in a phase transition, what changes is the state that minimizes the free energy of the system under the imposed external conditions.

To establish this point more clearly, we should recall some basic concepts of statistical thermodynamics. In a thermodynamic system, we can characterize the macroscopic state by extensive variables such as internal energy U , entropy S , volume V and particle number N , and intensive variables such as temperature T , pressure p and chemical potential μ . When we move to the statistical description, a central quantity is the partition function. In the canonical ensemble, it is defined by

$$Z(T, V, N) = \sum_s e^{-\beta E_s}, \quad (2)$$

where the sum runs over the microstates s of the system, E_s is the energy of the corresponding microstate and

$$\beta = \frac{1}{k_B T}. \quad (3)$$

Throughout this thesis, we will often use natural units, in which $k_B = 1$, so that $\beta = 1/T$.

The partition function contains the fundamental statistical information of the system in equilibrium. It determines the probability of finding the system in a microstate of energy E_s at a fixed temperature.

Lower-energy states are favored by the Boltzmann factor, but entropy also plays an essential role. It is precisely from this competition that the structure of the free energy emerges.

The Helmholtz free energy is defined as

$$F(T, V, N) = U - TS, \quad (4)$$

where U is the internal energy and S is the entropy of the system. Free energy represents the energy available to perform work at fixed temperature, volume and particle number. The term TS measures the entropic contribution: the higher the temperature or the higher the entropy of a state, the greater the thermodynamic weight associated with disorder. Thus, minimizing F does not mean only minimizing energy, but finding the best balance between energy and entropy.

The relation between the partition function and the free energy is given by

$$F(T, V, N) = -T \ln Z(T, V, N). \quad (5)$$

This equation is the bridge between the microscopic description and macroscopic thermodynamics: from Z , we can obtain the free energy, entropy, mean energy, specific heat and the other relevant thermodynamic quantities.

The equilibrium state of the system, for a given temperature, is the one that minimizes the free energy. If the order parameter is M , then the equilibrium condition can be written as

$$\frac{\partial F(T, M)}{\partial M} = 0, \quad \frac{\partial^2 F(T, M)}{\partial M^2} > 0, \quad (6)$$

where the first condition identifies stationary points and the second selects local minima. When there is more than one minimum, the global minimum represents the thermodynamically stable

phase, while local minima can represent metastable phases.

Now that we understand these concepts, we can return to the phase transition itself. An intrinsic aspect of many phase transitions is that they appear associated with a change of symmetry of the equilibrium state. At high temperatures, the system is generally in a more symmetric and disordered phase. As the temperature decreases, the minimum of the free energy can become degenerate, and the system spontaneously chooses one among several equivalent configurations that have the same free energy. This choice of a specific minimum is what we call **spontaneous symmetry breaking**.

Physically, the low-temperature phase usually has a smaller symmetry than the high-temperature phase. A gas or a liquid, for example, are invariant under continuous translations: if we shift the whole system by an arbitrary distance, its macroscopic appearance does not change. A solid, on the other hand, has only invariance under discrete translations associated with its crystal lattice. Therefore, when going from liquid to solid, the system loses part of its continuous translational symmetry. In the magnetic case, the idea is even more direct: at high temperatures, the spins are disordered and there is no preferred spatial direction; at low temperatures, the spontaneous magnetization chooses a direction, breaking the original rotational symmetry of the system.

Landau theory provides a simple and powerful way to describe this phenomenon [19, 20]. It starts from the idea that, close to the transition, the free energy can be expanded in powers of the order parameter. Let us consider, for simplicity, a scalar order parameter M . If the system has the symmetry $M \rightarrow -M$, as happens in many magnetic systems in the absence of an external field, then states with magnetization M and $-M$ must be physically equivalent. This means that the free energy must satisfy

$$F(T, M) = F(T, -M). \quad (7)$$

As a consequence, only even powers of M appear in the expansion:

$$F(T, M) = F_0(T) + \alpha_2(T)M^2 + \alpha_4(T)M^4 + \dots \quad (8)$$

Here, $F_0(T)$ is the part of the free energy independent of the order parameter, while the coefficients $\alpha_2(T)$ and $\alpha_4(T)$ encode the dependence on temperature and on the microscopic physics of the system.

For a continuous transition described by the expansion up to fourth order, we must impose $\alpha_4 > 0$. This condition guarantees that the free energy is bounded from below for large values of M , that is, that $F(T, M)$ is convex for $M \gg 1$. If α_4 were negative, the expansion up to fourth order would not be sufficient to stabilize the system, and we would need to keep higher-order terms, such as $\alpha_6 M^6$, with $\alpha_6 > 0$.

Let us now analyze the minima of the Landau free energy. The equilibrium condition is

$$\frac{\partial F}{\partial M} = [2\alpha_2(T) + 4\alpha_4(T)M^2] M = 0. \quad (9)$$

This equation has the solutions

$$M = 0 \quad \text{or} \quad M = \pm \sqrt{-\frac{1}{2} \frac{\alpha_2(T)}{\alpha_4(T)}}. \quad (10)$$

Now we see clearly how the phase transition appears mathematically.

For $T > T_c$, we want the only stable physical solution to be $M = 0$, corresponding to the symmetric or disordered phase. This happens if

$$\alpha_2(T > T_c) > 0, \quad (11)$$

because, in that case, the solution with $M \neq 0$ becomes imaginary and therefore does not represent a real physical state. For $T < T_c$, on the other hand, we want the stable solution to correspond to $M \neq 0$, that is, to an ordered or symmetry-broken phase. For this, we need

$$\alpha_2(T < T_c) < 0. \quad (12)$$

In a continuous transition, the order parameter must vary smoothly from zero as we cross the critical temperature. Therefore,

$$\alpha_2(T_c) = 0. \quad (13)$$

The simplest choice, valid sufficiently close to the transition, is to take

$$\alpha_2(T) = \alpha_0(T - T_c), \quad \alpha_0 > 0, \quad (14)$$

and to consider α_4 approximately constant near T_c .

With these assumptions, the equilibrium order parameter becomes

$$M_{\text{eq}}(T) = \begin{cases} 0, & T \geq T_c, [0.3\text{cm}] \pm \sqrt{\frac{\alpha_0}{2\alpha_4}(T_c - T)}, \\ T < T_c. \end{cases} \quad (15)$$

Thus, below the critical temperature, the order parameter grows continuously from zero. Near T_c , we have

$$|M_{\text{eq}}(T)| \propto |T - T_c|^{\beta_{\text{crit}}}, \quad \beta_{\text{crit}} = \frac{1}{2}, \quad (16)$$

where β_{crit} is the critical exponent associated with the order parameter in the Landau mean-field approximation. It is important to note that this β_{crit} should not be confused with $\beta = 1/T$, used earlier in the partition function.

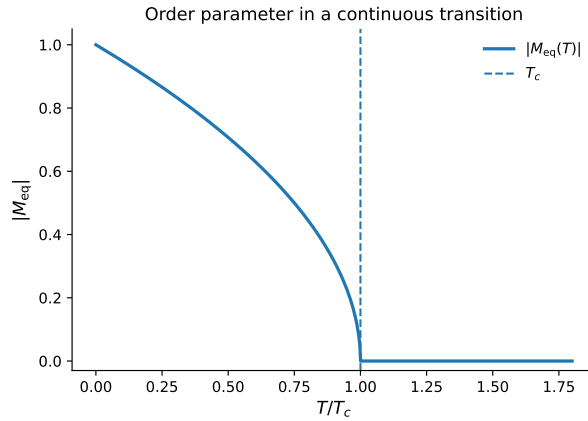


Figure 6: Mean-field behavior of the order parameter in a continuous phase transition. Above T_c , the stable phase has $M_{\text{eq}} = 0$. Below T_c , the ordered phase appears continuously, with $|M_{\text{eq}}| \propto (T_c - T)^{1/2}$ in the Landau approximation.

We can also compute the free energy at the minimum. For $T > T_c$, the minimum is at $M = 0$, so

$$F_{\text{min}}(T) = F_0(T). \quad (17)$$

For $T < T_c$, using $M^2 = -\alpha_2/(2\alpha_4)$, we obtain

$$F_{\text{min}}(T) = F_0(T) - \frac{\alpha_2^2(T)}{4\alpha_4}. \quad (18)$$

Therefore,

$$F_{\min}(T) = \begin{cases} F_0(T), & T \geq T_c, \\ [0.3cm] F_0(T) - \frac{\alpha_0^2}{4\alpha_4}(T - T_c)^2, & T < T_c. \end{cases} \quad (19)$$

This expression shows an essential feature of second-order transitions: the free energy and its first derivative are continuous at T_c , but higher derivatives can present discontinuities. For example, the specific heat at constant volume can be written as

$$C_V = -T \frac{\partial^2 F_{\min}}{\partial T^2}. \quad (20)$$

Since the free energy below T_c has a contribution proportional to $(T - T_c)^2$, the specific heat presents a finite jump at T_c :

$$\Delta C_V = C_V(T_c^-) - C_V(T_c^+) = \frac{T_c \alpha_0^2}{2\alpha_4}. \quad (21)$$

This result is one of the thermodynamic signatures of a continuous transition: the order parameter does not jump, but some responses of the system can present discontinuities or singularities at the critical temperature.

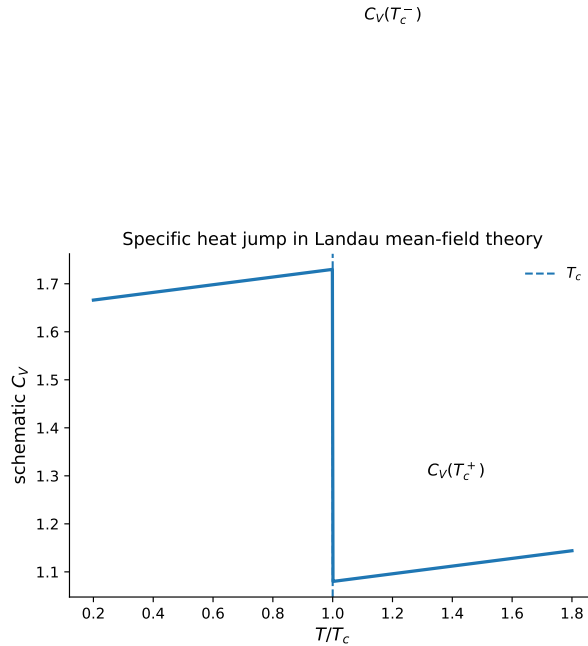


Figure 7: Schematic jump of the specific heat in the Landau mean-field description. In a continuous transition, the order parameter is continuous at T_c , but second derivatives of the free energy, such as the specific heat, may be discontinuous.

Another important example is the susceptibility. To define it, we can imagine that the system is coupled to an external field h , so that the free energy receives a term

$$F(T, M; h) = F(T, M) - hM. \quad (22)$$

The susceptibility is then

$$\chi = \left. \frac{\partial M}{\partial h} \right|_{h \rightarrow 0}. \quad (23)$$

In the Landau model, it diverges as T approaches T_c , indicating that the system becomes extremely sensitive to small external perturbations. This divergence is tied to the growth of correlations among microscopic degrees of freedom and is one of the most important signatures of the critical temperature.

A typical example of a continuous transition is the ferromagnetic transition, often studied through the Ising model. In that case, the order parameter is the magnetization: above the critical temperature, the spontaneous magnetization is zero; below it, the system chooses one of the possible magnetized phases. Graphically, this appears as a deformation of the free energy: for $T > T_c$, there is only one minimum at $M = 0$; at $T = T_c$, the minimum begins to flatten; and for $T < T_c$, two degenerate minima appear at $M \neq 0$.

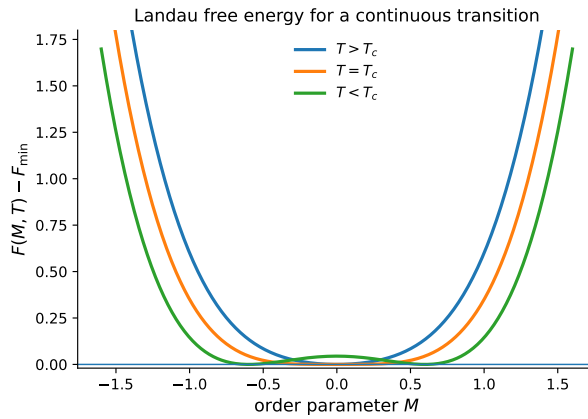


Figure 8: Landau free energy for a continuous phase transition. For $T > T_c$, the stable minimum is at $M = 0$, corresponding to the symmetric or disordered phase. At $T = T_c$, the minimum becomes flatter, and for $T < T_c$ two degenerate minima appear continuously at $M \neq 0$.

This image is fundamental for us to understand the electroweak transition later. In the Higgs case, the order parameter will be associated with the mean value of the scalar field, and the change of phase will be associated with the displacement of the minimum of the effective

potential. In a simplified description, at high temperatures the minimum may be at $\phi = 0$, corresponding to the symmetric phase; at low temperatures, the minimum may shift to $\phi \neq 0$, characterizing a symmetry-broken phase. This is the structure that appears in the well-known “Mexican hat” potential of the Higgs mechanism, which will be discussed in more detail in the next sections.

However, not every phase transition is continuous. Landau theory also allows us to understand, in a simple way, what characterizes a first-order transition. For this, we need the free energy to have two minima separated by a barrier in a certain temperature interval. A simple way to obtain this structure, while keeping the symmetry $M \rightarrow -M$, is to include the sixth-order term:

$$F(T, M) = F_0(T) + \alpha_2(T)M^2 + \alpha_4 M^4 + \alpha_6 M^6, \quad (24)$$

with

$$\alpha_4 < 0, \quad \alpha_6 > 0. \quad (25)$$

The sixth-order term stabilizes the free energy for large values of M , while the negative quartic term allows a barrier to appear between minima. The equilibrium condition is now

$$\frac{\partial F}{\partial M} = 2M [\alpha_2(T) + 2\alpha_4 M^2 + 3\alpha_6 M^4] = 0. \quad (26)$$

Besides the solution $M = 0$, we can have solutions with $M \neq 0$. Defining $y = M^2$, the nonzero extrema satisfy

$$3\alpha_6 y^2 + 2\alpha_4 y + \alpha_2(T) = 0, \quad (27)$$

that is,

$$y = M^2 = \frac{-\alpha_4 \pm \sqrt{\alpha_4^2 - 3\alpha_6 \alpha_2(T)}}{3\alpha_6}. \quad (28)$$

The first-order transition occurs when the minimum at $M = 0$ and the minima at $M \neq 0$ have

the same free energy. This is the condition of phase coexistence:

$$F(T_c, 0) = F(T_c, M_c). \quad (29)$$

Using simultaneously the minimum condition and the degeneracy condition, we find

$$M_c^2 = -\frac{\alpha_4}{2\alpha_6}, \quad (30)$$

and

$$\alpha_2(T_c) = \frac{\alpha_4^2}{4\alpha_6}. \quad (31)$$

This result highlights the main difference relative to the continuous transition: in a first-order transition, the order parameter does not grow smoothly from zero. It jumps from

$$M = 0 \quad (32)$$

to

$$|M| = M_c = \sqrt{-\frac{\alpha_4}{2\alpha_6}}. \quad (33)$$

Therefore, the order parameter is discontinuous at T_c .

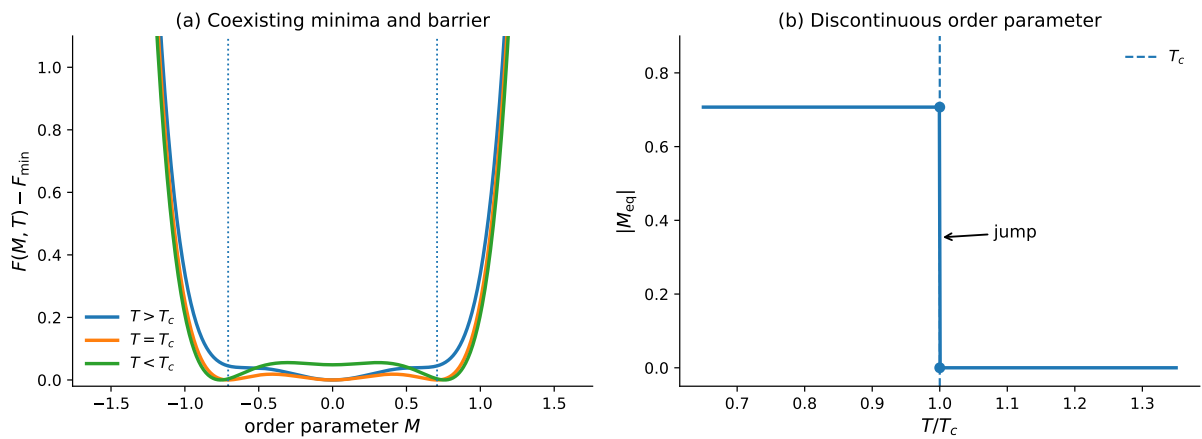


Figure 9: Landau picture of a first-order phase transition. The left panel shows the coexistence of minima separated by a barrier near the critical temperature. At $T = T_c$, the symmetric and broken minima are degenerate. The right panel shows the discontinuous jump of the order parameter, which is the defining thermodynamic feature of a first-order transition.

Physically, this means that two distinct phases coexist at the critical temperature. One of them can be metastable, that is, it can correspond to a local minimum of the free energy, but not to the global minimum. In that case, the final state of the system depends not only on the temperature, but also on the thermal history and on the path followed by the system. This is the origin of phenomena such as hysteresis, supercooling and metastability.

Another fundamental characteristic of a first-order transition is the presence of latent heat. Since the entropy is given by

$$S = -\frac{\partial F_{\min}}{\partial T}, \quad (34)$$

a discontinuity in the first derivative of the free energy implies a discontinuity in the entropy. The latent heat can be written as

$$L = T_c \Delta S. \quad (35)$$

This is a crucial difference relative to second-order transitions: in a continuous transition, the first derivative of the free energy is continuous; in a first-order transition, it jumps.

It is also important to highlight a subtlety. In an ideal second-order transition, quantities such as the susceptibility and the correlation length can diverge at T_c . In a first-order transition, on the other hand, the most characteristic thermodynamic signal is the discontinuity of the order parameter and the existence of latent heat. Phase coexistence, the barrier between minima and metastability make the dynamics of the transition much more complex.

When the symmetry $M \rightarrow -M$ is not an exact symmetry of the problem, odd terms can also appear in the Landau expansion. A cubic term, for example,

$$F(T, M) = F_0(T) + a(T)M^2 - b(T)M^3 + c(T)M^4, \quad c(T) > 0, \quad (36)$$

can directly generate a barrier between the symmetric and the broken phase. This structure is especially important in finite-temperature field theories, where thermal corrections can produce effective terms that favor a first-order transition.

Thus, the Landau language already anticipates the geometry that will be essential for us. A continuous transition corresponds to a smooth deformation of the minimum of the free energy. A first-order transition corresponds to the coexistence of minima separated by a barrier,

with a jump in the order parameter and possible metastability. In field theory, this free energy will be replaced by the finite-temperature effective potential, $V_{\text{eff}}(\phi, T)$, and the minima of this potential will represent the phases of the early Universe.

2.3 From thermodynamic phases to scalar fields

In the last section we established the thermodynamic language that will be used throughout this chapter: a phase is determined by the minimization of a free energy, and the change of phase can be tracked by an order parameter. Now, the natural question is: how does this language appear when the physical system is no longer water, a gas or a magnetic material, but rather quantum fields?

The first step is to leave classical thermodynamics and write the partition function in quantum language. In the canonical ensemble, the partition function ceases to be merely a sum over classical configurations and becomes the trace of an operator in Hilbert space:

$$Z(T, V, N) = \text{Tr} \left(e^{-\beta \hat{H}} \right), \quad \beta \equiv \frac{1}{T}, \quad (37)$$

Writing this trace explicitly in a complete basis of Hamiltonian eigenstates $|n\rangle$, we have

$$Z(T, V, N) = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle. \quad (38)$$

This expression makes the conceptual shift clear: quantum thermodynamics does not simply sum over classical states, but over quantum states weighted by the Boltzmann operator $e^{-\beta \hat{H}}$.

From Z , the free energy is still given by

$$F(T, V, N) = -T \ln Z(T, V, N). \quad (39)$$

Therefore, the physics remains the same: to find which phase is favored, we need to find which configuration minimizes the free energy.

The most organized way to express this is to introduce the density operator. In the canonical ensemble, we define

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}. \quad (40)$$

This operator contains the statistical information of the system. The thermal average of any

quantum observable $\hat{O}$ is then

$$\langle \hat{O} \rangle_T = \text{Tr} \left(\hat{\rho} \hat{O} \right) = \frac{1}{Z} \text{Tr} \left(e^{-\beta \hat{H}} \hat{O} \right). \quad (41)$$

This equation shows that, in statistical quantum mechanics, measuring an observable means taking a trace weighted by the density operator, that is, looking at the statistical mixture of quantum states.

Now we can translate this structure to a field theory. Our transition object is a scalar field, which we will denote generically by ϕ . In QFT, particles are interpreted as excitations of fields. The scalar field is, therefore, the fundamental physical object that describes scalar degrees of freedom of a particle theory. In the Standard Model, the central example is the Higgs field.

Thus, studying a phase transition in a scalar theory means studying how the mean value of this field changes when the temperature varies. The order parameter becomes

$$M \longrightarrow \langle \phi \rangle_T. \quad (42)$$

More explicitly,

$$\langle \phi(x) \rangle_T = \frac{1}{Z} \text{Tr} \left(e^{-\beta \hat{H}} \hat{\phi}(x) \right). \quad (43)$$

This mean value is the analogue, in field theory, of the average magnetization in a magnetic system. At zero temperature, we talk about the vacuum expectation value; at finite temperature, we talk about a thermal mean value.

In the homogeneous case, that is, when the equilibrium state does not depend explicitly on position, we can represent this value by a mean field ϕ_c . The symmetric phase is usually associated with $\phi_c = 0$, while a symmetry-broken phase is associated with $\phi_c \neq 0$. For the Higgs, this change is tied to electroweak symmetry breaking and to the generation of the masses of the Standard Model particles.

We now arrive at the central point of this section. The quantum partition function contains the operator

$$e^{-\beta \hat{H}}. \quad (44)$$

On the other hand, in quantum mechanics, the time-evolution operator is

$$\hat{U}(t_f, t_i) = e^{-i(t_f - t_i)\hat{H}}. \quad (45)$$

The similarity is too large to be a coincidence and immediately raises a suspicion. The partition function looks like a time evolution performed over an imaginary time interval, that is,

$$t_f - t_i = -i\beta. \quad (46)$$

It seems that there is an intimate connection between the temperature of a system and imaginary time.

This observation is the entry point to the finite-temperature path-integral formulation. By rotating to Euclidean time and compactifying the time direction to the interval $0 \leq \tau \leq \beta$, we can write, schematically, [21–23]

$$Z = \int_{\phi(\tau+\beta)=\phi(\tau)} \mathcal{D}\phi, e^{-S_E[\phi]}. \quad (47)$$

This expression will not be demonstrated in detail here, since this formalism is rather laborious and has a very rich interpretation, which must be explored carefully. It will be taken up later in this dissertation, in Chapter 3. For now, its role is to highlight the bridge between classical thermodynamics and QFT: the thermodynamic partition function can be written as a functional integral over fields, weighted by the Euclidean action.

To turn this idea into a practical tool, we introduce an external source $J(x)$ coupled to the field. The motivation is similar to the magnetic case: in a ferromagnet, an external field h forces the magnetization and allows us to measure the response of the system and thus find the susceptibility when we take the field to $h = 0$. In a scalar theory, the source $J(x)$ forces the field $\phi(x)$ to a mean value and allows us to study its averages and correlations.

In the thermal Euclidean formalism, we write

$$Z_\beta[J] = \int_{\phi(\tau+\beta)=\phi(\tau)} \mathcal{D}\phi, \exp \left[-S_E[\phi] + \int_0^\beta d\tau \int d^3x, J(x)\phi(x) \right]. \quad (48)$$

Here, $J(x)$ is an external source. It should be viewed as a mathematical tool: by differentiating $Z_\beta[J]$ with respect to J , we obtain mean values and correlation functions of the field. Note that

differentiating with respect to J gives precisely:

$$\langle \phi(x) \rangle_T = \frac{1}{Z} \int_{\phi(\tau+\beta)=\phi(\tau)} \mathcal{D}\phi, \exp \left[-S_E[\phi] + \int_0^\beta d\tau \int d^3x, J(x)\phi(x) \right] \phi(x) \quad (49)$$

$$= \frac{1}{Z} \frac{\delta Z[J]}{\delta J} \Big|_{J=0} \quad (50)$$

$$= \frac{\delta \ln Z[J]}{\delta J} \Big|_{J=0} \quad (51)$$

Defining the connected generating functional,

$$W_\beta[J] = \ln Z_\beta[J], \quad (52)$$

we have

$$\phi_c(x) = \frac{\delta W_\beta[J]}{\delta J(x)} = \langle \phi(x) \rangle_J. \quad (53)$$

This field $\phi_c(x)$ is called the classical field or the mean field. The name does not mean we are in a classical theory; it only means that we are looking at the statistical average of the field after summing over its quantum and thermal fluctuations.

Now the path is clear. The source J allows us to control the mean value of the field. The functional $W_\beta[J]$ generates the connected correlation functions: derivatives of this functional give the mean field, its second derivative gives the two-point function, and so on. But note that the quantity we want to minimize is not directly $W_\beta[J]$. To find the physical field configuration, the one that minimizes the available free energy, we need to trade the dependence on J for a dependence on the mean field itself ϕ_c , that is, we need to find the potential associated with this partition function. This trade is made by a Legendre transform and defines the effective action.

It is the effective action that, for homogeneous fields, will give rise to the effective potential.

The next step is to define the effective action carefully and show how, from it, the effective potential at zero temperature arises. After that, we will introduce finite temperature in a complete way and build $V_{\text{eff}}(\phi, T)$.

2.4 The zero-temperature effective potential

In the previous section, we saw that, when translating thermodynamics into quantum language, the central object ceases to be only a classical partition function and becomes a gen-

erating functional. We introduced an external source $J(x)$ coupled to the field and saw that, schematically, this source works as an “external field” that forces the system to assume a certain mean value. In analogy with the magnetic case, where an external field h forces an average magnetization M , the source $J(x)$ forces an average value for the scalar field.

Now, the natural question is: if our order parameter is the mean value of the field, how do we construct a function that depends on this order parameter?

This question is crucial. In Landau theory, we wrote the free energy as a function of the order parameter, $F(T, M)$. Equilibrium was obtained by minimizing this function with respect to M . In field theory, we want an analogous construction. The order parameter is the mean field, $\phi_c(x)$. Therefore, we want to construct an object that depends on ϕ_c , and not on the external source J .

This object is the **effective action**.

To make this clear, let us start from the Euclidean generating functional, obtained in the previous section,

$$Z[J] = \int \mathcal{D}\phi, \exp \left[-S_E[\phi] + \int d^d x, J(x)\phi(x) \right], \quad (54)$$

where $S_E[\phi]$ is the Euclidean action of the theory. We are using the Euclidean formalism because it makes the connection with free energy and with the statistical weight more transparent.

The functional $W[J]$, defined in the previous section, depends on the source J . But the physics of phases must be described in terms of the mean field ϕ_c , that is, in terms of the order parameter. We need, then, to exchange the independent variable: instead of a function of J , we want a function of ϕ_c .

In this way, we define the Euclidean effective action by [23–25]

$$\Gamma_E[\phi_c] = -W[J] + \int d^d x, J(x)\phi_c(x). \quad (55)$$

After the Legendre transform, the source J must be understood as a function of ϕ_c . In other words, J is eliminated in favor of the mean field, and this field is now our main variable.

This definition is completely analogous to thermodynamics. When we pass from a free energy written as a function of an external field to a free energy written as a function of the order parameter, we perform a Legendre transform. The same thing happens here: J is the external

variable that forces the field; ϕ_c is the mean response of the system. The effective action is the function that describes the cost of keeping the system with that mean value of the field.

Let us now derive the fundamental equation of the effective action. Varying (55), we have

$$\delta\Gamma_E = -\delta W[J] + \int d^d x, \delta J(x) \phi_c(x) + \int d^d x, J(x) \delta\phi_c(x). \quad (56)$$

But, by the definition of ϕ_c ,

$$\delta W[J] = \int d^d x, \frac{\delta W[J]}{\delta J(x)} \delta J(x) = \int d^d x, \phi_c(x) \delta J(x). \quad (57)$$

Therefore, the first two terms cancel and we are left with

$$\delta\Gamma_E = \int d^d x, J(x) \delta\phi_c(x). \quad (58)$$

Thus,

$$\frac{\delta\Gamma_E[\phi_c]}{\delta\phi_c(x)} = J(x). \quad (59)$$

This equation is one of the most important in this section. It says that the external source is the derivative of the effective action with respect to the mean field.

Now we switch off the source. The physical configuration of the theory, that is, the physical vacuum, must be obtained when there is no external force holding the field at an artificial value. Therefore, we set

$$J(x) = 0. \quad (60)$$

From equation (59), we obtain

$$\frac{\delta\Gamma_E[\phi_c]}{\delta\phi_c(x)} = 0. \quad (61)$$

This is the quantum equation of motion. The classical action generates the classical equations of motion; the effective action generates the equations of motion corrected by quantum fluctuations.

Before moving on, it is worth clarifying why this Legendre transform is well defined. The

second functional derivative of $W[J]$ is

$$\frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} = \langle \phi(x) \phi(y) \rangle_J - \langle \phi(x) \rangle_J \langle \phi(y) \rangle_J. \quad (62)$$

That is, it is the connected two-point correlation function. For any test function $f(x)$, we have

$$\int d^d x, d^d y, f(x) \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} f(y) = \left\langle \left(\int d^d x, f(x) \phi(x) - \left\langle \int d^d x, f(x) \phi(x) \right\rangle_J \right)^2 \right\rangle_J \geq 0. \quad (63)$$

Therefore, $W[J]$ is convex in J . This is exactly what we need to perform the Legendre transform, since it is only well defined for functions with definite convexity properties. Since the functional $W[J]$ and its derivatives are correlation functions, its derivatives are well defined.

Now we can define the effective potential. The effective action can be expanded in a derivative expansion of the mean field:

$$\Gamma_E[\phi_c] = \int d^d x \left[V_{\text{eff}}(\phi_c) + \frac{1}{2} Z_{\text{eff}}(\phi_c) \partial_\mu \phi_c, \partial_\mu \phi_c + \dots \right]. \quad (64)$$

The first term, with no derivatives, is the effective potential. The next terms describe kinetic corrections and higher-derivative terms. To study homogeneous phases, we take a constant background field,

$$\phi_c(x) = \phi_c. \quad (65)$$

In that case, all derivative terms vanish, and the effective action reduces to

$$\Gamma_E[\phi_c] = V_d V_{\text{eff}}(\phi_c), \quad (66)$$

where

$$V_d = \int d^d x \quad (67)$$

is the Euclidean volume of the spacetime region considered.

The equilibrium condition with no external source becomes, for constant fields,

$$\frac{\partial V_{\text{eff}}(\phi_c)}{\partial \phi_c} = 0. \quad (68)$$

Therefore, the vacua of the theory are found by minimizing the effective potential. This is the quantum generalization of the classical rule of minimizing the potential.

The connection with free energy also becomes clear. At finite temperature, the Euclidean volume is

$$V_d = \beta V, \quad (69)$$

where V is the spatial volume and $\beta = 1/T$. Since the partition function behaves, in the homogeneous-field limit, as

$$Z \sim e^{-\Gamma_E[\phi_c]} = e^{-\beta V V_{\text{eff}}(\phi_c, T)} \quad (70)$$

we have

$$F(\phi_c, T) = -T \ln Z = T\beta V, V_{\text{eff}}(\phi_c, T) = V V_{\text{eff}}(\phi_c, T). \quad (71)$$

Thus,

$$V_{\text{eff}}(\phi_c, T) = \frac{F(\phi_c, T)}{V}. \quad (72)$$

In words: the effective potential is the free-energy density as a function of the mean field. At zero temperature, the free energy reduces to the energy of the ground state, and $V_{\text{eff}}(\phi)$ represents the vacuum energy density, now including quantum corrections.

Let us now compute this object in a loop expansion. The idea is simple: if we want to know how quantum fluctuations modify the energy of the background field, we write the full field as

$$\phi(x) = \phi_c(x) + \eta(x) \quad (73)$$

where $\phi_c(x)$ is the classical background, or mean field, and $\eta(x)$ represents quantum fluctua-

tions around that background.

It is important to state what is being integrated. The original field $\phi(x)$ is the integration variable; however, the background $\phi_c(x)$ is held fixed. What is integrated are the fluctuations $\eta(x)$ around that background. Thus,

$$\int D\phi \longrightarrow \int D\eta \quad (74)$$

We expand the Euclidean action around ϕ_c :

$$S_E[\phi_c + \eta] = S_E[\phi_c] + \int d^d x, \left. \frac{\delta S_E}{\delta \phi(x)} \right|_{\phi_c} \eta(x) + \frac{1}{2} \int d^d x, d^d y, \eta(x) \Delta(x, y; \phi_c) \eta(y) + \dots \quad (75)$$

where

$$\Delta(x, y; \phi_c) = \left. \frac{\delta^2 S_E}{\delta \phi(x) \delta \phi(y)} \right|_{\phi_c} \quad (76)$$

is the quadratic fluctuation operator.

Substituting this expansion into the generating functional,

$$Z[J] = \int \mathcal{D}\eta \exp \left[-S_E[\phi_c + \eta] + \int d^d x J(x) (\phi_c(x) + \eta(x)) \right] \quad (77)$$

and keeping terms up to second order in η , we obtain

$$Z[J] \simeq \exp \left[-S_E[\phi_c] + \int d^d x J(x) \phi_c(x) \right] \int \mathcal{D}\eta \exp \left[-\frac{1}{2} \int d^d x d^d y \eta(x) \Delta(x, y; \phi_c) \eta(y) \right] \quad (78)$$

provided that the linear term in η is cancelled. This linear term is proportional to

$$\int d^d x \left[J(x) - \left. \frac{\delta S_E}{\delta \phi(x)} \right|_{\phi_c} \right] \eta(x). \quad (79)$$

Thus, at the classical level of the expansion, the source that sustains the background must satisfy

$$J(x) = \left. \frac{\delta S_E}{\delta \phi(x)} \right|_{\phi_c}. \quad (80)$$

The physical interpretation is exactly that of an external force. If we have a classical particle in a potential $U(q)$ and we want to study small oscillations around an arbitrary point q_0 , a linear term $U'(q_0)\eta$ appears. This means that q_0 is not an equilibrium point: the particle wants to fall somewhere else. If we apply an external force $f = U'(q_0)$, the particle is held there. The same happens here. The source J sustains the mean field ϕ_c . After constructing the effective action, we switch off the source. The physical value of ϕ_c is the one that remains an extremum even when $J = 0$.

Now we need to understand carefully what this Gaussian integral is doing. To avoid hiding the physics in a very compact notation, let us first recall the finite-dimensional case. If A is a positive symmetric matrix, then

$$\int d^N \eta \exp \left[-\frac{1}{2} \eta_i A_{ij} \eta_j \right] = \left(\frac{(2\pi)^N}{\det A} \right)^{1/2}. \quad (81)$$

In field theory, the matrix A_{ij} is replaced by a functional operator $\Delta(x, y; \phi_c)$, and the sum over indices is replaced by integrals in x and y . For this reason we write, in abbreviated form,

$$\eta \Delta \eta \equiv \int d^d x d^d y \eta(x) \Delta(x, y; \phi_c) \eta(y). \quad (82)$$

Thus,

$$\int \mathcal{D}\eta \exp \left[-\frac{1}{2} \int d^d x d^d y \eta(x) \Delta(x, y; \phi_c) \eta(y) \right] \propto (\det \Delta)^{-1/2}. \quad (83)$$

Taking the logarithm, we use the identity

$$\ln \det \Delta = \text{Tr} \ln \Delta. \quad (84)$$

This identity is the functional version of the familiar relation between determinants and eigenvalues: the determinant is the product of eigenvalues, the logarithm of the determinant is the sum of the logarithms of eigenvalues, and this sum is precisely the trace of $\ln \Delta$.

Therefore, keeping terms up to one loop, we find

$$W[J] = -S_E[\phi_c] + \int d^d x J(x) \phi_c(x) - \frac{1}{2} \text{Tr} \ln \Delta + \text{constant} + \dots. \quad (85)$$

Using the definition of the effective action,

$$\Gamma_E[\phi_c] = -W[J] + \int d^d x J(x) \phi_c(x) \quad (86)$$

we obtain

$$\Gamma_E[\phi_c] = S_E[\phi_c] + \frac{1}{2} \text{Tr} \ln \Delta + \text{counterterms} + \dots \quad (87)$$

This is the general form of the one-loop effective action for a real bosonic field. The term $S_E[\phi_c]$ is the classical contribution. The term $\frac{1}{2} \text{Tr} \ln \Delta$ is the first quantum correction. It represents the sum of the effect of all small Gaussian fluctuations around the background ϕ_c .

Now let us apply this construction to the simplest case: a real scalar field. Consider the Euclidean action

$$S_E[\phi] = \int d^4 x \left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + V_0(\phi) \right], \quad (88)$$

with classical potential

$$V_0(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (89)$$

The condition $\lambda > 0$ guarantees that the potential is bounded from below. The first derivative is

$$\frac{dV_0}{d\phi} = m^2 \phi + \frac{\lambda}{6} \phi^3. \quad (90)$$

Stationary points satisfy

$$\phi \left(m^2 + \frac{\lambda}{6} \phi^2 \right) = 0. \quad (91)$$

Thus,

$$\phi = 0 \quad \text{or} \quad \phi^2 = -\frac{6m^2}{\lambda}. \quad (92)$$

If $m^2 > 0$, only $\phi = 0$ is a stable real solution. If $m^2 < 0$, the origin becomes unstable and two

degenerate minima appear,

$$\phi = \pm v, \quad v^2 = -\frac{6m^2}{\lambda}. \quad (93)$$

This is the simplest example of spontaneous symmetry breaking in a scalar theory with symmetry $\phi \rightarrow -\phi$.

The second derivative of the potential is

$$\frac{d^2 V_0}{d\phi^2} = m^2 + \frac{\lambda}{2}\phi^2. \quad (94)$$

This quantity defines the field-dependent mass squared:

$$M^2(\phi) = m^2 + \frac{\lambda}{2}\phi^2. \quad (95)$$

This object has an important physical interpretation. When evaluated at the minimum of the potential, it is the mass squared of small fluctuations around that vacuum. When evaluated at an arbitrary point in field space, it is the background-dependent effective mass, that is, the mass that fluctuations feel when the mean field is held at that value.

We can see this explicitly by writing

$$\phi(x) = \phi_c + \eta(x) \quad (96)$$

with ϕ_c constant. Expanding the potential,

$$V_0(\phi_c + \eta) = V_0(\phi_c) + V_0'(\phi_c)\eta + \frac{1}{2}V_0''(\phi_c)\eta^2 + \dots \quad (97)$$

The quadratic term in η is the mass term for the fluctuations. Therefore,

$$m_\eta^2 = V_0''(\phi_c) = M^2(\phi_c). \quad (98)$$

In the broken vacuum, $\phi_c = v$, we have

$$M^2(v) = m^2 + \frac{\lambda}{2}v^2 = m^2 - 3m^2 = -2m^2 = \frac{\lambda}{3}v^2. \quad (99)$$

Since $m^2 < 0$, this mass squared is positive. The curvature of the potential at the minimum is

positive, as it must be for a stable state.

The parameter m^2 that appears in the classical potential should not automatically be confused with the physical mass measured in a laboratory or collider. It is a parameter of the Lagrangian. When $m^2 < 0$, the origin $\phi = 0$ has negative curvature:

$$V_0''(0) = m^2 < 0. \quad (100)$$

This means that the origin is not a minimum, but an unstable direction of the potential. Sometimes we call this a “tachyonic mass”, a negative mass referring to the unstable point. It does not mean that a physical particle measured in the laboratory has negative mass squared. It only means that we are expanding around the wrong point of the potential.

The physical mass of the scalar excitations must be read from the curvature around the correct vacuum, the true minimum of the potential. When we are in the broken vacuum, $\phi = v$, we find

$$V_0''(v) = M^2(v) > 0. \quad (101)$$

Therefore, the physical particle associated with the fluctuation around the minimum has positive mass squared.

For the real scalar field, the quadratic fluctuation operator is

$$\Delta(\phi_c) = -\partial^2 + M^2(\phi_c), \quad (102)$$

Since we are considering a constant background ϕ_c , the operator Δ is diagonal in momentum space. The eigenfunctions are plane waves,

$$\eta_p(x) \sim e^{ip_E \cdot x}, \quad (103)$$

and, on them,

$$\Delta(\phi_c)\eta_p = [p_E^2 + M^2(\phi_c)] \eta_p. \quad (104)$$

Thus,

$$\text{Tr} \ln \Delta = V_4 \int \frac{d^4 p_E}{(2\pi)^4} \ln [p_E^2 + M^2(\phi_c)] , \quad (105)$$

where $V_4 = \int d^4 x$ is the Euclidean volume. Dividing the effective action by the volume, we obtain the one-loop contribution to the effective potential, given by

$$V_1(\phi_c) = \frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \ln [p_E^2 + M^2(\phi_c)] . \quad (106)$$

This integral is ultraviolet divergent, that is, divergent for very large momenta. This is not a surprise: we are in a quantum field theory, and fluctuations of all wavelengths contribute to the vacuum energy. To make sense of this expression, we need to regularize and renormalize.

In dimensional regularization and in the $\overline{\text{MS}}$ scheme, the renormalized contribution of a real scalar field is [See Appendix A for more details]

$$V_1^{\overline{\text{MS}}}(\phi_c) = \frac{M^4(\phi_c)}{64\pi^2} \left[\ln \left(\frac{M^2(\phi_c)}{\mu^2} \right) - \frac{3}{2} \right] , \quad (107)$$

where μ is the renormalization scale. Therefore, the one-loop effective potential becomes

$$V_{\text{eff}}(\phi_c) = V_0(\phi_c) + \frac{M^4(\phi_c)}{64\pi^2} \left[\ln \left(\frac{M^2(\phi_c)}{\mu^2} \right) - \frac{3}{2} \right] , \quad (108)$$

with

$$M^2(\phi_c) = m^2 + \frac{\lambda}{2} \phi_c^2. \quad (109)$$

This correction is the well-known Coleman–Weinberg correction [24, 25] . The physical interpretation is: even at zero temperature, the vacuum is not empty in the classical sense. It has quantum fluctuations. These fluctuations change the energy associated with each value of the background field and, therefore, deform the classical potential. In some cases, this deformation may be only a small quantitative correction; in others, it can qualitatively change the structure of the minima of the theory.

This is the central idea of the effective potential: it is the potential that already knows, perturbatively, that the field has quantum corrections. It carries the memory of quantum fluctuations around each background ϕ_c .

In a general theory with several particles coupled to the scalar field, the Coleman–Weinberg correction can be written schematically as [18, 26–28]

$$\Delta V_{\text{CW}}(\phi) = \sum_i \frac{n_i}{64\pi^2} m_i^4(\phi) \left[\ln \left(\frac{m_i^2(\phi)}{\mu^2} \right) - c_i \right]. \quad (110)$$

Here, i sums over the particles in the theory, $m_i^2(\phi)$ are the field-dependent masses, n_i counts the internal degrees of freedom of each particle, including a negative sign for fermions, and c_i depends on the type of particle and on the renormalization scheme. In the $\overline{\text{MS}}$ scheme, we usually have $c_i = 3/2$ for scalars and fermions, and $c_i = 5/6$ for gauge bosons, in appropriate gauges such as Landau gauge.

This formula shows why the effective potential is so important for cosmological phase transitions. The background field ϕ changes the masses of particles; these masses change the loop corrections; the loop corrections change the shape of the potential; and the shape of the potential determines the vacua of the theory. Therefore, studying the effective potential is studying how the microscopic particle physics reorganizes the phase structure of the Universe.

In the Higgs case, this idea gains a direct interpretation. The expectation value of the Higgs field determines the masses of Standard Model particles. Thus, when we compute the Higgs effective potential, we are asking which value of the field is energetically favored after taking into account the quantum fluctuations of all relevant fields: the Higgs itself, gauge bosons, fermions, quarks and, depending on the model, new fields beyond the Standard Model.

2.5 Quantum field theory at finite temperature

To understand the finite-temperature potential, we must understand the behavior of the bosons and fermions coupled to the field in question and how they affect its geometry.

For bosonic fields this gives periodic boundary conditions,

$$\phi(\tau + \beta, \mathbf{x}) = \phi(\tau, \mathbf{x}) \quad (111)$$

while for fermionic fields, the trace must be represented using Grassmann coherent states. The anticommuting nature of these variables produces an extra minus sign when the trace is closed,

leading to antiperiodic boundary conditions,

$$\psi(\tau + \beta, \mathbf{x}) = -\psi(\tau, \mathbf{x}). \quad (112)$$

This difference is not decorative; it is exactly what distinguishes bosonic thermal statistics from fermionic thermal statistics.

With these boundary conditions, the thermal path integral takes the Euclidean form

$$Z = \int_{\text{thermal}} \mathcal{D}\phi e^{-S_E[\phi]} \quad (113)$$

where “thermal” means that the fields obey the appropriate periodic or antiperiodic condition in the compact Euclidean time direction.

Because the time direction is compact, the Fourier modes in τ are discrete. For bosons,

$$\phi(\tau, \mathbf{x}) = \sum_{n=-\infty}^{+\infty} e^{i\omega_n^B \tau} \phi_n(\mathbf{x}), \quad \omega_n^B = 2\pi n T \quad (114)$$

while for fermions,

$$\psi(\tau, \mathbf{x}) = \sum_{n=-\infty}^{+\infty} e^{i\omega_n^F \tau} \psi_n(\mathbf{x}), \quad \omega_n^F = (2n + 1)\pi T. \quad (115)$$

These are the Matsubara frequencies. Thus, the finite-temperature theory looks like a theory where the continuous energy integral is replaced by a discrete sum:

$$\int \frac{dp_0}{2\pi} \longrightarrow T \sum_{n=-\infty}^{+\infty}. \quad (116)$$

Now we can understand the one-loop thermal correction in the same spirit as the zero-temperature Coleman–Weinberg correction. At zero temperature, the one-loop term for a bosonic fluctuation with field-dependent mass $m^2(\phi)$ was controlled by

$$\frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \ln [p_E^2 + m^2(\phi)]. \quad (117)$$

At finite temperature, the Euclidean energy is replaced by Matsubara frequencies [See Ap-

pendix B for more details]:

$$V_1^B(\phi, T) = \frac{T}{2} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \ln [(\omega_n^B)^2 + \mathbf{k}^2 + m^2(\phi)] . \quad (118)$$

This expression contains both the zero-temperature contribution and the genuinely thermal contribution. After performing the Matsubara sum, it can be written as

$$V_1^B(\phi, T) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k(\phi) + T \int \frac{d^3k}{(2\pi)^3} \ln [1 - e^{-\beta\omega_k(\phi)}] , \quad (119)$$

where

$$\omega_k(\phi) = \sqrt{\mathbf{k}^2 + m^2(\phi)} . \quad (120)$$

The first term is the zero-point energy and belongs to the zero-temperature Coleman–Weinberg sector after renormalization. The second term is the finite-temperature correction.

