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On the bouncing completion of eternal inflation

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"ON THE BOUNCING COMPLETION OF ETERNAL INFLATION"

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Nothing in life is to be feared, it is only to be understood.

Marie Curie

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To my mother, father and sister, whom I love dearly and whose support has always been essential. The Universe might not be eternal, but my gratitude to all of you certainly is.

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Abstract

The duration of our Universe, all of its content, and whether it had a beginning or has always existed has long been the subject of intense investigation. As free particles trajectories are given by geodesics, the inquisition of *godesic completeness* and extensions become crucial for the investigation of possible eternal Universes. Nevertheless, General Relativity's invariance under diffeomorphisms imposes an additional difficulty to realize whether the incompleteness has physical significance or if it is merely an inappropriate coordinate choice. In this context, the singularity theorems provide sufficient conditions for geodesic incompleteness without recurring to coordinate charts. However, kinematic alternatives for classifying incomplete space-times that are expanding have been proposed, leading to the *Borde-Guth-Vilenkin* (BGV) theorem, where no restriction on the matter fields are necessary, such as energy conditions. Notwithstanding, whether the space time admits a metric extension that is compatible with General Relativity, i.e, a $\mathcal{C}^2$ extension, needs to be addressed.

In this dissertation, using the pivotal example of the flat patch of the de Sitter space, we manage to find a new global chart for this space - which without considerations of extensibility would be diagnosed as geodesically incomplete. Furthermore, we developed a general protocol for a C^2 extension of a flat Friedmann-Lemaître-Robertson-Walker metric, and the necessary conditions for its application by exhausting all the possible cases in the asymptotic limit, finding necessary and sufficient conditions for extensibility. The incomplete spaces that violate the assumptions have either a scalar curvature singularity or a parallelly propagated singularity, in which cases no $\mathcal{C}^2$ extension is allowed. Moreover, we discuss results for possible cyclic scenarios proposed in the literature. The results obtained in this work were published in Phys. Rev. **D** 111, 123531 (2025).

Key Words: GEODESIC COMPLETENESS, COSMOLOGICAL MODELS,

SINGULARITY THEOREMS, ANALYTIC EXTENSION.

Chapter 1

Introduction

The question of whether the Universe had a beginning has long pervaded human inquisitions, either in philosophy or in physics [1, 2, 3, 4]. In the physical description of reality, there are four fundamental interactions in our Universe: the strong interaction, responsible for the cohesion of the atomic nucleus, the electromagnetic, which mediates interactions between charged particles, the weak interaction, which is responsible for the decay of particles and gravity, an attractive interaction, described by General Relativity, where the trajectory of particles is dictated by the curvature of the spacetime manifold. On large scales, due to the average electric neutrality of bodies and the large distances considered, the most relevant interaction for the dynamical evolution is gravity and, thus, as far as Cosmological inquiries are concerned, such as the beginning of the Cosmos, Einstein's Field Equations (EFEs) are essential. In this framework, the whole Universe is described by a four dimensional *manifold* $\mathcal{M}$ endowed with a Lorentzian metric $g_{\mu\nu}$, whose geometry is described by the set of coupled non-linear second order Einstein's Field Equations [5],

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where, on the left side of (1.1), we have contractions of the Riemann curvature tensor, i.e, the Ricci tensor and scalar, which associate the curvature of the spacetime manifold $\mathcal{M}$ with the matter and energy content filling the spacetime, given by $T_{\mu\nu}$. The set $(\mathcal{M}, g_{\mu\nu})$ is what we regard as the *spacetime*. The solutions $g_{\mu\nu}$ of EFEs will give us the local notion of interval between two points in $\mathcal{M}$, which shall be extremized over *geodesic* curves. In a sense, geodesics are the generalization of straight lines in Euclidean space, and as such, are used to describe the trajectory of free particles on $\mathcal{M}$. Since all free particles follow geodesic curves on $\mathcal{M}$, they are a natural candidate for the investigation of the eternity of the Universe: if all the free particles in the Universe observe an eternal duration, hence, if all particle trajectories are *complete*, the Universe did not have a beginning, as it is eternal for every observer. This can be conceptualized in terms of the *invariant interval* along geodesics and the conditions for their completeness shall be the investigation of this work. However, in what concerns Cosmology, a few philosophical assumptions, based on observation[1] are inferred, such as the Copernican[2] Principle. According to this principle, we are not in a privileged position in the universe: in other words, what we observe from Earth must be, on large scales, about the same at any other point in the spacetime manifold. Therefore, since we observe the background of the universe to be spatially homogeneous and isotropic [6, 7, 8], we extrapolate this observational fact to the entirety of the spacetime.