This result is exactly the field-theory version of a collection of harmonic oscillators. A free bosonic field is a set of independent oscillators, one for each momentum mode. For a single bosonic oscillator with frequency ω , the energy levels are

$$E_n = \omega \left(n + \frac{1}{2} \right) , \quad n = 0, 1, 2, \dots \quad (121)$$

Its partition function is

$$Z_{\text{bho}} = \sum_{n=0}^{\infty} e^{-\beta\omega(n+1/2)} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} , \quad (122)$$

and therefore

$$F_{\text{bho}} = -T \ln Z_{\text{bho}} = \frac{\omega}{2} + T \ln (1 - e^{-\beta\omega}) . \quad (123)$$

The first term is the zero-point energy; the second is the thermal occupation of the bosonic mode. Summing this result over all momenta, we obtain precisely the thermal contribution above. Hindmarsh presents this logic explicitly by treating a free bosonic field as a collection of harmonic oscillators and extracting the free-energy density from the partition function [19].

For a bosonic species with n_b degrees of freedom, the finite-temperature contribution is

$$\Delta V_T^B(\phi, T) = n_b, T \int \frac{d^3 k}{(2\pi)^3} \ln [1 - e^{-\beta \omega_k(\phi)}] . \quad (124)$$

Changing variables to

$$x = \frac{|\mathbf{k}|}{T}, \quad y_b(\phi, T) = \frac{m_b^2(\phi)}{T^2}, \quad (125)$$

we obtain

$$\Delta V_T^B(\phi, T) = n_b \frac{T^4}{2\pi^2} J_B(y_b), \quad (126)$$

where

$$J_B(y) = \int_0^\infty dx, x^2 \ln [1 - e^{-\sqrt{x^2+y}}] . \quad (127)$$

The fermionic case follows the same physical logic, but with a crucial difference: fermions obey the Pauli exclusion principle. A single fermionic mode can be occupied by only 0 or 1 particle. This changes the partition function of each oscillator and replaces the Bose–Einstein logarithm by the Fermi–Dirac one.

For a fermionic oscillator, the thermal part of the free energy has the form

$$F_{\text{fho}}^T = -T \ln (1 + e^{-\beta \omega}) . \quad (128)$$

Therefore, for a fermionic species with n_f positive degrees of freedom,

$$\Delta V_T^F(\phi, T) = -n_f, T \int \frac{d^3 k}{(2\pi)^3} \ln [1 + e^{-\beta \omega_k(\phi)}] . \quad (129)$$

Equivalently,

$$\Delta V_T^F(\phi, T) = -n_f \frac{T^4}{2\pi^2} J_F(y_f), \quad (130)$$

with

$$J_F(y) = \int_0^\infty dx, x^2 \ln \left[1 + e^{-\sqrt{x^2+y}} \right]. \quad (131)$$

This sign convention keeps $n_f > 0$ as the number of fermionic degrees of freedom and writes the minus sign explicitly. An equivalent convention, widely used in effective-potential codes, is to absorb this minus sign into a signed degeneracy factor n_i .

Combining bosons and fermions, the one-loop thermal correction can be written as [21, 22, 29]

$$\Delta V_T(\phi, T) = \sum_b n_b \frac{T^4}{2\pi^2} J_B \left(\frac{m_b^2(\phi)}{T^2} \right) - \sum_f n_f \frac{T^4}{2\pi^2} J_F \left(\frac{m_f^2(\phi)}{T^2} \right). \quad (132)$$

This equation tells us how the thermal bath modifies the effective potential. The key point is that the masses are field-dependent. For example, if a scalar field ϕ couples to another scalar χ and to a fermion ψ through terms such as

$$\mathcal{L} * \text{int} \supset -\frac{1}{2} g^2 \phi^2 \chi^2 - y, \phi, \bar{\psi} \psi, \quad (133)$$

then the masses of χ and ψ depend on the background value of ϕ :

$$m * \chi^2(\phi) = m_{\chi,0}^2 + g^2 \phi^2, \quad m_\psi^2(\phi) = y^2 \phi^2, \quad (134)$$

up to model-dependent normalization factors. Therefore, the free energy of the plasma is different for different values of ϕ . This is precisely why the thermal bath can displace the minimum of the potential and why the Higgs background gives effective masses to particles in the plasma.

For physical intuition, it is useful to expand the thermal functions at high temperature, $m/T \ll 1$. For bosons, one finds

$$J_B(y) = -\frac{\pi^4}{45} + \frac{\pi^2}{12} y - \frac{\pi}{6} y^{3/2} - \frac{1}{32} y^2 \ln \left(\frac{y}{a_B} \right) + \dots, \quad (135)$$

while for fermions,

$$J_F(y) = \frac{7\pi^4}{360} - \frac{\pi^2}{24}y - \frac{1}{32}y^2 \ln\left(\frac{y}{a_F}\right) + \dots \quad (136)$$

Here a_B and a_F are numerical constants that depend on the convention used in the high-temperature expansion.

Substituting these expansions into (132), we see the most important physical effects. First, there are temperature-dependent constant terms proportional to T^4 , which affect the pressure and radiation energy density, but do not by themselves determine where the minimum in ϕ is located. Second, there are terms proportional to $T^2 m_i^2(\phi)$. Since the masses usually contain terms proportional to ϕ^2 , these terms generate thermal masses for the scalar background. Schematically,

$$V_{\text{eff}}(\phi, T) \supset \frac{1}{2}c, T^2\phi^2. \quad (137)$$

This is the simplest way to understand thermal symmetry restoration: at high temperatures, the positive thermal mass can stabilize the origin $\phi = 0$ even if the zero-temperature potential favors a broken phase.

The third effect is especially important for first-order phase transitions. The bosonic expansion contains a non-analytic cubic term,

$$\Delta V_T^B \supset -\frac{n_b T}{12\pi} m_b^3(\phi). \quad (138)$$

If $m_b(\phi) \propto \phi$, this term behaves as

$$-E, T, \phi^3. \quad (139)$$

This cubic term can create a barrier between the symmetric and broken minima. This is one of the simplest perturbative mechanisms by which a thermal phase transition becomes first order. Fermions do not generate such a cubic term in the high-temperature expansion. This is why bosonic degrees of freedom are particularly important for strengthening first-order transitions.

At low temperature, $m/T \gg 1$, the thermal functions are exponentially suppressed. Physically, this is exactly what we expect: particles much heavier than the temperature are rarely

excited in the plasma, and their thermal contribution becomes Boltzmann suppressed. Thus,

$$\Delta V_T(\phi, T) \longrightarrow 0 \quad \text{when} \quad T \rightarrow 0. \quad (140)$$

In this limit, the zero-temperature effective potential is recovered.

Finite temperature changes the potential because the particles in the plasma have field-dependent masses. As the Universe cools, the thermal part of the free energy changes, and the minimum of $V_{\text{eff}}(\phi, T)$ can move from a symmetric phase to a broken phase. The functions J_B and J_F are the quantitative way of encoding this statement. See Fig. 10.

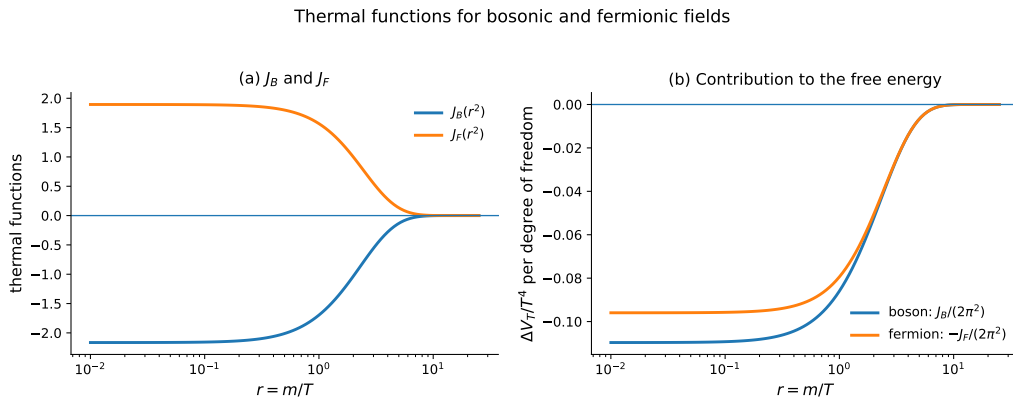


Figure 10: Thermal functions $J_B(y)$ and $J_F(y)$ as functions of $y = m^2/T^2$. At low temperature, the contributions are suppressed; at high temperature, they admit analytic expansions that reveal thermal masses and, for bosons, the cubic term responsible for perturbative barriers.

2.6 The finite-temperature effective potential

Now we can assemble the finite-temperature effective potential. Summing all contributions, we can write

$$V_{\text{eff}}(\phi, T) = V_0(\phi) + \Delta V_{\text{CW}}(\phi) + \Delta V_T(\phi, T) + V_{\text{daisy}}(\phi, T). \quad (141)$$

Each term has a physical meaning.

The first term, $V_0(\phi)$, is the tree-level potential. It is the classical vacuum landscape. The second term, $\Delta V_{\text{CW}}(\phi)$, is the Coleman–Weinberg correction: it contains the zero-temperature quantum fluctuations, including the coupling of each particle. The fourth term, $\Delta V_T(\phi, T)$, is the one-loop thermal correction discussed in the previous subsection. Finally, $V_{\text{daisy}}(\phi, T)$ is a thermal resummation term, necessary because ordinary perturbation theory becomes unreliable for certain bosonic modes at high temperature.

Let us understand why this last term appears.

At high temperatures, bosonic Matsubara zero modes can become sensitive to the infrared. In practice, this means that the naive loop expansion may treat long-wavelength thermal fluctuations with insufficient care. These are precisely the soft modes that can become important near a phase transition, especially near the origin in field space where some masses may be small. If we ignore this point, the perturbative potential may contain unphysical imaginary parts or overestimate the strength of the transition.

The standard cure is to resum a class of diagrams known as ring or daisy diagrams [18, 30, 31]. The name is visual: these diagrams look like loops decorated by repeated self-energy insertions. Physically, the resummation takes into account the fact that particles in a plasma do not propagate as if they were in the vacuum. They acquire thermal self-energies, or Debye masses.

At the practical level, the main effect is the substitution

$$m_b^2(\phi) \longrightarrow \mathcal{M}_b^2(\phi, T) = m_b^2(\phi) + \Pi_b(T), \quad (142)$$

where $\Pi_b(T)$ is the thermal self-energy of the bosonic mode. This substitution should be applied only to the bosonic modes that develop infrared sensitivity. Fermions do not have Matsubara zero modes, since their frequencies are

$$\omega_n^F = (2n + 1)\pi T, \quad (143)$$

and therefore they do not require the same daisy resummation.

In the Arnold–Espinosa prescription, the daisy contribution is written as

$$V_{\text{daisy}}(\phi, T) = -\frac{T}{12\pi} \sum_b n_b [\mathcal{M}_b^3(\phi, T) - m_b^3(\phi)], \quad (144)$$

where the sum is over the bosonic modes to be resummed. The subtraction by $m_b^3(\phi)$ avoids double counting of the cubic term already present in the one-loop thermal correction.

This expression has an important characteristic: it is linear in T and cubic in the thermal masses. This is exactly the structure associated with bosonic modes sensitive to the infrared. In a phase transition, these modes are important because they help control the barrier and the strength of the transition.

There are different prescriptions for thermal resummation. In the Parwani prescription, one replaces

$$m_b^2(\phi) \rightarrow \mathcal{M}_b^2(\phi, T) \quad (145)$$

directly inside the one-loop thermal functions. In the Arnold–Espinosa prescription, one keeps the one-loop functions with the original field-dependent masses and adds the separate daisy term (144). Both approaches are common in the literature, and comparing resummation schemes is part of the theoretical uncertainty in perturbative studies of cosmological phase transitions. Athron discusses precisely these finite-temperature resummation subtleties and their role in effective-potential calculations [18].

For the purposes of this thesis, the important point is not to turn this section into a complete review of resummation schemes. The point is to understand what must be included in a consistent finite-temperature potential. The plasma changes the masses; the masses change the potential; and, for bosons, the soft thermal modes require a resummation.

Putting the pieces together, the one-loop finite-temperature potential can be written as

$$\begin{aligned}
 V_{\text{eff}}(\phi, T) = & \underbrace{V_0(\phi)}_{\text{tree level}} + \underbrace{\Delta V_{\text{CW}}(\phi)}_{\text{zero-temperature quantum corrections}} \\
 & + \underbrace{\sum_b n_b \frac{T^4}{2\pi^2} J_B \left(\frac{m_b^2(\phi)}{T^2} \right) - \sum_f n_f \frac{T^4}{2\pi^2} J_F \left(\frac{m_f^2(\phi)}{T^2} \right)}_{\text{one-loop finite-temperature corrections}} \\
 & + \underbrace{V_{\text{daisy}}(\phi, T)}_{\text{thermal resummation}} .
 \end{aligned} \quad (146)$$

This equation is the main object that closes the theoretical part of this chapter. Once a model provides the field-dependent masses $m_i^2(\phi)$ and the thermal self-energies $\Pi_i(T)$, the finite-temperature effective potential tells us which phase is favored at each temperature.

At high temperatures, thermal masses tend to restore symmetries and favor the origin. At low temperatures, the zero-temperature vacuum structure dominates again. Between these limits, new minima can appear, disappear, become degenerate or be separated by a barrier. In this way, the cooling of the Universe is translated into a geometric problem: following the minima of $V_{\text{eff}}(\phi, T)$ as T decreases.

This is exactly the picture we need for first-order phase transitions. If the effective poten-

tial changes smoothly and the minimum moves continuously, the transition is a crossover or a continuous transition. If, instead, the potential develops two competing minima separated by a barrier, the Universe may remain trapped in a metastable false vacuum before transitioning to the true vacuum through bubble nucleation. The potential tells us where the phases are; the next chapter will tell us if and when the transition actually happens.

2.7 Crossover, continuous transitions and the Standard Model Higgs

Having established the finite-temperature effective potential in Eq. (146), we can finally connect the formal construction with the physical example that motivates a large part of this thesis: the Higgs field. Up to this point, the discussion has been deliberately general. We introduced phases as minima of a free-energy-like quantity, translated the order parameter into a scalar background field, and constructed the effective potential that determines which configuration is thermodynamically preferred at a given temperature.

The natural question now is: what happens when this scalar field is the Standard Model Higgs field?

At tree level, the Higgs potential can be written as

$$V_0(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2. \quad (147)$$

Choosing the neutral radial direction of the Higgs doublet,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \quad (148)$$

the tree-level potential becomes

$$V_0(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4. \quad (149)$$

At zero temperature, the minimum occurs at

$$v^2 = \frac{\mu^2}{\lambda}, \quad (150)$$

and the physical Higgs mass is related to the curvature of the potential at the minimum,

$$m_h^2 = \left. \frac{d^2 V_0}{d\phi^2} \right|_{\phi=v} = 2\lambda v^2. \quad (151)$$

Thus,

$$\lambda = \frac{m_h^2}{2v^2}, \quad \mu^2 = \lambda v^2 = \frac{m_h^2}{2}. \quad (152)$$

The measured values are approximately

$$v \simeq 246 \text{ GeV}, \quad m_h \simeq 125 \text{ GeV}, \quad (153)$$

so that

$$\lambda \simeq 0.13. \quad (154)$$

These numbers are not arbitrary parameters. They are fixed by laboratory measurements, especially by electroweak precision data and by the discovery and subsequent measurements of the Higgs boson at the LHC [32].

The importance of the Higgs VEV comes from the fact that several Standard Model particles obtain their masses through their coupling to the Higgs field. Along the radial direction ϕ , the field-dependent masses of the W and Z bosons, mediators of the weak interaction, are, schematically,

$$m_W^2(\phi) = \frac{g^2}{4}\phi^2, \quad m_Z^2(\phi) = \frac{g^2 + g'^2}{4}\phi^2, \quad (155)$$

and, for a fermion f ,

$$m_f^2(\phi) = \frac{y_f^2}{2}\phi^2. \quad (156)$$

In particular, the top quark gives the dominant fermionic contribution because its Yukawa coupling is the largest in the Standard Model.

At $\phi = v$, these expressions reproduce the physical masses,

$$m_W = \frac{gv}{2} \simeq 80.36 \text{ GeV}, \quad m_Z = \frac{\sqrt{g^2 + g'^2} v}{2} \simeq 91.19 \text{ GeV}, \quad m_t = \frac{y_t v}{\sqrt{2}} \simeq 173.1 \text{ GeV}. \quad (157)$$

This is the central physical meaning of the Higgs phase: when the Higgs field has a nonzero expectation value, electroweak symmetry is broken and the particles coupled to it acquire the masses that we measure today.

The early Universe can be understood as scanning the Higgs potential with temperature. At high temperature, the plasma favors the origin; at low temperature, the vacuum structure measured in laboratories is recovered.

However, an important point must be made carefully. For the physical value of the Higgs mass, the electroweak transition of the Standard Model is not a strong first-order phase transition. It is a crossover [15, 16]. This means that the Higgs expectation value changes smoothly with temperature. There is no true coexistence of two separated phases, no potential barrier separating a false and a true vacuum, and no bubble nucleation in the sense required for a cosmological first-order phase transition.

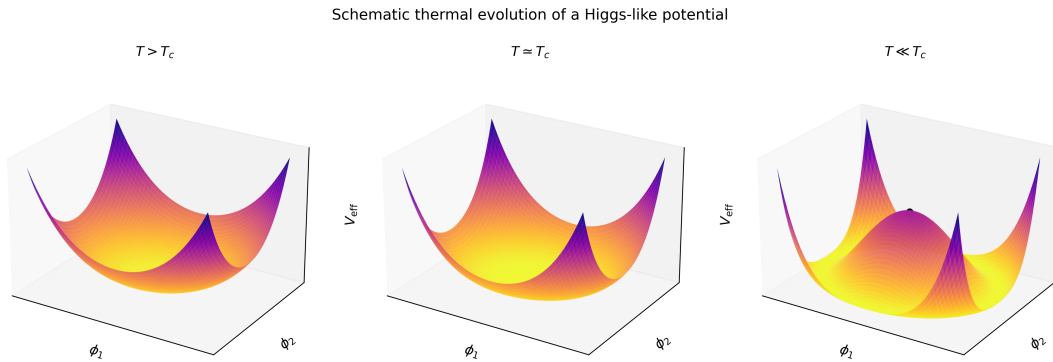


Figure 11: Schematic representation of the thermal history of the Higgs potential. At high temperatures, the symmetric minimum at $\phi = 0$ is favored. As the Universe cools, the potential structure deforms and, at low temperatures, the Mexican hat-shaped geometry associated with the phase with broken symmetry emerges. The figure is qualitative and serves to visualize the restoration and breaking of symmetry.

The lesson of the Standard Model is, therefore, twofold. On the one hand, it gives us the natural scalar field whose thermal history we want to understand. On the other hand, it tells us that the minimal theory is not sufficient. If we want a first-order electroweak phase transition capable of generating gravitational waves, the Higgs sector must be modified, or new degrees

of freedom must participate in the thermal potential.

2.8 How new physics can make the Higgs transition first order

Now we can ask the central question:

How can the Higgs transition become first order if the Standard Model predicts only a crossover?

The answer is that the shape of the effective potential must change. More precisely, the finite-temperature potential must develop two competing minima separated by a barrier. One minimum corresponds to the symmetric phase, usually near $\phi = 0$, and the other to the broken phase, at $\phi \neq 0$. At the critical temperature, these two phases are degenerate:

$$V_{\text{eff}}(\phi_{\text{false}}, T_c) = V_{\text{eff}}(\phi_{\text{true}}, T_c). \quad (158)$$

Below T_c , the broken minimum becomes deeper, but the Universe does not necessarily transition immediately. The old phase can remain as a local minimum, separated from the true vacuum by a barrier.

This is the defining physical feature of a first-order phase transition: the transition is not homogeneous. The field cannot simply roll smoothly from one phase to the other throughout all space. Instead, the system must cross or tunnel through a barrier. In the early Universe, this happens through the nucleation of bubbles of the true phase inside the false phase.

Thus, in a first-order transition, three ideas always appear together:

$$\text{barrier} + \text{metastability} + \text{bubble nucleation}. \quad (159)$$

The false vacuum is locally stable, but not globally stable. The true vacuum is energetically favored, but it is separated from the false vacuum by a barrier. The transition begins when thermal fluctuations or quantum tunneling create bubbles of the true vacuum large enough to grow instead of collapsing.

This is why the critical temperature is not the end of the story. It only tells us when the two minima are degenerate. The real transition occurs at the nucleation temperature T_n , when the probability of bubble nucleation becomes large enough, typically of order one bubble per Hubble volume. This distinction will be developed carefully in the next chapter.

It is also useful to introduce the spinodal temperature. Roughly speaking, a spinodal temperature marks the point at which a local minimum ceases to exist or becomes unstable. In a first-order transition, we can have a temperature interval in which both minima coexist, another interval in which one of them is metastable, and eventually a point at which the metastable minimum disappears. These temperatures organize the thermal history of the transition, but the most important one dynamically is T_n .

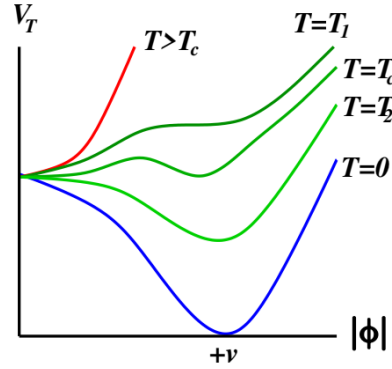


Figure 12: Qualitative representation of the characteristic temperatures of a first-order phase transition. At T_c , the minima are degenerate; at T_n , bubble nucleation becomes efficient; and at T_{sp} , the metastable minimum disappears. The curve at $T = 0$ represents the limit at which the vacuum structure should recover the results measured today. Figure made by [19]

A familiar analogy helps. Water can remain liquid below its freezing temperature if no efficient nucleation site is present. The solid phase is thermodynamically favored, but the system can remain trapped in a metastable liquid phase. A perturbation can trigger crystallization, and ice crystals then grow through the system, just as when we shake a beer and it quickly freezes. The same idea appears, in a much more fundamental language, in a first-order Higgs transition: the Universe can remain trapped in a false vacuum until bubbles of the true vacuum nucleate and expand.

Of course, the Higgs field is not literally water, and bubbles of the Higgs phase are not ice crystals. But the physical logic is the same: a barrier prevents a smooth transition, metastability delays the change of phase, and nucleation makes the transition proceed locally before completing globally.

This bubble dynamics is precisely what makes first-order phase transitions interesting for gravitational waves. Expanding bubbles disturb the plasma. Their walls accelerate or reach a terminal velocity, transfer energy to the surrounding fluid, collide and generate long-lasting sound waves and turbulence. These processes create anisotropic stresses in the energy-

momentum tensor, which act as sources of gravitational radiation. The detailed calculation of this signal belongs to later chapters, but the origin of the signal is already visible here: a first-order transition is violent, inhomogeneous and out of equilibrium [18, 19].

Now the practical question becomes: what kind of new physics can change the Higgs potential enough to create this barrier?

There are several broad mechanisms.

Tree-level barriers

The first possibility is to modify the scalar potential itself at zero temperature. In this case, the barrier is already present, at least partially, at tree level. This can happen if the scalar sector contains additional fields or higher-dimensional operators.

A simple effective example is [2, 17]

$$V_0(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{8\Lambda^2}\phi^6. \quad (160)$$

Depending on the signs and sizes of the parameters, this type of deformation can change the global structure of the potential and allow two minima to coexist. The scale Λ represents the energy scale at which new physics becomes important.

This possibility is especially attractive because it keeps the theory close to the Standard Model at low energies, while still allowing the Higgs potential to be different at field values and temperatures relevant to the early Universe. However, there is an important restriction: at $T = 0$, the model must still reproduce what we measure today. In particular, the minimum must remain at

$$\phi = v \simeq 246 \text{ GeV}, \quad (161)$$

and the curvature at the minimum must reproduce

$$m_h \simeq 125 \text{ GeV}. \quad (162)$$

Thus, any modification of the Higgs potential must be large enough to affect the early Universe, but controlled enough to recover collider measurements at zero temperature.

Another possible example is the one carried out in [1], in which the modifications appear in the integration measure of the effective potential, that is, in $D\phi$. The results of that article were also reproduced in the code developed by me and can be found on [GitHub](#).

Thermal barriers from bosonic degrees of freedom

The second possibility is that the barrier is generated thermally. As discussed in the previous section, bosonic thermal corrections contain a cubic contribution in the high-temperature expansion:

$$\Delta V_T^B \supset -\frac{n_b T}{12\pi} m_b^3(\phi). \quad (163)$$

If

$$m_b(\phi) \propto \phi, \quad (164)$$

this gives a term of the form

$$-ET\phi^3. \quad (165)$$

This term can create a barrier between $\phi = 0$ and $\phi \neq 0$. In the Standard Model, such a term is present from the gauge bosons, but for the physical Higgs mass the effect is too weak to produce a strong first-order transition. Historically, for a much lighter Higgs, the perturbative electroweak transition would have been stronger, but the measured Higgs mass places the Standard Model in the crossover regime.

New bosonic degrees of freedom coupled to the Higgs can strengthen this thermal cubic effect. This is why extra scalar fields are so common in models of strong electroweak phase transitions. Fermions, in contrast, do not generate the same cubic term in the high-temperature expansion. They contribute importantly to the thermal potential, especially through the top quark, but they do not create a perturbative barrier in the same way.

Thermal resummations and reorganized perturbation theory

A third layer of physics concerns how perturbation theory is organized at finite temperature. Near a phase transition, bosonic infrared modes can become important, and naive perturbation theory can become unreliable. Daisy or ring resummation is the standard treatment

included in many calculations:

$$m_b^2(\phi) \rightarrow m_b^2(\phi) + \Pi_b(T). \quad (166)$$

This substitution takes into account thermal self-energies in the plasma and improves the behavior of the perturbative expansion.

Other reorganizations are possible. Optimized perturbation theory, screened perturbation theory and related methods try to reorganize the expansion by introducing thermal or variational masses and fixing them through an optimization prescription. These approaches do not merely add a new term to the potential; they change the perturbative bookkeeping [30, 31, 33, 34]. In this thesis, this point is especially relevant because one of our objectives is to show how the daisy resummation can affect the phase structure and the resulting gravitational-wave parameters.

The important message is that the existence and strength of a first-order transition can be sensitive to how finite-temperature effects are treated.

Putting these mechanisms together, the general picture is simple:

to obtain a first-order Higgs transition, new physics must modify the effective potential so that a barrier separates the symmetric and broken phases at high energy scales.

This modification can appear at tree level; it can appear through bosonic thermal effects; or it can be affected by the resummation scheme used to treat the plasma. In all cases, the central object remains the same:

From the point of view of gravitational waves, this is the key reason why first-order phase transitions are so powerful. They connect physics through a huge chain of ideas:

$$\text{new scalar physics} \rightarrow V_{\text{eff}}(\phi, T) \rightarrow \text{bubble nucleation} \rightarrow \text{plasma motion} \rightarrow \Omega_{\text{GW}}(f). \quad (167)$$

Future space-based interferometers can, therefore, test sectors of particle physics that are extremely difficult to access directly in terrestrial colliders. Collider measurements tell us what the Higgs sector looks like today, at $T = 0$. Gravitational waves may tell us how this sector behaved when the Universe was hot.

With this, we close the theoretical construction of Chapter 2.

The next chapter begins exactly where this one ends. Once the potential is defined, we need to answer dynamical questions: what is the critical temperature? At what temperature

does the transition actually happen? How do we compute the bubble nucleation rate? How do we extract the parameters of the transition strength α and characteristic transition time β/H_* ? And, finally, how do these quantities feed the gravitational-wave spectrum?

Thus, Chapter 2 built the free energy and the field. Chapter 3 will study their dynamics.

3 Chapter Three

From bubble nucleation to thermal parameters

In the previous chapter we saw how the effective potential is intimately connected to the phase structure of a theory. In particular, we saw that, once quantum and thermal corrections are included, the effective potential begins to describe the free energy associated with different field configurations. In this way, its minima indicate which phases can exist for a given temperature and how these phases change as the Universe cools.

However, knowing the shape of the potential is not enough to describe the transition. If there is a true minimum separated from the false minimum by a barrier, we still need to answer a dynamical question: how does the system go from one phase to the other? In a first-order transition, this passage does not occur continuously. It happens through the nucleation of bubbles of the true phase inside the false phase, followed by the expansion of these bubbles in the primordial plasma.

The goal of this chapter is precisely to study this dynamics. Starting from an effective potential $V_{\text{eff}}(\phi, T)$, we want to understand how to calculate the decay rate of the false vacuum, how to determine the temperature at which nucleation becomes efficient, and how to extract the parameters that are the signature of the phase transition. In particular, we will be interested in the strength of the transition, usually represented by α , and in its characteristic time scale, represented by β . These parameters will be fundamental later on, because they condense the thermodynamic information of the transition and enter directly into the estimates of the gravitational-wave spectrum.

To get to them, let us begin with the most basic point: the transition amplitude between two states. In quantum mechanics, the probability that a system prepared in an initial state $|\psi_i\rangle$ is found in a final state $|\psi_f\rangle$ after a time interval is controlled by an amplitude of the form

$$\mathcal{A}_{i \rightarrow f} = \langle \psi_f | e^{-i\hat{H}(t_f - t_i)} | \psi_i \rangle, \quad (168)$$

so that the associated probability is proportional to

$$P_{i \rightarrow f} \propto |\mathcal{A}_{i \rightarrow f}|^2. \quad (169)$$

What we will do now is rewrite this amplitude in the language of path integrals. This step is essential because it will show, in a natural way, how the transition rate is related to the Euclidean action, to the partition function, and, finally, to temporal compactification at finite temperature.

3.1 Wick rotation, Euclidean time and temporal compactification

Let us consider, first, a quantum particle. The transition amplitude between an initial position q_i at time t_i and a final position q_f at time t_f is given by

$$K(q_f, t_f; q_i, t_i) = \langle q_f | e^{-i\hat{H}(t_f - t_i)} | q_i \rangle. \quad (170)$$

We can decompose the time evolution into many infinitesimal steps. In the limit in which the number of intervals goes to infinity and the size of each interval goes to zero, the transition amplitude can be written as a sum over all possible paths that connect q_i to q_f .

To make this statement more explicit, we divide the total time interval into N small equal intervals,

$$\epsilon = \frac{t_f - t_i}{N}, \quad t_n = t_i + n\epsilon, \quad x_0 = x_i, \quad x_N = x_f. \quad (171)$$

Inserting the completeness relation of the position eigenstates,

$$\mathbb{I} = \int dx_n, |x_n\rangle \langle x_n|, \quad (172)$$

at each intermediate time, we can write the amplitude as

$$K(x_f, t_f; x_i, t_i) = \langle x_f | e^{-\frac{i}{\hbar} \hat{H}(t_f - t_i)} | x_i \rangle = \lim_{N \rightarrow \infty} \int dx_1 \cdots dx_{N-1} \prod_{n=1}^N \langle x_n | e^{-\frac{i}{\hbar} \epsilon \hat{H}} | x_{n-1} \rangle.$$

Notice that, already at this point, a deeply non-classical idea appears. The probability amplitude is not being associated with a single trajectory between x_i and x_f , but with a sum over all possible intermediate points. In other words, even before taking the continuous limit, the particle seems to explore all paths compatible with the initial and final points, including paths that cross classically forbidden regions.

Now we look at each small factor of time evolution. For this, we also insert the completeness relation of the momentum eigenstates,

$$\mathbb{I} = \int \frac{dp_n}{2\pi\hbar} |p_n\rangle \langle p_n|, \quad (173)$$

and we use

$$\langle x | p \rangle = e^{\frac{i}{\hbar} p x}. \quad (174)$$

Thus, for an infinitesimal interval, we obtain

$$\begin{aligned} \langle x_n | e^{-\frac{i}{\hbar} \epsilon \hat{H}} | x_{n-1} \rangle &= \int \frac{dp_n}{2\pi\hbar} \langle x_n | e^{-\frac{i}{\hbar} \epsilon \hat{H}} | p_n \rangle \langle p_n | x_{n-1} \rangle \\ &\simeq \int \frac{dp_n}{2\pi\hbar} \exp \left\{ \frac{i}{\hbar} p_n (x_n - x_{n-1}) - \frac{i}{\hbar} \epsilon H(x_n, p_n) \right\}. \end{aligned} \quad (175)$$

In the last step, we used that, to first order in ϵ , the Hamiltonian operator can be replaced by the classical function $H(x_n, p_n)$ inside the matrix element. This passage involves a choice of operator ordering. With Weyl ordering, for example, the Hamiltonian appears evaluated at the midpoint,

$$H \left(\frac{x_n + x_{n-1}}{2}, p_n \right), \quad (176)$$

without changing the final result in the continuum limit. A more formal discussion of this step can be found in [21–23] .

Substituting this identity in all factors of the amplitude, we find

$$K(x_f, t_f; x_i, t_i) = \lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int dx_1 \cdots dx_{N-1} \int \frac{dp_1}{2\pi\hbar} \cdots \frac{dp_N}{2\pi\hbar} \times \exp \left\{ \frac{i\epsilon}{\hbar} \sum_{n=1}^N \left[p_n \frac{x_n - x_{n-1}}{\epsilon} - H(x_n, p_n) \right] \right\}. \quad (177)$$

Here there is a detail worth commenting on: there is one more integral over momentum than over position. The reason is simple. The amplitude was written between position states, so the endpoint positions are fixed: $x_0 = x_i$ and $x_N = x_f$. Therefore, among the $N + 1$ position points, only the $N - 1$ intermediate points are integrated over. The momenta, however, have no fixed boundary conditions in this problem. Each small time interval carries its own momentum p_n , and that is why N momentum integrals appear.

This difference also has an interesting physical interpretation. Since we are asking for the amplitude to go from an initial position to a final position, we fix where the particle starts and where it ends, but we do not fix with which momentum it travels along each intermediate segment. The momentum works as the conjugate variable that connects two neighboring points of the trajectory. In the classical limit, once the trajectory $x(t)$ is known, the momentum becomes related to the velocity; but, in the path integral, before taking the semiclassical limit, all these intermediate momenta must still be summed over. We can also interpret that, having two fixed position points, there is an associated average momentum, so the momentum loses one degree of freedom out of the $N + 1$ degrees of freedom of the problem.

Now taking the continuous limit, we have

$$\frac{x_n - x_{n-1}}{\epsilon} \longrightarrow \dot{x}(t), \quad \epsilon \sum_{n=1}^N \longrightarrow \int_{t_i}^{t_f} dt. \quad (178)$$

Therefore,

$$\begin{aligned} \lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{i\epsilon}{\hbar} \sum_{n=1}^N \left[p_n \frac{x_n - x_{n-1}}{\epsilon} - H(x_n, p_n) \right] &= \frac{i}{\hbar} \int_{t_i}^{t_f} dt, [p(t)\dot{x}(t) - H(x(t), p(t))] \\ &= \frac{i}{\hbar} \int_{t_i}^{t_f} dt, L(x, \dot{x}) = \frac{i}{\hbar} S[x]. \end{aligned} \quad (179)$$

The term in brackets,

$$p\dot{x} - H(x, p), \quad (180)$$

is precisely the Legendre transform that takes us from the Hamiltonian to the Lagrangian. Therefore, the sum over trajectories in phase space turns into a sum weighted by the classical action. When the momentum integral can be performed, it contributes only a normalization factor to the functional measure. Thus, we can write, schematically,

$$K(q_f, t_f; q_i, t_i) = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q, e^{iS[q]}. \quad (181)$$

Here,

$$S[q] = \int_{t_i}^{t_f} dt \left[\frac{1}{2} \dot{q}^2 - V(q) \right] \quad (182)$$

is the classical action evaluated along each path. This expression summarizes one of the central ideas of the path integral: the quantum amplitude is not determined by a single classical trajectory, but by a coherent sum over all possible trajectories. Among these trajectories there are, in particular, paths that cross classically forbidden regions. That is exactly why this formalism is so natural for studying tunneling.

For a field theory, the idea is the same, but the fundamental degree of freedom stops being a coordinate $q(t)$ and becomes a field configuration $\phi(t, \mathbf{x})$. Thus, the transition amplitude between an initial configuration $\phi_i(\mathbf{x})$ and a final configuration $\phi_f(\mathbf{x})$ is written as

$$\langle \phi_f | e^{-i\hat{H}(t_f-t_i)} | \phi_i \rangle = \int_{\phi(t_i, \mathbf{x})=\phi_i(\mathbf{x})}^{\phi(t_f, \mathbf{x})=\phi_f(\mathbf{x})} \mathcal{D}\phi, e^{iS[\phi]}, \quad (183)$$

where, for a real scalar field,

$$S[\phi] = \int_{t_i}^{t_f} dt \int d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right]. \quad (184)$$

In this thesis we use $\hbar = 1$. Even so, it is useful to remember that the semiclassical regime corresponds to the limit in which the action is such that $S \gg \hbar$. In this limit, the path integral is dominated by stationary configurations of the action, that is, by configurations that satisfy the classical equations of motion. Later on, we will see that the *bounce* will be precisely one of these stationary configurations, but now in the Euclidean formalism.

The connection with thermodynamics appears when we compare quantum time evolution with the canonical partition function. The partition function is defined by

$$Z(\beta) = \text{Tr}, e^{-\beta \hat{H}}, \quad \beta = \frac{1}{T}. \quad (185)$$

The formal similarity between $e^{-i\hat{H}\Delta t}$ and $e^{-\beta \hat{H}}$ suggests that thermal evolution can be interpreted as evolution in imaginary time. To make this relation precise, we introduce the Wick rotation,

$$t = -i\tau, \quad dt = -i, d\tau. \quad (186)$$

Applying this transformation to the action of a particle, we obtain

$$iS[q] = i \int dt \left[\frac{1}{2} \left(\frac{dq}{dt} \right)^2 - V(q) \right] = - \int d\tau \left[\frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right] \equiv -S_E[q].$$

Therefore,

$$e^{iS[q]} \longrightarrow e^{-S_E[q]}. \quad (187)$$

The Euclidean action for a scalar field becomes

$$S_E[\phi] = \int d\tau \int d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]. \quad (188)$$

This is an important conceptual point. In the Minkowski formalism, the path integral has an oscillatory weight, e^{iS} , and different configurations interfere with each other. After the Wick rotation, the weight becomes damped, e^{-S_E} . This makes the path integral formally similar to a statistical sum, in which configurations with smaller Euclidean action are less suppressed.

Now we can return to the partition function. Using a basis of field eigenstates, the trace can be written as

$$Z(\beta) = \int \mathcal{D}\phi_0, \langle \phi_0 | e^{-\beta \hat{H}} | \phi_0 \rangle. \quad (189)$$

This expression tells us that, to compute the trace, we sum amplitudes in which the initial configuration and the final configuration are the same. In other words, we are summing closed trajectories in configuration space. Since $e^{-\beta \hat{H}}$ is a Euclidean evolution over an interval $\Delta\tau = \beta$, we obtain

$$Z(\beta) = \int_{\phi(0, \mathbf{x}) = \phi(\beta, \mathbf{x})} \mathcal{D}\phi, e^{-S_E[\phi]}. \quad (190)$$

This is the functional representation of the finite-temperature partition function. The condition

$$\phi(\tau + \beta, \mathbf{x}) = \phi(\tau, \mathbf{x}) \quad (191)$$

shows that Euclidean time has been compactified into a circle of length β . For bosonic

fields, like the scalar field we are considering, the boundary conditions are periodic. For fermionic fields they would be antiperiodic, but this detail will not be central in the present discussion.

Thus, finite temperature appears geometrically: Euclidean spacetime ceases to be $\mathbb{R}^4$ and takes the structure

$$S^1_\beta \times \mathbb{R}^3, \quad (192)$$

where S^1_β represents the compact time direction of length $\beta = 1/T$.

It is important to separate two ideas that are sometimes mixed together. The Wick rotation is the analytic continuation that transforms the oscillatory weight e^{iS} into the damped weight e^{-S_E} . A more careful discussion about the validity of this analytic continuation, including the role of the contour in the complex plane, is presented in Appendix C. Temporal compactification, on the other hand, appears when we compute the thermal partition function, because the trace requires that the initial and final configurations coincide after a Euclidean interval β . Together, these two ingredients allow us to treat quantum systems at finite temperature through a Euclidean path integral.

Physically, this does not mean that the bubble evolves in imaginary time the way it would evolve in real time. Euclidean time is a mathematical tool that allows us to compute the semi-classically dominant configuration for the transition process. In the false-vacuum decay problem, this configuration will be the *bounce*. It does not directly describe the real expansion of the bubble in Minkowski time; it computes the exponential suppression associated with the nucleation of the critical bubble.

This distinction will be essential for the rest of the chapter. First we will find the Euclidean configuration responsible for nucleation. Then, when discussing bubble expansion and plasma hydrodynamics, we will return to real-time physics. Therefore, the Euclidean formalism provides the bubble creation rate; the later dynamics of growth and collision belongs to another stage of the transition.

3.2 False vacuum decay and the bounce solution

In the last section we discussed the Euclidean action and saw how it is directly linked to the transition probability between states. The Wick rotation transformed the oscillatory weight

of the path integral, e^{iS} , into a damped weight, e^{-S_E} . This step is essential because, from that point on, the tunneling problem stops looking like a highly oscillatory sum and starts to look like a variational problem: finding which configurations dominate the Euclidean integral.

The question we want to answer now is: *which configuration dominates the decay of the false vacuum?* In the previous chapter, the effective potential allowed us to identify the minima of the theory. If the potential has two minima separated by a barrier, we call the minimum with higher free energy the false vacuum, ϕ_f , and the minimum with lower free energy the true vacuum, ϕ_t . Classically, a system trapped in the false vacuum would not be able to cross the potential barrier. Quantum mechanically, however, there is a tunneling probability. What we want is to turn this qualitative statement into a quantitative expression for the decay rate [35–37].

Schematically, the decay rate per unit volume will be written as

$$\frac{\Gamma}{V} = Ae^{-B}, \quad (193)$$

at zero temperature, and as

$$\frac{\Gamma(T)}{V} = A(T)e^{-B(T)/T}, \quad (194)$$

at finite temperature. The goal of this section is to understand the physical and mathematical origin of these expressions: where the exponent comes from, why it is associated with the Euclidean action, what the so-called *bounce* is, and what the role of the prefactor A is.

The semi-classical approach and the steep descent method

Let us begin with the simplest case: a particle in a one-dimensional potential $V(q)$, initially located in the false vacuum q_0 . We want to calculate the Euclidean amplitude for the particle to start at q_0 and return to q_0 after an interval of Euclidean time $\mathcal{T} = \tau_f - \tau_i$:

$$\langle q_0 | e^{-\hat{H}\mathcal{T}} | q_0 \rangle = \int_{q(\tau_i)=q_0}^{q(\tau_f)=q_0} \mathcal{D}q e^{-S_E[q]}, \quad (195)$$

where

$$S_E[q] = \int_{\tau_i}^{\tau_f} d\tau \left[\frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right]. \quad (196)$$

This integral is still impossible to compute exactly in general. The idea, then, is to use a semiclassical approximation. This means temporarily restoring $\hbar$ in the exponent,

$$e^{-S_E[q]} \longrightarrow e^{-S_E[q]/\hbar}, \quad (197)$$

and considering the regime in which $S_E \gg \hbar$. In this limit, small variations of the action produce large variations in the exponent. Thus, paths for which the action varies at first order interfere destructively or are exponentially suppressed. The dominant paths are those that make the action stationary.

This is the steepest-descent method, or saddle-point method. The idea is to expand the action around a classical configuration $q_c(\tau)$,

$$q(\tau) = q_c(\tau) + \eta(\tau), \quad (198)$$

with fluctuations $\eta(\tau)$ that satisfy the boundary conditions

$$\eta(\tau_i) = \eta(\tau_f) = 0. \quad (199)$$

Expanding the Euclidean action around q_c , we have

$$S_E[q_c + \eta] = S_E[q_c] + \int d\tau \frac{\delta S_E}{\delta q(\tau)} \Big|_{q_c} \eta(\tau) + \frac{1}{2} \int d\tau d\tau' \eta(\tau) \frac{\delta^2 S_E}{\delta q(\tau) \delta q(\tau')} \Big|_{q_c} \eta(\tau') + \dots \quad (200)$$

The configuration q_c is chosen precisely to extremize the action, that is,

$$\left. \frac{\delta S_E}{\delta q(\tau)} \right|_{q_c} = 0. \quad (201)$$

Therefore, the linear term disappears. This is the same spirit as what appeared in the previous chapter when we discussed the effective action: the classical field, or mean field, is the one around which we organize fluctuations. But here there is an important subtlety. At this moment, we are not simply saying that q_c is the equilibrium expectation value of the field. We are looking for the saddle configuration that dominates a transition amplitude. In other words, q_c is not just a static minimum of the potential; it is an entire Euclidean trajectory.

Keeping terms up to second order, we obtain

$$S_E[q_c + \eta] = B + \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \eta(\tau) \mathcal{M} \eta(\tau) + \mathcal{O}(\eta^3), \quad (202)$$

where we define

$$B \equiv S_E[q_c], \quad (203)$$

and

$$\mathcal{M} = -\frac{d^2}{d\tau^2} + V''(q_c). \quad (204)$$

The leading exponent of the decay rate will be given by B , that is, by the Euclidean action evaluated on the classical solution. The prefactor will come from the Gaussian integral over the fluctuations η . Substituting the expansion into the path integral,

$$\int \mathcal{D}q e^{-S_E[q]} \approx e^{-B} \int \mathcal{D}\eta \exp \left[-\frac{1}{2} \int d\tau \eta(\tau) \mathcal{M} \eta(\tau) \right]. \quad (205)$$

This expression clearly shows the physical separation of the problem. The factor e^{-B} carries the exponential suppression of tunneling. The remaining term, which we will call gener-

ically C , measures the quantum fluctuations around the dominant configuration:

$$C \equiv \int \mathcal{D}\eta \exp \left[-\frac{1}{2} \int d\tau \eta \mathcal{M} \eta \right]. \quad (206)$$

The semiclassical approximation corresponds, therefore, to writing

$$\int \mathcal{D}q e^{-S_E[q]} \approx C e^{-B}. \quad (207)$$

This is the first concrete appearance of the structure we wanted: a rate exponentially suppressed by an action. The approximation is equivalent, in spirit, to the WKB approximation discussed in Appendix D: in both cases, the tunneling effect is dominated by a semiclassical exponent, while finer quantum details appear in the prefactor.

The functional determinant and the meaning of the prefactor

Let us now understand the term C better. For this, we treat the space of fluctuations $\eta(\tau)$ as a vector space of functions that vanish at the endpoints. Since the operator $\mathcal{M}$ is self-adjoint with respect to the inner product

$$\langle f, g \rangle = \int_{\tau_i}^{\tau_f} d\tau f(\tau) g(\tau), \quad (208)$$

we can expand any fluctuation in a basis of eigenfunctions of $\mathcal{M}$:

$$\eta(\tau) = \sum_n c_n \eta_n(\tau), \quad (209)$$

with

$$\mathcal{M} \eta_n(\tau) = \lambda_n \eta_n(\tau), \quad \int d\tau \eta_n(\tau) \eta_m(\tau) = \delta_{nm}. \quad (210)$$

Substituting this expansion into the quadratic term, we obtain

$$\frac{1}{2} \int d\tau \eta \mathcal{M} \eta = \frac{1}{2} \sum_n \lambda_n c_n^2. \quad (211)$$

The functional integral then becomes a product of ordinary Gaussian integrals:

$$C = \prod_n \int dc_n \exp \left[-\frac{1}{2} \lambda_n c_n^2 \right] \propto \prod_n \lambda_n^{-1/2}. \quad (212)$$

Since the product of eigenvalues is, formally, the determinant of the operator $\mathcal{M}$, we write

$$C \propto (\det \mathcal{M})^{-1/2}. \quad (213)$$

This result is the functional version of the usual Gaussian integral. In a finite-dimensional integral, when we expand a function around a saddle point, the determinant of the Hessian matrix appears. Here the same thing happens, but the Hessian matrix has been replaced by a differential operator, and its determinant is the product of its eigenvalues.

However, there is an important subtlety that cannot be ignored. The Gaussian formula

$$\int dc_n e^{-\frac{1}{2} \lambda_n c_n^2} \propto \lambda_n^{-1/2} \quad (214)$$

only makes sense when $\lambda_n \neq 0$ and, more precisely, when the direction is stable. If some eigenvalue is zero, the corresponding integral stops being a damped Gaussian. It becomes simply

$$\int dc_0, \quad (215)$$

that is, an integral over a flat direction of the action. This is not a small technical detail: it is telling us that there exists a transformation that changes the configuration but does not change the value of the action.

In the case of the bounce, this flat direction has a very clear physical origin. If $q_b(\tau)$ is a solution, then

$$q_b(\tau - \tau_0) \quad (216)$$

is also a solution for any value of τ_0 . That is, we can shift the center of the bounce in Euclidean time without changing its action. The action does not know at which Euclidean instant the tunneling happened. It only knows that it happened.

Mathematically, this translational invariance appears as a zero mode of the fluctuation operator. To see this, we first use the Euclidean equation of motion satisfied by the bounce,

$$\frac{d^2 q_b}{d\tau^2} = V'(q_b). \quad (217)$$

Differentiating this equation with respect to τ , we obtain

$$\frac{d^3 q_b}{d\tau^3} = V''(q_b) \dot{q}_b. \quad (218)$$

Now we apply the fluctuation operator $\mathcal{M}$ to $\dot{q}_b$:

$$\begin{aligned} \mathcal{M} \dot{q}_b &= \left[-\frac{d^2}{d\tau^2} + V''(q_b) \right] \dot{q}_b \\ &= -\frac{d^3 q_b}{d\tau^3} + V''(q_b) \dot{q}_b \\ &= 0. \end{aligned} \quad (219)$$

Therefore, $\dot{q}_b$ is an eigenfunction of $\mathcal{M}$ with zero eigenvalue. Physically, it represents the freedom to shift the bounce in Euclidean time.

Since this direction cannot be integrated as an ordinary Gaussian, we remove this eigenvalue from the determinant. This is what the symbol $\det' \mathcal{M}$ means: the determinant is computed over all modes except the zero mode. In place of the integral over the coefficient c_0 , we change variables to the center of the bounce, τ_0 . This change introduces a Jacobian proportional

to $\sqrt{B/2\pi}$ and an integral over all possible instants at which the bounce could occur:

$$\int d\tau_0 = \mathcal{T}. \quad (220)$$

Thus, the quadratic contribution around a bounce becomes, schematically,

$$C = \mathcal{T} \left(\frac{B}{2\pi} \right)^{1/2} (\det' \mathcal{M})^{-1/2}. \quad (221)$$

To prevent infinite normalizations of the measure from contaminating the physical result, we usually compare this determinant with the determinant associated with the trivial configuration, i.e., the particle sitting at the false vacuum q_0 :

$$\mathcal{M}_0 = -\frac{d^2}{d\tau^2} + V''(q_0). \quad (222)$$

Thus, the physical prefactor appears as a ratio of determinants,

$$A \sim \left(\frac{B}{2\pi} \right)^{1/2} \left(\frac{\det' \mathcal{M}}{\det \mathcal{M}_0} \right)^{-1/2}, \quad (223)$$

where the factor involving B comes precisely from the treatment of the translational zero mode.

This point is important because it anticipates the physical interpretation of the rate: the factor $\mathcal{T}$ appears because the bounce could have occurred at any instant within the Euclidean interval considered. When we pass from an amplitude to a rate per unit time, this factor will be removed.

The bounce equation and the inverted potential

The configuration that dominates the integral must extremize the Euclidean action. Applying the Euler–Lagrange equations to the action (196), we obtain

$$\frac{d^2 q}{d\tau^2} = \frac{\partial V}{\partial q}. \quad (224)$$

This equation looks simple, but it contains the main physical intuition of tunneling. In ordinary classical mechanics, the equation would be

$$\frac{d^2 q}{dt^2} = -\frac{\partial V}{\partial q}. \quad (225)$$

After the Wick rotation, the effective sign of the potential changes. The Euclidean equation is equivalent to a classical equation in time τ , but with an inverted potential,

$$V(q) \longrightarrow -V(q). \quad (226)$$

This is the beauty of the formalism. Tunneling, which has no direct classical description in real time, gains a semiclassical description in Euclidean time. The particle does not cross the barrier in $V(q)$; it rolls classically in the inverted potential $-V(q)$.

That is why the solution is called a *bounce*. It starts at the false vacuum, moves away from it, reaches close to the region of the true vacuum, and returns to the false vacuum in Euclidean time. In terms of boundary conditions,

$$q_b(\tau \rightarrow -\infty) = q_0, \quad q_b(\tau \rightarrow +\infty) = q_0, \quad (227)$$

Since the Euclidean action is invariant under translations in τ , we can freely choose where to place the center of the bounce. By convention, we choose this center at $\tau = 0$. Moreover, the bounce solution is symmetric around this point: it leaves the false vacuum, reaches the turning point, and returns along the same path in Euclidean time. Thus,

$$q_b(-\tau) = q_b(\tau). \quad (228)$$

Differentiating this relation, we see that $\dot{q}_b(\tau)$ is an odd function. Therefore, at the center of the bounce,

$$\dot{q}_b(0) = 0. \quad (229)$$

Physically, this point $\tau = 0$ is the turning point of the Euclidean trajectory. In the analogy with the particle rolling in the inverted potential, it is the instant at which we release the particle with zero velocity.

The solution is, therefore, a trajectory that leaves and returns to the same point. This “back and forth” is not a real physical evolution in Minkowski time. It is a trajectory in Euclidean time that computes the exponential suppression of the transition.

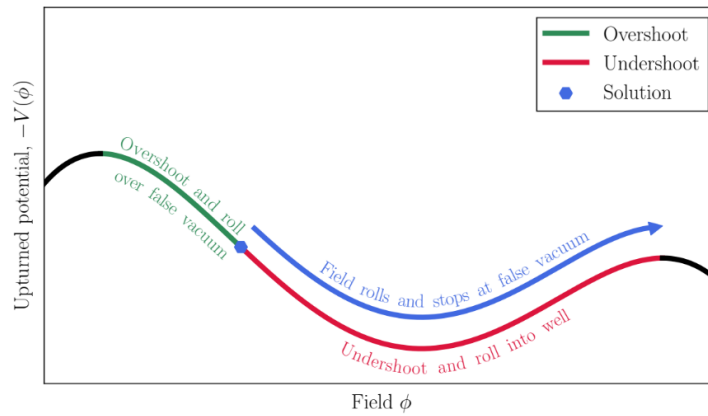


Figure 13: Schematic representation of the bounce problem in the inverted potential. In the effective Euclidean potential, the tunneling problem can be understood as the classical motion of a particle in $-V$, with special boundary conditions. *Overshooting* solutions pass beyond the false vacuum, while *undershooting* solutions stop before reaching it. The bounce solution is the critical trajectory between these two behaviors. Source [18]

This figure will be important to fix the intuition. In the inverted potential, we want to release the particle from a point **close to the true vacuum** and make it arrive at the false vacuum with zero velocity. If we release the particle from an inadequate point, it may pass beyond the false vacuum: this is *overshooting*. If we release it from another point, it may stop before reaching the false vacuum: this is *undershooting*. The correct bounce solution is the boundary between these two behaviors.

From bounce to decay rate

So far we have found the structure

$$\int \mathcal{D}q e^{-S_E[q]} \sim C e^{-B}. \quad (230)$$

However, a decay rate is not just an amplitude. We need to understand how the decay appears on the spectral side of the theory.

Applying the Wick rotation to the left-hand side of the amplitude, we can expand in eigenstates of the Hamiltonian:

$$\langle q_0 | e^{-\hat{H}\mathcal{T}} | q_0 \rangle = \sum_n e^{-E_n \mathcal{T}} \langle q_0 | n \rangle \langle n | q_0 \rangle. \quad (231)$$

In the limit $\mathcal{T} \rightarrow \infty$, the dominant term is the lowest-energy state compatible with the amplitude:

$$\sum_n e^{-E_n \mathcal{T}} |\psi_n(q_0)|^2 \longrightarrow e^{-E_0 \mathcal{T}} |\psi_0(q_0)|^2. \quad (232)$$

This result is one of the reasons the Euclidean formalism is so useful: large Euclidean time intervals project the theory onto the lowest-energy state. In the false-vacuum problem, however, there is a subtlety. The false vacuum is not the true ground state of the theory; it is a metastable state. This means it has a finite lifetime. A convenient way to describe this is to allow its effective energy to have an imaginary part:

$$E_0 = E_R - i\frac{\Gamma}{2}. \quad (233)$$

In real time,

$$e^{-iE_0 t} = e^{-iE_R t} e^{-\Gamma t/2}, \quad (234)$$

and, therefore, the probability decays as

$$|\psi(t)|^2 \propto e^{-\Gamma t}. \quad (235)$$

Thus,

$$\Gamma = -2 \operatorname{Im} E_0. \quad (236)$$

The subtle point is that this imaginary part does not come from the trivial solution $q(\tau) = q_0$. The trivial solution only describes persistence in the false vacuum. The imaginary part associated with the decay comes from the non-trivial configurations: the bounces.

There is still an important technical subtlety. The fluctuation operator around the bounce has a negative eigenvalue. This may seem strange at first sight: if we are looking for the configuration that dominates the integral, why is it not simply a minimum of the Euclidean action?

The answer is that the bounce is not an absolute minimum of the action. It is a saddle point. There are many directions in which small fluctuations increase the action, but there is a special direction in which the action decreases. Physically, this direction is associated with changing the “size” of the critical configuration. If the bubble is slightly smaller than critical, it tends to disappear; if it is slightly larger, it tends to grow. The bounce is exactly at the threshold between these two behaviors.

Mathematically, a negative direction means that, for some mode η_- ,

$$\mathcal{M}\eta_- = \lambda_- \eta_-, \quad \lambda_- < 0. \quad (237)$$

The Gaussian contribution of this direction would have the form

$$\int dc_- \exp \left[-\frac{1}{2} \lambda_- c_-^2 \right]. \quad (238)$$

Since $\lambda_- < 0$, the exponent grows rather than decays. This integral is not an ordinary

convergent Gaussian. The correct treatment involves deforming the integration contour along the steepest-descent direction. This deformation is precisely what produces an imaginary part in the false-vacuum energy.

And this is physically perfect: if there were no imaginary part, there would be no decay. The negative mode is, therefore, the mathematical signature that the false vacuum is unstable. It turns the bounce contribution into a real decay rate.

If we consider the possibility of multiple well-separated bounces in Euclidean time, we enter the so-called dilute instanton gas approximation. Since each bounce is localized in time, a configuration with n bounces contributes approximately an action nB .

Now, why can we sum configurations with multiple bounces? The idea is that, over a large Euclidean interval $\mathcal{T}$, the system can tunnel once, twice, three times, and so on. Each bounce is localized in a small region of Euclidean time. If they are well separated from each other, we can approximate the total action as the sum of the individual actions:

$$S_E^{(n)} \approx nB. \quad (239)$$

This is the dilute instanton gas approximation: bounces are treated as rare events and sufficiently separated, so that they barely interact with each other.

For a single bounce, we saw that a contribution of the form appears

$$A\mathcal{T}e^{-B}. \quad (240)$$

The factor $\mathcal{T}$ comes from the fact that the center of the bounce could be anywhere in the Euclidean interval. For two bounces, we would have two integrals over centers, something like

$$\int d\tau_1 d\tau_2. \quad (241)$$

But the two bounces are indistinguishable. Swapping τ_1 with τ_2 does not generate a new physical configuration. That is why we divide by $2!$. For n bounces, by the same reasoning, we divide by $n!$. Thus, the contribution of n bounces is

$$\frac{1}{n!} (A\mathcal{T}e^{-B})^n. \quad (242)$$

Summing all possibilities, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} (A\mathcal{T}e^{-B})^n = \exp(A\mathcal{T}e^{-B}). \quad (243)$$

Comparing with the expected exponential behavior for an unstable state, we arrive at the central result:

$$\Gamma = |A|e^{-B}. \quad (244)$$

This result summarizes the physics of false-vacuum decay at zero temperature. The exponent B is the Euclidean action of the bounce. The prefactor A contains the quantum fluctuations around the bounce, the determinant ratio, the treatment of the zero mode, and the unstable direction. In practice, the exponent strongly dominates the rate, since small changes in B produce enormous changes in Γ .

The physical interpretation is quite direct: the larger the barrier, the larger the Euclidean action needed to cross it and, therefore, the smaller the decay rate. Tunneling is rare because it is exponentially suppressed. The barrier does not need to be crossed classically; it is crossed semiclassically by the dominant Euclidean configuration. [35–37]

From quantum mechanics to field theory

We can now translate this idea to field theory. Instead of a coordinate $q(\tau)$, we have a field configuration $\phi(\tau, \mathbf{x})$. The effective Euclidean action, in the simplest derivative approximation, is

$$S_E[\phi] = \int d\tau d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V_{\text{eff}}(\phi, T) \right]. \quad (245)$$

Here we are using the effective potential as an effective free energy for homogeneous field

configurations. Technically, this corresponds to truncating the effective action in a derivative expansion, keeping the canonical kinetic term and the effective potential. This is the usual approximation in phenomenological studies of cosmological phase transitions.

For the bounce action to be finite, it is convenient to subtract the false-vacuum energy:

$$\Delta V_{\text{eff}}(\phi, T) = V_{\text{eff}}(\phi, T) - V_{\text{eff}}(\phi_f, T). \quad (246)$$

Thus, the false vacuum has zero energy in this normalization, and the action measures only the energetic cost of the non-trivial configuration.

The Euclidean equation of motion is

$$\partial_\mu \partial_\mu \phi = \frac{\partial V_{\text{eff}}}{\partial \phi}. \quad (247)$$

Again, this equation is equivalent to a classical problem in the inverted potential. The difference is that now the system's "position" is not a coordinate q , but an entire field configuration. Tunneling between two minima of the effective potential is described by a non-trivial solution of this equation.

At zero temperature, the Euclidean action has $O(4)$ symmetry, since the four Euclidean coordinates appear on the same footing. The lowest-action solution can then be taken as a function only of the four-dimensional radius

$$\rho = \sqrt{\tau^2 + \mathbf{x}^2}. \quad (248)$$

This is not just a convenient choice; it is the choice that maximizes symmetry and minimizes the action for the dominant bounce. With this symmetry,

$$\phi(\tau, \mathbf{x}) = \phi_b(\rho), \quad (249)$$

and the Euclidean Laplacian can be written in hyperspherical coordinates. In four Euclidean dimensions, it takes the form

$$\partial_\mu \partial_\mu = \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_{S^3}, \quad (250)$$

where Δ_{S^3} is the angular Laplacian on the unit three-sphere. Since we are assuming an $O(4)$ -symmetric solution, the field does not depend on angles, but only on ρ . Therefore,

$$\Delta_{S^3} \phi_b = 0, \quad (251)$$

and we obtain

$$\partial_\mu \partial_\mu \phi_b = \frac{d^2 \phi_b}{d\rho^2} + \frac{3}{\rho} \frac{d\phi_b}{d\rho}. \quad (252)$$

Thus, the zero-temperature bounce equation is

$$\frac{d^2 \phi_b}{d\rho^2} + \frac{3}{\rho} \frac{d\phi_b}{d\rho} = \frac{\partial V_{\text{eff}}}{\partial \phi} \Big|_{\phi=\phi_b}. \quad (253)$$

The boundary conditions are

$$\frac{d\phi_b}{d\rho} \Big|_{\rho=0} = 0, \quad \phi_b(\rho \rightarrow \infty) = \phi_f. \quad (254)$$

The first condition guarantees regularity at the center of the bounce. If the derivative were not zero at $\rho = 0$, the solution would have a preferred direction and the action could become singular. The second condition says that, far from the bubble, the field must return to the false vacuum. This is fundamental: the bounce describes the nucleation of a localized bubble inside a region that is still in the false vacuum.

With $O(4)$ symmetry, the action reduces to

$$S_4[\phi_b] = 2\pi^2 \int_0^\infty d\rho \rho^3 \left[\frac{1}{2} \left(\frac{d\phi_b}{d\rho} \right)^2 + \Delta V_{\text{eff}}(\phi_b) \right]. \quad (255)$$

The factor $2\pi^2\rho^3$ comes from the angular volume of a three-sphere in four Euclidean dimensions. This expression is the field-theory version of the action B discussed earlier. At zero temperature, therefore, [35]

$$B = S_4[\phi_b], \quad (256)$$

and the rate per unit volume takes the form

$$\frac{\Gamma}{V} \simeq Ae^{-S_4}. \quad (257)$$

Interpreting the bounce solution

Equation (253) has an extremely useful mechanical interpretation. Imagine a fictitious particle moving in the inverted potential $-V_{\text{eff}}$, where the variable “time” is ρ . The equation

$$\frac{d^2\phi_b}{d\rho^2} + \frac{3}{\rho} \frac{d\phi_b}{d\rho} = \frac{\partial V_{\text{eff}}}{\partial \phi} \quad (258)$$

can be read as an equation of motion with ρ -dependent friction. The term

$$\frac{3}{\rho} \frac{d\phi_b}{d\rho} \quad (259)$$

acts as a friction. At the beginning, near $\rho = 0$, this friction is large; for large ρ , it decreases. We must choose the initial value $\phi_b(0)$ such that the particle rolls in the inverted potential and stops exactly at the false vacuum as $\rho \rightarrow \infty$.

If we choose $\phi_b(0)$ too close to the true vacuum, the particle may have too much energy and pass the false vacuum: overshooting. If we choose $\phi_b(0)$ insufficiently, it loses energy due

to friction and stops before reaching the false vacuum: undershooting [35, 38]. The bounce is the critical solution between these two regimes.

Physically, the result of the bounce is the spatial configuration of the bubble at the instant of nucleation. When we rotate back to Minkowski time, the $\tau = 0$ slice of the Euclidean solution provides the initial configuration of the bubble in real time:

$$\phi_M(t = 0, \mathbf{x}) = \phi_b(\tau = 0, \mathbf{x}). \quad (260)$$

This must be interpreted with care. The bounce does not describe the real expansion of the bubble. It describes the critical configuration that appears via tunneling. After nucleation, the evolution of the bubble wall occurs in Minkowski time and must be studied using the real equations of motion, including the interaction with the thermal plasma.

The physical picture is the following: the false vacuum fills space. At some point, a critical bubble of the true phase appears. This bubble does not grow because the bounce “evolves” in Euclidean time; it grows because, after nucleation, the free-energy difference between phases pushes the bubble wall in real time.

Finite temperature and temporal compactification

We can now return to the central point of this thesis: phase transitions in the early Universe happen at finite temperature. In this case, the fundamental quantity is not just the persistence amplitude of the false vacuum, but the thermal partition function.

As we discussed earlier, since we are computing a thermal trace, we must sum only over configurations periodic in Euclidean time:

$$Z(\beta) = \int_{\phi(\tau+\beta, \mathbf{x})=\phi(\tau, \mathbf{x})} \mathcal{D}\phi e^{-S_E[\phi]}. \quad (261)$$

Thus, at finite temperature, Euclidean time is compactified into a circle of length β . This is the direct connection between the statistical physics of the previous chapter and the tunneling formalism of this chapter.

The decay rate at finite temperature is related to the imaginary part of the free energy,

$$Z(\beta) = e^{-\beta F(\beta)}. \quad (262)$$

Analogously to the zero-temperature case, where the rate was associated with the imaginary part of the metastable-state energy, now the rate is associated with the imaginary part of the free energy. Schematically,

$$\Gamma(T) \propto \text{Im } F(T). \quad (263)$$

This statement has two complementary physical interpretations. For low temperatures, the process still looks like quantum tunneling, but now thermally weighted by the available states. For high temperatures, the process approaches a thermal fluctuation over the barrier. In this case, the particle does not need to tunnel through the barrier; it can receive enough thermal energy to go over it. The Euclidean formalism unifies these two possibilities.

In the high-temperature limit, β is small. Since the Euclidean time direction becomes compact and short, the dominant configuration tends not to depend on τ [39, 40]. Thus,

$$\phi(\tau, \mathbf{x}) \approx \phi(\mathbf{x}). \quad (264)$$

In this limit, the Euclidean action factorizes:

$$\Gamma_E[\phi] = \int_0^\beta d\tau \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + V_{\text{eff}}(\phi, T) \right] = \beta S_3[\phi]. \quad (265)$$

Since $\beta = 1/T$, we obtain

$$\Gamma_E[\phi] = \frac{S_3[\phi]}{T}, \quad (266)$$

where

$$S_3[\phi] = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + \Delta V_{\text{eff}}(\phi, T) \right]. \quad (267)$$

This is one of the most important equations in the chapter. It shows why, in cosmological phase transitions, we often compute $S_3(T)/T$ rather than S_4 . In the relevant thermal regime, the exponential suppression of the rate is given by [37, 40]

$$\frac{\Gamma(T)}{V} \simeq A(T) \exp \left[-\frac{S_3(T)}{T} \right]. \quad (268)$$

In general, the prefactor $A(T)$ has dimensions of energy to the fourth power and is often estimated to be of order T^4 , although its full calculation involves functional determinants and fluctuation modes. In most phenomenological applications, the exponent dominates the rate, and that is why the computation of $S_3(T)/T$ is the central element.

The thermal bounce

At finite temperature, in the limit in which the solution is independent of Euclidean time, we look for a solution with spherical symmetry in three-dimensional space. We write

$$\phi(\mathbf{x}) = \phi_b(r), \quad r = |\mathbf{x}|. \quad (269)$$

The three-dimensional Laplacian in spherical coordinates is

$$\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr}. \quad (270)$$

Thus, the thermal bounce equation is

$$\frac{d^2\phi_b}{dr^2} + \frac{2}{r} \frac{d\phi_b}{dr} = \frac{\partial V_{\text{eff}}(\phi, T)}{\partial\phi} \Big|_{\phi=\phi_b}. \quad (271)$$

The boundary conditions are

$$\left. \frac{d\phi_b}{dr} \right|_{r=0} = 0, \quad \phi_b(r \rightarrow \infty) = \phi_f. \quad (272)$$

The corresponding three-dimensional action is

$$S_3(T) = 4\pi \int_0^\infty dr r^2 \left[\frac{1}{2} \left(\frac{d\phi_b}{dr} \right)^2 + \Delta V_{\text{eff}}(\phi_b, T) \right]. \quad (273)$$

Compare this expression with (255). The difference between S_4 and S_3 is not just a mathematical detail: it expresses the physical change between quantum tunneling in four Euclidean dimensions and thermal nucleation in three spatial dimensions. At zero temperature, the bounce is a configuration localized in Euclidean time and in space. At high finite temperature, Euclidean time is compact and short, and the dominant configuration becomes approximately constant along that direction. The problem reduces to the formation of a critical spatial bubble.

What does the bounce calculate and what does it not calculate?

It is important to end this section by making clear the physical meaning of what has been constructed. The bounce is not the real-time trajectory of the bubble. It does not describe the bubble wall expanding in the plasma, nor its collision with other bubbles, nor the formation of sound waves or turbulence. The bounce computes the nucleation probability of the critical bubble.

In other words, the Euclidean formalism answers the question:

What is the exponential suppression for a critical configuration of the true phase to appear inside the false phase?

After this configuration appears, the question changes. From then on, we must return to Minkowski time and study the real evolution of the bubble. This evolution involves the pressure difference between phases, the friction of the wall with the plasma, the wall velocity, and, finally, the hydrodynamical dynamics responsible for the sources of gravitational waves.

Therefore, the logical chain built so far is:

$$V_{\text{eff}}(\phi, T) \implies \phi_f, \phi_t \implies \phi_b(r, T) \implies S_3(T) \implies \frac{\Gamma(T)}{V} \sim A(T)e^{-S_3(T)/T}. \quad (274)$$

With this, we have turned the geometry of the effective potential into a physical nucleation rate. It still remains, however, to answer a cosmological question: at what temperature does this rate become large enough for the transition to actually happen in the expanding Universe? This question will take us to the characteristic temperatures of the transition, in particular to the critical temperature, the nucleation temperature, and the percolation temperature.

3.3 Characteristic temperatures of the transition

In the last section we saw that there exists a bounce solution associated with the transition between vacua. This solution is the one that extremizes the Euclidean action and lies precisely at the threshold between a configuration that is too small, which collapses back to the false vacuum, and a configuration large enough to grow after nucleation. It is important to repeat this point carefully: this “growth” is still not the real expansion of the bubble in Minkowski time. The bounce captures the critical configuration in Euclidean space. The real dynamics starts afterward, when we perform the inverse rotation and move on to study the evolution of the bubble in the plasma.

However, since we are dealing with a finite-temperature transition, one essential question is still missing: at which temperature should we compute the bounce?

The short answer is: for each temperature at which there exists a false vacuum separated from a true vacuum by a barrier, there exists a bounce problem. That is, for each temperature T , we can solve the bounce equation, compute $S_3(T)$, and estimate the nucleation rate

$$\frac{\Gamma(T)}{V} \simeq A(T) \exp \left[-\frac{S_3(T)}{T} \right]. \quad (275)$$

But this still does not tell us when the transition “starts”. For that we need to turn the nucleation rate into a cosmological criterion. In other words, it is not enough for a tunneling probability to exist; we need to know when this probability becomes large enough for bubbles of the true phase to appear in a relevant way in the Universe.

Before defining that temperature, it is worth organizing the characteristic temperatures that appear in the thermal history of the transition. They are important not only conceptually, but also computationally. It is from them that we can build the numerical pipeline, which is used in the Github code: tracking the minima of the effective potential, identifying when they coexist, computing the bounce at the relevant temperatures, and extracting the thermodynamic parameters that will be used later in the gravitational-wave spectrum.

Critical temperature

Perhaps the most intuitive temperature is the critical temperature, T_c . It is defined as the temperature at which two distinct minima of the effective potential have the same free energy. We write:

$$V_{\text{eff}}(\phi_f, T_c) = V_{\text{eff}}(\phi_t, T_c), \quad (276)$$

with

$$\left. \frac{\partial V_{\text{eff}}}{\partial \phi} \right|_{\phi=\phi_f, T=T_c} = 0, \quad \left. \frac{\partial V_{\text{eff}}}{\partial \phi} \right|_{\phi=\phi_t, T=T_c} = 0. \quad (277)$$

Here ϕ_f represents the minimum associated with the false phase and ϕ_t the minimum associated with the true phase.

Physically, T_c marks the moment at which the two phases are thermodynamically degenerate. Above this temperature, the old phase is favored. Below it, the new phase begins to have lower free energy. In this sense, T_c marks the beginning of the thermodynamic possibility of the transition.

But there is an important subtlety: the transition does not necessarily happen at T_c . Even below T_c , there is still a barrier separating the minima. The Universe can remain trapped in the false vacuum for some time. This delay is what we call supercooling. Therefore,

$$T_n < T_c \quad (278)$$

in a usual first-order transition during cosmic cooling.

Thus, T_c tells us when the true phase becomes favored, but it does not tell us when bubbles of that phase start to appear efficiently. For that we will need the nucleation temperature.

Spinodal temperature

Before talking about T_n , it is worth defining another useful temperature: the spinodal temperature, which we will denote by T_s . This temperature is associated with the disappearance or appearance of a local minimum. In a one-field theory, it is defined by the conditions

$$\left. \frac{\partial V_{\text{eff}}}{\partial \phi} \right|_{\phi=\phi_s, T=T_s} = 0, \quad \left. \frac{\partial^2 V_{\text{eff}}}{\partial \phi^2} \right|_{\phi=\phi_s, T=T_s} = 0. \quad (279)$$

The first equation says that we are at a stationary point of the potential. The second says that the curvature at that point vanishes. In other words, the point is exactly ceasing to be a minimum, maximum, or ordinary saddle point. It is the moment when local stability changes.

This temperature is useful because it delimits the region of coexistence between phases. In a first-order transition, there is typically an interval of temperatures in which two minima coexist separated by a barrier. Within this interval, it makes sense to talk about a false vacuum, a true vacuum, and a bounce solution. Outside it, one of the minima may simply no longer exist.

A computational way to reconstruct this thermal history is to track the minima of the effective potential. They are defined by

$$\frac{\partial}{\partial \phi} V_{\text{eff}}(\phi_{\text{min}}(T), T) = 0. \quad (280)$$

Differentiating this equation with respect to temperature, we obtain

$$\frac{d}{dT} \left[\frac{\partial V_{\text{eff}}}{\partial \phi} \right] = \frac{\partial^2 V_{\text{eff}}}{\partial^2 \phi} \bigg|_{\phi_{\text{min}}} \frac{d\phi}{dT} \bigg|_{\phi_{\text{min}}} + \frac{\partial^2 V_{\text{eff}}}{\partial T \partial \phi} \bigg|_{\phi_{\text{min}}} = 0. \quad (281)$$

This equation summarizes the following idea for tracking phases: knowing where a minimum is at one temperature, we can predict where it shifts when the temperature changes. Numerically, this is exactly the kind of idea that appears in phase-transition codes, such as

CosmoTransitions and similar phase-tracking methods.

The physical point, however, is simple: the effective potential changes with T , so its minima move. The thermal history of the transition is precisely the history of these minima.

Nucleation temperature

Now we can define the nucleation temperature, T_n . Unlike T_c and T_s , it does not come only from a local geometric condition on the potential. It comes from a physical requirement: we want to know when the nucleation rate becomes large enough to produce relevant bubbles in the Universe.

The question is:

When can we say that bubble nucleation really started?

The answer is: when we expect at least one bubble nucleated per Hubble volume. The physical reason is as follows. The Hubble volume represents the characteristic causal scale of the Universe at that instant. If there is, on average, one bubble per Hubble volume, then a typical observer in the early Universe would have at least one region of the true phase forming within the false phase.

This idea sounds very cosmological, but it has a very general intuition. When do we say that water started to freeze? It is not when the solid phase becomes thermodynamically favored in the abstract. It is when ice crystals appear in a macroscopic amount of water. In the Universe, the natural “macroscopic amount” is the Hubble volume.

The nucleation rate per unit volume is

$$\frac{\Gamma(T)}{V} \simeq A(T) \exp \left[-\frac{S_3(T)}{T} \right]. \quad (282)$$

Since the prefactor has dimensions of energy to the fourth power, a common dimensional estimate is

$$A(T) \sim T^4. \quad (283)$$

The criterion of one bubble per Hubble volume in a Hubble time can be written, approximately, as

$$\frac{\Gamma(T_n)}{H^4(T_n)} \sim 1. \quad (284)$$

Substituting the rate expression,

$$\frac{T_n^4}{H^4(T_n)} \exp \left[-\frac{S_3(T_n)}{T_n} \right] \sim 1. \quad (285)$$

Taking the logarithm, we obtain

$$\frac{S_3(T_n)}{T_n} \simeq 4 \ln \left(\frac{T_n}{H(T_n)} \right). \quad (286)$$

During the radiation-dominated era,

$$H(T) \simeq 1.66 \sqrt{g_*} \frac{T^2}{M_{\text{Pl}}}, \quad (287)$$

where g_* is the effective number of relativistic degrees of freedom and M_{Pl} is the Planck mass. Thus,

$$\frac{T}{H(T)} \simeq \frac{M_{\text{Pl}}}{1.66 \sqrt{g_*} T}. \quad (288)$$

For a transition at the electroweak scale, with

$$T_n \sim 100 \text{ GeV}, \quad g_* \sim 100, \quad (289)$$

we have, approximately, [14, 18, 19]

$$\frac{S_3(T_n)}{T_n} \sim 140. \quad (290)$$

This is the most commonly used practical criterion to find the nucleation temperature in electroweak-scale transitions. In a code, the idea is simple: we solve the bounce for several temperatures, compute $S_3(T)/T$, and look for the temperature at which this ratio crosses approximately the value 140.

Of course, the number 140 is not a fundamental constant of nature. It depends on the energy scale of the transition, the prefactor, the number of relativistic degrees of freedom, and the assumed cosmological history. For much colder, much hotter, or strongly supercooled transitions, this criterion must be treated more carefully. Still, for transitions near the electroweak scale, it provides an excellent phenomenological approximation.

In this thesis, we will adopt T_n as the characteristic temperature of the transition for the calculation of thermodynamic parameters. This choice is common and adequate when the transition is sufficiently fast, so that nucleation, percolation, and gravitational-wave generation occur at nearby temperatures. However, it is important to make clear that, in a more complete analysis of post-nucleation dynamics, other temperatures may be more appropriate, especially the percolation temperature.

3.4 Bubble expansion and plasma hydrodynamics

And after nucleation, what happens?

So far, the bounce gave us the critical bubble configuration and the nucleation rate. But after this bubble appears, the evolution stops being Euclidean. We go back to real Minkowski time.

The initial configuration of the bubble can be identified with the $\tau = 0$ slice of the bounce solution:

$$\phi_M(t = 0, \mathbf{x}) = \phi_b(r), \quad r = |\mathbf{x}|. \quad (291)$$

This equation is the physical picture of the bounce: it is the initial profile of the critical bubble at the instant it appears.

The characteristic radius of the bubble can be defined from the profile itself. A practical choice is to take the point where the field is halfway between the two vacuum values,

$$\phi_b(R_b) = \frac{\phi_t + \phi_f}{2}. \quad (292)$$

This definition is not unique, but it is useful to visualize the thickness and size of the bubble. In a thin-wall approximation, the free energy of a spherical bubble of radius R can be written as

$$F(R) = 4\pi R^2 \sigma - \frac{4\pi}{3} R^3 \Delta V, \quad (293)$$

where σ is the surface tension of the wall and

$$\Delta V(T) = V_{\text{eff}}(\phi_f, T) - V_{\text{eff}}(\phi_t, T) > 0 \quad (294)$$

is the free-energy difference between the false and true vacuum.

This expression has a nice interpretation. The first term is the cost of creating the bubble wall: the larger the area, the larger the cost. The second term is the volume gain: inside the bubble, the field is in the true phase, which has lower free energy. For small bubbles, the surface cost dominates and the bubble collapses. For large bubbles, the volume gain dominates and the bubble grows.

The critical radius is obtained by imposing

$$\frac{dF}{dR} = 0. \quad (295)$$

Thus,

$$8\pi R_c \sigma - 4\pi R_c^2 \Delta V = 0, \quad (296)$$

and therefore

$$R_c = \frac{2\sigma}{\Delta V}. \quad (297)$$

This is the simplest version of the critical-bubble idea. If $R < R_c$, the bubble tends to collapse. If $R > R_c$, it tends to grow. The bounce provides exactly the configuration at the threshold between these two behaviors.

The force that drives the expansion of the bubble comes from the pressure difference between phases. Since pressure is minus the free energy,

$$p(\phi, T) = -V_{\text{eff}}(\phi, T), \quad (298)$$

we have

$$\Delta p = p_t - p_f = -V_{\text{eff}}(\phi_t, T) + V_{\text{eff}}(\phi_f, T) = \Delta V(T). \quad (299)$$

Therefore, below T_c , when $\Delta V > 0$, the pressure inside the true-phase bubble is higher than outside it. It is this pressure difference that pushes the wall outward. The true-vacuum bubble grows inside the false vacuum for the same intuitive reason why a thermodynamically favored phase tends to occupy the system.

Below the nucleation temperature, more and more bubbles are formed, and each of them expands inside the plasma. The picture is similar to ice crystals growing inside liquid water: first small regions of the new phase appear, then these regions grow, collide, and finally occupy the entire system.

While bubbles expand, space also expands. If a bubble nucleates at an instant t' and is observed at a later instant t , its physical radius can be written as

$$R(t, t') = a(t) \int_{t'}^t \frac{v_w(t'')}{a(t'')} dt''. \quad (300)$$

This equation shows the competition between two effects. The wall grows with velocity v_w , but this expansion happens inside a space that is also being stretched by the scale factor

$a(t)$. In fast transitions, we can often neglect cosmic expansion during the transition. In slow or strongly supercooled transitions, this may cease to be true.

Moreover, the bubble wall does not move in vacuum. It travels through a plasma of particles. In the case of an electroweak transition, for example, the Higgs field changes its value across the wall, and particles coupled to it acquire different effective masses in different regions of space. Mathematically $m_i = m_i(\phi)$.

Thus, when a particle crosses the wall, its effective energy changes. This process transfers momentum between the wall and the plasma. The macroscopic result is a friction force that opposes the indefinite acceleration of the wall.

This was a delicate point for me when studying bubble expansion. We should not imagine only individual particles “gaining mass” inside the bubble in an isolated way. What exists is a relativistic fluid of energy and momentum interacting with a scalar field whose configuration varies in space. The bubble wall perturbs this fluid, and this perturbation propagates as collective motion of the plasma: sound waves, shocks, and, eventually, turbulence.

It is precisely this interaction between wall and plasma that makes the post-nucleation dynamics complicated. To determine it fully, one would need to solve simultaneously the equations of motion of the scalar field and the Boltzmann equations for the distributions of the plasma particles. This is a rather sophisticated hydrodynamical and kinetic problem.

Percolation and completion

Nucleating a bubble does not mean that the transition has completed. After that, the bubble must grow, encounter other bubbles, and occupy a significant fraction of the Universe.

To describe this process, one defines the fraction of volume still trapped in the false vacuum,

$$P_f(t) = e^{-I(t)}. \quad (301)$$

The function $I(t)$ measures the effective volume converted to the true phase, taking into account nucleation and bubble growth. In an expanding Universe, it can be written schematically as

$$I(t) = \frac{4\pi}{3} \int_{t_c}^t dt' \Gamma(t') a^3(t') \left[\int_{t'}^t \frac{v_w(t'')}{a(t'')} dt'' \right]^3. \quad (302)$$

Here $a(t)$ is the scale factor and v_w is the bubble-wall velocity. The interpretation is direct: we sum over all bubbles nucleated at earlier times t' , compute how much they have grown by time t , and estimate what fraction of space has been converted to the true phase.

The percolation temperature, T_p , is defined when bubbles have grown and collided enough to form a connected network of the true phase. A common convention is

$$P_f(t_p) \simeq 0.71, \quad (303)$$

which corresponds to a fraction of about 29% of the volume in the true phase. This number comes from percolation theory for randomly distributed spherical bubbles. In other words, percolation marks the moment when bubbles stop being isolated regions and start forming a connected structure.

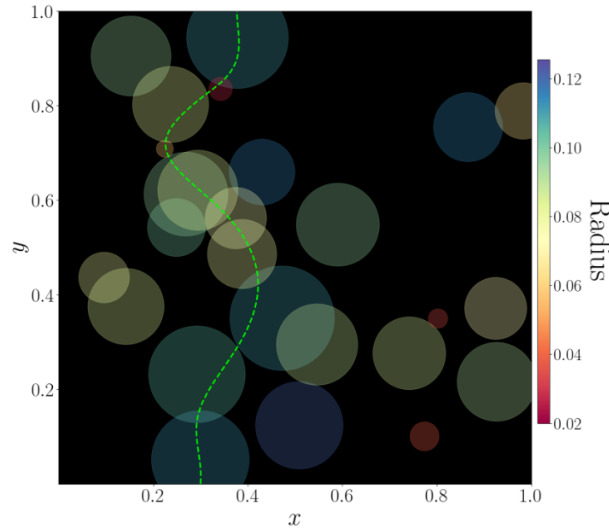


Figure 14: Schematic representation of percolation. At the beginning, bubbles of the true phase appear isolated inside the false vacuum. With time, they grow, collide, and form a connected network. The percolation temperature marks the moment when this network begins to span the relevant causal volume. Source [18]

The completion temperature, in turn, is a stronger notion and also more convention-dependent. It is usually associated with a moment at which the false-vacuum fraction has

become very small, for example

$$P_f(t_{\text{comp}}) \ll 1, \quad (304)$$

or, in some practical choices,

$$P_f(t_{\text{comp}}) \sim 0.01. \quad (305)$$

For the purposes of this thesis, we will not compute T_p nor T_{comp} , since that requires tracking the spatial evolution of bubbles and the plasma hydrodynamics. We will use T_n as the characteristic temperature at which gravitational waves are produced. Still, it is important to mention them because they indicate a limitation of the approximation we will adopt: by using T_n as the characteristic temperature, we are assuming that the transition is fast enough that nucleation, percolation, and gravitational-wave production occur on nearby thermal scales.

Deflagration, detonation and hybrid regimes

The expansion of the bubble wall in the plasma can occur in different ways. The most common classification compares the wall velocity, v_w , with the sound speed in the plasma, c_s . For an ideal relativistic plasma,

$$c_s \simeq \frac{1}{\sqrt{3}}. \quad (306)$$

The first regime is deflagration. In this case, the wall is subsonic,

$$v_w < c_s. \quad (307)$$

The wall moves more slowly than sound perturbations. For this reason, the plasma in front of the wall already “knows” that the bubble is coming: a shock wave forms ahead of the wall, heating and perturbing the false phase before it is converted.

The second regime is detonation. In this case, the wall is supersonic,

$$v_w > c_s, \quad (308)$$

and, more precisely, fast enough that the plasma in front of the wall remains almost unperturbed. The wall arrives before sound can “warn” the plasma. The disturbance is mostly behind the wall, in the true phase.

Between these two limits there is the hybrid regime, or supersonic deflagration. In this case, the wall moves faster than sound, but there is still a hydrodynamical structure with a shock in front and a rarefaction behind the wall. This regime is harder to visualize. From Fig. 15 we can better look at the representation of these 3 regimes.

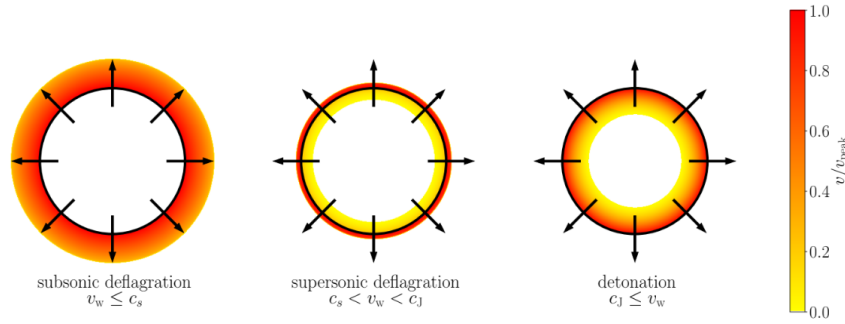


Figure 15: Schematic representation of hydrodynamic bubble-expansion regimes. In a deflagration, the wall is subsonic and there is a shock wave in front. In a detonation, the wall is supersonic and the plasma in front remains almost unperturbed. The hybrid regime combines features of both cases. Source [19]

For the purposes of this thesis, we will assume

$$v_w = 1. \quad (309)$$

This corresponds to an ultra-relativistic wall, moving close to the speed of light. This choice simplifies the analysis and is common in phenomenological estimates when one does not solve the microscopic wall dynamics. Physically, it is closer to a strong detonation regime or to a very fast wall.

This choice will be sufficient for the goal of this thesis, this is, extracting the main parameters of the transition and using them to estimate the gravitational-wave spectrum. [19, 41, 42]

3.5 Thermal parameters: transition strength and duration

Now we finally arrive at the most important point of the chapter. We now need to connect all the physics we have built to gravitational waves, and associate thermal quantities of the phase transition to the future gravitational-wave spectrum.

These quantities are the thermodynamic signatures of the transition. They summarize how strong the transition was, how fast it happened, and with what velocity the bubbles expanded. These are the parameters that will enter directly into the calculation of the gravitational-wave spectrum.

In this thesis, we will evaluate these parameters at the characteristic temperature of the transition, which we will take to be the nucleation temperature,

$$T_* \equiv T_n. \quad (310)$$

Transition strength

The first parameter is the strength of the transition, usually denoted by α . Physically, α measures how much energy is released during the transition compared to the energy of the radiation bath of the Universe.