Assuming the Copernican Principle, and hence a global foliation of spacetime where the space hypersurface are homogeneous and isotropic everywhere, we can decompose the (3+1) manifold as the product $\mathcal{I} \times \mathbb{R}^3$, or any other homogeneous and isotropic 3-dimensional space, where the cosmic time $t \in \mathcal{I}$. Despite being only a few empirical assumptions, by imposing them on cosmological models we end up with very strong restrictions on the possible geometries of the space sections, as we shall see now.

1.1 The Geometry of a Cosmological Spacetime

By assuming the Copernican principle for the cosmological solutions of EFEs, we assume that there exists a foliation of the spacetime manifold in which the spatial section, for being homogeneous and isotropic, is maximally symmetric and, therefore, possess the maximum number of Killing vectors: due to homogeneity, translations in any of the three spatial directions must leave the metric $g_{\mu\nu}$ invariant, and due to isotropy rotations around the three spatial axis must also keep $g_{\mu\nu}$ unaltered. However, by imposing such a strong principle on the spacetime geometry, we highly restrict the possible scenarios that satisfies it[3].

[1]Some observations such as the Cosmic Microwave Background

[2]Nicolau Copérnico (1473-1543)

[3]For the case of homogeneity, there are only 8 possible 3 dimensional geometries, as stated by the Thurston Conjecture [9]. Additionally imposing isotropy reduces it to 3 possibilities.

In the two-dimensional case, two trivial geometries with the above properties come to mind: the two-sphere $\mathbb{S}^2$ and the plane. We can generalize this notion to a 3-dimensional sphere, embedded in a 4-dimensional space, whose line element will be given by the constraint

$$x^2 + y^2 + z^2 \pm w^2 = \pm\kappa^2, \tag{1.2}$$

where κ^2 is a positive constant parameter. For the positive signs in the constraint equation (1.2) we can easily identify it as a 3-sphere $\mathbb{S}^3$ embedded in the 4 dimensional Euclidean space, while for the negative signs, we have an hyperboloid $\mathbb{H}^3$ embedded in a (3+1)-Lorentzian space $\mathbb{R}^{1,3}$. By differentiating (1.2), we get a relation between infinitesimal displacements

$$\mathrm{d}w = \mp\frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{w} = \mp\frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{\sqrt{\kappa^2 \mp x^2 \mp y^2 \mp z^2}}. \tag{1.3}$$

Therefore, we can write the line element $\mathrm{d}l^2$ in function of the 3 coordinates (x, y, z):

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \pm \frac{(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z)^2}{\kappa^2 \mp x^2 \mp y^2 \mp z^2}. \tag{1.4}$$

However, we see from (1.2) that Cartesian coordinates are degenerate since in these coordinates equation (1.2) does not uniquely specifies the point, as for any coordinate (x, y, z), there corresponds a distinct antipodal point. We can make a coordinate transformation into the more suitable spherical coordinates in the standard manner:

$$\begin{aligned} x &= r\cos\phi\sin\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\theta. \end{aligned} \tag{1.5}$$

In these coordinates, since $x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = r\mathrm{d}r$, the line element is

$$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \pm \frac{r^2\mathrm{d}r^2}{\kappa^2 \mp r^2} = \left[\frac{\mathrm{d}r^2}{1 \mp \frac{r^2}{\kappa^2}}\right] + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1.6}$$

Defining the curvature of the spatial section as $k \equiv 1/\kappa^2$, we see that, as the radius goes to infinity $\kappa \to \infty$, the curvature k goes to zero, and we recover the metric of