In a first-order transition, the Universe leaves a false phase, with higher free energy, and goes to a true phase, with lower free energy. The energy difference between these phases is converted into bubble-wall energy, kinetic energy of the plasma, reheating, sound waves, turbulence, and, finally, a small fraction into gravitational waves. Therefore, the larger the released energy, the stronger the transition.

To define this precisely, we begin by recalling that the finite-temperature effective potential acts as a free-energy density. Thus, for a phase described by a minimum $\phi_i(T)$, the pressure is given by

$$p_i(T) = -V_{\text{eff}}(\phi_i, T). \quad (311)$$

The energy density associated with this phase is obtained from the thermodynamic relation

$$\rho_i(T) = V_{\text{eff}}(\phi_i, T) - T \frac{\partial V_{\text{eff}}(\phi_i, T)}{\partial T}. \quad (312)$$

This equation is important because it shows that the energy released in the transition is not just the direct difference between the values of the potential. Since the potential depends on temperature, there is also an entropic contribution. That is why the term with the derivative with respect to T appears, associated with the energy lost due to the ordering of the system, its symmetry breaking.

We then define the difference in energy density between the false and true phases as

$$\Delta\rho(T) = \rho_f(T) - \rho_t(T). \quad (313)$$

Using (312), we obtain

$$\Delta\rho(T) = \Delta V_{\text{eff}}(T) - T \frac{d\Delta V_{\text{eff}}(T)}{dT}, \quad (314)$$

where

$$\Delta V_{\text{eff}}(T) = V_{\text{eff}}(\phi_f, T) - V_{\text{eff}}(\phi_t, T). \quad (315)$$

Here ϕ_f and ϕ_t are the false and true minima evaluated at the temperature under consideration. The total derivative in (314) should be understood along the minima of the theory, i.e., taking into account that ϕ_f and ϕ_t also depend on T .

The natural reference energy in the early Universe is the radiation energy density,

$$\rho_{\text{rad}}(T) = \frac{\pi^2}{30} g_*(T) T^4, \quad (316)$$

where $g_*(T)$ is the effective number of relativistic degrees of freedom. Thus, we define

$$\alpha = \frac{\Delta\rho(T)}{\rho_{\text{rad}}(T)} \Big|_{T=T_*}. \quad (317)$$

Or, more explicitly:

$$\alpha = \frac{\Delta V_{\text{eff}}(T_*) - T_* \frac{d\Delta V_{\text{eff}}}{dT} \Big|_{T_*}}{\frac{\pi^2}{30} g_*(T_*) T_*^4}. \quad (318)$$

This is the operational definition of the transition strength. It turns the shape of the effective potential into a physical number. If α is small, the released energy is small compared to the radiation plasma, and the transition is weak. If α is large, the transition releases a significant fraction of the available energy, becoming more violent.

A larger barrier tends to hinder nucleation, because it increases S_3/T and decreases the rate. This can lead to more supercooling: the Universe remains trapped in the false vacuum for longer, the energy difference between minima increases, and when the transition finally happens, it can be stronger. That is why, frequently, slower and more supercooled transitions are associated with larger values of α and stronger gravitational signals. The quantity that directly measures the strength of the transition is not the barrier height itself; it is the energy released between the phases, normalized by radiation.

Inverse duration of the transition

The second fundamental parameter is β . It measures the time scale of the transition. More precisely, β^{-1} is the characteristic time over which the nucleation rate changes significantly.

The nucleation rate is

$$\frac{\Gamma(T)}{V} \simeq A(T) \exp \left[-\frac{S_3(T)}{T} \right]. \quad (319)$$

Since the exponent dominates the time dependence, we can approximate the rate near the characteristic temperature by

$$\Gamma(t) \simeq \Gamma(t_*)e^{\beta(t-t_*)}. \quad (320)$$

This equation defines β :

$$\beta = \left. \frac{d}{dt} \ln \Gamma(t) \right|_{t=t_*}. \quad (321)$$

Now we use the semiclassical form of the rate. Neglecting the slow time variation of the prefactor $A(T)$, we have

$$\ln \Gamma(t) \simeq -\frac{S_3(T)}{T}. \quad (322)$$

Thus,

$$\beta \simeq -\left. \frac{d}{dt} \left(\frac{S_3(T)}{T} \right) \right|_{t=t_*}. \quad (323)$$

This expression already shows the physical interpretation of β : if S_3/T changes very quickly with time, the nucleation rate grows rapidly and the transition is short. If S_3/T changes slowly, nucleation spreads over a larger interval and the transition lasts longer.

To write this quantity in terms of temperature, we use the relation between time and temperature in a radiation-dominated Universe. If the expansion is approximately adiabatic and g_* does not vary rapidly,

$$\frac{dT}{dt} \simeq -H(T)T. \quad (324)$$

Substituting into (323), we obtain

$$\begin{aligned}\beta &\simeq - \frac{d}{dT} \left(\frac{S_3}{T} \right) \frac{dT}{dt} \Big|_{T=T_*} \\ &\simeq H(T) T \frac{d}{dT} \left(\frac{S_3}{T} \right) \Big|_{T=T_*}.\end{aligned}\tag{325}$$

Thus, the dimensionless quantity normally used is

$$\frac{\beta}{H_*} = T \frac{d}{dT} \left(\frac{S_3(T)}{T} \right) \Big|_{T=T_*},\tag{326}$$

where

$$H_* \equiv H(T_*).\tag{327}$$

This is the practical definition we will use. After finding T_n , we solve the bounce around that temperature, compute $S_3(T)/T$ for nearby temperatures, and extract the derivative in (326).

The interpretation is also direct. Since H_*^{-1} is the time scale of cosmic expansion, β/H_* compares the duration of the transition with the Hubble time. If

$$\frac{\beta}{H_*} \gg 1,\tag{328}$$

the transition is much faster than the expansion of the Universe. If β/H_* is small, the transition lasts a significant fraction of the cosmological time.

Here it is worth correcting a common intuition: β is not a length, but a frequency, i.e., an inverse time scale. The characteristic time is

$$\Delta t_{\text{PT}} \sim \beta^{-1}.\tag{329}$$

Since we use natural units, $c = 1$, time and length have the same units. That is why the characteristic spatial scale associated with bubble growth is of order

$$R_* \sim \frac{v_w}{\beta}, \quad (330)$$

up to numerical factors that depend on the precise definition of the mean separation between bubbles. This is one of the reasons why β/H_* is so important for gravitational waves: it controls the typical scale of the sources. Slower transitions, with smaller β/H_* , produce larger bubbles and tend to generate signals at lower frequencies and with larger amplitudes. Very fast transitions, with large β/H_* , generate smaller bubbles and shift the signal to higher frequencies. [14, 18, 19, 43]

Closing the chapter

We thus reach the natural closing of this chapter. We began with a simple question. Given an effective potential with two minima separated by a barrier, how does the Universe go from one minimum to the other? The answer led us to the Euclidean action, the bounce, the nucleation rate, the characteristic temperatures of the transition, and the initial dynamics of bubbles in the plasma.

In the end, this whole construction can be summarized in a small set of parameters:

$$T_*, \quad \alpha, \quad \frac{\beta}{H_*}, \quad v_w. \quad (331)$$

Where we take, for simplicity:

$$T_* = T_n, \quad v_w = 1, \quad (332)$$

and we will compute α and β/H_* from the effective potential and the bounce action.

These parameters are the macroscopic signature of the transition. They carry the physical information we need to move from the microscopic field theory to gravitational-wave phenomenology. In a sense, they are what translate the modification of the scalar sector into an observable prediction.

The next step is to understand how this transition generates gravitational waves. There

are three main mechanisms: bubble collisions, sound waves in the plasma, and magnetohydrodynamic turbulence. Each of these mechanisms produces a contribution to the final spectrum,

$$\Omega_{\text{GW}}(f) = \Omega_{\text{col}}(f) + \Omega_{\text{sw}}(f) + \Omega_{\text{turb}}(f). \quad (333)$$

The next chapter will be devoted precisely to this final bridge: how, starting from T_* , α , β/H_* and v_w , we can build the stochastic gravitational-wave spectrum associated with a first-order phase transition.

4 Chapter Four

Gravitational Waves

In the previous chapters we built, step by step, the microscopic part of the transition. In Chapter 2, we saw how the finite-temperature effective potential can be interpreted as a free-energy density for homogeneous field configurations. From it, we can follow how the vacuum landscape changes with temperature and, in particular, how a barrier between phases can arise. In Chapter 3, we advanced from the geometry of the potential to the dynamics of the transition: we calculated the bounce solution, the nucleation rate, the characteristic temperatures and, finally, the thermal parameters that summarize the transition.

In other words, we went from the most raw description of the theory, given by

$$V_{\text{eff}}(\phi, T),$$

and arrived at a small set of macroscopic parameters:

$$T_*, \quad \alpha, \quad \frac{\beta}{H_*}, \quad v_w.$$

In this thesis, as discussed previously, we will take

$$T_* = T_n, \quad v_w = 1,$$

so that the physical information of the transition is condensed mainly into its characteristic temperature, its strength, and its inverse duration. These parameters are precisely the bridge between the effective potential and the possible observational signature of the transition. [14, 18]

However, we still need to answer the main question of this chapter: how can a microscopic transition, described by an effective potential, leave a macroscopic signature in the geometry of spacetime? Or, more directly, how can we go from a first-order phase transition in the early Universe to a gravitational-wave spectrum, $\Omega_{\text{GW}}(f)$?

The goal of this chapter is to build this final bridge. First, we will see how gravitational waves arise from Einstein's equations when we study small perturbations of the metric. Next,

we will understand why gravitational radiation is not generated by just any motion of mass-energy, but rather by non-spherical and non-trivial variations of the energy distribution, with the quadrupole as the first radiative term. This observation will be essential to connect gravitational waves with phase transitions: perfectly spherical and isolated bubbles would not, by themselves, be efficient sources; the radiation arises from the collective dynamics, from collisions, from deformations, and from the anisotropic stresses generated in the primordial plasma.

After that, we will move from the wave itself to its energy spectrum. This will be the quantity directly used in the phenomenology:

$$\Omega_{\text{GW}}(f) = \frac{1}{\rho_c} \frac{d\rho_{\text{GW}}}{d \ln f}.$$

Finally, we will discuss how the phenomenological templates used for first-order phase transitions translate the thermal parameters into contributions from **bubble collisions**, **sound waves**, **and turbulence**, allowing us to compare the predicted signal with the sensitivity curves of future detectors such as LISA, DECIGO and BBO.

4.1 From Einstein equations to gravitational radiation

To understand gravitational waves, we need to start with the object that relates matter, energy and the geometry of spacetime, in other words, Einstein's equations.

The total action of general relativity can be written as the sum of the Einstein-Hilbert gravitational action and the matter action,

$$S = S_E + S_M, \tag{334}$$

where

$$S_E = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R. \tag{335}$$

Here R is the Ricci scalar, g is the determinant of the metric $g_{\mu\nu}$, G is Newton's constant and c is the speed of light. Matter appears through S_M , and the energy-momentum tensor is defined by the variation of this action with respect to the metric:

$$\delta S_M = \frac{1}{2c} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}. \quad (336)$$

This definition shows that $T^{\mu\nu}$ is not only an “energy density” in the usual sense, it also measures how matter responds to a small deformation of the geometry of spacetime, that is, of the metric. In a gravitational theory, this is exactly what we want to know, because the source of curvature is not only mass, but every form of energy, pressure and momentum flux.

Varying the total action with respect to the metric, we obtain Einstein’s equations, given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (337)$$

These equations say, in a compact way, that matter determines geometry and that geometry determines the motion of matter. To study gravitational waves, however, we do not need to solve the full nonlinear theory of general relativity. The relevant regime for the propagation of weak waves, which is our case, can be obtained by expanding the metric around flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \quad (338)$$

The tensor $h_{\mu\nu}$ represents a small perturbation of the Minkowski metric. The condition $|h_{\mu\nu}| \ll 1$ defines the linearized regime. When inserting this decomposition into Einstein’s equations, we keep only terms of first order in $h_{\mu\nu}$. Thus, gravity starts to look, locally, like a field theory propagating on a fixed background.

There is an important conceptual subtlety here. By writing the metric in the form (338), we have already chosen a reference frame in which spacetime is almost flat. However, this choice does not remove all the coordinate freedom of general relativity. We can still perform an infinitesimal transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x), \quad (339)$$

as long as $\partial_\mu \xi_\nu$ is of the same order of smallness as $h_{\mu\nu}$. Under this transformation, the perturbation changes as

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \quad (340)$$

Therefore, not all components of $h_{\mu\nu}$ are physical. Part of them describes only our freedom to choose coordinates, our gauge freedom. This point is essential. A gravitational wave is not just any apparent oscillation of the metric, because some oscillations can be removed by a change of coordinates. The role of a good gauge choice is to separate what is physical from what is only a redundancy of the description.

In addition to these local infinitesimal transformations, the linearized theory also preserves the global Lorentz transformations,

$$x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu, \quad (341)$$

which by definition do not change $\eta^{\mu\nu}$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}. \quad (342)$$

Under these transformations, the perturbation behaves as a Lorentz tensor,

$$h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x). \quad (343)$$

Spatial rotations do not compromise the condition $|h_{\mu\nu}| \ll 1$. For boosts, however, we should only restrict the analysis to reference frames that preserve the perturbative regime.

To write the linearized Einstein equations, we first need to expand the geometric objects to first order in $h_{\mu\nu}$. The linearized Riemann tensor is [44]

$$R_{\mu\nu\rho\sigma}^{(1)} = \frac{1}{2} (\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}). \quad (344)$$

From it we obtain the linearized Ricci tensor and Ricci scalar. It is convenient to introduce the trace

$$h = \eta^{\mu\nu} h_{\mu\nu}, \quad (345)$$

and the trace-reversed perturbation,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \quad (346)$$

With these definitions, the linearized Einstein equations can be written as

$$\square \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \quad (347)$$

where

$$\square = \eta^{\mu\nu} \partial_\mu \partial_\nu \quad (348)$$

is the d'Alembert operator in flat space.

Now we can use the gauge freedom to simplify the equation. Under the transformation (340), the trace-reversed perturbation transforms as

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho). \quad (349)$$

Thus,

$$\partial^\nu \bar{h}_{\mu\nu} \rightarrow \partial^\nu \bar{h}'_{\mu\nu} = \partial^\nu \bar{h}_{\mu\nu} - \square \xi_\mu. \quad (350)$$

If the initial configuration satisfies

$$\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x), \quad (351)$$

we can choose ξ_μ such that

$$\square \xi_\mu = f_\mu(x). \quad (352)$$

In this way, it is possible to impose

$$\partial^\nu \bar{h}_{\mu\nu} = 0. \quad (353)$$

This condition is called the harmonic gauge, Hilbert gauge, de Donder gauge or, by analogy with electromagnetism, Lorenz gauge.

In the harmonic gauge, equation (347) takes a much simpler form [44] :

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \quad (354)$$

This is a wave equation with source. The analogy with electromagnetism becomes clear. Just as the electromagnetic potential satisfies a wave equation when we choose the Lorenz gauge, the gravitational perturbation also satisfies a wave equation when we choose the harmonic gauge.

Since $h_{\mu\nu}$ is symmetric, it initially has ten independent components. The harmonic gauge condition imposes four constraints, reducing the number of apparent components to six. These four removed components were not physical degrees of freedom, but only redundancies associated with the choice of coordinates. The goal now is to go further and identify, among the remaining components, those that truly correspond to gravitational radiation.

For this, let us first consider the source-free region,

$$T_{\mu\nu} = 0. \quad (355)$$

In this case,

$$\square \bar{h}_{\mu\nu} = 0. \quad (356)$$

This is the propagation equation of gravitational waves in the linearized regime. Even after imposing the harmonic gauge, there is still a residual freedom: we can perform transformations with

$$\square \xi_\mu = 0, \quad (357)$$

without destroying condition (353). This residual freedom allows us to choose an even more physical gauge, called the transverse-traceless gauge, or TT gauge.

In the TT gauge, the components of the perturbation satisfy

$$h_{0\mu}^{\text{TT}} = 0, \quad \partial_i h_{ij}^{\text{TT}} = 0, \quad h_{ii}^{\text{TT}} = 0. \quad (358)$$

The first condition says that the radiative perturbation is entirely in the spatial part of the metric. The second says that it is transverse to the direction of propagation. The third says that it has no trace. Thus, from all the initial components of $h_{\mu\nu}$, only two independent physical components remain.

For a gravitational wave propagating in the z direction, we can write

$$h_{ij}^{\text{TT}}(t, z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - z/c)]. \quad (359)$$

The two functions h_+ and $h_\times$ are the two physical polarizations of the gravitational wave, called the plus mode and the cross mode.

This form makes something very important clear: a gravitational wave is a transverse and traceless deformation of geometry. If a wave propagates in the z direction, it deforms the perpendicular plane (x, y) . A set of test particles initially at rest, arranged in a circle in this plane, starts to be stretched in one direction and compressed in the perpendicular direction. In the $+$ mode, when the x direction is stretched, the y direction is compressed, and half a cycle later the effect is reversed. In the $\times$ mode, the same structure occurs, but rotated by 45° .

ωt	h_+	$h_\times$
0		
$\pi/2$		
π		
$3\pi/2$		

Figure 16: Schematic representation of the effect of the $+$ and $\times$ polarizations of a gravitational wave on a ring of test particles in the plane transverse to the direction of propagation. Ref. [44]

This image also helps us understand why we associate gravitational radiation with a spin-

2 field. The physical pattern of deformation does not need a full rotation of 2π to return to itself. A rotation of π already reproduces the same configuration. This contrasts with electromagnetism, associated with a spin-1 field, whose transverse polarizations return to the same pattern after a full rotation. In the case of spin-1/2 fermions, the situation is even more subtle: a rotation of 2π changes the sign of the state, $|\psi\rangle \rightarrow -|\psi\rangle$, and only a rotation of 4π returns exactly to the original state. In the gravitational case, the tensorial structure of the perturbation h_{ij}^{TT} precisely reflects the helicity ± 2 character of gravitational waves.

So far we have studied free propagation. But, to connect with phase transitions, we need to return to the original question: how does a source produce gravitational radiation?

In the linearized regime, the solution of the equation with source can be written in terms of the energy-momentum tensor. For a weak and localized source, far from the observer, the radiation is obtained by expanding the energy distribution in multipoles. The essential point is that not every multipole can radiate gravitationally.

The monopole term would be associated with the total mass-energy of the source. However, a static source or a conserved quantity cannot radiate. Since total energy is conserved, there is no monopole radiation. The dipole term would be associated with the center of mass and with the total linear momentum of the source. By conservation of linear momentum, there is also no gravitational dipole radiation. Therefore, the first moment that can contribute to gravitational radiation is the quadrupole.

This conclusion has a simple physical interpretation. Gravitational waves are generated by changes in the non-spherical part of the energy distribution. A perfectly spherical contraction, however violent it may be, does not produce gravitational radiation. To radiate, the source needs to have a dynamics that breaks spherical symmetry in a time-dependent way.

We define the mass quadrupole moment by

$$Q_{ij}(t) = \int d^3x \rho(t, \mathbf{x}) \left(x_i x_j - \frac{1}{3} \delta_{ij} r^2 \right), \quad (360)$$

where $\rho = T^{00}/c^2$ is the mass density in the nonrelativistic limit. In the quadrupole approximation, the radiative part of the metric is given by [44]

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{c^4 r} \ddot{Q}_{ij}^{\text{TT}} \left(t - \frac{r}{c} \right). \quad (361)$$

The double dot over Q_{ij} indicates that the radiation depends on the second time derivative of the quadrupole. Thus, an energy distribution may be large, but if its quadrupole does not change in time, it does not radiate. Equivalently, a dynamics may be fast, but if it is perfectly spherical, it also does not generate gravitational waves.

The total power emitted in gravitational waves, in the quadrupole approximation, can be written as

$$P_{\text{GW}} = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle, \quad (362)$$

where the average is taken over several characteristic periods of the source. This expression shows two fundamental properties. First, gravitational radiation is extremely weak, because it is suppressed by G/c^5 . Second, it is favored by violent, relativistic and highly non-spherical sources, in which the quadrupole varies rapidly.

Now we can see the first conceptual link with first-order phase transitions. During a FOPT, bubbles of the true phase nucleate, expand in the plasma and eventually collide. A perfectly spherical and isolated bubble would have too much symmetry to generate significant gravitational radiation. However, the real process is not a single spherical bubble in the vacuum. It is a network of bubbles nucleating at different positions, colliding, deforming the plasma, generating sound waves and turbulence. This dynamics produces time-dependent anisotropic stresses, exactly the type of source capable of feeding the modes h_{ij}^{TT} .

Thus, what we should absorb so far, which is central for the rest of the chapter, is:

$$\text{time-varying anisotropic stresses} \implies h_{ij}^{\text{TT}} \implies \text{gravitational waves.}$$

In the next section, we will move to the quantity that will be used in the results of this thesis, the spectral energy density in gravitational waves, $\Omega_{\text{GW}}(f)$.

4.2 Energy density and the stochastic spectrum

In the previous section we saw two essential points. First, once produced, gravitational waves can propagate even in source-free regions, because they satisfy a wave equation for the radiative modes h_{ij}^{TT} . Second, for them to be generated, it is not enough for energy or motion to exist: there must be a time variation of the non-spherical part of the energy-momentum distribution. In the limit of weak and nonrelativistic sources, this appears as the well-known

statement that the first radiative moment is the quadrupole.

Now we need to turn this idea into an observable quantity. In other words, we want to know how much energy is carried by these waves and how this energy is distributed in frequency. This is precisely the information encoded in

$$\Omega_{\text{GW}}(f).$$

Let us begin, then, with the energy carried by a gravitational wave. In the linearized regime, and in the transverse-traceless gauge, the energy density associated with gravitational waves is given by [5, 6, 44]

$$\rho_{\text{GW}} = \frac{c^2}{32\pi G} \left\langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \right\rangle. \quad (363)$$

Here the spatial indices are summed over, the dot denotes derivative with respect to physical time and the brackets indicate a time average, or, in the case of a stochastic background, a statistical average. This expression has an important physical message: the energy of the gravitational wave is second order in the perturbation. This was expected, because a wave that oscillates around zero cannot carry energy linearly in h_{ij} ; the energy must be associated with the intensity of the oscillation, that is, with something proportional to the square of its variation.

For an isolated astrophysical source, such as the coalescence of two black holes, we are often interested in the waveform $h(t)$. In this case, the signal has a well-defined temporal structure, determined by the specific dynamics of the source. For a cosmological stochastic background, however, the situation is different. The observed signal is not a single coherent wave, but the superposition of an enormous number of waves produced in causally disconnected regions of the early Universe. What reaches us is a random fluctuation of spacetime, described statistically and not as a well-defined, deterministic waveform.

For this reason, a stochastic background is characterized by simple statistical hypotheses: It is, to first approximation, stationary, Gaussian, isotropic and unpolarized. We can write the tensor perturbation as a superposition of plane waves,

$$h_{ij}(t, \mathbf{x}) = \sum_{A=+, \times} \int_{-\infty}^{+\infty} df \int d^2\hat{n} \tilde{h}_A(f, \hat{n}) e_{ij}^A(\hat{n}) e^{-2\pi i f(t - \hat{n} \cdot \mathbf{x}/c)} \quad (364)$$

where $A = +, \times$ denotes the two polarizations, $\hat{n}$ the direction of propagation and e_{ij}^A the polarization tensors.

The natural cosmological unit to normalize an energy density is the critical density of the Universe,

$$\rho_c = \frac{3c^2 H_0^2}{8\pi G}, \quad (365)$$

where H_0 is the Hubble constant today. The critical density represents, in simple terms, the energy scale needed for the spatial geometry of the Universe to be flat. Thus, we define the fraction of the total energy that is in gravitational waves by

$$\Omega_{\text{GW}} = \frac{\rho_{\text{GW}}}{\rho_c}.$$

But one crucial piece of information is still missing. Detectors do not measure only a total energy: they are sensitive to frequencies. Therefore, we want to know how the energy of the gravitational waves is distributed along the spectrum. For this we write

$$\rho_{\text{GW}} = \int_0^\infty d \ln f \frac{d\rho_{\text{GW}}}{d \ln f}. \quad (366)$$

The choice of $d \ln f$, and not simply df , is particularly convenient because we are interested in logarithmic intervals of frequency. In a log-log plot, for example, each decade of frequency occupies the same visual space; so it makes sense to ask how much energy exists per decade of frequency. With this, we define

$$\Omega_{\text{GW}}(f) = \frac{1}{\rho_c} \frac{d\rho_{\text{GW}}}{d \ln f}. \quad (367)$$

This is the central quantity in the phenomenology of cosmological gravitational waves. The normalized total energy is recovered by

$$\Omega_{\text{GW}} = \int_0^\infty d \ln f \Omega_{\text{GW}}(f). \quad (368)$$

Notice that $\Omega_{\text{GW}}(f)$ is still dimensionless. This happens precisely because $d \ln f$ is also dimensionless. In other words, $\Omega_{\text{GW}}(f)$ is not the energy per hertz, but the energy per logarithmic interval of frequency.

There is still an important convention. Since

$$H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad (369)$$

and since there is an observational uncertainty associated with the determination of H_0 , it is common to present results in terms of

$$h^2 \Omega_{\text{GW}}(f),$$

instead of only $\Omega_{\text{GW}}(f)$. This choice does not alter the physics of the gravitational background; it only removes from the normalization a conventional uncertainty linked to the current value of the Hubble constant.

Before continuing, it is worth emphasizing the physical difference between an individual astrophysical wave and a cosmological stochastic background. An astrophysical wave, for example coming from a binary merger, has a characteristic shape in time. We can look for a chirp, for a phase, for a set of source parameters. A primordial stochastic background, on the other hand, does not generally have an identifiable individual event. It is the collective result of many sources, distributed throughout the early Universe, with random phases. Thus, its signature is not a single waveform, but a statistical excess of energy at certain frequencies.

4.3 Gravitational waves from first-order phase transitions

In the previous section, we saw that the gravitational-wave spectrum is related to the energy density carried by the tensor modes. In the section before that, we saw that gravitational waves are generated by non-spherical variations of the energy-momentum distribution. Now we need to put these two ideas together: how does the dynamics of a first-order phase transition produce anisotropic stresses capable of generating a stochastic background of gravitational waves?

During a FOPT, bubbles of the true phase nucleate inside the false phase, expand in the primordial plasma and eventually collide. These bubbles arise at different points, at different times, grow, deform the plasma, collide and generate collective motions. Spherical symmetry is then broken, and this breaking appears in the energy-momentum tensor as anisotropic stresses varying in time. This is one of the sources of gravitational waves.

In a Friedmann-Robertson-Walker spacetime perturbed by tensor modes, we can write

$$ds^2 = a^2(\eta) \left[-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right], \quad (370)$$

where η is conformal time and h_{ij} satisfies the TT conditions,

$$\partial_i h_{ij} = 0, \quad h_{ii} = 0. \quad (371)$$

The equation of motion of the tensor modes is

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G a^2 \Pi_{ij}^{\text{TT}}, \quad (372)$$

where a prime denotes derivative with respect to conformal time, $\mathcal{H} = a'/a$, and Π_{ij}^{TT} is the transverse-traceless part of the anisotropic stress of the source.

The left-hand side describes the propagation of the gravitational wave in an expanding Universe. The right-hand side describes the physical source. For a phase transition, this source comes from the dynamics of the scalar field and of the plasma. In a schematic way, the total energy-momentum tensor can be separated as

$$T_{\text{tot}}^{\mu\nu} = T_{\phi}^{\mu\nu} + T_{\text{fluid}}^{\mu\nu}. \quad (373)$$

The first term is the contribution of the scalar field that describes the transition; the second is the contribution of the relativistic plasma. In a phenomenological description, the spatial part that acts as a source for gravitational waves can be written as

$$\Pi_{ij} \sim w\gamma^2 v_i v_j + \partial_i \phi \partial_j \phi, \quad (374)$$

before the TT projection. Here $w = \rho + p$ is the enthalpy of the fluid, γ is the Lorentz factor associated with the motion of the plasma, v_i is the fluid velocity and ϕ is the scalar field. The first term represents the contribution of the collective motion of the plasma; the second represents the contribution of the field gradients, that is, of the bubble walls.

This form already anticipates the physics of the three main mechanisms. If the energy remains mainly in the walls, bubble collisions are important. If the energy is transferred to the plasma, the collective motion of the fluid dominates. If this motion becomes nonlinear and

turbulent, an additional contribution from magnetohydrodynamic turbulence appears [11–14] .

Solving this dynamics directly is extremely difficult. In principle, it would be necessary to evolve simultaneously the scalar field, the relativistic plasma and the perturbed metric, including the nucleation of many bubbles and their collisions. In a schematic way, the system can be represented by an equation of motion for the field with effective friction,

$$\square\phi + \frac{\partial V_{\text{eff}}}{\partial\phi} = \mathcal{F}_{\text{fric}}, \quad (375)$$

together with the total conservation of energy-momentum,

$$\nabla_\mu T_{\text{tot}}^{\mu\nu} = 0. \quad (376)$$

This conservation is often written by separating the energy exchange between the field and the fluid:

$$\nabla_\mu T_\phi^{\mu\nu} = J^\nu, \quad \nabla_\mu T_{\text{fluid}}^{\mu\nu} = -J^\nu, \quad (377)$$

where J^ν phenomenologically represents the friction between the bubble wall and the plasma. The important point is not to solve this system here, but to understand what it teaches us: the gravitational source depends on the anisotropic part of the energy-momentum tensor, and this part arises naturally from the field–fluid dynamics during the transition.

Since the goal of this thesis is to build a phenomenological pipeline, we will not directly simulate this hydrodynamical system. Instead, we will use results fitted from simulations and analytic arguments present in the literature. The idea is to translate the microphysics of the effective potential into a small set of thermal parameters obtained previously and, from them, build the spectrum observed today.

The general structure of the spectrum

Before separating the mechanisms, it is useful to understand the common structure of the formulas. The spectrum observed today can be thought of as [14, 18, 43]

$$h^2\Omega_i(f) = \underbrace{\mathcal{R}_i}_{\text{redshift}} \times \underbrace{\mathcal{A}_i\left(\alpha, \frac{\beta}{H_*}, v_w, \kappa_i, g_*\right)}_{\text{scaling}} \times \underbrace{S_i(f)}_{\text{shape}}, \quad (378)$$

where i denotes the production mechanism: bubble collisions, sound waves or turbulence.

The first block, $\mathcal{R}_i$, contains the cosmological redshift. The signal is produced in the early Universe, at a temperature T_* , and observed today, at a much lower temperature. For this reason, both frequencies and amplitudes are redshifted. A characteristic frequency produced at the time of the transition appears today, typically, as

$$f_0 \simeq 1.65 \times 10^{-5} \text{ Hz} \left(\frac{f_*}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \quad (379)$$

The amplitude, in turn, receives a typical factor

$$h^2\Omega_0 \simeq 1.67 \times 10^{-5} \left(\frac{100}{g_*} \right)^{1/3} \Omega_*. \quad (380)$$

Here g_* is the effective number of relativistic degrees of freedom at the time of the transition.

The second block, $\mathcal{A}_i$, contains the scaling of the source. It says how the amplitude depends on the available energy and on the time scale of the transition. Two factors appear recurrently:

$$\frac{\kappa_i \alpha}{1 + \alpha}, \quad \frac{H_*}{\beta}. \quad (381)$$

The first measures the fraction of the total energy that is effectively placed in the channel i . The parameter α measures the strength of the transition, while κ_i is an efficiency: not all released energy becomes gravitational waves; part of it heats the plasma, part becomes kinetic energy of the fluid, part may remain in the walls and part may become turbulence. The factor $1 + \alpha$ appears because the released energy must be compared with the total energy of the system.

The second factor, H_*/β , measures the duration of the source in cosmological units. Since β^{-1} is the typical time scale of the transition, a fast transition has large β/H_* and small H_*/β . A slower transition, on the other hand, has larger bubbles and a longer-lived source, which tends to increase the amplitude of the spectrum.

Finally, the third block, $S_i(f)$, describes the spectral shape. In general, spectra generated

by FOPTs are well described by broken power laws: they grow as a power of f at low frequencies, reach a peak and then decay with another power at high frequencies. At low frequencies, causality arguments favor behaviors of the type f^3 for causal cosmological sources. At high frequencies, the decay depends on the details of the source: thin walls, acoustic waves, turbulence, lifetime and correlation scales.

With this structure in mind, we can discuss the three main mechanisms:

$$\Omega_{\text{GW}}(f) = \Omega_{\text{coll}}(f) + \Omega_{\text{sw}}(f) + \Omega_{\text{turb}}(f). \quad (382)$$

Bubble collisions

The most intuitive mechanism is bubble collision. During the transition, the true phase nucleates in localized regions, forming bubbles that grow inside the false phase. While an isolated bubble is approximately spherical, it is not an efficient source of gravitational waves. The situation changes when many bubbles collide. The collisions break spherical symmetry, deform the walls and generate anisotropic stresses.

Historically, this process was modeled by the so-called envelope approximation. In this approximation, one assumes that the energy is concentrated in very thin bubble walls, and that the part of the wall that has already collided stops contributing to the source. Thus, only the external envelope of the regions that have not yet overlapped continues to radiate. This approximation is simple and captures the central idea of symmetry breaking, but it has important limitations, especially when a thermal plasma exists.

A usual parametrization for the collision contribution is

$$h^2 \Omega_{\text{coll}}(f) = 1.67 \times 10^{-5} \left(\frac{100}{g_*} \right)^{1/3} \left(\frac{H_*}{\beta} \right)^2 \left(\frac{\kappa_{\text{coll}} \alpha}{1 + \alpha} \right)^2 \left(\frac{0.11 v_w^3}{0.42 + v_w^2} \right) S_{\text{coll}}(f), \quad (383)$$

with peak frequency

$$f_{\text{coll}} = 16.5 \times 10^{-6} \text{ Hz} \left(\frac{0.62}{1.8 - 0.1 v_w + v_w^2} \right) \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}. \quad (384)$$

The spectral shape can be approximated by

$$S_{\text{coll}}(f) = \frac{3.8 (f/f_{\text{coll}})^{2.8}}{1 + 2.8 (f/f_{\text{coll}})^{3.8}}. \quad (385)$$

This function is normalized to be one at the peak. It grows approximately as $f^{2.8}$ at low frequencies and decays approximately as f^{-1} at high frequencies. [43, 45, 46]

The physical dependence is transparent. The amplitude grows with

$$\left(\frac{H_*}{\beta}\right)^2 \left(\frac{\kappa_{\text{coll}}\alpha}{1+\alpha}\right)^2. \quad (386)$$

Therefore, bubble collisions produce larger signals when the transition releases a lot of energy and when it lasts long enough for the bubbles to grow. However, in thermal transitions like those relevant to the electroweak scale, the walls normally do not accelerate indefinitely. They interact with the plasma and reach a terminal velocity. In this case, a large part of the energy does not remain in the walls; it is transferred to the fluid. For this reason, in many FOPT scenarios in the plasma, the collision contribution is subdominant when compared to sound waves.

Sound waves

The second mechanism, and usually the most important one in thermal transitions, comes from sound waves in the plasma. The bubble wall does not only separate two phases of the field; it also pushes the primordial fluid, due to the coupling of the particle (Higgs) to the plasma (bosons and fermions). As the bubble grows, energy is transferred to the plasma, creating velocity and pressure profiles. After the bubbles collide, these profiles do not disappear immediately. They continue to propagate as relativistic acoustic waves.

This is a crucial difference relative to bubble collisions. The collision of the walls is a relatively short event, controlled by the duration of the transition itself. Sound waves, on the other hand, can persist for a much longer time, possibly a significant fraction of the Hubble time. Since the accumulated gravitational energy depends on the time during which the source remains active, this mechanism can receive a large gain in amplitude.

The contribution from sound waves is usually written as

$$h^2\Omega_{\text{sw}}(f) = 2.65 \times 10^{-6} \left(\frac{100}{g_*}\right)^{1/3} \left(\frac{H_*}{\beta}\right) \left(\frac{\kappa_{\text{sw}}\alpha}{1+\alpha}\right)^2 v_w \Upsilon_{\text{sw}} S_{\text{sw}}(f), \quad (387)$$

where κ_{sw} is the efficiency of converting the energy released in the transition into collective motion of the plasma. The peak frequency observed today is

$$f_{\text{sw}} = 1.9 \times 10^{-5} \text{ Hz} \frac{1}{v_w} \left(\frac{\beta}{H_*}\right) \left(\frac{T_*}{100 \text{ GeV}}\right) \left(\frac{g_*}{100}\right)^{1/6}. \quad (388)$$

The spectral shape is

$$S_{\text{sw}}(f) = \left(\frac{f}{f_{\text{sw}}} \right)^3 \left[\frac{7}{4 + 3(f/f_{\text{sw}})^2} \right]^{7/2}. \quad (389)$$

This form grows as f^3 at low frequencies and decays as f^{-4} at high frequencies. The power f^3 at low frequencies is the expected signature for a causal source on scales larger than the characteristic scale of the transition.

The factor Υ_{sw} deserves a comment. The first estimates assumed that sound waves lasted approximately a Hubble time. However, in many models, acoustic waves can become nonlinear and form shocks before that. To take this effect into account, one introduces a suppression factor,

$$\Upsilon_{\text{sw}} = \min(1, H_* \tau_{\text{sw}}), \quad (390)$$

with

$$H_* \tau_{\text{sw}} \simeq (8\pi)^{1/3} \frac{v_w}{\beta/H_*} \left[\frac{4(1 + \alpha)}{3\kappa_{\text{sw}}\alpha} \right]^{1/2}. \quad (391)$$

This correction is important because it prevents the acoustic source from being artificially kept active longer than the plasma dynamics allows. [14, 47, 48]

Physically, the sound-wave contribution summarizes very well the bridge between the phase transition and gravitational waves. The effective potential determines the released energy; the released energy accelerates the plasma; the moving plasma has anisotropic stresses; these stresses feed h_{ij}^{TT} ; and the result is a stochastic background of gravitational waves.

Turbulence

The third mechanism is magnetohydrodynamic turbulence. After the collisions and the acoustic phase, the evolution of the fluid can become nonlinear. Sound waves can form shocks, vorticity can be generated in the collision regions and part of the kinetic energy of the plasma can be transferred to turbulent motions. Since the primordial plasma is ionized, magnetic fields can also be amplified, and for this reason the relevant turbulence is, in general, MHD.

The turbulent contribution is more uncertain than the acoustic contribution. This happens because turbulence is difficult to simulate, depends on nonlinear scales and involves the

coupling between the fluid and magnetic fields. Even so, it is common to parametrize this contribution by

$$h^2 \Omega_{\text{turb}}(f) = 3.35 \times 10^{-4} \left(\frac{100}{g_*} \right)^{1/3} \left(\frac{H_*}{\beta} \right) \left(\frac{\kappa_{\text{turb}} \alpha}{1 + \alpha} \right)^{3/2} v_w S_{\text{turb}}(f). \quad (392)$$

The peak frequency is

$$f_{\text{turb}} = 2.7 \times 10^{-5} \text{ Hz} \frac{1}{v_w} \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6}, \quad (393)$$

and the spectral shape is

$$S_{\text{turb}}(f) = \frac{(f/f_{\text{turb}})^3}{[1 + f/f_{\text{turb}}]^{11/3} (1 + 8\pi f/h_*)}, \quad (394)$$

where

$$h_* = 1.65 \times 10^{-5} \text{ Hz} \left(\frac{T_*}{100 \text{ GeV}} \right) \left(\frac{g_*}{100} \right)^{1/6} \quad (395)$$

is the Hubble scale redshifted to today. [14, 43, 49]

The efficiency κ_{turb} represents the fraction of the released energy that becomes turbulence. Unlike κ_{sw} , it is less well known. A common assumption is to write

$$\kappa_{\text{turb}} = \epsilon \kappa_{\text{sw}}, \quad (396)$$

with ϵ of order a few percent, although larger or smaller values are also used to test the theoretical uncertainty. For this reason, turbulence should be understood as an important contribution, but one that is less controlled from the phenomenological point of view. [41, 42]

Dependence on α and β/H_*

With the three mechanisms in hand, we can finally understand why α and β/H_* are the central parameters of the phenomenology.

The parameter α controls the available energy. The larger α is, the larger the fraction of the plasma energy associated with the transition. In the amplitudes, this dependence appears through factors of the type

$$\frac{\kappa_i \alpha}{1 + \alpha}.$$

For $\alpha \ll 1$, this factor grows approximately as $\kappa_i \alpha$. For $\alpha \gg 1$, it tends to saturate, because the released energy already dominates the total energy of the plasma. Thus, a stronger transition tends to produce a larger spectrum, but this tendency does not grow indefinitely.

The parameter β/H_* , on the other hand, controls the duration and characteristic size of the source. Since

$$\beta = - \frac{d}{dt} \left(\frac{S_3(T)}{T} \right) \Big|_{t=t_*}, \quad \frac{\beta}{H_*} \simeq T \frac{d}{dT} \left(\frac{S_3(T)}{T} \right) \Big|_{T=T_*}, \quad (397)$$

we see that β/H_* measures how quickly the nucleation rate changes on a cosmological scale. A transition with large β/H_* happens quickly: many bubbles nucleate in a short time, their mean separations are smaller and the source has a small characteristic scale. A transition with smaller β/H_* lasts longer: the bubbles grow more before colliding, the source has a larger scale and the signal tends to be more intense.

In the amplitudes, this appears as

$$\Omega_{\text{coll}} \propto \left(\frac{H_*}{\beta} \right)^2, \quad \Omega_{\text{sw}} \propto \left(\frac{H_*}{\beta} \right), \quad \Omega_{\text{turb}} \propto \left(\frac{H_*}{\beta} \right), \quad (398)$$

disregarding, for now, the efficiency and suppression factors. The peak frequencies, in turn, obey, schematically,

$$f_{\text{peak}} \propto \left(\frac{\beta}{H_*} \right) T_*. \quad (399)$$

Therefore, decreasing β/H_* increases the amplitude and shifts the peak to lower frequencies. This is one of the reasons why slower transitions are especially interesting for gravitational waves.

This observation also explains why the separation between T_c and T_n will be important in the results chapter. If nucleation occurs far below the critical temperature, that is, if

$$T_n \ll T_c,$$

the transition underwent supercooling. During this interval, the energy difference between the two phases tends to grow, increasing the released energy and, therefore, favoring larger values of α . At the same time, later nucleation is often associated with a slower transition, that is, with smaller values of β/H_* . Thus, in a qualitative way, the larger the separation between T_c and T_n ,

the larger we expect the intensity of the gravitational-wave spectrum to be.

This relation, however, should be understood carefully. It is not a universal law independent of the model. In very supercooled transitions, for example, the usual assumptions of radiation dominance, efficient percolation and standard redshift may cease to be valid. Even so, for the class of thermal transitions that we will study in this thesis, the intuition is extremely useful: larger signals tend to come from strong and relatively slow transitions.

We can summarize the content of this section in a single physical chain:

$$V_{\text{eff}}(\phi, T) \implies T_*, \alpha, \frac{\beta}{H_*}, v_w \implies \Pi_{ij}^{\text{TT}} \implies h_{ij}^{\text{TT}} \implies \Omega_{\text{GW}}(f). \quad (400)$$

This is the final bridge between the microphysics of the effective potential and the cosmological observable. In the next section, we will use this structure to discuss how the predicted spectrum can be compared with sensitivity curves of future detectors.

4.4 Phenomenological templates and detector sensitivity

Before moving on to the results, it is useful to make clear what exactly we will compare the obtained spectra with. So far, we have built the physical path that leads from a first-order phase transition to the spectrum

$$h^2 \Omega_{\text{GW}}(f).$$

However, to discuss detectability, we need to compare this spectrum with the expected sensitivity of future detectors. In this thesis, we use the peak-integrated sensitivity curves for phase-transition signals presented in Ref. [7]. In particular, we use the semi-analytic expressions for LISA, DECIGO and BBO, implemented in the numerical code associated with this dissertation.

The idea of these curves is simple. Given a detector, its sensitivity is not equally good at all frequencies. There is a range in which the instrumental noise is lower and, therefore, in which a stochastic gravitational-wave background would be more easily observed. Thus, when we plot $h^2 \Omega_{\text{GW}}(f)$ together with the detector sensitivity curve, we can directly visualize whether the predicted signal has sufficient amplitude in the appropriate frequency range.

Following the convention of Ref. [7], we define

$$x \equiv \frac{f}{1 \text{ mHz}}. \quad (401)$$

That is, when f is written in mHz, we simply have $x = f$ numerically. The curves used in this

thesis correspond to the s channel, associated with the spectral shape of sound waves, which is precisely the dominant contribution expected in many scenarios of thermal transitions in the plasma.

For LISA, we use

$$h^2\Omega_{\text{PIS}}^{\text{LISA}}(f) = 10^{-14} \left[3.58 \times 10^{-3} x^{-4} + 3.26 \times 10^{-1} x^{-3} + 1.20 x^{-2} + 2.48 x^{-1} + 2.85 \times 10^{-1} x + 1.81 \times 10^{-2} x^2 + 1.50 \times 10^{-3} x^3 \right]. \quad (402)$$

For DECIGO, we use

$$h^2\Omega_{\text{PIS}}^{\text{DECIGO}}(f) = 10^{-14} \left[3.82 \times 10^{-1} x^{-4} + 2.26 x^{-3/2} + 1.10 \times 10^{-3} + 2.56 \times 10^{-6} x + 2.91 \times 10^{-8} x^2 + 7.54 \times 10^{-12} x^3 \right]. \quad (403)$$

And, finally, for BBO, we use

$$h^2\Omega_{\text{PIS}}^{\text{BBO}}(f) = 10^{-14} \left[1.77 \times 10^{-1} x^{-4} + 1.06 x^{-3/2} + 1.35 \times 10^{-4} + 2.23 \times 10^{-6} x + 1.29 \times 10^{-9} x^2 + 2.99 \times 10^{-12} x^3 \right]. \quad (404)$$

These expressions are experimental comparison curves; they indicate, approximately, what amplitude of $h^2\Omega_{\text{GW}}$ would be necessary for a signal with spectral shape compatible with a phase transition to be detectable by each interferometer [7–10].

We use LISA, DECIGO and BBO because the frequency range expected for phase transitions close to the electroweak scale naturally falls between mHz and fractions of Hz. This can be seen directly from the peak frequencies presented in the previous section, which scale as

$$f_{\text{peak}} \propto \left(\frac{\beta}{H_*} \right) \left(\frac{T_*}{100 \text{ GeV}} \right). \quad (405)$$

For $T_* \sim 100 \text{ GeV}$ and typical values of β/H_* , the peak of the spectrum usually appears precisely in the region explored by space interferometers. Therefore, these detectors are the natural candidates to test first-order electroweak phase transitions.

Minimizing the curves above, we obtain a simple estimate of the best sensitivity of each experiment in the channel considered. For LISA, the minimum sensitivity occurs around

$$f \simeq 2.8 \text{ mHz}, \quad h^2\Omega_{\text{PIS}}^{\text{LISA}} \sim 2 \times 10^{-14}. \quad (406)$$

For DECIGO, the most sensitive region is around

$$f \simeq 0.15 \text{ Hz}, \quad h^2 \Omega_{\text{PIS}}^{\text{DECIGO}} \sim 3 \times 10^{-17}, \quad (407)$$

while for BBO we find approximately

$$f \simeq 0.19 \text{ Hz}, \quad h^2 \Omega_{\text{PIS}}^{\text{BBO}} \sim 10^{-17}. \quad (408)$$

These numbers should be read only as reference scales, because real detectability depends on the complete shape of the spectrum, the observation time, the statistical treatment and the combination between signal and noise. Even so, they help interpret the results. Signals with a peak close to the region of best sensitivity and amplitude above these curves are the most promising.

In the plots of the next chapter, we will then plot the total spectrum

$$h^2 \Omega_{\text{GW}}(f) = h^2 \Omega_{\text{coll}}(f) + h^2 \Omega_{\text{sw}}(f) + h^2 \Omega_{\text{turb}}(f), \quad (409)$$

together with the sensitivity curves of LISA, DECIGO and BBO [14, 50] . In this way, we will be able to evaluate which regions of parameter space produce a first-order phase transition strong enough to leave a potentially observable signature.

With this, we close the theoretical construction of the thesis. We started with an effective potential $V_{\text{eff}}(\phi, T)$, used this potential to extract the thermal history of the transition, reduced this dynamics to the thermal parameters and, finally, transformed these parameters into a gravitational-wave spectrum comparable with future detectors. The next chapter will be devoted precisely to the numerical application of this pipeline to the potentials studied in this dissertation.

5 Chapter Five

Results and conclusions

After all the theoretical construction carried out in the previous chapters, we finally arrive at the point where we study and verify *what happens* when this construction is applied to concrete potentials.

The goal of this chapter is to present the numerical results obtained from the code developed throughout this project, the github.com/IgorStellet/GW_FOPT_Simulation.

We will divide the analysis of the results into three fronts. First, we study the sector associated with the contribution of the functional measure, motivated by the logarithmic potential discussed in Ref. [1]. Then, we study the polynomial deformation of the scalar sector by higher-dimensional irrelevant operators, in particular terms of the type ϕ^6 and ϕ^8 , following the effective structure discussed in Ref. [2]. Finally, we investigate the combined effect of these two deformations.

The goal is to understand the physics behind the calculated spectrum and what these curves are trying to tell us. When a barrier appears, when the transition nucleates, how the strength of the transition grows, how the characteristic time changes and, finally, when the gravitational signal enters the sensitivity region of detectors such as LISA, DECIGO and BBO.

We also investigate the impact of daisy thermal resummation. This comparison is important because the infrared bosonic modes, discussed in Chapter 2, can modify the geometry of the finite-temperature potential. As a consequence, the inclusion or removal of this contribution can alter the minima, the width of the barrier, the bounce action, the thermodynamic parameters and, at the end of the chain, the gravitational-wave spectrum itself. Thermal resummation, therefore, changes the structure of the finite-temperature potential and, consequently, in the end, alters the observed spectrum. The goal is to verify the effect of this change and whether it is relevant or not for the study of the modified potentials. [30, 31, 33, 34]

5.1 Numerical pipeline

Before presenting the specific results, it is worth describing how the code organizes the physical information. Starting from a given effective potential, the numerical pipeline identifies the phase structure, finds the critical temperature, calculates the nucleation temperature, solves

the bounce profile and, finally, builds the parameters that feed the gravitational-wave spectrum. [18, 38]

In practical terms, for each point in parameter space, the code can generate a sequence of figures that tells the complete history of the transition. The first important piece of information is the shape of the potential at zero temperature. This figure shows whether the potential correctly recovers the observed electroweak vacuum and allows us to visualize the effect of the one-loop corrections and of the new deformations.

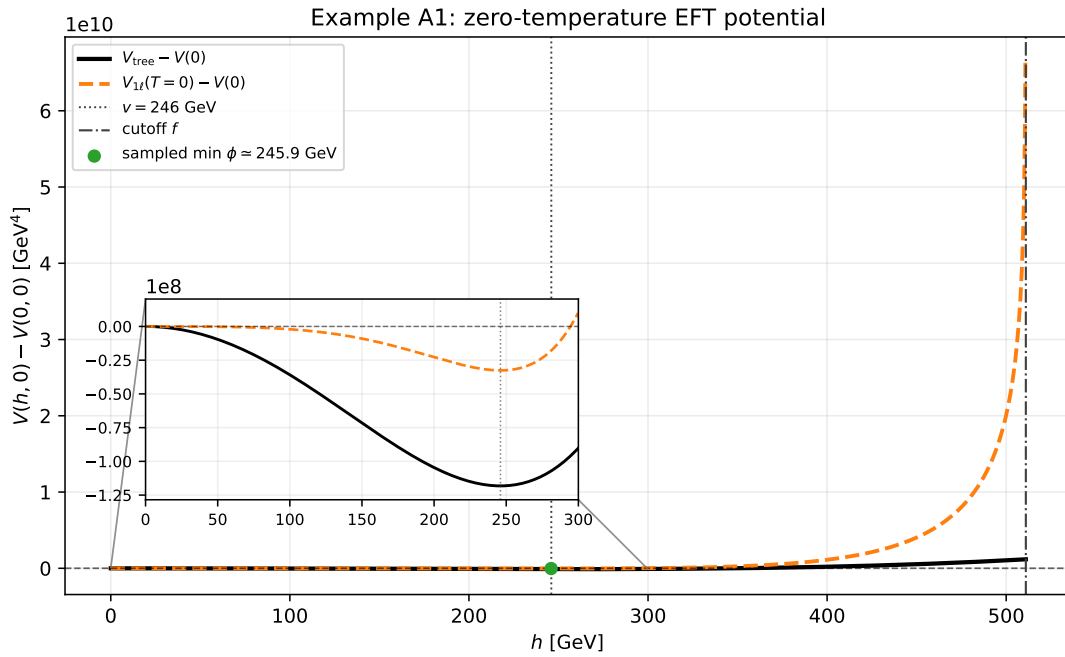


Figure 17: Geometry of $V(\phi, T = 0)$. The black curve is the tree-level potential V_{tree} , while the dashed orange curve is the potential with the 1-Loop correction. The dashed black vertical line represents the location of the electroweak minimum $\phi = v$ while the horizontal one represents $V(\phi) = 0$

Next, the code follows the potential at different temperatures. This figure is one of the most important to build physical intuition: it shows how the minima appear, disappear, become degenerate or become separated by a barrier. The code finds the relevant temperatures, that is, the nucleation (T_n), critical (T_c), and minima appearance and disappearance temperatures ($T_s^{\text{true}}, T_s^{\text{false}}$).

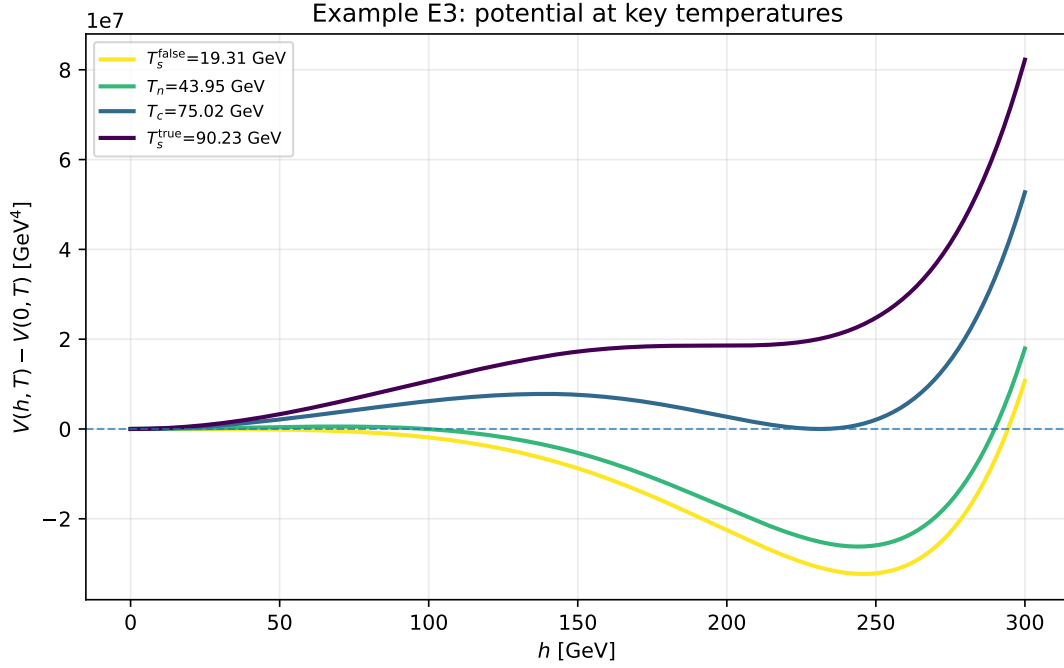


Figure 18: Example of the thermal evolution of the finite-temperature effective potential. As the temperature decreases, the geometry of $V_{\text{eff}}(\phi, T)$ can develop minima competing with each other and a barrier between the phases. The graph shows $T_n, T_c, T_s^{\text{true}}, T_s^{\text{false}}$.

From the identification of the nucleation temperature, the next relevant figure is the potential at T_n . This is the geometry effectively used in the bounce problem. The width of the barrier, the horizontal separation between the minima and the vertical difference between the vacua already give a first intuition of the strength and duration of the transition. A more supercooled transition, that is, with a larger relative separation between T_c and T_n , tends to release more energy, due to the larger difference between the minima, and, therefore, to produce larger values of α . The larger horizontal spacing between the minima, as well as the vertical size of the barrier, is also directly associated with the decay rate and, therefore, with the duration of the transition, measured by β/H . [18, 51]

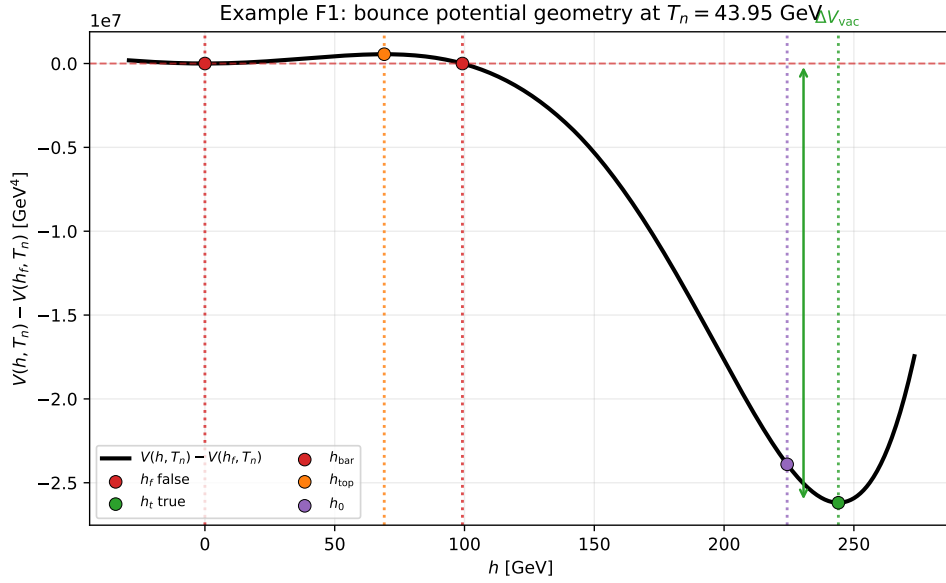


Figure 19: Geometry of the potential at the nucleation temperature. Here we have a clear image of the barrier that the vacuum must cross to generate nucleation

It is also useful to visualize the inverted potential. As discussed in Chapter 3, the bounce solution can be interpreted as the motion of a fictitious particle in the inverted potential, with radial friction. This figure makes the semiclassical intuition more concrete. Here we want to roll from a point near the true vacuum, starting from rest, found by the bounce solution at $\phi_b(r = 0) = \phi_0$, with $\frac{d\phi_b}{dr}|_{r=0} = 0$ and stop at the false vacuum also at rest, that is, $\phi_b(r \gg 1) = \phi_{false}$, with $\dot{\phi}_b(r \gg 1) = 0$. The figure below shows this intuition, whose bounce solution is in purple and the black arrows indicate the direction of increasing r . [35, 38, 40]

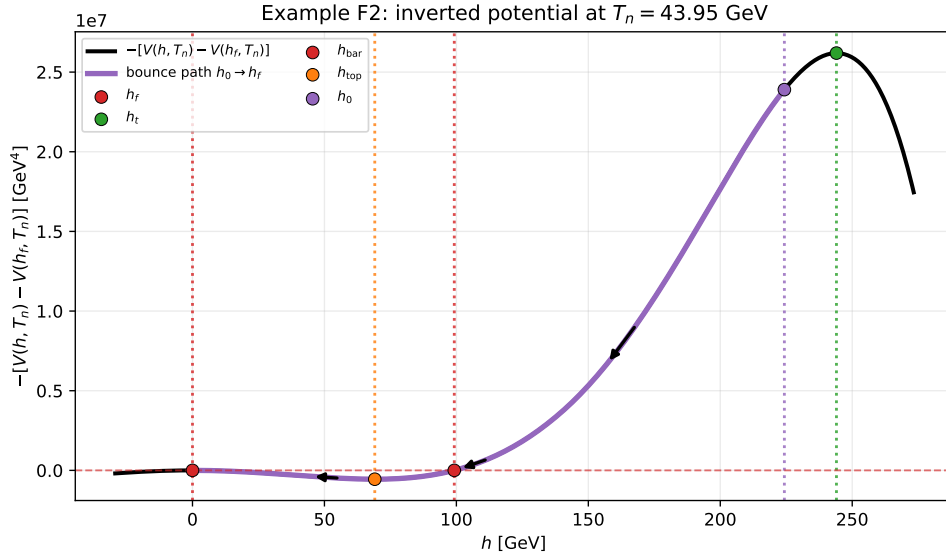


Figure 20: Inverted potential used to interpret the bounce solution. The effective trajectory connects the region near the true vacuum to the false vacuum, represented by the purple curve, the bounce solution $\phi_b(r)$. The black arrows indicate the region of radius growth.

Once the bounce solution is found, we obtain the radial profile $\phi_b(r)$ of the critical bubble. This profile describes how the field varies from the interior of the bubble, close to the true phase, to the exterior, which remains in the false phase.

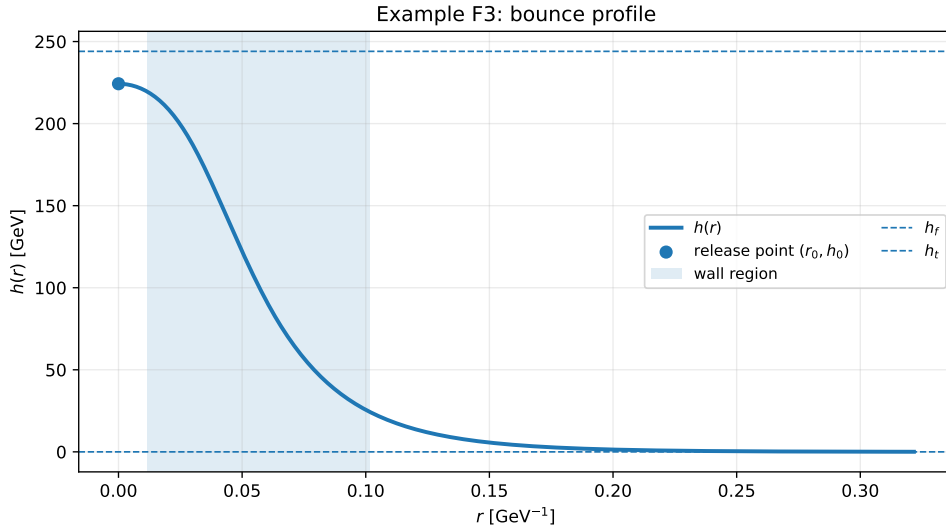


Figure 21: Radial profile of the bounce solution at the nucleation temperature. The interior of the bubble is close to the true phase, while the exterior remains in the false vacuum.

The same solution can be represented as a spatial section of the bubble. This visualization is less necessary for the quantitative analysis, but it is useful to remember that the bounce is not just a mathematical curve: it represents the critical configuration from which the true phase begins to occupy real space in the Universe.

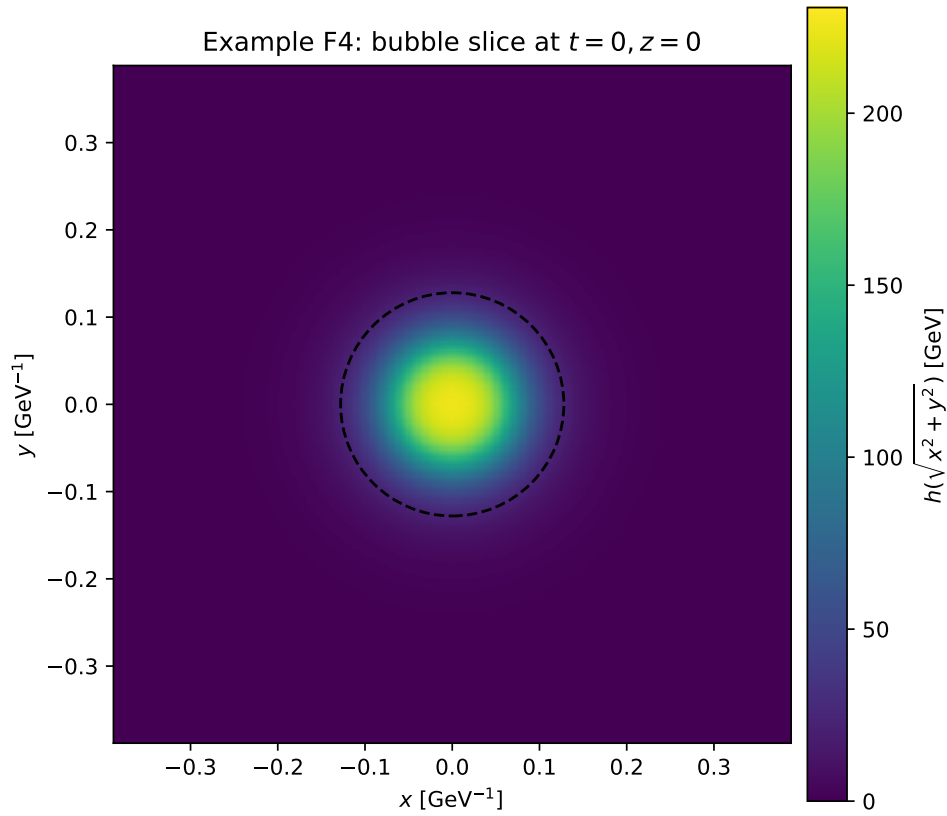


Figure 22: Spatial slice of the critical bubble reconstructed from the bounce profile. The color represents the local value of the scalar field.

With $\phi_b(r)$ and T_n , the code calculates the relevant thermodynamic parameters, in particular α and β/H_* . From them, we build the gravitational-wave spectrum and compare it with the sensitivity curves of the detectors. [7, 14, 50]

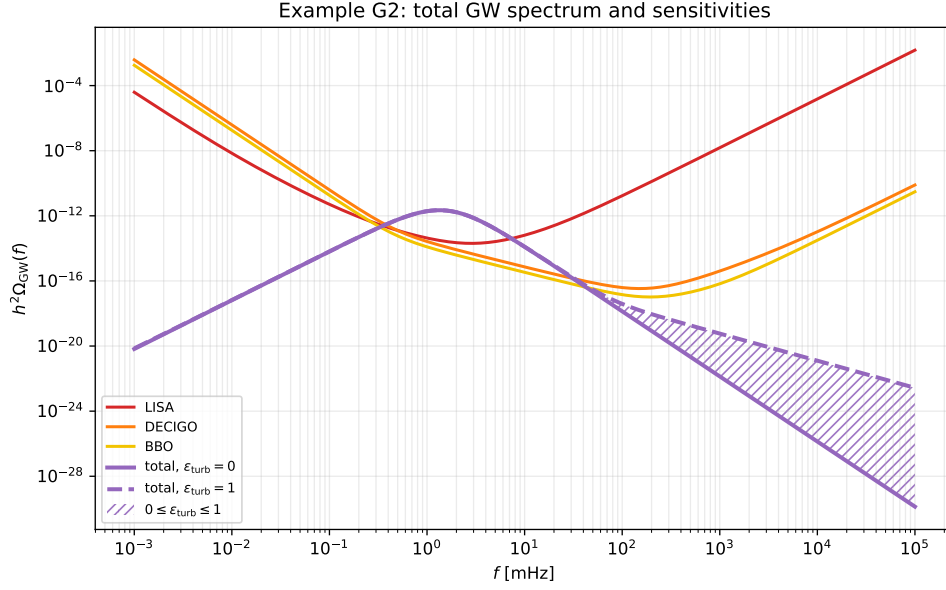


Figure 23: Example of a gravitational-wave spectrum produced by a first-order phase transition. The spectrum is compared with sensitivity curves of space interferometers. The dotted curve represents the case in which we treat turbulence with $\epsilon = 1$, while the solid curve, $\epsilon = 0$.

The case demonstrated above corresponds to the Glauber potential with configuration $C = 3.83$ and $\Lambda = 1000 \text{ GeV}$. The code can reproduce this sequence of figures for any implemented potential or, alternatively, save only the numerical diagnostics in `.csv` and `.json` tables with all the parameters found. Throughout the chapter, we will use only the subset of figures that best translates the physics we want to discuss. Besides the figures shown, figures such as the evolution of the masses and of the minima along temperature, the evolution of the action and many others can also be plotted by the code. It serves as a large didactic study, for those who are going deeper into the study of first-order phase transitions and the corresponding generation of gravitational waves.

In larger scans, in which many points of parameter space are evaluated, we do not always save the complete spectrum for each point. In these cases, the post-processing reconstructs smooth curves from the saved thermal parameters and anchors the result in the peak diagnostics, that is, in f_{peak} and Ω_{peak} , extracted during the main execution. This choice makes the process lighter without losing the essential information to compare detectability and peak movement in the plane $(f_{\text{peak}}, \Omega_{\text{peak}})$.

5.2 Numerical setup

In all cases, we adopt the same numerical pipeline. First, we build the finite-temperature effective potential,

$$V_{\text{eff}}(\phi, T) = V_0(\phi) + \Delta V_{\text{CW}}(\phi) + \Delta V_T(\phi, T) + V_{\text{daisy}}(\phi, T) + \Delta V_{\text{new}}(\phi), \quad (410)$$

where V_0 is the tree-level potential, ΔV_{CW} is the Coleman-Weinberg correction, ΔV_T contains the one-loop thermal corrections, V_{daisy} is included or removed depending on the analyzed case, and ΔV_{new} represents the scalar deformation studied in each subsection. [1, 2, 25, 29, 30]

In all scenarios, we fix the zero-temperature parameters so that the potential reproduces the observed electroweak vacuum,

$$v = 246 \text{ GeV}, \quad m_h = 125 \text{ GeV}. \quad (411)$$

This condition is important: we want to modify the thermal history of the Universe without destroying what we already know at low energies.

To find the nucleation temperature T_n , we search for solutions in the interval

$$1 \text{ GeV} \leq T \leq 250 \text{ GeV}. \quad (412)$$

For each candidate temperature, the code identifies the relevant minima, calculates the bounce solution and evaluates the three-dimensional Euclidean action $S_3(T)$. The nucleation temperature is then defined by the condition [18, 19, 40]

$$\frac{S_3(T_n)}{T_n} \simeq 140. \quad (413)$$

Numerically, we adopt the tolerance

$$\left| \frac{S_3(T_n)}{T_n} - 140 \right| < 10^{-2} \quad (414)$$

and the search procedure uses up to 160 iterations to locate the temperature that satisfies

this criterion. After finding T_n , the bounce is recalculated at this temperature, and the thermodynamic parameters relevant for gravitational waves are obtained.

In this analysis, we assume

$$T_* = T_n, \quad v_w = 1. \quad (415)$$

We also use [32]

$$g_* = 106.75, \quad (416)$$

corresponding to the effective number of relativistic degrees of freedom of the electroweak plasma in the Standard Model.

Following the discussion of Chapter 4, the main contribution to the spectrum comes from sound waves in the plasma [13, 14, 52, 53]. Turbulence is treated as a phenomenological uncertainty, parametrized by

$$\kappa_{\text{turb}} = \epsilon \kappa_{\text{sw}}, \quad 0 \leq \epsilon \leq 1, \quad (417)$$

being represented in the graphs as an auxiliary region or curve. This choice reflects the fact that the turbulent contribution is physically important, especially on high-frequency scales, but less controlled from the theoretical point of view [14, 49, 53].

The direct contribution from bubble collisions is neglected, since we estimate that it contributes very little to the spectrum [14, 41, 42], we are being conservative in this analysis, so that:

$$\Omega_{\text{coll}}(f) = 0, \quad (418)$$

and the total spectrum used in the main analysis becomes

$$h^2\Omega_{\text{GW}}(f) = h^2\Omega_{\text{sw}}(f) + h^2\Omega_{\text{turb}}(f). \quad (419)$$

Finally, we compare the obtained spectrum with the sensitivity curves of LISA, DECIGO and BBO already presented in Chapter 4.

Because β/H_* is obtained from a temperature derivative of the bounce action, [14, 18, 19]

$$\frac{\beta}{H_*} = T \frac{d}{dT} \left(\frac{S_3(\phi_b(T))}{T} \right)_{T=T_*}, \quad (420)$$

it is more sensitive to numerical noise than α [33, 34]. Small numerical fluctuations in the bounce action can be amplified by the derivative, especially in parameter scans where many nearby points are compared.

For this reason, in the scan plots we display a smoothed estimate of β/H_* obtained by fitting $\log(\beta/H_*)$ as a low-degree polynomial in the scan parameter. This is used only to visualize the global trend. The raw numerical values are still used as diagnostics of each individual point, while the fitted curve is meant to guide the physical interpretation of the scan.

5.3 Tree-level constraints and preservation of the electroweak vacuum

Before entering the specific results, it is useful to make clear how the tree-level parameters are fixed. For the polynomial deformation, we write

$$V_0(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{8}\frac{c_6}{f^2}\phi^6 + \frac{1}{16}\frac{c_8}{f^4}\phi^8. \quad (421)$$

Defining

$$k_6 \equiv \frac{c_6}{f^2}, \quad k_8 \equiv \frac{c_8}{f^4}, \quad (422)$$

the zero-temperature conditions are [2, 32]

$$\left. \frac{\partial V_0}{\partial \phi} \right|_{\phi=v} = 0, \quad \left. \frac{\partial^2 V_0}{\partial \phi^2} \right|_{\phi=v} = m_h^2. \quad (423)$$

From these conditions, we obtain

$$\lambda = \frac{m_h^2}{2v^2} - \frac{3}{2}k_6v^2 - \frac{3}{2}k_8v^4, \quad (424)$$

and

$$\mu^2 = \frac{m_h^2}{2} - \frac{3}{4}k_6v^4 - k_8v^6. \quad (425)$$

These expressions guarantee that the deformation by higher-dimensional operators does not shift the electroweak minimum nor alter the physical Higgs mass at $T = 0$. The code calculates these coefficients for each point of parameter space and checks whether the value of λ remains in an acceptable perturbative regime.

In the case of the functional-measure contribution, we adopt the renormalized form so that the additional terms also do not alter the local conditions at the electroweak vacuum. Thus, the modification of the measure changes the global geometry of the potential, but preserves the point of arrival at low energies.

5.4 Functional-measure contribution: Glauber sector

We begin with the functional-measure sector. The motivation comes from Ref. [1], where it is shown that a nontrivial choice of the measure in field space can be rewritten as a logarithmic contribution to the effective scalar potential. In terms of the radial Higgs field, we write this contribution as [1]

$$\Delta V_{\text{measure}}(\phi) = -\frac{\Lambda^4}{8\pi^2} \log \left(1 - \frac{C\phi^2}{\Lambda^2} \right) + V_{\text{ren}}^{\text{measure}}(\phi). \quad (426)$$

Here, C controls the intensity of the deformation and Λ represents the scale associated with the effective cutoff of the theory. The term $V_{\text{ren}}^{\text{measure}}$ contains the counterterms necessary

to preserve, at zero temperature, the minimum at $\phi = v$ and the physical curvature m_h^2 .

In the numerical implementation, it is convenient to write the contribution already in the subtracted form. Defining

$$t(\phi) = \frac{C\phi^2}{\Lambda^2}, \quad t_0 = \frac{Cv^2}{\Lambda^2}, \quad (427)$$

we use

$$\Delta V_{\text{measure}}(\phi) = -\frac{\Lambda^4}{8\pi^2} \left[\log(1 - t(\phi)) + \frac{(1 - 2t_0)t(\phi) + \frac{1}{2}t^2(\phi)}{(1 - t_0)^2} \right]. \quad (428)$$

This form makes the role of the modification clearer. It changes the global geometry of the potential without changing the basic observables already measured in the electroweak vacuum.

In the limit

$$\left| \frac{C\phi^2}{\Lambda^2} \right| \ll 1, \quad (429)$$

the logarithm can be expanded as [1]

$$-\log\left(1 - \frac{C\phi^2}{\Lambda^2}\right) = C\frac{\phi^2}{\Lambda^2} + \frac{C^2}{2}\frac{\phi^4}{\Lambda^4} + \frac{C^3}{3}\frac{\phi^6}{\Lambda^6} + \frac{C^4}{4}\frac{\phi^8}{\Lambda^8} + \dots. \quad (430)$$

This expansion is conceptually interesting, because it shows that the logarithmic correction effectively contains a tower of even operators in the scalar field. This anticipates a natural connection with the polynomial deformation studied in the next section.

Since there is already in the literature a study of the relevant values of C and Λ , our additional goal in reproducing this sector was to verify how the inclusion or removal of daisy thermal resummation affects the results found. In other words, we use this potential as the first laboratory to test the code. It is already capable of generating first-order transitions, and we want to understand how much the reorganized thermal physics changes the final prediction for gravitational waves.

In the scans presented below, we fix

$$\Lambda = 1000 \text{ GeV} \quad \Lambda = 2000 \text{ GeV} \quad (431)$$

and vary C in the intervals

$$C \in [3.64, 3.85] \quad \text{for} \quad \Lambda = 1000 \text{ GeV} \quad (432)$$

$$C \in [6.23, 6.65] \quad \text{for} \quad \Lambda = 2000 \text{ GeV} \quad (433)$$

The first comparison shows the spectra obtained for three representative values of C , chosen to go from the least detectable case to the most detectable case.

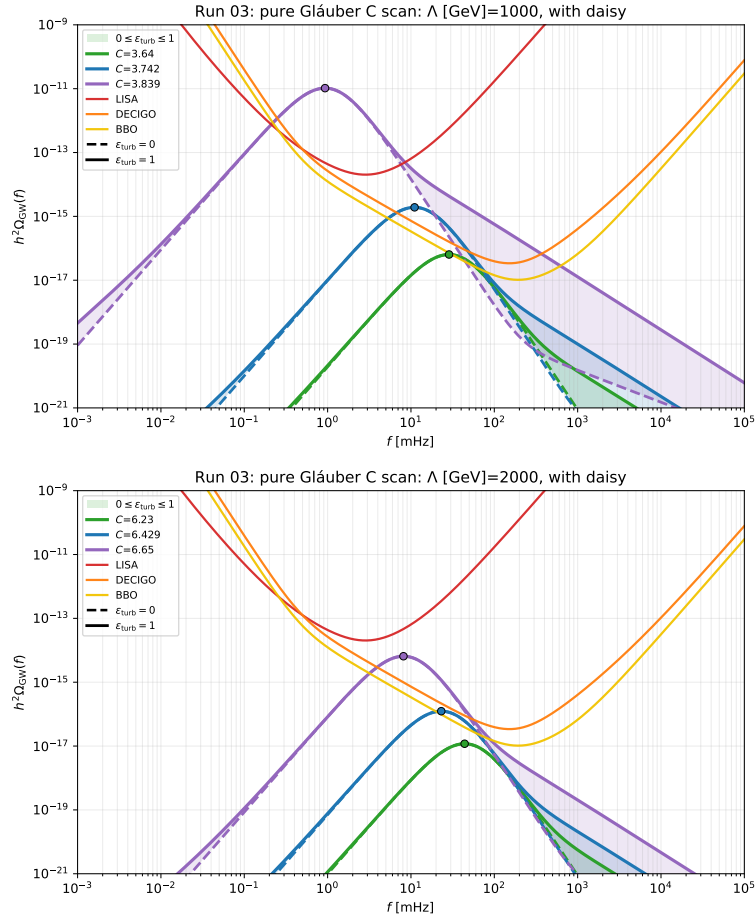


Figure 24: Gravitational-wave spectra in the functional-measure sector with daisy resummation. The upper panel shows $\Lambda = 1000$ GeV with $C = 3.64, 3.742$ and 3.839 , while the lower panel shows $\Lambda = 2000$ GeV with $C = 6.23, 6.429$ and 6.65 . The colors go from the least to the most detectable spectrum: green, blue and purple. Solid curves include maximal turbulence, $\epsilon_{\text{turb}} = 1$, and dotted curves show the case without turbulence, $\epsilon_{\text{turb}} = 0$; the markers indicate the spectral peaks. The detector curves are LISA in red, DECIGO in orange and BBO in yellow. Increasing C strengthens the transition and moves the signal toward the region probed by future space-based detectors.

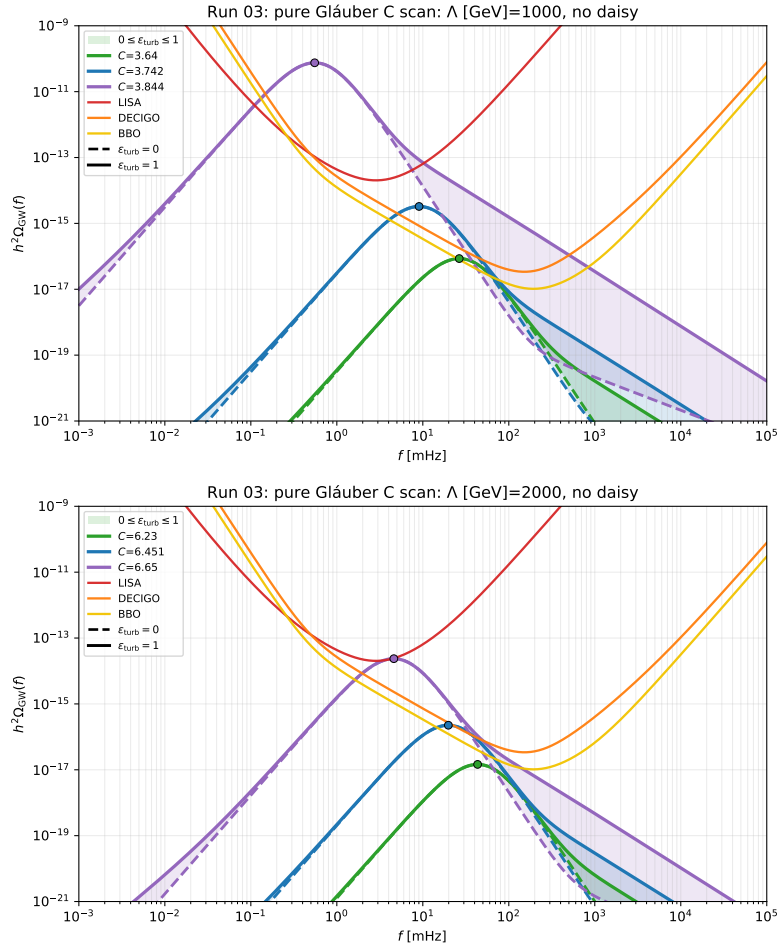


Figure 25: Gravitational-wave spectra in the functional-measure sector without daisy resummation. The upper panel shows $\Lambda = 1000$ GeV with $C = 3.64, 3.742$ and 3.844 , while the lower panel shows $\Lambda = 2000$ GeV with $C = 6.23, 6.451$ and 6.65 . The spectra are ordered from least to most detectable using green, blue and purple. Solid curves include maximal turbulence, $\epsilon_{\text{turb}} = 1$, whereas dotted curves neglect turbulence, $\epsilon_{\text{turb}} = 0$; the markers indicate the peaks. LISA, DECIGO and BBO sensitivities are shown in red, orange and yellow. Comparison with the daisy-resummed case shows how thermal resummation changes the peak amplitude and frequency.

The direct comparison between with and without daisy is shown in Fig. 26. The hatched region represents the difference between the spectra. This figure is useful because it visually isolates the effect of thermal resummation without mixing many values of C in the same graph, we observe the effect for a specific C .

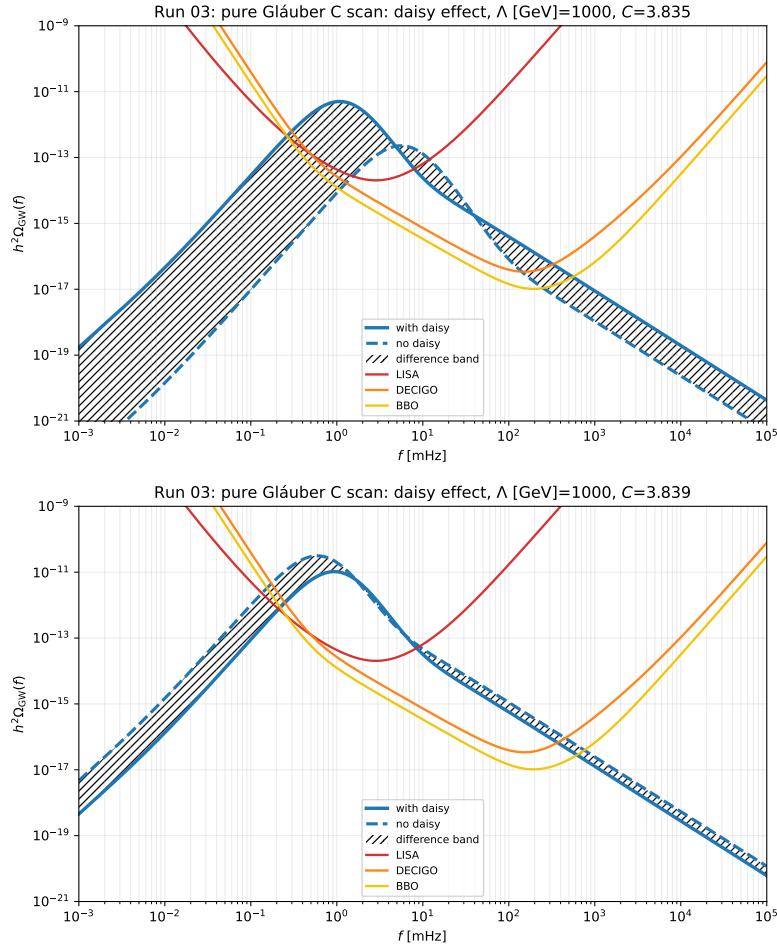


Figure 26: Direct comparison between spectra with and without daisy resummation in the functional-measure sector. Both curves include maximal turbulence, $\epsilon_{\text{turb}} = 1$. The solid curve includes daisy resummation, while the dotted curve does not. The hatched band marks the difference between the two spectra. The detector curves are LISA in red, DECIGO in orange and BBO in yellow. The figure isolates how thermal resummation changes the final gravitational-wave prediction.

Here we see directly how considering thermal resummation affects the potential and, therefore, the final spectrum. Taking it into account, we observe that it can both shift the peak upward or downward and, as we will see in the next results, also alter the range in which we find bubble nucleation from the adopted criterion. Since resummation affects the structure of the finite-temperature potential, we alter the bounce profile found $\phi_b(r)$ and the T_n found, consequently, we modify the thermodynamic parameters obtained from these quantities and, obviously, the observed spectrum. This effect is larger, in the case of the modification of the integration measure, for larger values of C 's, for values of C 's for which the spectrum is weakly detectable and the transition is weak we also observe little contribution from this change coming from thermal resummation.

It is also interesting to observe the parameters that control the transition. In particular, we

analyze α , β/H_* and the supercooling parameter,

$$\frac{T_c - T_n}{T_c}. \quad (434)$$

These quantities characterize the physics before the final spectrum.

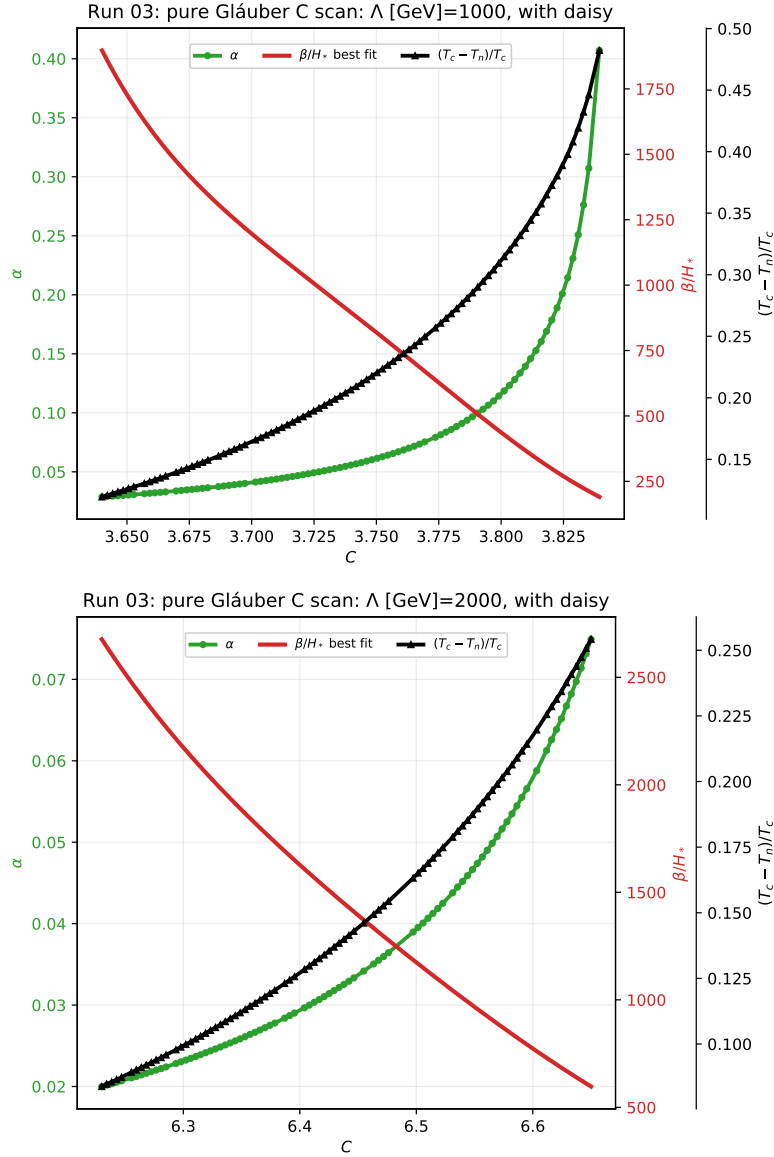


Figure 27: Thermodynamic parameters in the functional-measure sector with daisy resummation, varying C . The upper panel corresponds to $\Lambda = 1000$ GeV and the lower panel to $\Lambda = 2000$ GeV. The green curve shows α , the red curve shows the smoothed best-fit estimate of β/H , and the black curve shows the supercooling measure $(T_c - T_n)/T_c$. Different vertical scales are used because the three quantities have different numerical ranges. The main trend is that α grows together with the supercooling, while the fitted β/H curve tends to decrease in the stronger-transition region.

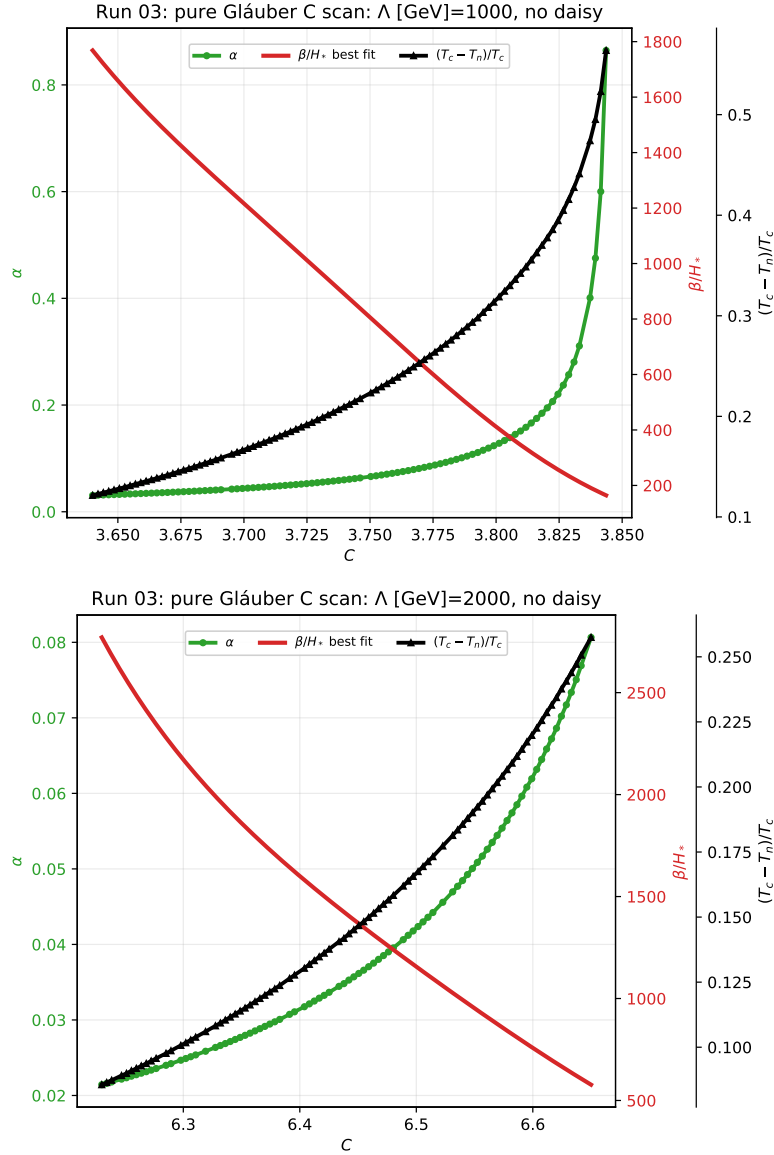


Figure 28: Thermodynamic parameters in the functional-measure sector without daisy resummation, varying C . The upper panel corresponds to $\Lambda = 1000$ GeV and the lower panel to $\Lambda = 2000$ GeV. The green curve shows α , the red curve shows the smoothed best-fit estimate of β/H , and the black curve shows $(T_c - T_n)/T_c$. Different vertical scales are used to display all quantities in the same plot. As in the daisy case, increasing C produces larger α and stronger supercooling, while the fitted β/H trend decreases in the strongest-transition region.