3-dimensional Euclidean space in spherical coordinates:

$$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{1.7}$$

where the range of the coordinates is as usual: $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. It is useful to re-scale the line element (1.6), such that the curvature of the spatial sections only assume the discrete values

$$k \in \{-1, 0, 1\}. \tag{1.8}$$

The absolute value of the curvature can later be incorporated into the *scale factor* of the 4 dimensional metric, which will give the size of each section at a given time. Additionally to the flat case, where $k = 0$, we can also obtain the metric for the scenario with spatial negative curvature $k = -1$. Defining coordinate χ, such that $\mathrm{d}\chi^2 = \frac{\mathrm{d}r^2}{1+r^2}$, by simple integration we have

$$\int \mathrm{d}\chi = \int \frac{\mathrm{d}r}{\sqrt{1+r^2}} \qquad \Rightarrow \qquad \chi(r) = \sinh^{-1}(r), \tag{1.9}$$

and the metric for the $k = -1$ case can be written as

$$\mathrm{d}l^2 = \mathrm{d}\chi^2 + \sinh^2(\chi)(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2). \tag{1.10}$$

Since r is defined over the positive branch of $\mathbb{R}$, the coordinates (χ, θ, ϕ) are defined over $\chi \in [0, \infty)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Finally, for the case where $k = +1$, we find

$$\int \mathrm{d}\chi = \int \frac{\mathrm{d}r}{\sqrt{1-r^2}} \qquad \Rightarrow \qquad \chi(r) = \sin^{-1} r, \tag{1.11}$$

$$\mathrm{d}l^2 = \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{1.12}$$

with coordinates limited to $\chi \in [0, \pi)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Now that the possible homogeneous and isotropic spatial geometries have been determined, the only way to add an evolution in time without loosing homogeneity and isotropy is to make the spatial geometry dependent on time through a function $a(t)$ which has no dependency on the spatial coordinates

$$\mathrm{d}l^2(t) = a^2(t)[\mathrm{d}\chi^2 + r(\chi)^2(\mathrm{d}\theta^2 + \sin^2\theta)]. \tag{1.13}$$

Note that this element is only a 3-dimensional spatial interval, hence, it is not an invariant under arbitrary coordinate transformations. This highly symmetric line element is the spatial section as measured by a very special observer, which we refer to as the *co-moving observer*, in whose coordinates all the symmetries of the space-time are made explicit through the full space-time metric

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}\chi^2 + r(\chi)^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2)]. \tag{1.14}$$

This is called a *Friedamann-Laimaître-Robertson-Walker* (FLRW) metric. As perceived through the element above, this metric is not necessarily invariant through time translation.

Backed up by the evidences of spatially flatness of the spacetime, let us restrict our attention to the flat FLRW case ($k = 0$) for the time being. In such cases, the co-moving metric is given by (1.14), with $\chi = r$:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)]. \tag{1.15}$$

We can investigate the behavior of geodesics co-moving to the frame with coordinates (t, r, θ, ϕ). Since the invariant interval along two events is the proper time measured by the observer in which these two events are at the same spatial coordinate, along a co-moving geodesic, we have

$$\mathrm{d}s^2 = -\mathrm{d}t^2 \qquad \Rightarrow \qquad \Delta s \propto \int_{t_i}^{t_f} \mathrm{d}t. \tag{1.16}$$

Thus we see that for flat FLRW, the invariant interval of co-moving observers will depend only on the interval of cosmic time for which the function $a(t)$ is defined. In particular, for models where $t \in (-\infty, t_0)$, the interval along any co-moving geodesic diverges, i.e, the co-moving observers experience no beginning for the Universe, rendering them the perception of an eternal Universe. The same, nevertheless, does not necessarily occurs for free particles in movement with relation to the frame of metric (1.15) (which we will refer to as the *non co-moving observer*). This would seem to be in contradiction with the *relativity principle*, in which all observers are equivalent. Let us consider a non space-like radial geodesic curve parametrized by λ, which is the proper time τ in the case of time-like geodesics or an affine parameter in the case of light-like geodesics. A vector tangent to the curve will be given by $u^\mu \equiv (\mathrm{d}t/\mathrm{d}\lambda, \mathrm{d}r/\mathrm{d}\lambda, \mathrm{d}\theta/\mathrm{d}\lambda, \mathrm{d}\phi/\mathrm{d}\lambda) = (u^t, u^r, 0, 0)$. Given metric (1.15) and the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0, \tag{1.17}$$

we can write the equations of motion for a non co-moving geodesic[4]. For the time component, $\mu = 0$, we have