We can also see the relative difference between these observables, in the figure below.

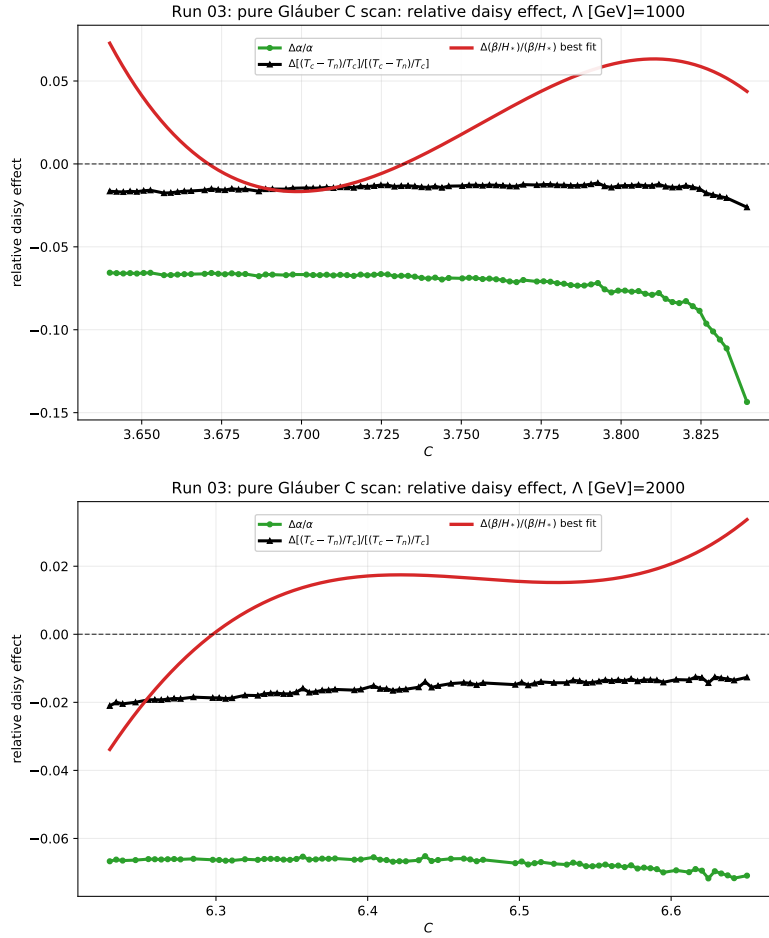


Figure 29: Relative effect of daisy resummation on the thermodynamic parameters of the functional-measure sector. For each observable X , the plotted quantity is $(X_{\text{daisy}} - X_{\text{no,daisy}})/X_{\text{no,daisy}}$, i.e. normalized by the result without daisy. The upper panel corresponds to $\Lambda = 1000$ GeV and the lower panel to $\Lambda = 2000$ GeV. The green, red and black curves show the relative changes in α , the smoothed best-fit β/H , and $(T_c - T_n)/T_c$, respectively. Daisy resummation tends to reduce α and the supercooling, while its effect on β/H is more non-monotonic.

Figs. 27 and 28 show an important physical relation: both in the case with daisy and in the case without daisy, the supercooling parameter and the strength of the transition, α , tend to follow the same growth curve. This has a direct interpretation. The larger the separation between T_c and T_n , the longer the Universe remains trapped in the false phase after the true phase has already become energetically favored, with the difference between the energy valleys becoming increasingly larger. With this, the larger the energy difference released in the transition tends to be, and, therefore, the larger the strength measured by α .

We also observe that, while α and $(T_c - T_n)/T_c$ grow, β/H_* tends to decrease. This relation is not so immediate at first sight, but it can be understood as follows: a more supercooled transition is generally associated with a more persistent potential barrier, that is, one that exists over a larger range of temperature, and with an action that varies more slowly with temperature near T_n . Since β/H_* essentially measures how fast the nucleation rate grows, a slower and more

prolonged transition tends to present smaller values of β/H_* , which is the inverse temporal scale – More long-lasting transitions tend to have smaller β/H_* . In physical terms: the transition becomes stronger, but also delayed.

Fig. 29 shows the relative difference between the results with and without daisy resummation, normalized by the result without daisy. In other words, for each observable X , we plot

$$\frac{X_{\text{daisy}} - X_{\text{no,daisy}}}{X_{\text{no,daisy}}}. \quad (435)$$

Consistently, the transition strength α is larger in the case without daisy resummation. This indicates that removing the thermal resummation leads to stronger transitions. On the other hand, the behavior of β/H_* is less monotonic, it increases in some regions and decreases in others.

This happens because, when we include thermal resummation we are changing the finite-temperature structure of the potential itself, and therefore the shape of the barrier that separates the minima at each temperature. Since the bounce action is determined by this barrier, the quantities extracted from it are naturally sensitive to this modification of the potential.

The decrease in α when daisy resummation is included can be understood from this point of view. Daisy resummation introduces thermal masses for the bosonic modes and reorganizes the infrared part of the finite-temperature potential. In practice this reduces the relative separation between T_c and T_n , as can also be seen consistently in the supercooling curve. Since a smaller supercooling generally means that the energy released in the transition is smaller, and this directly reduces α .

The parameter β/H_* is more delicate. Since it depends on a temperature derivative of the bounce action, its variation is not as simple to interpret as the variation of α . Small changes in the shape of the barrier can modify the slope of $S_3(T)/T$ near T_n , and therefore can either increase or decrease β/H_* depending on the region of the scan. This more non-monotonic behavior is precisely why the comparison shown in Fig. 26 is important: introducing daisy resummation can either enhance or suppress the final gravitational-wave spectrum.

This may be the difference between detecting or not detecting a signal of new physics. For this reason, the result reinforces the importance of understanding thermal resummation more precisely in effective-potential descriptions of cosmological phase transitions.

We can also follow the movement of the spectrum peak in the plane $(f_{\text{peak}}, \Omega_{\text{peak}})$. This graph summarizes very compactly the physical effect of the scan: as C varies, the transition moves in the space of gravitational observables.

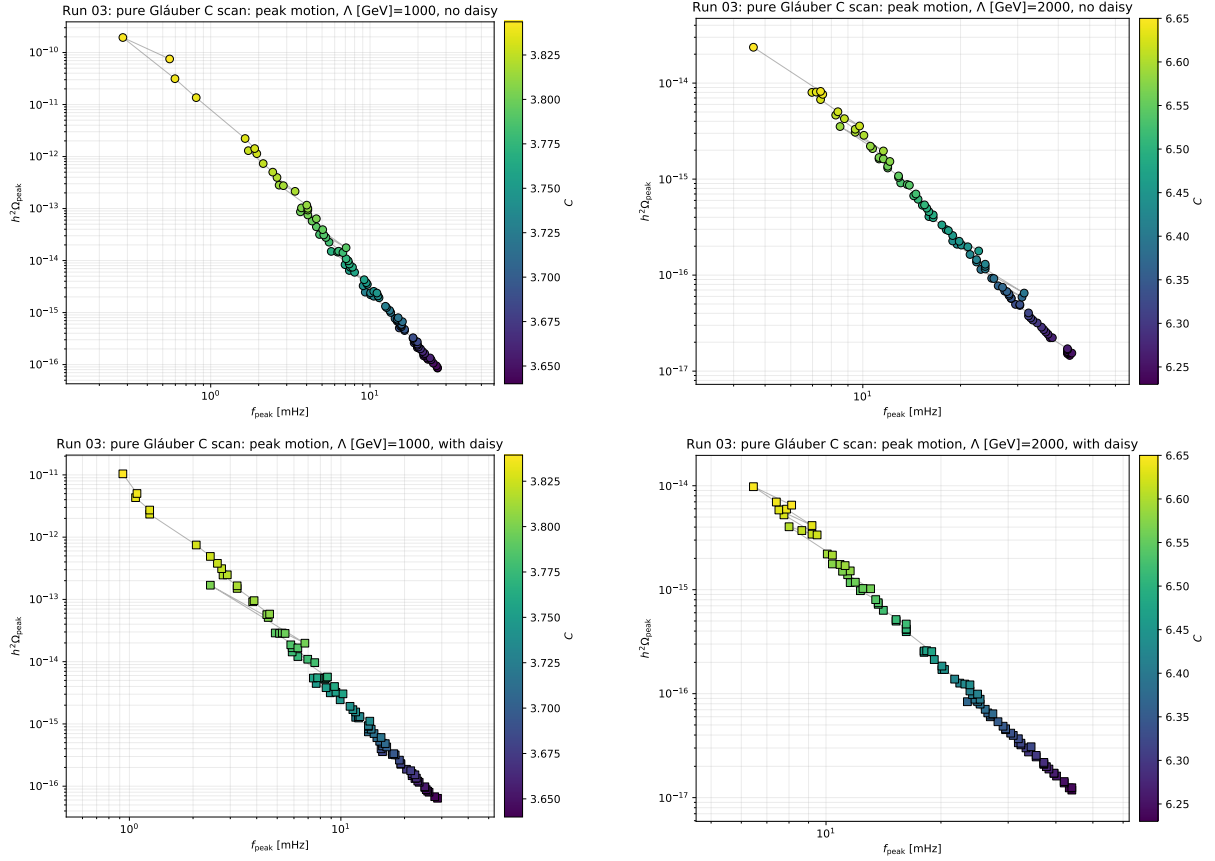


Figure 30: Movement of the gravitational-wave peak in the functional-measure sector. Each point gives the peak position in the plane $(f_{\text{peak}}, h^2\Omega_{\text{peak}})$, and the color bar indicates the scanned value of C . The upper panels show $\Lambda = 1000$ GeV and the lower panels show $\Lambda = 2000$ GeV, comparing the cases without and with daisy resummation. The scans cover approximately $C \in [3.64, 3.85]$ for $\Lambda = 1000$ GeV and $C \in [6.23, 6.65]$ for $\Lambda = 2000$ GeV. The general behavior is that increasing C moves the peak toward lower frequencies and larger amplitudes.

In the image above, we see that the effect of increasing C for the functional-measure modification is to increase the peak and shift it to lower frequencies, both for daisy and for no daisy. Below we can also see the graph showing the relative difference between the peaks and the peak frequencies when considering or not thermal resummation.

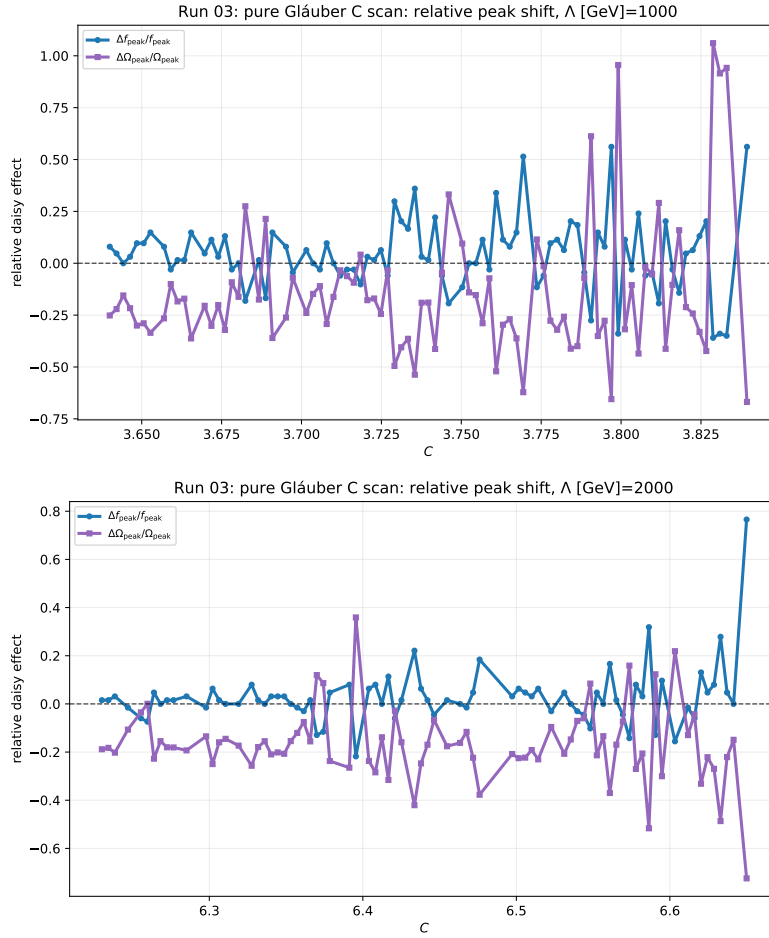


Figure 31: Relative effect of daisy resummation on the gravitational-wave peak in the functional-measure sector. For each peak observable Y , the plotted quantity is $(Y_{\text{daisy}} - Y_{\text{no,daisy}})/Y_{\text{no,daisy}}$, normalized by the result without daisy. The upper panel corresponds to $\Lambda = 1000$ GeV and the lower panel to $\Lambda = 2000$ GeV. The curves show the relative changes in f_{peak} and $h^2\Omega_{\text{peak}}$. Thermal resummation can shift both the frequency and the amplitude of the predicted signal.

With this, we conclude the first case study. The functional-measure sector shows how a logarithmic modification of the potential can generate first-order transitions and potentially detectable spectra. More than that, it shows that the thermal treatment of the potential can quantitatively alter the final prediction. The next step is to repeat the same logic for a polynomial deformation of the scalar sector.

5.5 Polynomial scalar deformation: $(\phi^6 + \phi^8)$ sector

Now we study the polynomial deformation of the scalar potential. The motivation comes from the idea that, if the Higgs sector is only the low-energy part of a more complete theory, higher-dimensional irrelevant operators can modify the geometry of the potential in field and temperature regions relevant to the primordial Universe.

Following Ref. [2], we consider an effective theory with operators

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{c_6}{f^2}(H^\dagger H)^3 + \frac{c_8}{f^4}(H^\dagger H)^4. \quad (436)$$

In this effective description, f represents the characteristic scale at which new physics responsible for the higher-dimensional operators becomes relevant. The coefficients c_6 and c_8 are dimensionless Wilson coefficients, while the combinations c_6/f^2 and c_8/f^4 control the actual size of the deformation in the scalar potential. Thus, smaller values of f , or larger values of the Wilson coefficients, correspond to stronger deviations from the Standard Model Higgs potential.

In this thesis we fix $f = 1000$ GeV and scan the effective combinations c_6/f^2 and c_8/f^4 , following the phenomenological setup of Ref. [2].

In the radial Higgs direction,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \quad (437)$$

this structure leads to the tree-level potential

$$V_0(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{8}\frac{c_6}{f^2}\phi^6 + \frac{1}{16}\frac{c_8}{f^4}\phi^8. \quad (438)$$

The physical role of these terms is direct: they modify the global geometry of the potential. Depending on the values of c_6/f^2 and c_8/f^4 , additional minima, barriers and metastable regions may arise, capable of making the electroweak transition first order.

In the results below, we fix

$$f = 1000 \text{ GeV}, \quad (439)$$

and study fixed values of c_8/f^4 , varying c_6/f^2 in selected intervals. In particular, we use

$$\frac{c_8}{f^4} = [0, 2, 5] \text{ TeV}^{-4}, \quad (440)$$

with

$$\frac{c_6}{f^2} \in [3.2, 3.6] \text{ TeV}^{-2} \quad \text{for} \quad \frac{c_8}{f^4} = 0 \text{ TeV}^{-4}. \quad (441)$$

$$\frac{c_6}{f^2} \in [3.0, 3.4] \text{ TeV}^{-2} \quad \text{for} \quad \frac{c_8}{f^4} = 2 \text{ TeV}^{-4}. \quad (442)$$

$$\frac{c_6}{f^2} \in [2.8, 3.2] \text{ TeV}^{-2} \quad \text{for} \quad \frac{c_8}{f^4} = 5 \text{ TeV}^{-4}. \quad (443)$$

As in the functional-measure sector, we compare the results with and without daisy resummation.

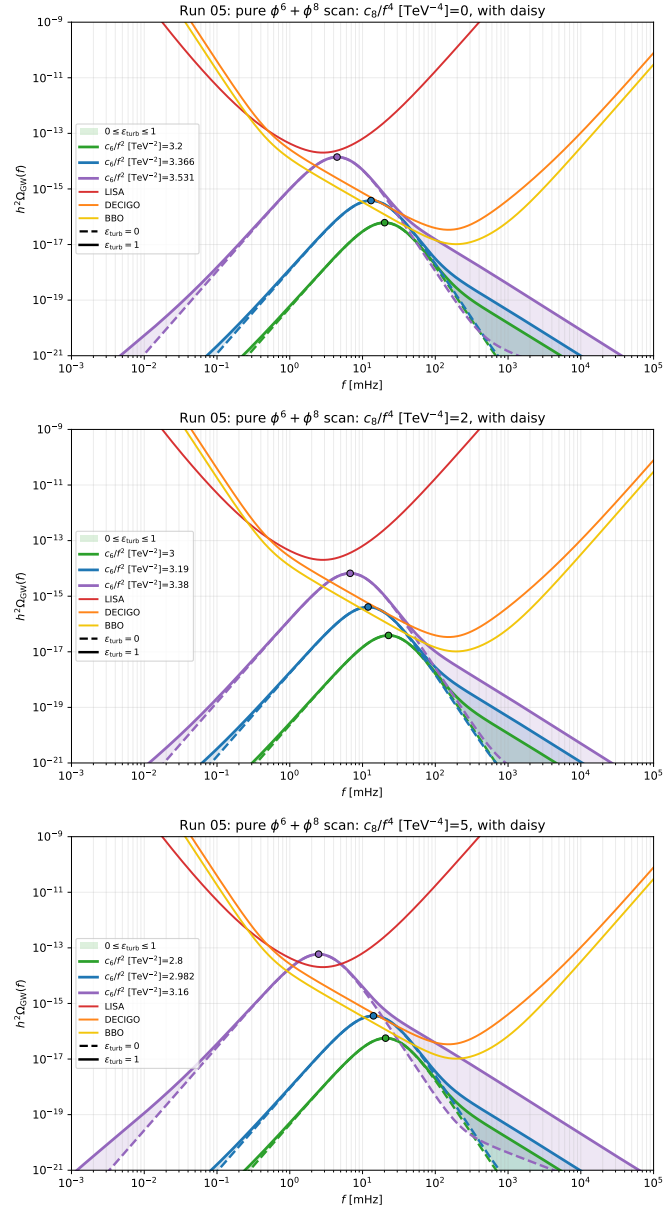


Figure 32: Gravitational-wave spectra in the $\phi^6 + \phi^8$ sector with daisy resummation. Each panel fixes c_8/f^4 and varies c_6/f^2 : from top to bottom, $c_8/f^4 = 0, 2$ and 5 TeV $^{-4}$. The representative c_6/f^2 values are $(3.2, 3.366, 3.531)$, $(3.0, 3.19, 3.38)$ and $(2.8, 2.982, 3.16)$ TeV $^{-2}$, respectively. Green, blue and purple go from least to most detectable. Solid curves include maximal turbulence, while dotted curves neglect turbulence; markers indicate the peaks. LISA, DECIGO and BBO are shown in red, orange and yellow.

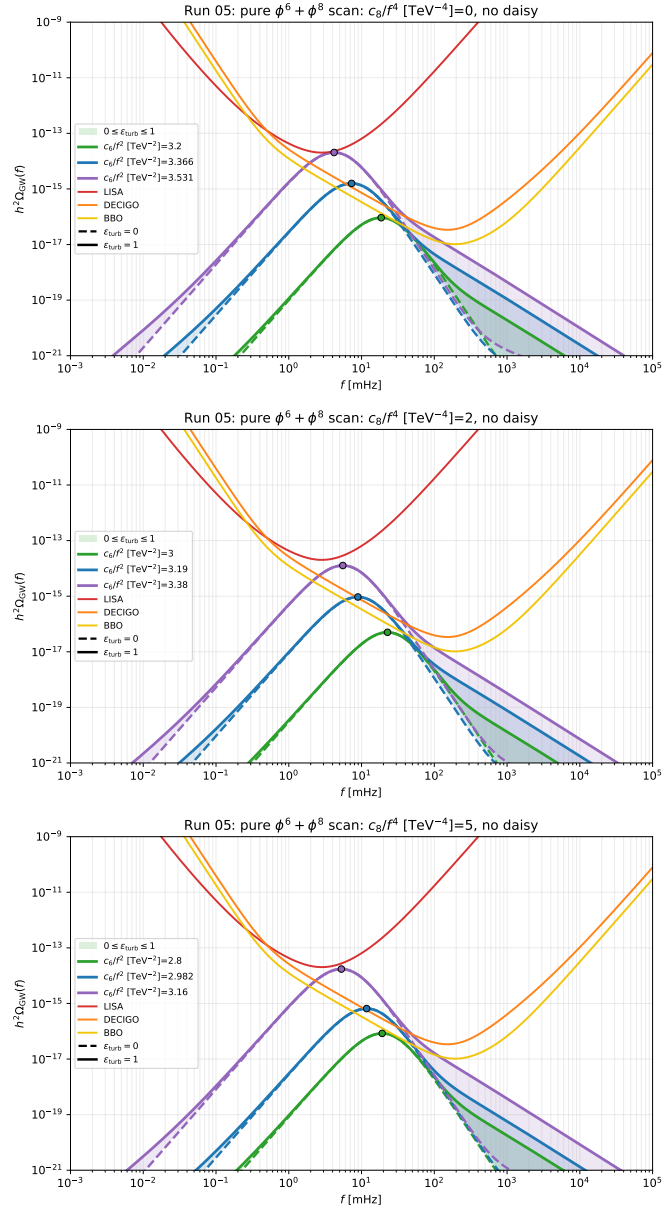


Figure 33: Gravitational-wave spectra in the $\phi^6 + \phi^8$ sector without daisy resummation. Each panel fixes $c_8/f^4 = 0, 2$ or 5 TeV^{-4} and varies c_6/f^2 . The plotted c_6/f^2 values are $(3.2, 3.366, 3.531)$, $(3.0, 3.19, 3.38)$ and $(2.8, 2.982, 3.16) \text{ TeV}^{-2}$ from top to bottom. The colors green, blue and purple order the spectra from least to most detectable. Solid curves include maximal turbulence and dotted curves omit it; the markers show the peaks. The red, orange and yellow curves correspond to LISA, DECIGO and BBO.

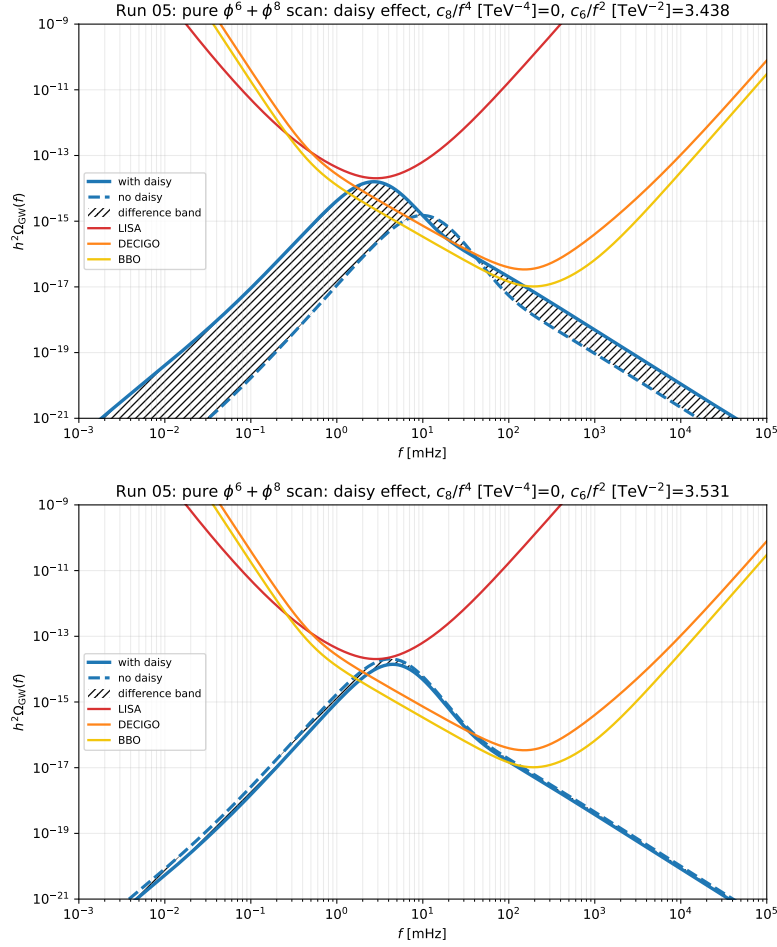


Figure 34: Direct comparison between spectra with and without daisy resummation in the $\phi^6 + \phi^8$ sector. Both curves include maximal turbulence, $\epsilon_{\text{turb}} = 1$. The solid curve includes daisy resummation, while the dotted curve does not. The hatched region indicates the difference between the two spectra. The detector curves are LISA in red, DECIGO in orange and BBO in yellow. The comparison shows that thermal resummation modifies both the amplitude and the position of the peak.

Thus, we see that the physical effect of this potential is to increase the strength of the transition as we increase c_6/f^2 , as well as c_8/f^4 , as already discussed in the reference article. Considering thermal resummation, we observe that, as in the functional-measure sector, the difference occurs mainly for values of parameter space for which the transition is stronger; the stronger the transition, the more sensitive it is to the change in the potential structure introduced by thermal resummation.

An important point must be highlighted. Since we are working at one loop, the inclusion of the irrelevant operators modifies the tree-level potential and the field-dependent masses, but the structure of the thermal resummation used remains organized by the infrared bosonic modes and their thermal masses. In other words, the resummation reorganizes the thermal perturbative expansion. Therefore, when comparing with and without daisy, we are comparing two thermal

approximations for the same low-energy effective deformation.

We also analyze the thermodynamic parameters that control the transition.

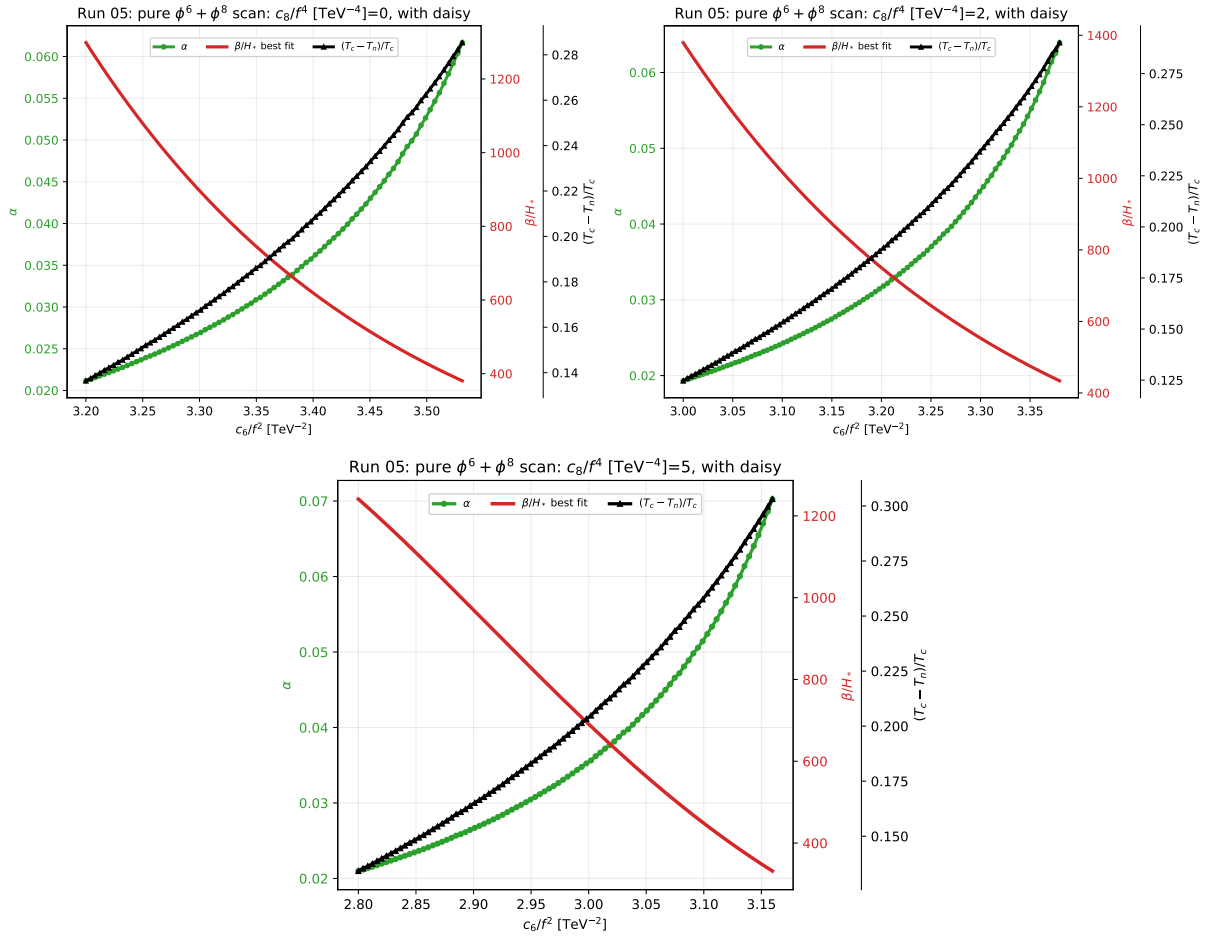


Figure 35: Thermodynamic parameters in the $\phi^6 + \phi^8$ sector with daisy resummation. The scan is performed in c_6/f^2 for fixed $c_8/f^4 = 0, 2$ and 5 TeV⁻⁴, shown from top to bottom. The green curve shows α , the red curve shows the smoothed best-fit estimate of β/H , and the black curve shows $(T_c - T_n)/T_c$. Different vertical scales are used for the three quantities. The main trend is that stronger supercooling is correlated with larger α , while the fitted β/H curve controls the transition time scale.

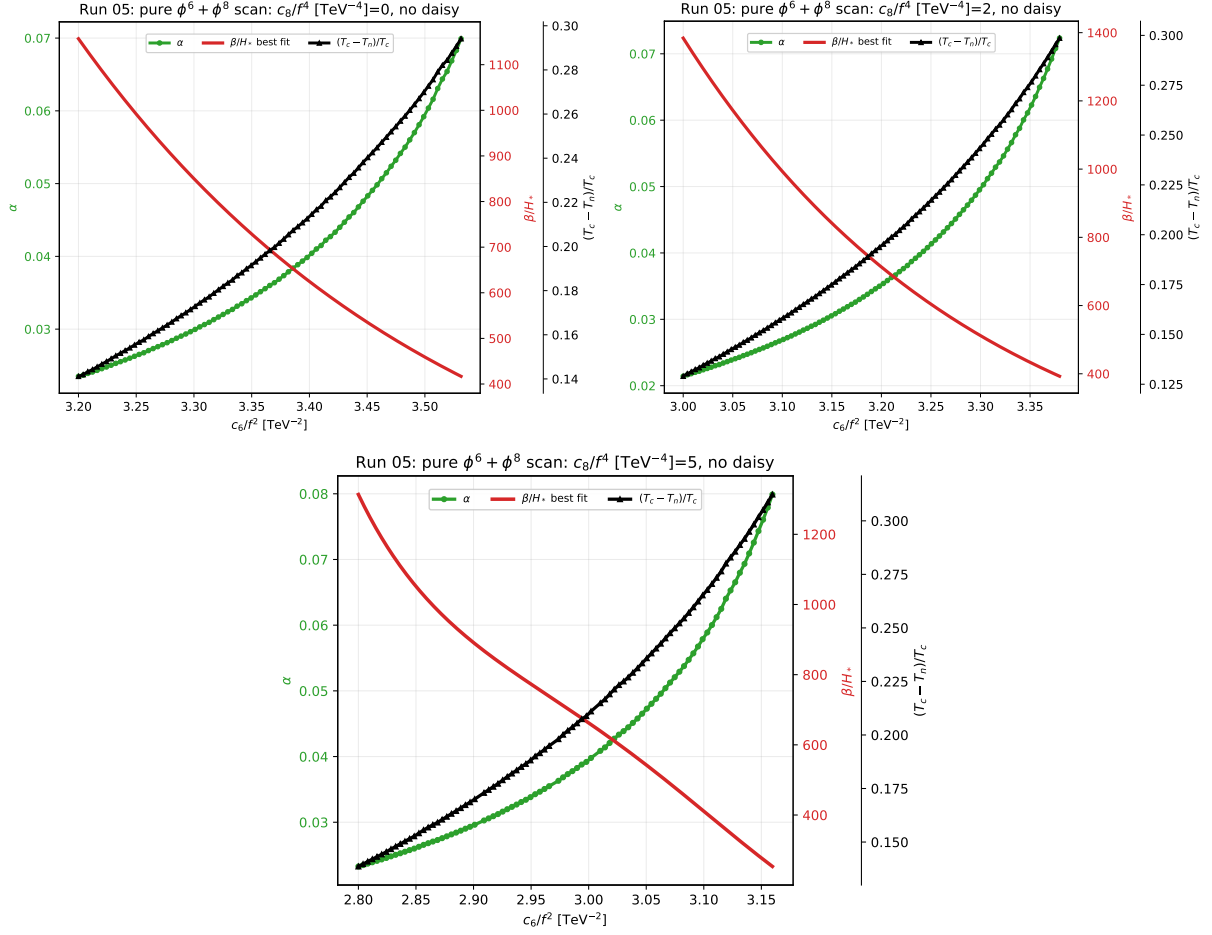


Figure 36: Thermodynamic parameters in the $\phi^6 + \phi^8$ sector without daisy resummation. The scan varies c_6/f^2 for fixed $c_8/f^4 = 0, 2$ and 5 TeV⁻⁴, from top to bottom. The green curve represents α , the red curve the smoothed best-fit estimate of β/H_* , and the black curve the supercooling measure $(T_c - T_n)/T_c$. The different vertical scales allow the three trends to be shown together. Increasing the polynomial deformation generally increases both α and the amount of supercooling.

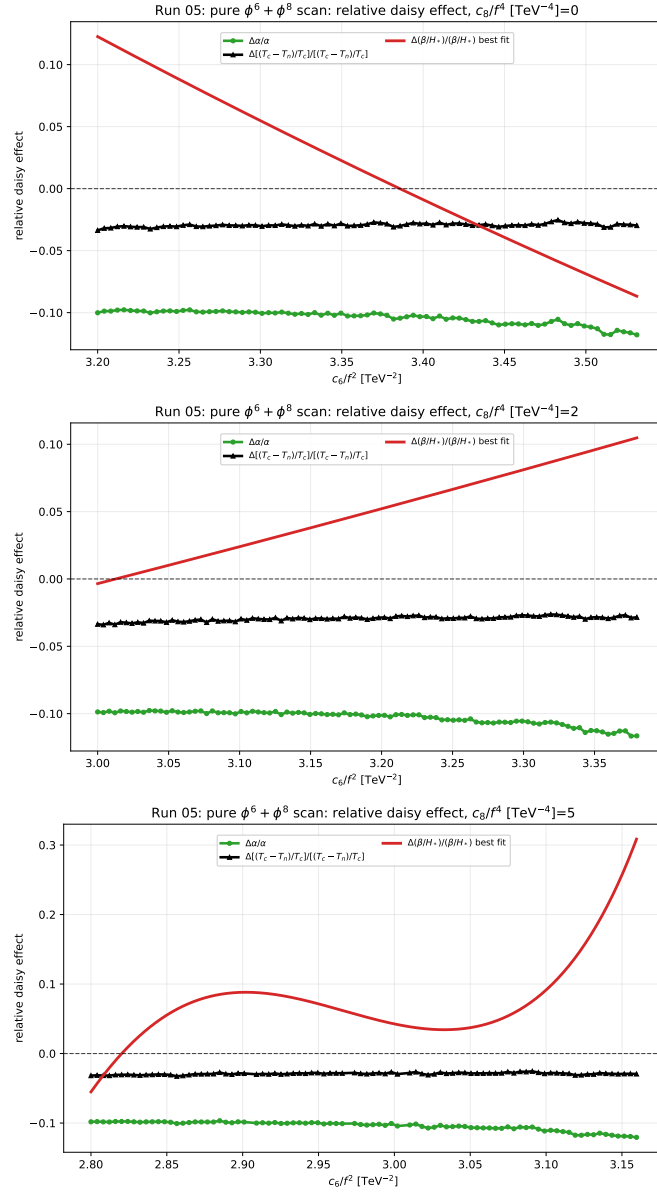


Figure 37: Relative effect of daisy resummation on the thermodynamic parameters of the $\phi^6 + \phi^8$ sector. For each observable X , the plotted quantity is $(X_{\text{daisy}} - X_{\text{no,daisy}})/X_{\text{no,daisy}}$. The three panels correspond to $c_8/f^4 = 0, 2$ and 5 TeV^{-4} , with c_6/f^2 on the horizontal axis. The green, red and black curves show the relative changes in α , the smoothed best-fit β/H_* , and $(T_c - T_n)/T_c$. Daisy resummation tends to reduce α , while the response of β/H_* is more sensitive to the local structure of the potential.

The same physical trend observed in the functional-measure sector reappears here: when the transition becomes more supercooled, α tends to grow, while β/H_* tends to decrease. This shows that the correlation between strength, supercooling and duration of the transition is not an accident of a specific potential. It reflects the very dynamics of a first-order transition: the longer the Universe remains trapped in the false vacuum, the larger the accumulated energy to be released, but the slower the transition tends to be.

Of course, the details depend on the potential. The $\phi^6 + \phi^8$ sector directly modifies

the polynomial part of the potential, while the functional measure introduces a logarithmic structure. Even so, both deformations share the same physics: creating or strengthening the barrier between the phases.

As in the functional-measure sector, we observe a consistent pattern: including daisy resummation tends to reduce the transition strength α , while the behavior of β/H_* is less predictable. Depending on the region of parameter space, the change in β/H_* can move the spectrum toward more detectable or less detectable regions.

The important point is that these relative changes are not universal. They depend both on the region of parameter space and on the specific structure of the new physics that deforms the scalar potential. In the $\phi^6 + \phi^8$ sector, as in the Glauber case, the common trend is that α decreases when daisy resummation is included. The parameter β/H_* , however, can increase, decrease, or even show a more oscillatory behavior from point to point in the scan.

This again reinforces the importance of understanding thermal resummation in effective-potential calculations. If the inclusion of daisy terms can already change the predicted gravitational-wave spectrum in a nontrivial way, it is natural to ask whether other higher-order thermal corrections or alternative resummation schemes could also modify, sometimes drastically, the signal that would be observed.

We also follow the movement of the spectrum peak in the plane $(f_{\text{peak}}, \Omega_{\text{peak}})$.

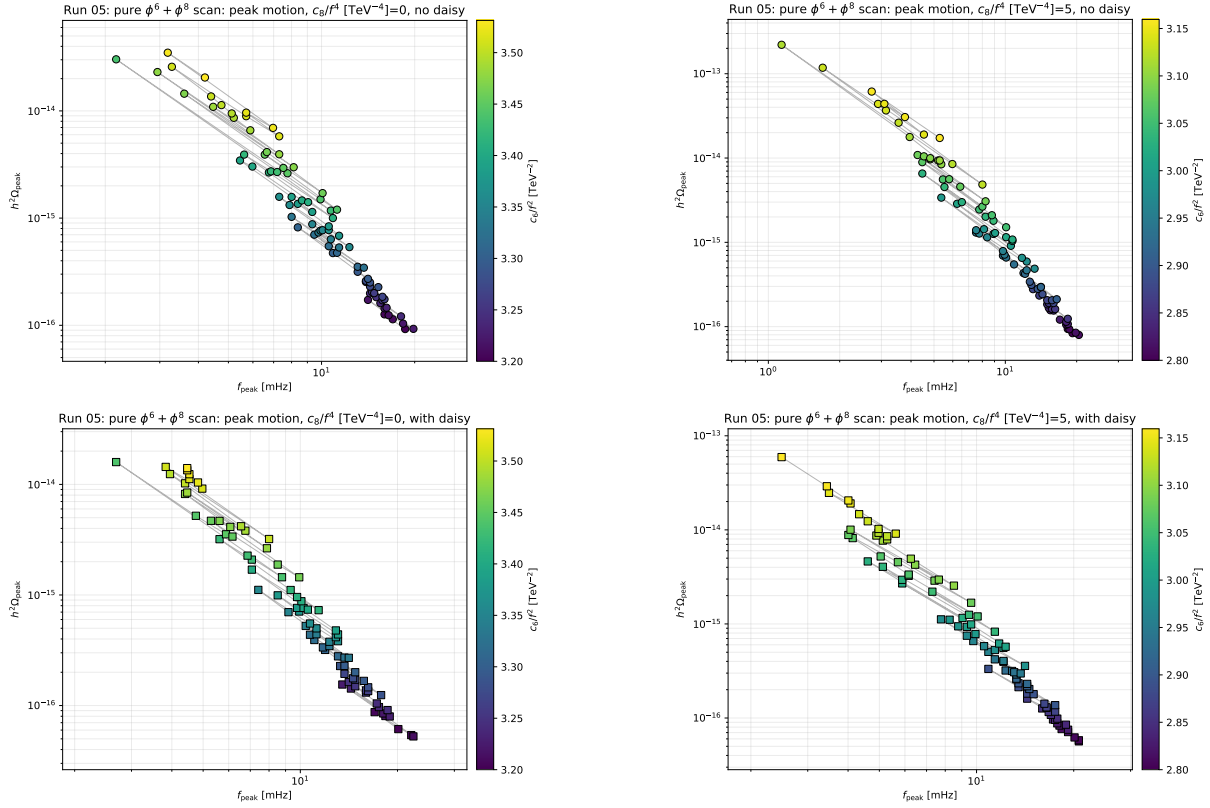


Figure 38: Movement of the gravitational-wave peak in the $\phi^6 + \phi^8$ sector. Each point gives the peak position in the plane $(f_{\text{peak}}, h^2\Omega_{\text{peak}})$, and the color bar indicates the scanned value of c_6/f^2 . The upper panels correspond to $c_8/f^4 = 0$, while the lower panels correspond to $c_8/f^4 = 5 \text{ TeV}^{-4}$, comparing the cases without and with daisy resummation. The scans cover approximately $c_6/f^2 \in [3.2, 3.531] \text{ TeV}^{-2}$ and $[2.8, 3.16] \text{ TeV}^{-2}$, respectively. Increasing the deformation shifts the peak toward lower frequencies and larger amplitudes.

As before, we note that increasing the coefficients of ϕ^6 , ϕ^8 , that is, c_6/f^2 and c_8/f^4 , produces spectra whose peak shifts to lower frequencies and becomes increasingly higher, therefore making the transition stronger.

We can also compare the relative difference between these results, shown in the graph below

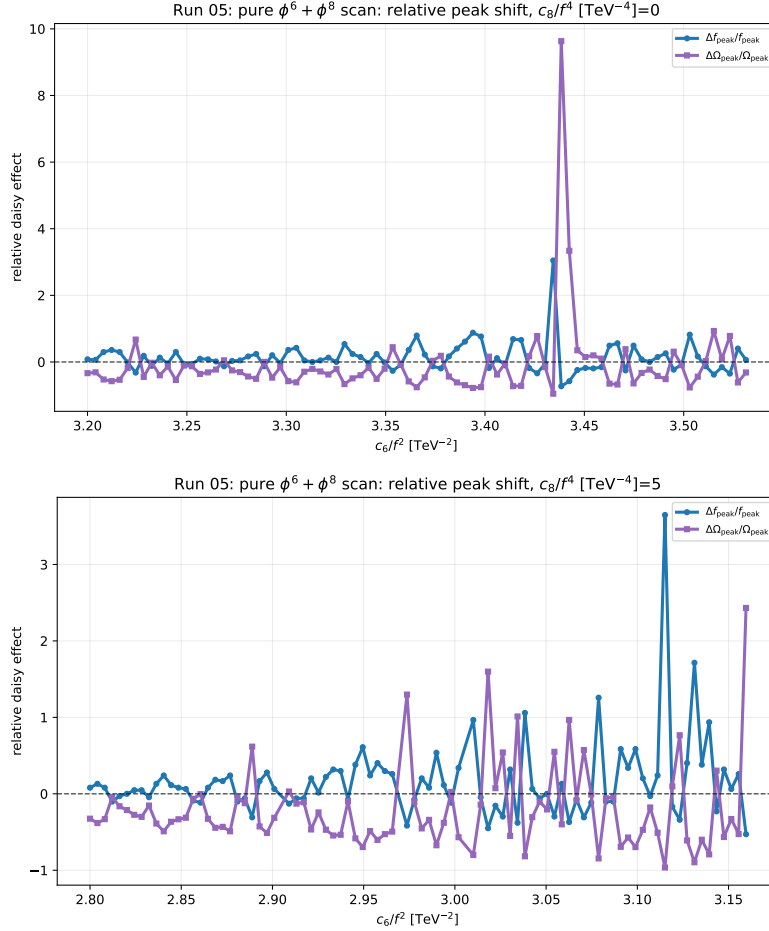


Figure 39: Relative effect of daisy resummation on the peak frequency and peak amplitude in the $\phi^6 + \phi^8$ sector. For each peak observable Y , the plotted quantity is $(Y_{\text{daisy}} - Y_{\text{no,daisy}})/Y_{\text{no,daisy}}$. The upper panel corresponds to $c_8/f^4 = 0$, while the lower panel corresponds to $c_8/f^4 = 5 \text{ TeV}^{-4}$. The curves show how daisy resummation shifts f_{peak} and $h^2\Omega_{\text{peak}}$ as c_6/f^2 varies. The result emphasizes that thermal resummation can affect both the amplitude and the characteristic frequency of the signal.

The conclusion of this section is that higher-dimensional operators in the scalar sector can generate a first-order transition sufficiently strong to produce an observable spectrum, depending on the region of parameters, as also for the functional measure. We again see the effect of thermal resummation on these observables and also how the expected physics is maintained for both results. Now we can ask how these results change when considering both modifications together.

5.6 Combined deformation: Glauber plus $(\phi^6 + \phi^8)$

Once the functional-measure sector and the polynomial deformation have been studied separately, a natural question arises: what happens if we consider both at the same time?

In principle, there is no physical impediment to this. The two deformations have different motivations. The first comes from a nontrivial contribution of the functional measure of the

path integral. The second comes from the possibility of higher-dimensional effective operators in the scalar sector of the Higgs. One modifies the potential through a logarithmic term; the other modifies the potential through even powers of the field. Nothing forces us to choose only one of them.

Thus, we consider a combined potential of the form [1, 2]

$$V_{\text{eff}}^{\text{comb}}(\phi, T) = V_{\text{eff}}^{\text{SM}}(\phi, T) + \Delta V_{\text{measure}}(\phi) + \frac{1}{8} \frac{c_6}{f^2} \phi^6 + \frac{1}{16} \frac{c_8}{f^4} \phi^8, \quad (444)$$

with the same counterterms and zero-temperature conditions discussed previously. The free parameters are, therefore,

$$C, \quad \Lambda, \quad \frac{c_6}{f^2}, \quad \frac{c_8}{f^4}, \quad f. \quad (445)$$

Since each sector separately can already produce a first-order transition, it would be natural to imagine that joining both would produce even stronger transitions. This intuition is partially correct, but it also hides an important physical point.

The first idea that comes to mind is to join the points that generate the largest transitions in each sector and obtain an extremely strong transition. However, this does not always produce a physically acceptable result. Why? Because a transition that is too strong may simply not happen.

When we increase the strength of the transition too much, the barrier between the minima can also become too large. In this case, the Universe may remain trapped in the false vacuum: the bounce action becomes large, the nucleation rate does not become efficient and we do not find a temperature T_n that satisfies the criterion

$$\frac{S_3(T_n)}{T_n} \simeq 140. \quad (446)$$

The physical meaning of this result is direct: there is not one bubble per Hubble volume, and the transition does not complete within the lifetime of the Universe. In other words, a very small barrier does not generate a strong FOPT, that is, there exists a lower limit of parameter

space capable of generating gravitational waves. On the other hand, a barrier that is too large or too thick can prevent the transition from happening, because there is no nucleation temperature that satisfies the given criterion. This is equivalent to an upper limit of parameter space. The Universe needs to cross the barrier, but the barrier cannot be neither too thin or too thick.

In this way, there is both a lower limit and an upper limit for a first-order transition to generate a physical gravitational-wave signal. The lower limit occurs when the barrier is too weak or nonexistent, and the transition becomes a crossover or a very weak transition. The upper limit occurs when the transition becomes so strong that the barrier prevents efficient nucleation and the transition does not complete. Numerically, this appears as the absence of T_n in the search interval.

The interest of the combined sector lies precisely in the middle of this path. It shows that individually small deformations can cooperate and generate an observable transition. In other words, perhaps no single mechanism needs to carry the entire weight of new physics by itself. Small contributions from different origins can add their effects and produce a detectable gravitational signature.

In the graphs below, we always define

$$f = \Lambda = 1000 \text{ GeV}, \quad (447)$$

and change the values of the coefficients C , c_6/f^2 and c_8/f^4 . We begin by showing an example of a gravitational-wave spectrum referring to a single configuration of the combined sector.

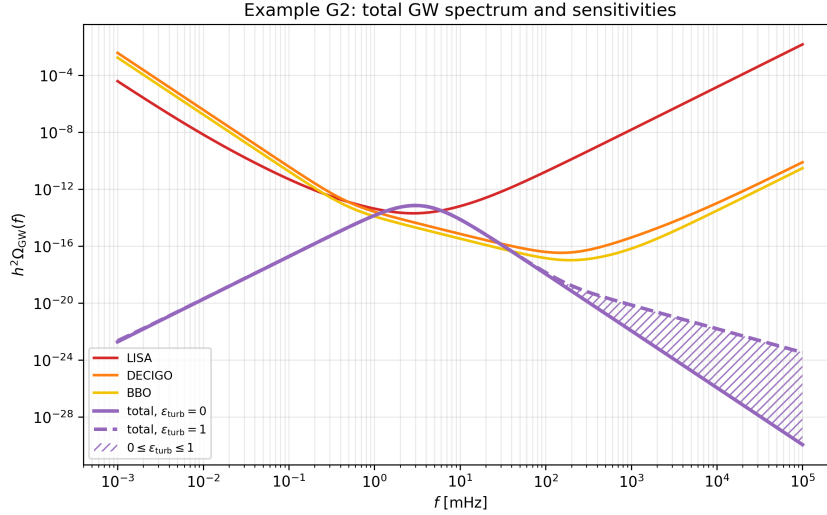


Figure 40: Example of a gravitational-wave spectrum in the combined sector, including both the functional-measure and the $\phi^6 + \phi^8$ deformations. The point shown has $f = \Lambda = 1000$ GeV, $C = 3.2$, $c_6/f^2 = 1.6$ TeV $^{-2}$ and $c_8/f^4 = 0$. The spectrum is compared with LISA, DECIGO and BBO sensitivities, shown in red, orange and yellow. The figure illustrates the cooperative effect of the combined sector: a value of C that is not sufficient by itself can become observable when a moderate polynomial deformation is added.

In Fig. 40, we see a first important result. In the pure functional-measure sector, for $\Lambda = 1000$ GeV, detectable transitions started only around $C = 3.65$, with the most intense peak near $C = 3.83$. However, by adding a moderate polynomial deformation, with $c_6/f^2 = 1.6$ TeV $^{-2}$ and $c_8/f^4 = 0.0$ TeV $^{-4}$, we already find a detectable spectrum for $C = 3.2$.

This result is the first physical message of the combined sector. The polynomial deformation does not need, by itself, to carry the whole transition; in the same way, the functional measure does not need, by itself, to push the system to the detectable region. When the two contributions act together, the parameter space shifts. Values that previously did not generate an observable signal now generate a transition strong enough to produce detectable gravitational waves.

This is physically relevant: if there are several possible mechanisms capable of modifying the thermal history of the scalar sector, the final signal may be the result of cooperation between them. The Universe does not need to “be” in a favorite and very specific parametrization capable of generating gravitational waves.

The transitions are out there, and so are the possible mechanisms that generate these transitions. If only some of them exist, or if several of them are small but act together, we may observe gravitational waves in the future.

To make the study more comprehensive, we present two types of scan. In the first, we fix c_6/f^2 and c_8/f^4 at different values and vary the parameter C of the functional-measure

contribution. This scan answers the question: how does the logarithmic sector shift an already deformed polynomial basis? In the second, we fix C and c_8/f^4 and vary c_6/f^2 . This scan answers the complementary question: how does the polynomial deformation alter a basis that already contains the functional-measure contribution?

We begin with the first case, in which we vary C while keeping fixed the coefficients of the $\phi^6 + \phi^8$ sector.

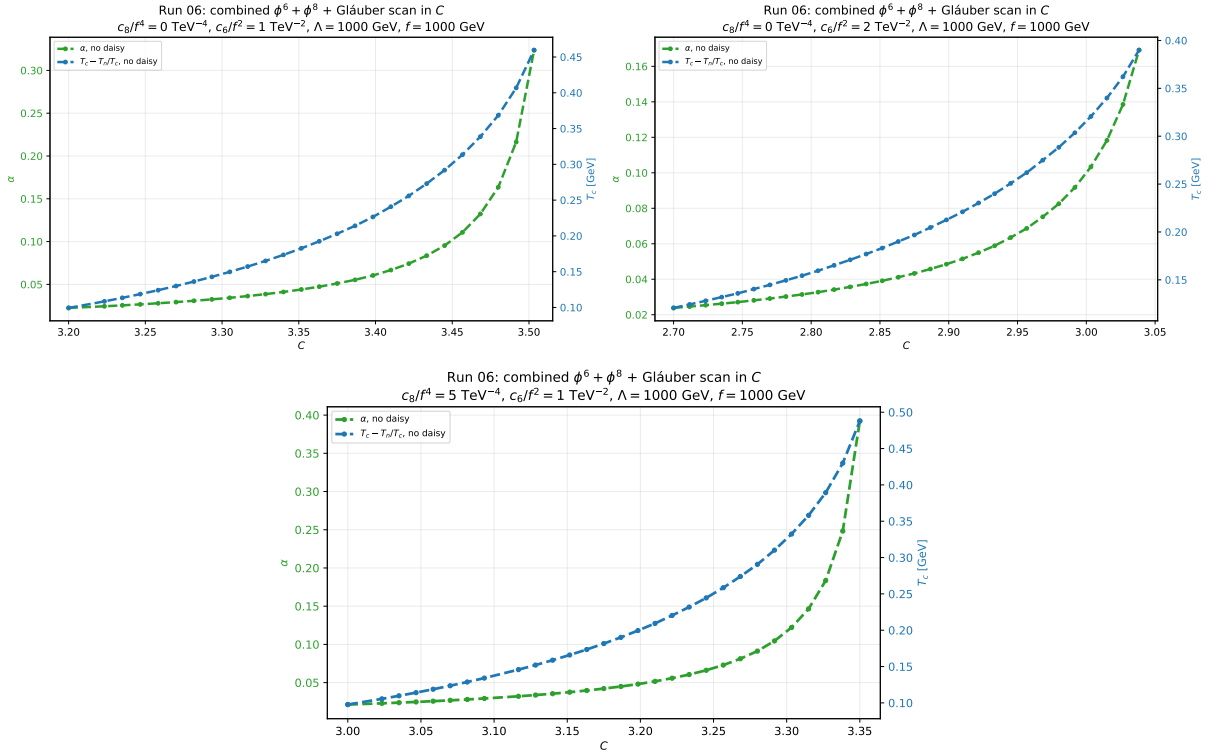


Figure 41: Thermodynamic parameters in the combined sector, varying C while keeping the polynomial deformation fixed. All panels use $f = \Lambda = 1000$ GeV and no daisy resummation. From top to bottom, the fixed choices are $(c_6/f^2, c_8/f^4) = (1 \text{ TeV}^{-2}, 0)$, $(2 \text{ TeV}^{-2}, 0)$ and $(1 \text{ TeV}^{-2}, 5 \text{ TeV}^{-4})$. The green curve shows α and the black curve shows $(T_c - T_n)/T_c$. The viable window in C shifts to smaller values once the polynomial deformation is included.

In Fig. 41, we observe the same physical structure found in the pure sectors: when the deformation increases the strength of the transition, supercooling also grows. In other words, α and $(T_c - T_n)/T_c$ continue to walk together. This reinforces the interpretation built in the previous sections: the longer the Universe remains trapped in the false vacuum after the minima become degenerate, the larger the energy released when the transition finally nucleates tends to be.

The difference now is that the parameter C no longer acts alone. It acts on a geometry that has already been altered by the ϕ^6 and ϕ^8 terms. For this reason, the interval of C in which we find viable transitions is not the same as in the pure functional sector. For $c_6/f^2 = 1.0 \text{ TeV}^{-2}$

and $c_8/f^4 = 0.0 \text{ TeV}^{-4}$, the viable interval shifts to smaller values of C , approximately

$$C \in [3.2, 3.5], \quad (448)$$

while in the pure functional-measure case, for $\Lambda = 1000 \text{ GeV}$, the detectable interval appeared approximately at

$$C \in [3.65, 3.83]. \quad (449)$$

When we increase to $c_6/f^2 = 2.0 \text{ TeV}^{-2}$, the same effect intensifies: the viable interval becomes approximately

$$C \in [2.7, 3.05]. \quad (450)$$

The inclusion of $c_8/f^4 = 5.0 \text{ TeV}^{-4}$ follows the same logic, again shifting the nucleation window to $C \in [3.0, 3.35]$. The physical result is clear: when turning on a polynomial deformation that already helps strengthen the transition, we need to decrease the logarithmic contribution so as not to cross the upper limit, that is, so as not to create a barrier so strong that the transition stops nucleating.

Below we show the movement of the spectrum peak when varying C in the combined sector.

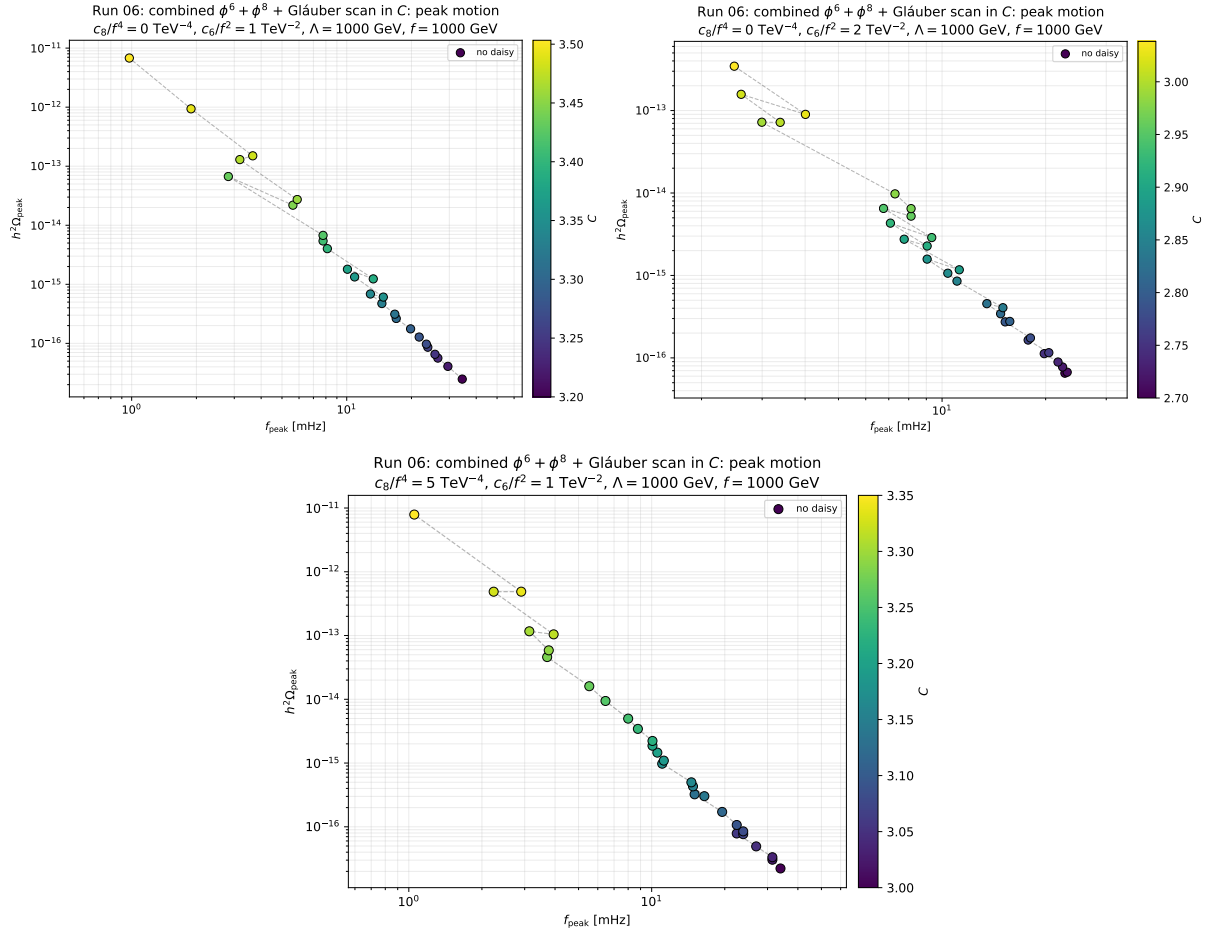


Figure 42: Movement of the gravitational-wave peak in the combined sector when varying C at fixed $\phi^6 + \phi^8$ deformation. Each point shows $(f_{\text{peak}}, h^2\Omega_{\text{peak}})$, and the color bar indicates the scanned value of C . The fixed choices are the same as in Fig. ?? : $(1 \text{ TeV}^{-2}, 0)$, $(2 \text{ TeV}^{-2}, 0)$ and $(1 \text{ TeV}^{-2}, 5 \text{ TeV}^{-4})$. The corresponding viable intervals are approximately $C \in [3.2, 3.5]$, $[2.7, 3.05]$ and $[3.0, 3.35]$. Increasing C moves the peak toward lower frequencies and larger amplitudes.

Fig. 42 translates the thermal parameters into observational language. As the transition becomes stronger, the peak tends to move to larger amplitudes and, in general, to lower frequencies. This behavior had already appeared in the pure sectors and now reappears in the combined sector. The interpretation is the same: larger values of α increase the energy available to the gravitational signal, while smaller values of β/H_* make the transition longer in Hubble units, shifting the characteristic scale of the spectrum.

The new point is that this movement of the peak now occurs in a shifted parameter window. We are not just increasing C as in the pure functional sector; we are increasing C on a potential that has already received polynomial deformations. For this reason, the peak reaches detectable regions for smaller values of C . This is precisely the signature of cooperation between the mechanisms.

We can observe the same idea in the complementary case, in which we fix C and c_8/f^4

and vary c_6/f^2 . The results are presented in Fig. 43.

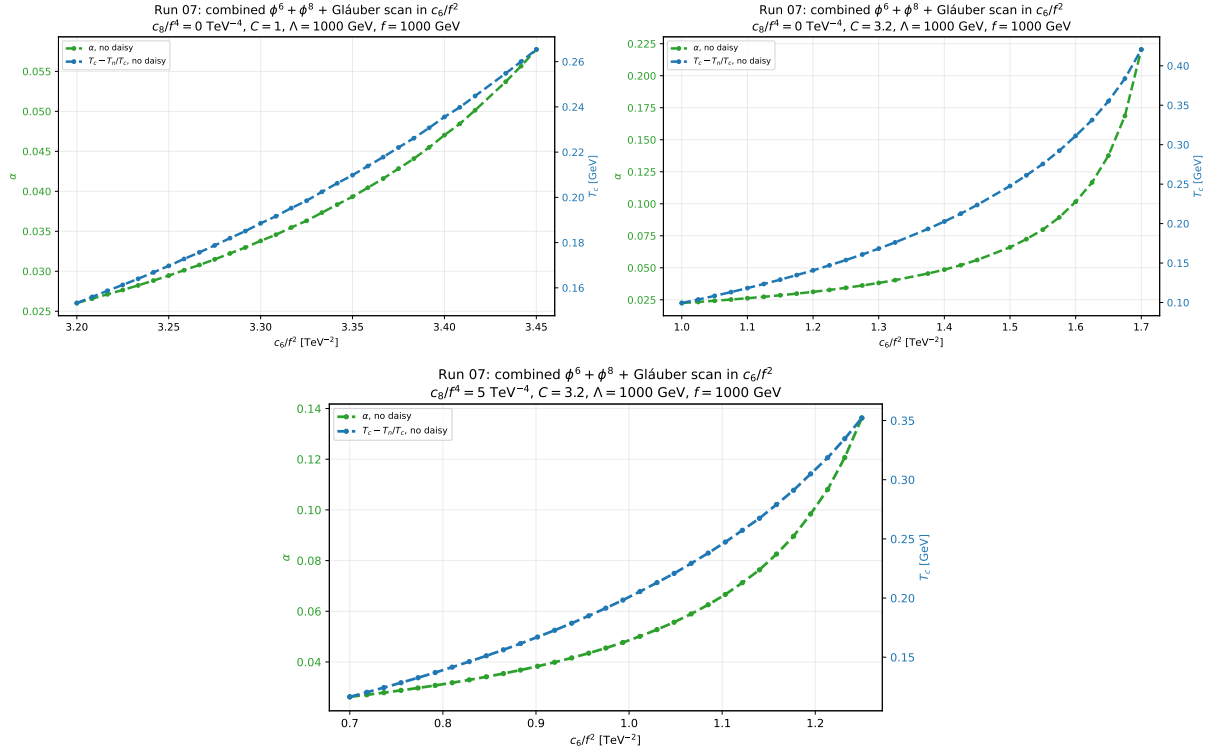


Figure 43: Thermodynamic parameters in the combined sector, varying c_6/f^2 while keeping C and c_8/f^4 fixed. All panels use $f = \Lambda = 1000 \text{ GeV}$ and no daisy resummation. From top to bottom, the fixed choices are $(C, c_8/f^4) = (1.0, 0)$, $(3.2, 0)$ and $(3.2, 5 \text{ TeV}^{-4})$. The green curve shows α and the black curve shows $(T_c - T_n)/T_c$. The figure shows that the polynomial deformation strengthens the transition only inside a finite nucleation window.

In this second scan, the physical logic becomes even more transparent. When C is small, the functional-measure contribution alone is not sufficient to produce a detectable transition in a broad region. However, by increasing c_6/f^2 , the polynomial deformation begins to collaborate with the formation of the barrier. Thus, regions that previously did not present efficient nucleation begin to present first-order transitions.

When we fix $C = 1.0$ and $c_8/f^4 = 0.0 \text{ TeV}^{-4}$, we observe that the transition appears only for

$$\frac{c_6}{f^2} \in [3.2, 3.45]. \quad (451)$$

For $C = 3.2$, keeping $c_8/f^4 = 0.0 \text{ TeV}^{-4}$, this window shifts to

$$\frac{c_6}{f^2} \in [1, 1.75]. \quad (452)$$

Finally, when considering $C = 3.2$ and $c_8/f^4 = 5.0 \text{ TeV}^{-4}$, we find

$$\frac{c_6}{f^2} \in [0.7, 1.25]. \quad (453)$$

The message is the same as seen in the scan in C , but now seen from the other side: when one mechanism already pushes the transition in the direction of a FOPT, the other does not need to be so large.

Below we show the movement of the peak in this case.

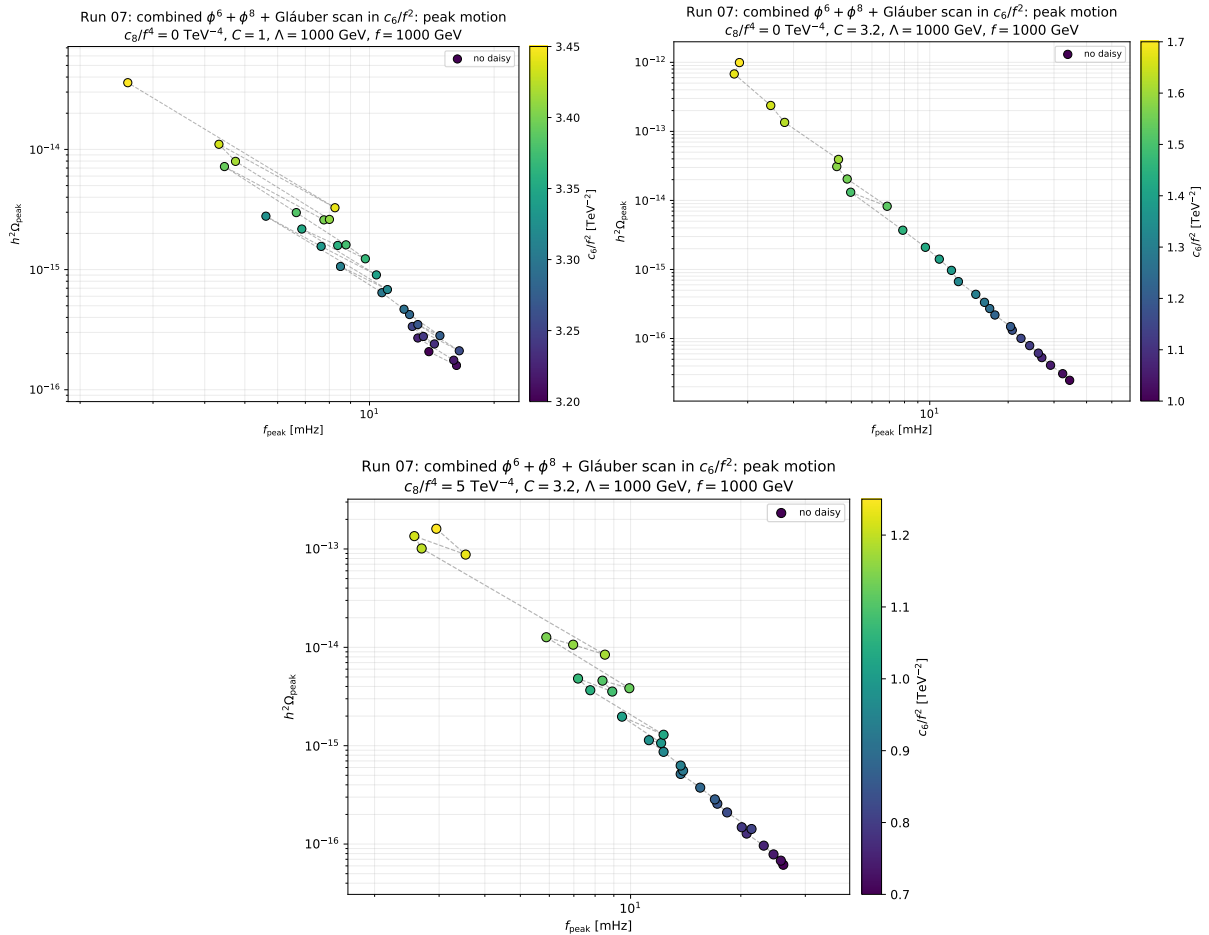


Figure 44: Movement of the gravitational-wave peak in the combined sector when varying c_6/f^2 at fixed C and c_8/f^4 . Each point shows $(f_{\text{peak}}, h^2 \Omega_{\text{peak}})$, and the color bar indicates the scanned value of c_6/f^2 . The fixed choices are $(C, c_8/f^4) = (1.0, 0)$, $(3.2, 0)$ and $(3.2, 5 \text{ TeV}^{-4})$. The viable intervals are approximately $c_6/f^2 \in [3.2, 3.45]$, $[1.0, 1.7]$ and $[0.7, 1.25] \text{ TeV}^{-2}$. Within the nucleation window, increasing c_6/f^2 tends to increase the peak amplitude and shift the signal to lower frequencies.

In Fig. 44, we see again that the peak shifts as the transition becomes stronger. For larger values of c_6/f^2 , within the window in which nucleation still happens, the amplitude tends to grow and the characteristic frequency tends to move according to the change in β/H_* . However, as in the scan in C , there is a limit: increasing the polynomial deformation too much does not mean producing an arbitrarily large signal. After a certain point, the transition stops nucleating in the interval studied.

This behavior organizes all the physics of the combined sector. The parameter space should not be seen as an infinite line toward increasingly stronger transitions. It looks more like a window: before it, the transition is too weak; after it, the transition is too strong to happen; inside it, the Universe manages to pass from one phase to another and, in this process, produce gravitational waves.

Therefore, the combined sector is not interesting only because it can increase the amplitude of the signal. It is interesting because it shows that detectability can arise from cooperation. A value of C that, alone, would be insufficient can become relevant when combined with higher-dimensional operators. A value of c_6/f^2 that, alone, might require a more extreme region can become viable when the functional measure also contributes. New physics does not need to come entirely from a single place.

This is perhaps the most important message of this subsection. Different small deformations of the scalar sector can add up and produce detectable GWs. If a future detection of gravitational waves associated with a phase transition is observed, it may not immediately point to a single isolated mechanism, but to the combination of several effects that, together, modified the geometry of the potential in the primordial Universe.

In other words, the gravitational signal may not be the signature of a single new piece. It may be the signature of a sum of small pieces that, together, finally made the Universe cross the barrier.

5.7 Conclusions and future directions

With these results in hand, we can extract some general lessons about first-order phase transitions and gravitational waves.

The first lesson is that the effective potential is, indeed, the central object of the story. Apparently small modifications in the scalar sector can alter the position of the minima, the height of the barrier, the nucleation temperature and the energy released in the transition. As a

consequence, these modifications leave direct marks on the gravitational-wave spectrum.

The second lesson is that the strength of the transition, measured by α , is closely linked to supercooling,

$$\frac{T_c - T_n}{T_c}. \quad (454)$$

In the sectors studied, when the Universe remains longer trapped in the false vacuum, the released energy tends to grow, and with it the amplitude of the spectrum grows. This relation expresses the fact that a later transition, relative to the critical temperature, accumulates a larger energy difference between the phases and therefore has more strength, which is directly associated with the released energy.

The third lesson is that β/H_* carries the complementary information. While α measures the strength of the transition, β/H_* measures how fast it happens. Stronger transitions tend, in many cases, to have smaller β/H_* , that is, a larger effective duration. This shifts and amplifies the gravitational-wave spectrum. Thus, the final signal does not depend only on how much energy is released, but also on how quickly this energy is released.

The fourth lesson is that thermal resummation cannot be completely ignored. By altering the bosonic thermal masses and the geometry of the potential, daisy resummation can change T_n , α , β/H_* and the peak of the spectrum and, especially, the range of parameter space in which the transition exists. In some points of parameter space, the effect can be small; in others, it can alter the physical interpretation of the result. This reinforces that perturbative predictions for cosmological transitions carry theoretical uncertainties that must be carefully declared.

The fifth lesson comes from the combined sector. New physics does not need to appear in a single dominant deformation. Different mechanisms can contribute moderately and, together, produce an observable signal. This enlarges the phenomenological space of interest and makes gravitational waves a particularly powerful tool: they can test any modifications existing in the scalar sector. Even if no wave is detected in the future, this will still be an extremely interesting physical result. That the Standard Model remains unshaken, even with many mechanisms still possible to exist.

We also learned that there is a physical upper limit to the strength of the transition. If the barrier becomes too large, the transition does not nucleate. Numerically, this appears as the absence of a temperature T_n satisfying the nucleation criterion. Physically, this means that the

Universe may remain trapped in the false vacuum. Therefore, a detectable transition needs to live in a window: strong enough to generate gravitational waves, but not so strong that it never happens.

These conclusions reinforce the idea that runs through this entire dissertation: gravitational waves can transform the primordial Universe into a laboratory of particle physics. Terrestrial colliders tell us how the Higgs sector behaves today, at $T = 0$. Gravitational waves can tell us how this same sector behaved when the Universe was hot, dense and much closer to the scales of new physics.

Limitations

Although this scenario is extremely interesting, there is an important limitation that we did not address deeply throughout this thesis. Different scenarios beyond the Standard Model can, in principle, generate the same effective values of α and β/H_* . Since the gravitational-wave spectrum used in this analysis is parametrized mainly by these thermal observables, different models that produce the same parameters can lead to very similar spectra, or even indistinguishable ones, within this description.

This means that there is a certain degeneracy between the models. The detection of a stochastic gravitational-wave background, by itself, would not necessarily uniquely identify which potential, which new sector or which microscopic mechanism was responsible for the transition.

However, since the space experiments will still be launched, the current concern still lies in verifying whether or not there are mechanisms capable of producing gravitational waves in the frequency and amplitude ranges in which these experiments will operate.

Once these data are detected, only then will we start worrying about breaking the degeneracy between the different physical models. From a possible detection, it will be necessary to combine the gravitational-wave spectrum with other physical information: collider measurements, cosmological constraints, potential structure, perturbative consistency, details of the plasma dynamics and possible complementary signatures. Only in this way will we be able to start distinguishing between models that, at the level of α , β/H_* and $\Omega_{\text{GW}}(f)$, look similar.

However, the study of cases that generate detectable transitions is still of utmost importance, especially considering the scenario in which the contribution of different models generates a detectable spectrum, which, as we saw throughout the thesis, may also be a possible

scenario. Finding these models allows us to already be prepared for when the data arrive to already limit our parameter space and also to justify why these experiments are so important. Even in the scenario in which nothing is measured, we have a great physical result, that the Standard Model remains stable.

Future work

This work opens several natural directions. A first direction concerns the treatment of the parameter β/H . In this thesis, we used a simplified analysis of β/H_* , fitting its behavior with a polynomial up to degree three in the scan parameter. This was done only to capture the general trend of a quantity that is numerically sensitive, since β/H_* is obtained from a temperature derivative of the bounce action. As future work, especially for the article of this study, we intend to perform a more systematic fit analysis by comparing different functional forms through their χ^2 values, in order to determine which curve describes this parameter in a more realistic way. In the present thesis, the fitted curve should therefore be understood as a preliminary visualization of the global behavior of β/H_* , sufficient for the physical discussion carried out here, but not as a final statistical treatment of this observable.

The second is to deepen the treatment of thermal resummation. Throughout the analysis, we saw that the effect of daisy resummation cannot be discarded a priori. It modifies the shape of the potential at T_n and, consequently, affects the observed gravitational-wave spectrum and the interval of parameters in which the theory produces a detectable first-order transition. As future work, we want to investigate how different treatments of thermal resummation affect the observable spectrum. Instead of using only the daisy prescription, we can compare approaches such as OPT, dimensional reduction with tools such as `DRalgo`, or other schemes that reorganize the thermal perturbative expansion.

The third direction is to broaden the study of the combined sector. In this thesis, it was used as a first physical exploration: to show that small deformations can cooperate. A natural next step is to carry out a systematic scan in the space

$$\left(C, \Lambda, \frac{c_6}{f^2}, \frac{c_8}{f^4} \right), \quad (455)$$

separating regions with weak transition, detectable transition, absence of nucleation and possible problems of effective validity.

The fourth direction is to connect these effective potentials to more complete microscopic models. The approach of this thesis was phenomenological and modular: modify the scalar sector, observe the thermal history and calculate the spectrum. The next step would be to ask which ultraviolet theories could generate these deformations and which other observables, in colliders or cosmology, would be correlated with the same signal.

Finally, it would also be interesting to incorporate more realistic uncertainties in the dynamics of the plasma, especially in the value of the wall velocity v_w and in the efficiency of energy conversion into sound waves and turbulence. In this work, we adopted controlled and transparent choices to build the first analysis. In future works, these choices can be relaxed to map the complete phenomenological uncertainty.

With this, we close the chain built throughout the dissertation. We began by asking how a phase transition in the primordial Universe could leave an observable signature. We built the effective potential, studied bubble nucleation, extracted the thermal parameters, translated these parameters into gravitational waves and, finally, applied the pipeline to concrete deformations of the scalar sector.

The final result is the physical message that guided the whole work: if the Higgs sector hides a thermal history richer than that of the minimal Standard Model, perhaps this history is not lost. Perhaps it is traveling through the Universe in the form of gravitational waves, waiting only for us to have the right instruments to listen to it.

A Appendix A | Dimensional regularization and Coleman-Weinberg correction

For a constant background, the operator Δ is diagonal in momentum space, and the one-loop contribution to the effective potential is

$$V_1(\phi_c) = \frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \ln [p_E^2 + M^2(\phi_c)] . \quad (456)$$

This integral is ultraviolet divergent. Physically, this means that fluctuations of arbitrarily high momenta contribute to the vacuum energy. Mathematically, we need to regularize the integral and then absorb the divergences into the counterterms of the theory.

We will use dimensional regularization. The idea is to extend the integral to

$$d = 4 - \epsilon \quad (457)$$

dimensions and introduce a scale μ to preserve the correct dimension of the potential [23–25] :

$$V_1(\phi_c) = \frac{1}{2} \mu^{4-d} \int \frac{d^d p_E}{(2\pi)^d} \ln [p_E^2 + M^2(\phi_c)] . \quad (458)$$

A practical way to compute (458) is to differentiate with respect to M^2 :

$$\frac{\partial V_1}{\partial M^2} = \frac{1}{2} \mu^{4-d} \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{p_E^2 + M^2} . \quad (459)$$

To evaluate this integral, we use the Schwinger representation,

$$\frac{1}{p_E^2 + M^2} = \int_0^\infty ds e^{-s(p_E^2 + M^2)} . \quad (460)$$

Substituting into (459),

$$\frac{\partial V_1}{\partial M^2} = \frac{1}{2} \mu^{4-d} \int_0^\infty ds e^{-sM^2} \int \frac{d^d p_E}{(2\pi)^d} e^{-sp_E^2} . \quad (461)$$

The Gaussian integral in d dimensions is

$$\int \frac{d^d p_E}{(2\pi)^d} e^{-sp_E^2} = \frac{1}{(4\pi s)^{d/2}} . \quad (462)$$

Thus,

$$\frac{\partial V_1}{\partial M^2} = \frac{1}{2} \frac{\mu^{4-d}}{(4\pi)^{d/2}} \int_0^\infty ds s^{-d/2} e^{-sM^2}. \quad (463)$$

Using the identity

$$\int_0^\infty ds s^{\alpha-1} e^{-sM^2} = \Gamma(\alpha) (M^2)^{-\alpha}, \quad (464)$$

with

$$\alpha = 1 - \frac{d}{2}, \quad (465)$$

we obtain

$$\frac{\partial V_1}{\partial M^2} = \frac{1}{2} \frac{\mu^{4-d}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) (M^2)^{\frac{d}{2}-1}. \quad (466)$$

Now we integrate back in M^2 . Since

$$\int d(M^2) (M^2)^{\frac{d}{2}-1} = \frac{2}{d} (M^2)^{d/2}, \quad (467)$$

we have

$$V_1(\phi_c) = \frac{1}{d} \frac{\mu^{4-d}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) (M^2(\phi_c))^{d/2} + \text{constant}. \quad (468)$$

Using the identity

$$\Gamma\left(1 - \frac{d}{2}\right) = -\frac{d}{2} \Gamma\left(-\frac{d}{2}\right), \quad (469)$$

we can write

$$V_1(\phi_c) = -\frac{1}{2} \frac{\mu^{4-d}}{(4\pi)^{d/2}} \Gamma\left(-\frac{d}{2}\right) (M^2(\phi_c))^{d/2} + \text{constant}. \quad (470)$$

Now we set $d = 4 - \epsilon$. Expanding the gamma function and the dimensional factors around

$\epsilon = 0$, we obtain the structure

$$V_1(\phi_c) = \frac{M^4(\phi_c)}{64\pi^2} \left[\Delta_{\overline{\text{MS}}} + \ln \left(\frac{M^2(\phi_c)}{\mu^2} \right) - \frac{3}{2} \right] + \text{constant}, \quad (471)$$

where

$$\Delta_{\overline{\text{MS}}} = \frac{2}{\epsilon} - \gamma_E + \ln(4\pi). \quad (472)$$

The term $\Delta_{\overline{\text{MS}}}$ contains the divergent pole as $\epsilon \rightarrow 0$, in addition to the conventional constants of the $\overline{\text{MS}}$ scheme. The counterterms are chosen precisely to absorb this divergent part. After subtraction in the $\overline{\text{MS}}$ scheme, the finite contribution remains

$$V_1^{\overline{\text{MS}}}(\phi_c) = \frac{M^4(\phi_c)}{64\pi^2} \left[\ln \left(\frac{M^2(\phi_c)}{\mu^2} \right) - \frac{3}{2} \right]. \quad (473)$$

The constant term independent of ϕ_c can be absorbed into a vacuum-energy constant. To study the phase structure, what matters is the dependence on ϕ_c , since it is what shifts minima, changes curvatures, and can modify the geometry of the potential.

The sum of the finite one-loop contribution with the counterterms is what we call, in general terms, the Coleman–Weinberg correction:

$$\Delta V_{\text{CW}}(\phi_c) = V_1^{\overline{\text{MS}}}(\phi_c) \quad (474)$$

For a general theory with several particles coupled to the background field, each species contributes with its own quadratic operator and, therefore, with its own functional determinant. The Coleman–Weinberg correction can be written as [25, 27, 28]

$$\Delta V_{\text{CW}}(\phi) = \sum_i \frac{n_i}{64\pi^2} m_i^4(\phi) \left[\ln \left(\frac{m_i^2(\phi)}{\mu^2} \right) - c_i \right]. \quad (475)$$

Here, i runs over the particle species of the theory, $m_i^2(\phi)$ are the background-dependent masses, and n_i counts the internal degrees of freedom, including the statistical sign. In particular,

$$n_i > 0 \quad \text{for bosons}, \quad n_i < 0 \quad \text{for fermions}. \quad (476)$$

The negative sign for fermions comes from the anticommuting nature of Grassmann variables: the fermionic Gaussian integral produces a determinant in the numerator, not in the denominator, flipping the sign of the contribution in the logarithm of the effective action.

For a colorless Dirac fermion, for example, we have four degrees of freedom, associated with the two helicities of particle and antiparticle. Thus, its typical contribution carries

$$n_{\text{Dirac}} = -4. \quad (477)$$

For quarks, we must also multiply by the color factor $N_c = 3$, so that, for the top quark,

$$n_t = -12. \quad (478)$$

The constants c_i depend on the particle type and on the renormalization scheme. In the $\overline{\text{MS}}$ scheme, in gauges such as Landau gauge, one often uses

$$c_i = \frac{3}{2} \quad \text{for scalars and fermions,} \quad (479)$$

while for massive gauge bosons one has

$$c_i = \frac{5}{6}. \quad (480)$$

B Appendix B | Matsubara sums and thermal logarithms

For a bosonic mode with energy

$$\omega = \sqrt{\mathbf{k}^2 + m^2(\phi)}, \quad (481)$$

the one-loop finite-temperature contribution involves the sum

$$I_B(\omega) = \frac{T}{2} \sum_{n=-\infty}^{+\infty} \ln [(\omega_n^B)^2 + \omega^2], \quad (482)$$

where the bosonic Matsubara frequencies are

$$\omega_n^B = 2\pi nT. \quad (483)$$

This sum looks ugly, but there is a simple trick: instead of summing the logarithm directly, we differentiate with respect to ω :

$$\frac{dI_B}{d\omega} = T \sum_{n=-\infty}^{+\infty} \frac{\omega}{(\omega_n^B)^2 + \omega^2}. \quad (484)$$

Therefore, the problem reduces to the sum

$$S_B(\omega) = T \sum_{n=-\infty}^{+\infty} \frac{1}{(\omega_n^B)^2 + \omega^2}. \quad (485)$$

This sum can be evaluated by methods of complex analysis. The central idea is to transform the sum over discrete frequencies into a contour integral. The bosonic frequencies

$$\omega_n^B = \frac{2\pi n}{\beta} \quad (486)$$

are exactly the poles of the function

$$\frac{1}{2} \cot \left(\frac{\beta z}{2} \right). \quad (487)$$

Indeed, near $z = \omega_n^B$,

$$\frac{1}{2} \cot \left(\frac{\beta z}{2} \right) \sim \frac{1}{\beta} \frac{1}{z - \omega_n^B}. \quad (488)$$

Since $T = 1/\beta$, this allows us to write, schematically,

$$T \sum_{n=-\infty}^{+\infty} f(\omega_n^B) = \oint_C \frac{dz}{2\pi i} \frac{1}{2} \cot\left(\frac{\beta z}{2}\right) f(z), \quad (489)$$

where the contour C encloses the Matsubara poles.

In our case,

$$f(z) = \frac{1}{z^2 + \omega^2}. \quad (490)$$

The function $f(z)$ has poles at

$$z = +i\omega, \quad z = -i\omega. \quad (491)$$

Deforming the contour, the sum of residues at these poles gives

$$S_B(\omega) = \frac{1}{2\omega} \coth\left(\frac{\beta\omega}{2}\right). \quad (492)$$

Using the identity

$$\coth\left(\frac{\beta\omega}{2}\right) = 1 + \frac{2}{e^{\beta\omega} - 1}, \quad (493)$$

we obtain

$$S_B(\omega) = \frac{1}{2\omega} [1 + 2n_B(\omega)], \quad (494)$$

where

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1} \quad (495)$$

is the Bose–Einstein distribution,

Substituting (494) into (484), we have

$$\frac{dI_B}{d\omega} = \omega S_B(\omega) = \frac{1}{2} + n_B(\omega). \quad (496)$$

Now we integrate in ω :

$$I_B(\omega) = \frac{\omega}{2} + T \ln(1 - e^{-\beta\omega}) + \text{constant}. \quad (497)$$

The constant does not depend on ω and, therefore, does not depend on the background field ϕ . It can be absorbed into a vacuum-energy constant [21, 22, 29].

Now we do the fermionic case. The Matsubara frequencies for fermions are

$$\omega_n^F = (2n + 1)\pi T. \quad (498)$$

They come from antiperiodic boundary conditions in Euclidean time. The basic sum is

$$S_F(\omega) = T \sum_{n=-\infty}^{+\infty} \frac{1}{(\omega_n^F)^2 + \omega^2}. \quad (499)$$

Now the Matsubara poles are captured by a different trigonometric function, associated with the tangent. The result of the contour integral is

$$S_F(\omega) = \frac{1}{2\omega} \tanh\left(\frac{\beta\omega}{2}\right). \quad (500)$$

Since

$$\tanh\left(\frac{\beta\omega}{2}\right) = 1 - \frac{2}{e^{\beta\omega} + 1}, \quad (501)$$

we have

$$S_F(\omega) = \frac{1}{2\omega} [1 - 2n_F(\omega)], \quad (502)$$

where

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1} \quad (503)$$

is the Fermi–Dirac distribution.

If we define, analogously,

$$I_F(\omega) = \frac{T}{2} \sum_{n=-\infty}^{+\infty} \ln [(\omega_n^F)^2 + \omega^2], \quad (504)$$

then

$$\frac{dI_F}{d\omega} = \omega S_F(\omega) = \frac{1}{2} - n_F(\omega). \quad (505)$$

Integrating,

$$I_F(\omega) = \frac{\omega}{2} + T \ln (1 + e^{-\beta\omega}) + \text{constant}. \quad (506)$$

Therefore, the fermionic thermal contribution to the effective potential appears as

$$\Delta V_T^F = -n_f T \int \frac{d^3k}{(2\pi)^3} \ln (1 + e^{-\beta\omega_k}), \quad (507)$$

where $n_f > 0$ counts fermionic degrees of freedom.