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + a\dot{a}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = 0, \tag{1.18}$$

and for the radial component, $\mu = 1$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + 2\frac{\dot{a}}{a}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right) = 0. \tag{1.19}$$

Since $\dot{a}\mathrm{d}t/\mathrm{d}\lambda = \mathrm{d}a/\mathrm{d}\lambda$, by multiplying the latter equation by $a^2(t)$, we can rewrite it as

$$a^2(t)\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + 2a(t)\frac{\mathrm{d}a}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[a^2(t)\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right] = 0. \tag{1.20}$$

Thus, we see that, along the non co-moving geodesic parametrized by λ, we have a constant of motion, which we will define as

$$a^2(t)\frac{\mathrm{d}r}{\mathrm{d}\lambda} \equiv v_0 = \mathrm{const}. \tag{1.21}$$

Thence, we can always write the radial component of the non co-moving observer as

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \frac{v_0}{a^2}. \tag{1.22}$$

By choosing the normalization of the tangent vector to be either $\beta \equiv 0, -1$, for the null and time-like case, respectively:

$$-\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + a^2(t)\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \frac{v_0^2}{a^2} = \beta. \tag{1.23}$$

[4]See Appendix A for Christoffel's symbols and curvature tensors evaluation.

Thereby, we can relate the co-moving time coordinate t with the affine parameter for null geodesics, yielding:

$$\mathrm{d}\lambda = \frac{1}{v_0} a(t) \mathrm{d}t. \tag{1.24}$$

For time-like non co-moving geodesics we find

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\sqrt{1 + \frac{v_0^2}{a^2}}}. \tag{1.25}$$

Given that the invariant interval $\Delta\tau$ and the affine parameter depend on the behavior of the function $a(t)$, we might have a finite invariant interval even in cases where the cosmological time is defined up to $t \to -\infty$, and, since the proper time $\Delta\tau$ is an invariant, i.e, it is unaltered under coordinate transformations, we say these curves are *incomplete*.[5] Kinematically what happens is that, depending on the behavior of the scale factor, as the non co-moving observer approaches the asymptotic past, it observes a time dilation of the co-moving interval if $a(t \to -\infty) \to 0$, which results in an infinite interval for the co-moving observer, while the non co-moving one reaches the asymptotic limit in a finite proper time. Additionally, due to spatial contraction, the spatial sections of a co-moving observer contract infinitely in the past boundary, but the same might not be true for a non co-moving frame, which might have finite, non null spatial section. Therefore, the volume observed by the non co-moving frame can be finite in the past boundary, in which cases it might be continuously extended across the past hypersurface.

1.2 Perfect Fluid with Linear Equation of State

An usual construction of the matter fields that fill up the homogeneous and isotropic cosmological space is that the matter can be modeled by a perfect fluid with density $\rho(t)$ and pressure $p(t)$[6]. In a frame co-moving to metric (1.14), the energy-momentum tensor is given by

[5]For the light case, λ is not an invariant. However, any other affine parameter will be of the form $\lambda' = a\lambda + b$, and thus, if λ is finite, so is λ'.

[6]Note that, in a general case $\rho = \rho(t, \vec{x})$ and $p = p(t, \vec{x})$. Nevertheless, due to isotropy, the fluid's properties cannot depend on the direction of $\vec{x}$ and due to homogeneity it cannot depend on the distance $|\vec{x}|$.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{1.26}$$

where u_μ is the four-velocity of a co-moving observer/fluid, which are at mutually at rest with respect to each other. A further step to determine the dynamics of the fluid's property is through energy conservation: although in general we lose the notion of global energy conservation, locally, we should always expect that the variation in the energy density in a infinitesimal element, in the absence of creation (or annihilation) of particles, is due solely to the escape/entrance through its border, i.e, the four-divergence of T^μ_0 should vanish:

$$\begin{gathered}\nabla_\mu T^\mu_0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\beta} T^\beta_0 - \Gamma^\beta_{\mu 0} T^\mu_\beta = 0,\\ -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + p) = 0.\end{gathered} \tag{1.27}$$