For bosons, the thermal contribution is

$$\Delta V_T^B = n_b T \int \frac{d^3k}{(2\pi)^3} \ln (1 - e^{-\beta\omega_k}). \quad (508)$$

Finally, introducing the dimensionless variable

$$x = \frac{|\mathbf{k}|}{T}, \quad y = \frac{m^2(\phi)}{T^2}, \quad (509)$$

and using

$$\omega_k = T \sqrt{x^2 + y}, \quad (510)$$

we obtain the thermal functions used in the main text:

$$J_B(y) = \int_0^\infty dx \, x^2 \ln (1 - e^{-\sqrt{x^2+y}}), \quad (511)$$

and

$$J_F(y) = \int_0^\infty dx x^2 \ln \left(1 + e^{-\sqrt{x^2+y}} \right). \quad (512)$$

Thus,

$$\Delta V_T^B(\phi, T) = n_b \frac{T^4}{2\pi^2} J_B \left(\frac{m_b^2(\phi)}{T^2} \right), \quad (513)$$

and

$$\Delta V_T^F(\phi, T) = -n_f \frac{T^4}{2\pi^2} J_F \left(\frac{m_f^2(\phi)}{T^2} \right). \quad (514)$$

C Appendix C | Wick Rotation

Important conditions and convergence criteria

Before applying the Wick rotation directly, it is important to recall two basic facts from complex integration that will be present throughout the formalism. The general idea is simple: if we want to replace an integral on the real axis by an integral along a complex direction, we need to know when this deformation of the integration path is allowed [21, 23].

The first fact is Cauchy's theorem. If a function $f(z)$ is analytic in a region of the complex plane and also in the interior of a closed contour $\mathcal{C}$, then

$$\oint_{\mathcal{C}} f(z) dz = 0. \quad (515)$$

In other words, if there are no singularities inside the contour, we can deform the integration path without changing the value of the integral. This is the mathematical basis behind the Wick rotation: the integral on the real axis can be compared with an integral along another axis or along another line in the complex plane, as long as the deformation does not cross singularities.

The second important fact is the criterion that allows us to discard integrals over circular arcs when the radius of the arc tends to infinity. In general, when we close a contour in the complex plane, contributions appear over arcs Γ_R . For the deformation to be useful, we need

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R} f(z) dz = 0. \quad (516)$$

A commonly used criterion for this is Jordan's lemma. In a brief form, we can, through inequalities for our integrand, show that, for functions of the type $f(z) = e^{iaz}g(z)$ and R^{-n} , with $n \geq 2$ and $\lim_{R \rightarrow \infty} g(z) = 0$, the integrals over circular arcs always go to 0. The proof of this lemma consists, above all, in observing the behavior of the real part of the function, when $z = Re^{i\theta}$:

$$\begin{aligned}
\left| \int_{\Gamma_R} f(z) dz \right| &\leq R \int_0^\pi e^{-aR \sin \theta} |g(Re^{i\theta})| d\theta \\
&\leq M_R \frac{\pi}{a} (1 - e^{-Ra}) = \frac{M_R \pi}{a}
\end{aligned}$$

where M_R is the maximum of $g(z)$ when $R \rightarrow \infty$. If the integrand contains a damped exponential factor, this expression can go to zero when $R \rightarrow \infty$. It is exactly this type of analysis that we will do below. The essential point is: the Wick rotation is only valid when the function is analytic in the deformed region and when the contributions from arcs at infinity vanish.

Wick-rotation contour

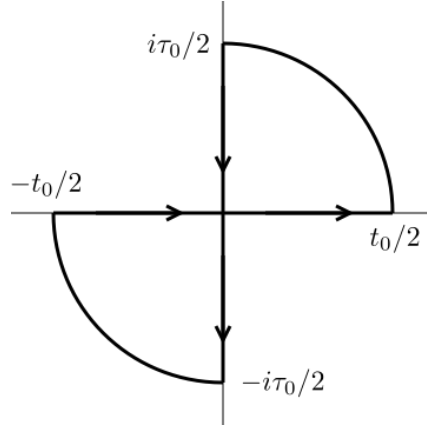
The motivation for the Wick rotation is quite similar in structure to the motivation for the residue method. In the residue method, we choose a clever contour in the complex plane to relate a real integral to a closed integral that will be the sum of residues. Here, since we want there to be no residues inside the contour, we will do something almost analogous: we want to compare an integral taken along the real axis with an integral taken along an imaginary direction, or more generally, along a rotated line in the complex plane.

Let us begin with an integral of the form

$$\int_{-\infty}^{\infty} f(t) dt$$

As in the residue method, using the analyticity of the function, we can extend its domain to the complex plane by taking t to a complex variable z , defined as $z = t + i\tau$. For simplicity, we will treat the case in which the extended function has no singularities and, therefore, its integral along a closed path in the complex plane is indeed 0, i.e.: $\oint f(z) dz = 0$.

A possible clever contour that we can choose to evaluate this function is:

Figure 45: Contour associated with a 90° rotation in the complex plane.

This contour has four contributions: one along the real axis, one along the imaginary axis, and two contributions over circular arcs. Schematically,

$$\oint f(z)dz = \int_{-R}^R f(t)dt + \int_{iR}^{-iR} f(z)dz + \int_{\Gamma^+} f(z)dz + \int_{\Gamma^-} f(z)dz$$

On the imaginary axis we write $z = i\tau$, so that $dz = i d\tau$. Since the chosen path goes from iR to $-iR$, we have

$$\int_{iR}^{-iR} f(z) dz = i \int_R^{-R} f(i\tau) d\tau = -i \int_{-R}^R f(i\tau) d\tau. \quad (517)$$

Therefore,

$$\oint f(z)dz = \int_{-R}^R f(t)dt - i \int_{-R}^R f(i\tau)d\tau + \int_{\Gamma^+} f(z)dz + \int_{\Gamma^-} f(z)dz$$

Now, as in the residue method, we take R to ∞ and analyze the possible behavior of the integrals over the quarter-circle arcs. Using the parametrization $z = Re^{i\theta}$, we find:

$$\int_{\Gamma^+} f(z)dz + \int_{\Gamma^-} f(z)dz = iR \left(\int_0^{\pi/2} f(Re^{i\theta})e^{i\theta}d\theta + \int_{3\pi/2}^{\pi} f(Re^{i\theta})e^{i\theta}d\theta \right)$$

We observe that functions that satisfy Jordan's lemma will have both integrals vanishing

when R is taken to infinity. Observe that we could have chosen another contour that encloses the quadrants not shown here; the only change would be an opposite sign in the results. The choice of contour depends on in which quadrants the quarter-circle integrals vanish, and this is precisely the part where we must be careful. Assuming that, for the given function, Jordan's lemma holds in quadrants 1 and 3, we can write:

$$\oint f(z)dz = \int_{-\infty}^{\infty} f(t)dt - i \int_{-\infty}^{\infty} f(i\tau)d\tau = 0$$

$$\int_{-\infty}^{\infty} f(t)dt = i \int_{-\infty}^{\infty} f(i\tau)d\tau$$

That is, evaluating the function on the real axis is equivalent to evaluating it on the imaginary axis. Basically, we can make the substitution $t \rightarrow i\tau$ and map the real axis onto the imaginary one. Note that both t and τ are real variables. Note also that this 90° rotation is equivalent to changing $t = i\tau$ in the integral, keeping the same integration limits. This is also equivalent to making the change of variable $t = e^{i\pi/2}\tau$, again keeping the integration limits.

It is important to emphasize that the choice of quadrants, which basically leads to an inversion in the direction of the path on the imaginary axis, must be analyzed with care. For each function, we must check whether, in the chosen quadrants, the contribution from the arc integrals is indeed 0.

The nicest part of this formalism is to realize that, in fact, we can generalize the result even further. Imagine that the rotation performed in the complex plane is not necessarily 90° , but rather an arbitrary angle, so that we still keep the closed contour as a sum of 4 integrals, two of them being circular arcs and one being a straight line, parametrized by a path C , in the complex plane. An image illustrating two particular paths can be seen below:

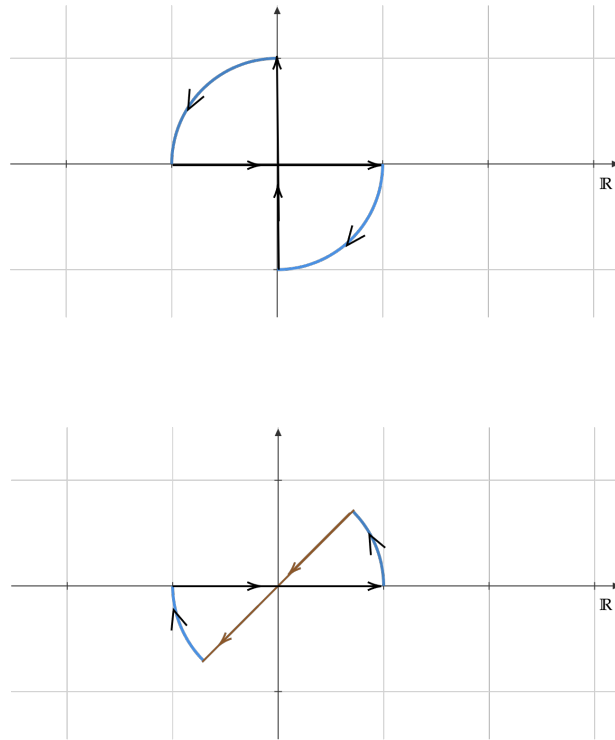


Figure 46: Examples of possible rotations in the complex plane. The validity of the deformation depends on the analyticity of the function and on the vanishing of the arc integrals at infinity.

Both paths, when the arc integrals go to 0, are also valid, so we can perform a rotation by an arbitrary angle of our choice. Thus, the general result of the formula found earlier is expressed as, when the arc integrals go to 0:

$$\int_{-\infty}^{\infty} f(t)dt = - \int_C f(z)dz$$

Here it is important to be attentive to the direction of the straight-line path C , the path taken by the line in the complex plane. We must note that C contains the orientation of the line and, therefore, also defines the integration limits. Choosing quadrants 1 and 3, the straight-line path C will be “opposite” to the real line, so the minus sign restores the usual limits. Choosing quadrants 2 and 4, it will have the same direction, so the minus sign leaves the limits reversed.

Application of the Wick rotation

Let us now consider a practical example of the Wick rotation, to make all concepts clearer.

For this, we can evaluate the integral:

$$\int_{-\infty}^{\infty} e^{iax^2} dx$$

Note that this integral has an oscillatory form and it is very difficult to quickly visualize whether it converges or not to some value. It is in this context that the Wick rotation becomes useful.

The first step to apply it is to determine in which quadrants the circular-arc integrals actually vanish and, thus, whether we can apply the formalism or not. In this way, we extend the function analytically and observe the behavior of the integral over the circular arcs, choosing the parametrization $z = Re^{i\theta}$, so that we have:

$$\begin{aligned} \int_{\Gamma} e^{iaz^2} dz &= \int_{\Gamma} e^{iaR^2 e^{i2\theta}} iRe^{i\theta} d\theta \\ &= i \int_{\Gamma} e^{-aR^2 \sin(2\theta)} e^{ia \cos(2\theta)} e^{i\theta} R d\theta \end{aligned}$$

Since the exponential contains powers of R^2 , in the limit $R \rightarrow \infty$, we only need to guarantee that the argument of the sin is positive, so that the integral goes to 0 and we can apply the described formalism.

It is easy to see that, for sin to be positive, $2\theta \in [0, \pi]$, that is, $\theta \in [0, \frac{\pi}{2}]$. We can also have $2\theta \in [2\pi, 3\pi]$, which gives $\theta \in [\pi, \frac{3\pi}{2}]$. In this way, the only quadrants in which the arc integrals go to 0, when R is taken to infinity, are the first and third quadrants. In quadrants 2 and 4, this integral diverges.

Thus, we can perform any Wick rotation within an angle in $[0, \frac{\pi}{2}]$ for this function. To simplify the integral, it is convenient to choose a 45° rotation; any other angle will bring us back to an integral that we cannot solve easily. This is equivalent to making the change of variable $x = e^{i\pi/4}y$, keeping the integration limits, but we will prove it in the formal way. The integral will be along the line $x = y$, which can be seen in Fig. 46, with $z = x + iy$. So we can write, remembering that always $dz = dx + i dy$:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{iax^2} dx &= - \int_C e^{iaz^2} dz \\
&= - \int_C e^{ia(x^2-y^2+2ixy)} (dx + idy) \\
&= - \int_{\infty}^{-\infty} e^{-2ax^2} (1+i) dx \\
&= (1+i) \underbrace{\int_{-\infty}^{\infty} e^{-2ax^2} dx}_{\text{Gaussian}} \\
\int_{-\infty}^{\infty} e^{iax^2} dx &= (1+i) \sqrt{\frac{\pi}{2a}} = \frac{1+i}{\sqrt{2}} \sqrt{\frac{\pi}{a}} \\
\int_{-\infty}^{\infty} e^{iax^2} dx &= e^{i\pi/4} \sqrt{\frac{\pi}{a}}
\end{aligned}$$

Note that the integration limits appeared reversed due to the direction of the line, which, deep down, is related to the quadrants in which we performed the rotation (1 and 3).

Important physical contexts

The Feynman propagator and the $i\epsilon$ prescription

Now, let us see the important physical examples in which this formalism appears and deserves to be illustrated. The first involves the Feynman propagator. What we seek to integrate is:

$$\Delta_F(x) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{ie^{-ip_0t}}{p_0^2 - E_{\mathbf{p}}^2 + i\epsilon}, \quad (518)$$

First, it is very important to note that the given integral has a singularity on the real axis, since, a priori, $|p_0| = E_{\mathbf{p}}$. There are two ways to treat this problem. We can follow the residue formalism, but now adding a small $i\epsilon$ in the denominator, as shown above, and make the two singularities that lie on the real line be shifted into the complex plane, taking $\epsilon \rightarrow 0$ at the end of the calculations. An important addendum, however, is that, to guarantee the mathematical formalism is correct, for the lower quadrants we must have $t > 0$, since this is necessary to ensure that the circular-arc integrals go to 0. Likewise, for the upper quadrants, $t < 0$. This is linked, physically, to advanced or retarded solutions that describe the propagator.

Another important point is that there are several ways to deform a singularity by adding $i\epsilon$. In principle, ϵ could be negative, or we could even deform as $p_0^2 \rightarrow (p_0 \pm i\epsilon)^2$. All these deformations are, in principle, mathematically valid. What guarantees that only one is correct is that, by computing the integral, we would obtain functions that violate our physical theory or what we consider as causal.

Note that, in each case, the singularities are placed in different quadrants of our contour, and that is what generates different solutions.

Roughly speaking, with negative ϵ , in the form presented, we would obtain solutions in which positive-energy particles evolve into the past, which is not physically plausible given what we observe. Likewise, we could add $i\epsilon$ in $p_0^2 \rightarrow (p_0 \pm i\epsilon)^2$; this would make both singularities lie below the real line or above it. However, as before, we would obtain solutions that do not correspond physically to the equations: we would obtain a superposition of positive- and negative-energy particles evolving together into the future or into the past, which is not plausible. Only the form presented is the one that corresponds to the solution we want. It is a good exercise to compute the integral for all deformations and check the different solutions. The correct solution, with the singularities in quadrants 2 and 4, is proportional to:

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{ie^{-ip_0 t}}{p_0^2 - E_p^2 + i\epsilon} = \frac{e^{-iE_p t}}{2E_p} \Theta(t) + \frac{e^{iE_p t}}{2E_p} \Theta(-t)$$

This can be found by the residue method.

Now, let us obtain these results through the Wick rotation. We can, cleverly, choose a contour that excludes the singularities, i.e., avoid quadrants 2 and 4, so we must rotate over quadrants 1 and 3, avoiding the poles and guaranteeing the formalism presented. In this way, we obtain:

$$\int d^3p \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{idp_0^E}{2\pi} \frac{ie^{p_0^E t}}{-(p_0^E)^2 - E_p^2} = \int d^3p \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0^E}{2\pi} \frac{e^{p_0^E t}}{(p_0^E)^2 + E_p^2}$$

What is the advantage of this? The ambiguity from the pole problem seems to disappear, making the function much better to work with (in fact, the poles of the integral move into the

complex plane and solving this integral involves the residue method, but without requiring the parameter ϵ). Performing the residue method on this integral, we obtain the same results as before, without needing to deform the singularities; this can be verified mathematically. The solution is, for $t > 0$, $\frac{e^{-iE_p t}}{2E_p}$ and, for $t < 0$, $\frac{e^{iE_p t}}{2E_p}$, as before. Remembering that the sign of t changes according to the quadrants. Thus, performing a 90° rotation in p_0 is a way to rewrite the problem in a more convenient form. Note that we could rotate by any angle in this problem, and all would give the same result; however, this one is the most convenient and allows us to write the integral in a better form. The reason for demonstrating all this will become clear right below, in the next example.

The Euclidean action

The second example, and the most important for this thesis, is the Wick rotation applied to the action. Consider the action of a real scalar field in Minkowski spacetime,

$$S[\phi] = \int dt d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right]. \quad (519)$$

Before, it was clear which contour to use, since we wanted to avoid the singularities. Now, in the action, it does not seem to be possible to know at which angle we can rotate and, in most books, we simply perform an arbitrary -90° rotation, with the only justification that it makes the equations more plausible. However, there is a deep reason why the rotation must be done in quadrants 2 and 4 and not in 1 and 3.

To find out which contour should be used, we need, at the very least, the rough form of the function ϕ that is being integrated. We just saw, earlier, that the Feynman-propagator integral is proportional to e^{-iEt} when t is positive and to e^{iEt} when $t < 0$. These solutions are related to the form of $S[\phi]$, which will be proportional to the given functions for each time.

Now observe the behavior of ϕ when we complexify t , so that $t \rightarrow z$, and take $R = |z|$ to infinity. For $t > 0$, which is restricted to quadrants 1 and 4 in the complexification of time, i.e., $\theta \in [\frac{3\pi}{2}, \frac{\pi}{2}]$, the function ϕ is proportional to e^{-iEz} , which, written in the parametrization $z = Re^{i\theta}$, is $e^{-iERe^{i\theta}}$. Since we only want to analyze the exponential decay of this function, to check which arc integrals go to 0, we can take only the real part of the exponent argument, so we obtain: $e^{ER\sin(\theta)}$. For this to decay exponentially and make the circle integral vanish, we need $\sin(\theta) < 0$, and this only happens in quadrants 3 and 4. Since this function is only

valid in quadrants 1 and 4, where $t > 0$ (positive reals), we conclude that the rotation must be performed within quadrant 4. Performing the same analysis for the function $t < 0$, when ϕ becomes proportional to e^{iEt} , we conclude that the rotation must be performed in quadrant 2, so that, as stated earlier, the rotation must be performed clockwise, opposite to the rotation we did for momentum. This is directly linked to the form of our field and to the locations of the singularities in p_0^E . And there is even a certain beauty in that.

In this way, the rotation proceeds as follows:

$$\begin{aligned} S[\phi] &= -i \int d\tau d^3x \left[-\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] \\ &= i \int d\tau d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]. \end{aligned} \quad (520)$$

$$iS[\phi] = -S_E = - \int d\tau d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] \quad (521)$$

Note that this leads us to a positive-definite $S_E \geq 0$, which will be very useful in the path-integral formulation, since it guarantees exponential damping and, therefore, a simpler analysis of convergence. This integral is equivalent to making the transformation $t = -i\tau$ while keeping the same integration limits.

Observe that there is a somewhat deep connection between the fact that the contour for p_0 (energy) and the one for time must be opposite. Mathematically, this ensures that the circular arcs go to 0, but we can interpret physically this need as being related to the physical causality of positive- and negative-energy wave functions.

D Appendix D | Transition Probability

First, to explicitly derive the structure of Eq. 193, which may look a bit “pulled out of a hat” when it appears for the first time, we can work with a simpler quantum case: a one-dimensional potential barrier, restricted to the interval $-a < x < a$, with a particle of energy $E < V(x)$ inside the classically forbidden region. This well-known problem allows us to compute the wavefunction before and after crossing the barrier and, thus, obtain the transition probability associated with tunneling.

The idea is simple. Before the barrier, we have an incident wave, and after the barrier, a transmitted wave. Schematically, we can write [37, 54–56]

$$\psi_{\text{in}}(x) \sim I(x)e^{i\varphi(x)}, \quad \psi_{\text{out}}(x) \sim T(x)e^{i\varphi(x)}. \quad (522)$$

Therefore, the transmissivity of the barrier, i.e. the tunneling probability, will be proportional to the ratio between the squared modulus of the transmitted wave and the squared modulus of the incident wave. Using the WKB approximation, we obtain

$$\begin{aligned} \text{prob} &\propto \frac{|\psi_{\text{out}}(x)|^2}{|\psi_{\text{in}}(x)|^2} \approx \left(\exp \left\{ -\frac{1}{\hbar} \int_{x_1}^{x_2} |p(x)| dx \right\} \right)^2 \\ \text{prob} &\propto e^{-2\gamma}, \\ |p(x)| &\equiv \sqrt{2m[V(x) - E]}, \\ \gamma &\equiv \frac{1}{\hbar} \int_{x_1}^{x_2} |p(x)| dx. \end{aligned} \quad (523)$$

Here x_1 and x_2 are the classical turning points, defined by

$$V(x_1) = V(x_2) = E. \quad (524)$$

In the example of an approximately rectangular barrier, we can take $x_1 = -a$ and $x_2 = a$. This calculation can be found explicitly in [54], section I, supplement H_I , or in [55], in chapter 8. I recommend looking at either of these references if one wants to see the detailed derivation of the calculations.

In this scenario, we interpret $p(x)$ as the classical momentum, with an important caveat: inside the barrier, this interpretation is only formal. In classical mechanics, a particle cannot be in a region where its total energy is smaller than the potential energy, since this would require a negative kinetic energy. In fact,

$$p(x) = \sqrt{2m [E - V(x)]} \quad (525)$$

becomes imaginary when $E < V(x)$. Therefore, in the classically forbidden region, we use

$$|p(x)| = \sqrt{2m [V(x) - E]}. \quad (526)$$

What this equation tells us is that tunneling is a purely quantum effect. The particle has a nonzero probability of crossing a region that would be classically forbidden. In this way, the result above is a semiclassical approximation: the WKB approximation. In intuitive terms, it consists of carrying out the quantum calculations assuming that $\hbar$ is small compared with the typical scale of variation of the action. Equivalently, we can say that the characteristic wavelength of the particle varies slowly compared to the scale on which the potential changes.

Still, it remains to clarify how to identify an action in Eq. 523, since the expression appears as an integral in dx and not as an integral in time. By a simple dimensional analysis, we already see that

$$\int |p(x)| dx \quad (527)$$

has units of action. But we can understand this more explicitly.

Inside the barrier, it is natural to introduce Euclidean time. The conserved energy can be written as

$$E = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 + V(x) \quad (528)$$

in real time. Since in the forbidden region $E < V(x)$, the real-time velocity would be imaginary. After the Wick rotation, $t = -i\tau$, we move to a Euclidean trajectory, for which

$$\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 = V(x) - E. \quad (529)$$

Thus,

$$\frac{dx}{d\tau} = \sqrt{\frac{2[V(x) - E]}{m}}. \quad (530)$$

Now we can rewrite the WKB integral:

$$\begin{aligned} \int_{x_1}^{x_2} |p(x)| dx &= \int \sqrt{2m[V(x) - E]} \frac{dx}{d\tau} d\tau \\ &= \int \sqrt{2m[V(x) - E]} \sqrt{\frac{2[V(x) - E]}{m}} d\tau \\ &= 2 \int [V(x) - E] d\tau. \end{aligned} \quad (531)$$

But, using

$$\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 = V(x) - E, \quad (532)$$

we can also write

$$2[V(x) - E] = \frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) - E. \quad (533)$$

Therefore,

$$\int_{x_1}^{x_2} |p(x)| dx = \int d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) - E \right]. \quad (534)$$

This result is very important. It shows that the WKB exponent is associated with a Euclidean action, but with the initial energy E subtracted. In other words, the quantity that appears in tunneling is the Euclidean action relevant for crossing the barrier, measured relative to the initial state:

$$S_E^{\text{red}} = \int d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) - E \right]. \quad (535)$$

Therefore,

$$\gamma = \frac{1}{\hbar} S_E^{\text{red}}. \quad (536)$$

The tunneling probability then becomes

$$\text{prob} \propto \exp \left[-\frac{2}{\hbar} S_E^{\text{red}} \right]. \quad (537)$$

This is the essential connection with what we did in the chapter. Tunneling is exponentially suppressed by a Euclidean action. The larger the barrier, whether in height or in width, the larger the action required to cross it and the smaller the transition probability.

The difference between this example and the false-vacuum case in field theory is that, in field theory, the coordinate x is replaced by an entire field configuration $\phi(\tau, \mathbf{x})$. The simple integral over a trajectory becomes a functional integral over field configurations. Still, the logic remains the same: in the semiclassical approximation, the transition probability is dominated by a classical Euclidean configuration, and the exponent is given by the Euclidean action evaluated on that configuration.

That is why, in the false-vacuum problem, we write schematically

$$\Gamma \sim A e^{-B}, \quad (538)$$

where

$$B = S_E[\text{bounce}] - S_E[\text{false vacuum}]. \quad (539)$$

That is, B is the difference between the Euclidean action of the bounce configuration and the action of the trivial false-vacuum configuration. This subtraction is analogous to the $-E$ term that appeared in the WKB problem: we are not interested in the absolute energy of the initial state, but in the additional Euclidean cost required to realize the transition.

Thus, we conclude that the physical motivation for the proposed formula comes directly from the WKB approximation. Tunneling, both in quantum mechanics and in quantum field theory, is controlled by a Euclidean action. The prefactor contains information about fluctuations around the dominant solution, while the exponent carries the leading semiclassical suppression.

For the finite-temperature case, the line of reasoning is the same, but the relevant action becomes $S_3(T)/T$, as discussed in the main text. A deeper discussion of these ideas can be found in [37] and [56]. The most relevant point for this thesis is to note that the tunneling probability is, indeed, associated with the action.

E Numerical benchmark tables

This appendix summarizes the main numerical benchmark points used in the discussion of Chapter 5. The full scan data are available in the output files of the numerical pipeline. Here, we include only the most relevant information for reconstructing the physical discussion: the scan intervals where a first-order phase transition with successful nucleation was found, and representative benchmark points selected by the largest value of the gravitational-wave peak amplitude in each subscan.

Only points with status `FOPT_with_nucleation` are included. Points with no first-order transition, no successful nucleation, or numerical transition errors were omitted from the benchmark tables. The quantities shown are the critical temperature T_c , the nucleation temperature T_n , the supercooling measure

$$\frac{T_c - T_n}{T_c}, \quad (540)$$

the transition strength α , the inverse duration parameter β/H_* , the peak frequency f_{peak} and the peak amplitude $h^2\Omega_{\text{peak}}$.

Unless otherwise stated, all temperatures are given in GeV, f_{peak} is given in mHz, and the higher-dimensional coefficients are given in powers of TeV,

$$\frac{c_6}{f^2}; [\text{TeV}^{-2}], \quad \frac{c_8}{f^4}; [\text{TeV}^{-4}]. \quad (541)$$

Viable scan intervals

Table 1 shows the intervals in which the numerical scan found a first-order phase transition with successful nucleation. These intervals should not be interpreted as absolute physical boundaries of the models, but as the viable regions found inside the finite scan windows explored in this thesis.

Sector	Fixed parameters	Daisy	Scan variable	Viable interval	Successful points	Total points
Glauber	$\Lambda = 1000$	no	C	[3.640, 3.844]	94	100
Glauber	$\Lambda = 1000$	yes	C	[3.640, 3.839]	89	100
Glauber	$\Lambda = 2000$	no	C	[6.230, 6.650]	90	100
Glauber	$\Lambda = 2000$	yes	C	[6.230, 6.650]	87	100
$\phi^6 + \phi^8$	$c_8/f^4 = 0$	no	c_6/f^2	[3.200, 3.531]	83	100
$\phi^6 + \phi^8$	$c_8/f^4 = 0$	yes	c_6/f^2	[3.200, 3.531]	83	100
$\phi^6 + \phi^8$	$c_8/f^4 = 2$	no	c_6/f^2	[3.000, 3.380]	95	100
$\phi^6 + \phi^8$	$c_8/f^4 = 2$	yes	c_6/f^2	[3.000, 3.380]	95	100
$\phi^6 + \phi^8$	$c_8/f^4 = 5$	no	c_6/f^2	[2.800, 3.160]	87	100
$\phi^6 + \phi^8$	$c_8/f^4 = 5$	yes	c_6/f^2	[2.800, 3.160]	90	100
Combined C -scan	$c_6/f^2 = 1, c_8/f^4 = 0$	no	C	[3.200, 3.503]	26	31
Combined C -scan	$c_6/f^2 = 2, c_8/f^4 = 0$	no	C	[2.700, 3.038]	30	31
Combined C -scan	$c_6/f^2 = 1, c_8/f^4 = 5$	no	C	[3.000, 3.350]	29	31
Combined c_6 -scan	$C = 1.0, c_8/f^4 = 0$	no	c_6/f^2	[3.200, 3.450]	30	31
Combined c_6 -scan	$C = 3.2, c_8/f^4 = 0$	no	c_6/f^2	[1.000, 1.700]	27	31
Combined c_6 -scan	$C = 3.2, c_8/f^4 = 5$	no	c_6/f^2	[0.700, 1.250]	31	31

Table 1: Viable scan intervals found in the numerical analysis. Only points with successful first-order phase transition and nucleation were included.

Benchmark points selected by peak amplitude

Table 2 lists, for each subscan, the point with the largest value of $h^2\Omega_{\text{peak}}$. These points are not meant to exhaust the scan, but to provide representative numerical benchmarks for the strongest gravitational-wave signals found in each case.

Sector	Daisy	Λ	C	c_6/f^2	c_8/f^4	T_c	T_n	$(T_c - T_n)/T_c$	α	β/H_*	f_{peak}	$h^2\Omega_{\text{peak}}$
Glauber	no	1000	3.842	–	–	73.13	34.88	0.523	0.600	41.7	0.281	1.94×10^{-10}
Glauber	yes	1000	3.839	–	–	74.28	38.46	0.482	0.407	125.9	0.931	1.04×10^{-11}
Glauber	no	2000	6.650	–	–	80.16	59.52	0.257	0.081	402.5	4.603	2.36×10^{-14}
Glauber	yes	2000	6.646	–	–	81.38	60.98	0.251	0.073	550.1	6.454	9.79×10^{-15}
$\phi^6 + \phi^8$	no	–	–	3.527	0	87.13	61.80	0.291	0.068	266.2	3.184	3.49×10^{-14}
$\phi^6 + \phi^8$	yes	–	–	3.438	0	90.57	70.05	0.227	0.041	199.0	2.689	1.59×10^{-14}
$\phi^6 + \phi^8$	no	–	–	3.360	2	87.15	62.49	0.283	0.065	130.4	1.570	1.27×10^{-13}
$\phi^6 + \phi^8$	yes	–	–	3.331	2	89.29	66.47	0.256	0.050	104.3	1.326	1.01×10^{-13}
$\phi^6 + \phi^8$	no	–	–	3.115	5	87.03	62.97	0.276	0.062	94.1	1.137	2.20×10^{-13}
$\phi^6 + \phi^8$	yes	–	–	3.160	5	87.59	61.04	0.303	0.070	211.9	2.490	5.93×10^{-14}
Combined C -scan	no	1000	3.503	1.000	0	76.19	41.17	0.460	0.324	123.5	0.975	6.79×10^{-12}
Combined C -scan	no	1000	3.038	2.000	0	80.03	48.82	0.390	0.169	265.3	2.490	3.45×10^{-13}
Combined C -scan	no	1000	3.350	1.000	5	76.41	39.12	0.488	0.392	139.3	1.053	7.89×10^{-12}
Combined c_6 -scan	no	1000	1.000	3.450	0	87.74	64.46	0.265	0.058	209.9	2.607	3.59×10^{-14}
Combined c_6 -scan	no	1000	3.200	1.700	0	78.67	45.58	0.421	0.220	212.8	1.859	9.92×10^{-13}
Combined c_6 -scan	no	1000	3.200	1.250	5	79.15	51.27	0.352	0.136	299.8	2.948	1.61×10^{-13}

Table 2: Representative benchmark points selected by the largest gravitational-wave peak amplitude in each subscan. The quantity f_{peak} is given in mHz. The peak amplitude corresponds to $h^2\Omega_{\text{peak}}$.

Comments on the benchmark selection

The strongest signal among the benchmark points appears in the pure Glauber sector with $\Lambda = 1000$ GeV and no daisy resummation, where the scan reaches

$$h^2\Omega_{\text{peak}} \simeq 1.94 \times 10^{-10}, \quad f_{\text{peak}} \simeq 0.281 \text{ mHz}. \quad (542)$$

This point is also associated with significant supercooling, $(T_c - T_n)/T_c \simeq 0.523$, and a relatively small value of $\beta/H_* \simeq 41.7$.

The inclusion of daisy resummation generally reduces the peak amplitude in the pure Glauber benchmarks shown here. For $\Lambda = 1000$ GeV, the largest peak changes from

$$h^2\Omega_{\text{peak}} \simeq 1.94 \times 10^{-10} \quad (543)$$

without daisy to

$$h^2\Omega_{\text{peak}} \simeq 1.04 \times 10^{-11} \quad (544)$$

with daisy. This reflects the fact that thermal resummation modifies the geometry of the potential, and therefore changes T_n , α , β/H_* and the final gravitational-wave prediction.

In the pure $\phi^6 + \phi^8$ sector, the largest benchmark peak in the table occurs for

$$c_8/f^4 = 5 \text{ TeV}^{-4}, \quad c_6/f^2 \simeq 3.115 \text{ TeV}^{-2}, \quad (545)$$

without daisy resummation. Although this signal is smaller than the strongest pure Glauber benchmark, it illustrates that polynomial scalar deformations can also generate first-order transitions with potentially interesting gravitational-wave spectra.

The combined sector shows explicitly the cooperative effect discussed in Chapter 5. In the C -scan, values of C smaller than those required in the pure Glauber case can still produce strong signals once polynomial deformations are included. For example,

$$C = 3.350, \quad c_6/f^2 = 1.0 \text{ TeV}^{-2}, \quad c_8/f^4 = 5.0 \text{ TeV}^{-4} \quad (546)$$

produces

$$h^2\Omega_{\text{peak}} \simeq 7.89 \times 10^{-12}. \quad (547)$$

This is the numerical expression of the main physical message of the combined analysis: different scalar-sector deformations can cooperate, shifting the viable region of parameter space and producing detectable signals without requiring a single mechanism to dominate the entire transition.

These tables should therefore be read as a compact numerical map of the thesis results. The full scans contain more points and should be used for detailed interpolation or reproduction of the figures, while the benchmark points listed here provide the values most useful for comparison, discussion and future extensions.

F Poems

On life

"I no longer expect anything from life. -but what does life expect from you?"

We must, throughout the course of our life, find a meaning for it, a meaning of life. However, along the way, we realize that life does not seem to have meaning by itself, there does not seem to exist a universal purpose for everyone, a rule that everyone must follow.

Thus, we are entrusted with a task that seems impossible, how to find meaning for something that does not have a single meaning?

Life seems to play a trick on us, we seek a meaning for being alive, here and now, but life gives us a freedom of meaning that makes us feel lost, without the map that tells us where to go and what we should be. It is precisely this freedom that makes the statement "I no longer expect anything from life" have no meaning, because life never gave you and will never give you anything, besides the freedom to live. But if there is freedom to live as one wishes, can we choose not to live? Can we choose that there is no meaning at all in life?

Here, we must ask ourselves, what does life expect from us? We cannot expect anything from it, but does it expect something from us?

Life gives us only the freedom to live. Therefore, if there is any meaning in existence it must be in this freedom. In the midst of this freedom of meaning, we have the responsibility to choose a direction, a purpose, which is not absolute and much less universal, but is ours. Your purpose, so to speak, is the answer to the question "what does life expect from you? Why do we not choose to die?" death ends freedom and also, with your responsibility over it.

Life, therefore, does not expect a single answer from all of us, of what we should do with it. However, paradoxically, life expects each one of us, individually, to give a purpose to our freedom. That each one of us has responsibility for our freedom and gives it meaning, whatever it may be.

Life then, does have a meaning, that of being responsible for your freedom, that of seeking your purpose and discovering what life wants from you.

Realizing that there is no absolute purpose, it becomes clear that we should not impose life conditions on others and tell them how to live, what it is to be happy and what it is to be sad. Our freedom therefore, in order to have an internal and external meaning, needs to limit itself to the freedom of the other, and the other's to your freedom.

Once the freedom of another is taken, we destroy our own, which becomes deprived of meaning. It is not necessary to have an omnipotent being for us to realize that this minimal goodness, that everyone should live freely, should be respected.

All of us carry the responsibility of freedom and, therefore, of giving meaning to it. The purpose of life itself then is the very search for purpose. Life loses its meaning when we empty ourselves of this freedom and think that life should give us a map, that we should expect something from it. It never gave us anything and, it is in this nothing, that we fill in whatever we want. This freedom, of choosing how to face life and how to deal with it, cannot be taken from us. We always have the freedom to choose who we are.

Who knows why to live, endures any how.

On suffering

Right from the beginning of our lives, for many fortunate ones, we are graced with love and realize that love gives meaning to our existence. To love and to be loved surely make us believe that one of the purposes of life is to expand this feeling. Later, we are introduced to creativity, arts and work, we also give life a meaning in our labor, in what we can do best and in what makes us happy. Researching, cooking, reading, creating, inventing. All activities also seem to give meaning to our existence and, therefore, beyond love, we learn that life also has meaning in what we can build.

However, at some moment of our life, for some very early and for some very late, we are placed before a situation of suffering. A situation that sometimes seems to have been imposed on us, that our freedom of choice was taken from us and we are being forced to go through terrible suffering. Sometimes we blame ourselves, sometimes we blame third parties for this suffering. In the midst of it, we begin to believe that it is not worth living a life of suffering, a life that seems to have taken some freedom from us. It is precisely this loss of freedom or the idea that we are trapped that seems to deprive our life of meaning and, therefore, a life that is not worth being lived.

However, for those who have already had the luck of leaving suffering and overcoming it, we learn that suffering makes us grow and mature. We learn that suffering is inevitable and, part of our life, is in the midst of suffering.

Thus, if there is meaning in living, there must also be meaning in suffering. Our mistake is thinking that we lose our freedom when suffering, even if some external agent has taken from

us some freedom of what to do – which is inevitable in a society – we still have the infinite internal freedom to choose how we will behave before this suffering, how we will face it. Even in the most extreme situations, no one is deprived of their deepest freedom, the freedom to face life as they wish, to face suffering and pain with pride and honor, and not in another way. To face suffering as something that has meaning. Just like a laboratory mouse, which is continuously pricked to test vaccines, perhaps spends its life without seeing meaning in its suffering, even if we see it. Just like it, we can spend life without understanding our suffering, if we do not realize that it is in it that life also happens. There is a purpose to what we suffer, sometimes we can only see this purpose after the suffering is far away, just as we seem to only see happiness when it has already passed. There is honor in suffering, after we overcome suffering, we see life as no one else sees it and find meaning and happiness where many find nothing.

The uniqueness of each individual, just like the uniqueness of each trajectory made by each individual, gives an ultimate meaning to suffering, that of bearing the responsibility of maintaining your freedom and of realizing your life in the way that only you can do. Suffering makes us look inward and realize, that even before external chaos, we possess an inner freedom. In the midst of suffering we are forced to elevate our spirit and, for many who have never gone through any suffering yet, they will hardly understand.

Life does not have meaning in only one place, but in all of them. Only someone who went through a certain situation has one more perspective on life, which no one else will have. Suffering shows us the uniqueness of our story, of our pain, of our joy, the uniqueness of our being.

In the midst of it, we can perceive our greater freedom, that of choosing how we will be, how we are going to be and how we are. That of realizing, even when a companion leaves, when longing exists, when someone hits us, that even so we can face life however we want and that there is meaning in everything.

Who knows why to suffer, endures any pain.

On happiness

We have spoken about life and suffering. It seems plausible to me to go on a little about what happiness is, for me.

We usually think that happiness is the opposite of sadness and suffering.

However, if the purpose of life is to be happy, happiness cannot be the opposite of suf-

fering, which is part of life. Believing that happiness is the opposite of sadness and that it is the purpose of life creates a situation in which, when there is no happiness, life is deprived of meaning. However, we have just seen that this is not true.

Thus, for me, the opposite of happiness is not suffering, but rather apathy.

Happiness also, for me, does not have the usual meaning, happiness for me, is not smiling and being cheerful.

If we understand happiness as the meaning that constitutes the feeling related to the understanding of your purpose for living, then happiness must be associated, before joy, with fullness.

Happiness is different from being well and smiling at everything, but rather being full with all situations. In conformity with what is imposed on you, because one knows that there is meaning in pain, and in conformity with your responsibility, with your freedom, in conformity that your life has a meaning, of seeking or even having your purpose. Happiness, thus, is the feeling when we are full toward life. When we accept it as it was given to us, by the social conditions that were imposed on us and also, joining this acceptance, we know of our inner freedom and of our responsibility of being able to face life however we want and, knowing, that the search for purpose exists and that this search already gives us a purpose. Happiness is the feeling of being full with the situations that surround us and knowing that, in the midst of what seems to have been destined to us, imposed and is unchangeable, there still exists a freedom to live, to choose how we want to face the situation and what we want to do to shape our destiny. Even if the future is not totally optional, there is still part of destiny that is not absolute. To face and to know how to join the opposite ideas of accepting what cannot be transformed with the power of choosing what still can, what one is free to do, is this feeling that I can call happiness.

Thus, there can be happiness even in the midst of suffering, of course. Even in extreme situations, we have moments and people who manage to overcome them, remembering that there is meaning in that and that, there is freedom in their existence. Being full with this, we manage to be happy even in the midst of chaos.

When we surrender to apathy, the sensation that the situation does not matter, that life no longer has meaning, it is then that we lose sight of happiness. When we think that it is found in the past, or will be found in the future, it is then that we become apathetic to the present and, thus, lose the meaning of living.

To be happy it is necessary to realize that, if we want to understand happiness as the

feeling that represents life and that life does not have a single meaning, happiness must be everywhere.

Happiness thus, is the medium in which life happens, it is everywhere. It is the air we breathe, the blue of the sky and the infinity of the sea. Happiness is, and must be, everywhere. If happiness gives meaning to existence then it itself is found everywhere, in suffering, in sadness and in joy, happiness can be obtained.

Paradoxically, in the moment we realize that we are happy, that we are happy in this moment and, for those who have already experienced the feeling of being full with life, this feeling seems to carry a certain apathy. Apathy not in the sentimental sense, but in the sense that we could endure any situation, that the situation does not matter, the fullness and recognition of it makes us think that we would be happy in any circumstances. That our happiness is in our hands, just like life. But this apathy does not come with the apathy of feeling, we still feel everything, we are joyful and sad, only apathetic to situations, because we realize that seeing happiness and having it depends only on ourselves. This apathy is different from sentimental apathy, the apathy of meaning, this one is indeed opposite to happiness.

Thus, being sad, you can be happy, if you want to associate happiness with the feeling of living it must be associated with the feeling of fullness of meaning.

Who knows how to be happy, knows why to live.

On love

For me, love is the most difficult of feelings. Maybe I am still too young to understand, or, maybe, love is truly impossible to explain.

Just like happiness, love seems to be found anywhere. Love seems to assume various forms and ways, love in the form of friendship, love for a work or result, love for people, love for the work of others.

Love seems, to me, the feeling of when we realize the greatness of our own existence and the infinite value of our life and also the infinite value of the life of the neighbor and of the other, of the work of the other. Love, just like happiness, is everywhere, thus love is also found when we realize the value of sharing the journey with others, the value of the road we travel. Love is the feeling that seems to attribute a certain value to you, to people and to the world. Paradoxically, the value of something loved is priceless and, one life alone has the same weight as any other.

Love is realizing that life has a value, yours and that of the neighbor, and this value is infinite. Love is the perception of the beauty of our world, of people and of situations.

If happiness is the medium in which life exists, and the purpose of life is the very search for purpose, then I understand love as that in which we are seeing our current purpose, in which we see what is worth living for. Love gives priceless value to the loved object and this loved object gives us the meaning of our existence, our purpose. Love changes with time, place, form and way, love can exist in more than one person in more than one moment and in more than one place, but it represents that in which we see the current purpose of our life.

Love represents the meaning through which we see our existence. This arbitrary meaning that changes, the reason we believe why we should live, is what we call love.

Being loved makes us realize the priceless value of our own life, of how deeply we are connected to the meaning of the existence of another.

Loving makes us realize the priceless value of the life of the neighbor and of the loved object, the feeling that we would die for others, because only our life covers the priceless value of another.

Loving the beauty of life, of other people's work and of your own, makes us realize the priceless value of purpose, that is, of each life itself.

Love, for me, represents in the last analysis, the current place where we put the meaning of our existence, that which we choose as why to live. Because happiness exists everywhere, love is a reflection of its priceless value, of life.

Who understands why they love and why they are loved, knows the priceless value of living.

Why physics?

Since I was a child, I catch myself thinking about the stars, about the world beyond this plane we inhabit, what might be beyond the horizon? Beyond the sky? Beyond what we can see? Could there be a distant land, where all the secrets of the universe are unveiled and nothing was left but love, the only necessary mystery.

I have always kept thinking, throughout life, how I wanted to discover everything, what everything is for, what everything is made for, what everything exists for and why. These questions, many times, seem to be beyond the real, perhaps in the metaphysical field, or even philosophical, questions for which there is no answer, unlike physics, which we believe to be

exact, is it not?

I faithfully disagree with this statement. For me, physics is as real as the imagination of a child in their fantasy games. And is physics not that? A great game? From early on, we are invited to imagine the most unreal and diverse problems; little balls falling, little balls spinning around other little balls, big balls of gas and little balls that are as tiny as imagination allows one to think. We are invited to imagine fields that possibly do not exist and even so form light. Invited to imagine that time is relative, that space is a fabric and worst of all, that there exists an inertia tensor.

If there is something I learned, in these years studying physics, it is that it is not actually an exact science. Physics, in large part, is also philosophy, arts and imagination. Doing physics is also having the capacity to imagine the most unbelievable situations and experiments, the most inconceivable theories to try to describe the reality we see, to try to describe this reality through concepts that are in truth imagined and invented; doing physics is having the gift of using imagination to get closer and closer to what we believe to be reality. A reality that always changes when a new concept emerges, this inconstant, unreal reality.

And is this contradiction not beautiful? For me the choice of physics was never about finding reality but rather about freeing myself, allowing myself to imagine, imagining while trying to get somewhere, ever closer to this reality, unreachable, the beauty of physics resides in this contradiction, in this paradox, in the fact that we will always have something new to discover, rediscover and change, to advance but never get there, perhaps physics for me reflects a little the beauty of what I believe life to be. It is not necessary for there to exist a purpose or a meaning in life, you are free to create something that has meaning and purpose for you. There is a certain beauty in insignificance, it allows you to place any meaning you like.

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