Additionally to the assumption of a perfect fluid, it is useful to consider an equation of state to relate the energy density and the pressure in the form

$$p(t, \vec{x}) = \omega\rho(t, \vec{x}), \qquad \omega = \text{const}. \tag{1.28}$$

In this case, the local conservation of energy yields

$$\begin{gathered}-\frac{\mathrm{d}\rho}{\mathrm{d}t} - 3\frac{\dot{a}}{a}(1 + \omega)\rho = 0,\\ \frac{\mathrm{d}\rho}{\rho} = -3(1+\omega)\frac{\mathrm{d}a}{a} \qquad \Rightarrow \qquad \rho(a) \propto a^{-3(1+\omega)}.\end{gathered} \tag{1.29}$$

Therefore, as expected, in an expanding Universe we have a decrease in the energy density due to the expansion of the volume. For instance, fluids such as dust, $\omega = 0$, the only way for the energy density to change is through expansion or contraction. As for radiation, $\omega = 1/3$, besides the expansion of space, the frequency of photons is redshifted, which causes the energy density to decrease even faster. So, for different fluids we have different densities contributions for an epoch in cosmic evolution. Therefore, for each epoch, we might consider one type of fluid to be dominant over the other.

However, to sort out the issue of geodesic completeness in such idealized models, we will need one last further step: it does not suffice to know the evolution of the energy density with the scale factor, we need to determine the evolution of the scale factor a *itself* with the cosmic time t. Evidently, the matter content will dictate the evolution of the geometry through EFE's. Therefore, for that, we turn to the dynamical Friedmann's equations to determine how the the dominant fluid in a model will impact the completeness of non co-moving test particles congruences.

1.3 Friedmann's Equations

Now that we have obtained the dependency of the invariant interval of non co-moving geodesics on the scale factor in a flat FLRW, we shall proceed to the investigation of the dynamics through EFEs to determine how the scale factor evolves depending on the matter content of the Universe, since for metric (1.14) the geometry of the manifold is fully determined with the scale factor $a(t)$ and the spatial curvature k. Given the line element in equation (1.14), we can compute the left side of EFEs (1.1) to obtain the Friedmann's equations. However, a more convenient way to portray EFE is by first obtaining the trace of the Ricci tensor as a function of the energy momentum tensor:

$$-R + 4\Lambda = 8\pi G T, \tag{1.30}$$

where T is the trace of $T_{\mu\nu}$ and R is the Ricci scalar. Then, the EFEs can be conveniently recast as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu}. \tag{1.31}$$

The resulting equations for the scale factor are[7]

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(T_{00} + \frac{1}{2}T\right) + \frac{\Lambda}{3}, \tag{1.32}$$

$$a\ddot{a} + 2\dot{a}^2 = 8\pi G \left(T_{11} - \frac{1}{2} T g_{11}\right) + \Lambda g_{11}. \tag{1.33}$$

For the case of a perfect fluid giving rise to a metric with flat spatial sections (1.15), whose components are given by (1.26), the dynamical equations (1.32) and (1.33) for the scale factor can be recast as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.34}$$

[7] See Appendix A.

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{1.35}$$

which are know as the *Friedmann's equations.* In order to solve equation (1.35), given the local energy conservation of a perfect fluid, it is reasonable to suppose that the energy density evolves as a power of the scale factor

$$\rho(a) \propto \rho_0 a^n, \qquad n \in \mathbb{R}. \tag{1.36}$$

Using equation (1.35), we can write

$$\dot{a} = \sqrt{\frac{8\pi G\rho}{3}}a \qquad \Rightarrow \qquad a^{-(n/2+1)}\mathrm{d}a = \sqrt{\frac{8\pi G\rho_0}{3}}\mathrm{d}t. \tag{1.37}$$

Then

$$\begin{gathered} \int_{a_0}^{a(t)} a^{-(n/2+1)}\mathrm{d}a \propto t - t_0, \\ a(t) \propto t^{-\frac{2}{n}}, \qquad n \neq 0. \end{gathered} \tag{1.38}$$

From (1.29), we see that $n = -3(1+\omega)$, yielding $a(t) \propto t^{2/3(1+\omega)}$. Hence, for any perfect fluid with an equation of state parameter ω, we can determine the evolution of $a(t)$ with the cosmic time as though it is the dominating matter component. Consequently, through equation (1.25), we can integrate to determine for which values of ω the non co-moving frame observes a finite proper time in the asymptotic past

$$\int_{\tau(t\to-\infty)}^{\tau} \mathrm{d}\tau = \int_{-\infty}^{t} \frac{\mathrm{d}t}{\sqrt{1 + v_0^2 t^{-4/3(1+\omega)}}}. \tag{1.39}$$

Considering space-times in which the cosmic time t is defined up to $-\infty$, we can divide the scale factor in two categories in terms of ω:

1. $a(t \to -\infty) \to \infty$, which occurs for $\omega > -1$.

2. $a(t \to -\infty) \to 0$, which occurs for $\omega < -1$.

In case 1, we have that

$$\lim_{t\to-\infty} \sqrt{1 + \frac{v_0^2}{a^2}} = 1. \tag{1.40}$$

Hence, the integral for the proper time of the non co-moving observer (1.39), if the Universe is dominated by a perfect fluid with $\omega > -1$, will coincide with the co-moving cosmic time in the asymptotic limit, rendering for both of them infinite proper time measurements along each respective geodesic. Thus, for $\omega > -1$, we have geodesic completeness in the asymptotic past. This is expected, as for $\omega > -1$ the picture is as follows: infinitely in the past, the spatial section of the Universe is essentially infinite and, as time goes on, the scale factor $a(t)$ decreases, describing a contracting phase. Since the scale factor diverges, there is no convergence in the congruence of time-like observers.

For the second case, nonetheless, let us define $\alpha \equiv 2/3(1+\omega)$, such that, for $\omega < -1$ we are working with $\alpha < 0$. In these cases, as $t \to -\infty$

$$\lim_{t\to-\infty} \sqrt{1+\frac{v_0^2}{a^2}} \approx \frac{|v_0|}{a}. \tag{1.41}$$

Thus, integral (1.39) can be written as

$$\int_{\tau(t\to-\infty)}^{\tau} \mathrm{d}\tau \approx \frac{1}{|v_0|}\int_{-\infty}^{t_0} t^{-|\alpha|}\mathrm{d}t = \frac{1}{1-|\alpha|}\left[t^{1-|\alpha|}\right]\Big|_{-\infty}^{t_0}. \tag{1.42}$$

We see that, if $|\alpha| < 1$, the proper time of non co-moving observers diverges $|\Delta\tau| \to \infty$ for all t_0. In terms of the fluid equation of state, since we are considering $\omega < -1$, the geodesically complete models correspond to $\omega < -5/3$, for

$$\frac{2}{3} < |1+\omega|, \tag{1.43}$$

which is only satisfied for $\omega < -5/3$. However, when we consider the case $|\alpha| > 1$, equation (1.42) converges, thus the non co-moving observer reaches the past boundary in a finite proper time $|\Delta\tau| < \infty$. Since $|\alpha| > 1$ only for $-5/3 < \omega < -1$, we have that this interval is geodesically incomplete.

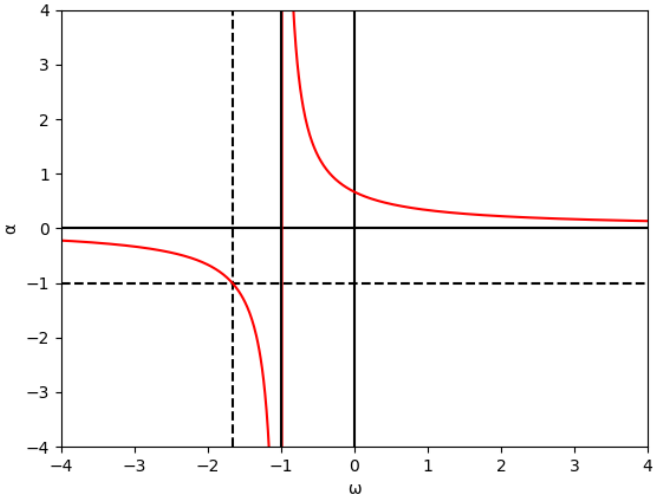

Figure 1.1: For $\omega > -1$, α is always positive, and hence, the integral diverges. For the cases where $\alpha < 0$, there are 2 sub-cases: the incomplete interval, $-5/3 < \omega < -1$; and the complete one: $\omega < -5/3$.

Despite the integral here evaluated being dependent on the coordinate system, which, in turn, might not cover the entire space-time manifold, we shall see in Chapter 4 that this incomplete interval portrays a deeper physical problem: it does not admit a metric $\mathcal{C}^2$-extension, i.e, it is not possible to extend the metric components (1.15) with $\mathcal{C}^2$ functions for the scale factor $a(t) \propto t^{2/3(1+\omega)}, \quad -5/3 < \omega < -1$. These models correspond to some *pre Big-Bang* models [10, 11, 12, 13]. Note that, space-times in this interval possess no scalar curvature singularity, since, for the flat case[8]

$$\begin{aligned} \lim_{t\to-\infty} \mathcal{R} &= \lim_{t\to-\infty} 6(\dot{H} + 2H^2) \propto \lim_{t\to-\infty} \frac{1}{t^2} \to 0, \\ \lim_{t\to-\infty} \mathcal{K} &= \lim_{t\to-\infty} 12(\dot{H}^2 + 2\dot{H}H^2 + 2H^4) \propto \lim_{t\to-\infty} \frac{1}{t^4} \to 0, \end{aligned} \tag{1.44}$$

where $\mathcal{R}$ and $\mathcal{K}$ are the Ricci and Kretschmann scalars, respectively. Additionally, notice that, several of the cases considered geodesically complete here violate some type of energy condition. For the time being, we verify the geodesically completeness of models dominated by a perfect fluid regardless of the energy conditions. Furthermore, a case that was purposefully left out of the discussion is the case in which $p = -\rho$. Since this case is, in particular, extremely relevant for Cosmology and for the main results in this work, we discuss it in the next section.

[8] Appendix A.

1.4 The de Sitter Spacetime

A case of special relevance to the cosmological investigation is one of the exact solution to EFE's, known as the de Sitter[9] spacetime, either in the context of the late acceleration observed in the Universe [14] or in the inflationary phase [15, 16], and whose properties will be central to this work. This space is not only spatially maximally symmetric, but 4-dimensionally *maximally symmetric*, i.e, it possesses the maximum number of Killing vectors. Physically, the de Sitter spacetime describes a vacuum solution with a cosmological constant Λ. Therefore, to obtain the line element that describes this geometry, let us consider the solutions to equation (1.35) with cosmological constant Λ but no matter content ($\rho = 0$). We have that Friedmann's equation reduce to:

$$H^2 = \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{1.45}$$

There are 3 solutions for the scale factor $a(t)$ depending on the value of k, which will alter not only the scale factor evolution in time, but also the spatial 3-dimensional sub-manifold of the co-moving observer. First, let us consider the flat case, $k = 0$, we have that the Friedmann's equation is simply

$$\begin{gathered} H^2 = \frac{\Lambda}{3}, \\ \dot{a} = \sqrt{\frac{\Lambda}{3}} a \qquad \Rightarrow \qquad a(t) = e^{\sqrt{\frac{\Lambda}{3}} t}, \end{gathered} \tag{1.46}$$

yielding the line element:

$$\begin{gathered} \mathrm{d}s^2_{\text{flat}} = -\mathrm{d}t^2 + e^{2\sqrt{\Lambda/3}t}[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \\ t \in \mathbb{R}, \quad r \in [0, \infty), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi). \end{gathered} \tag{1.47}$$

This first scenario describes a spacetime in which the co-moving observer experiences an exponential expansion of its flat spatial sections. This spacetime, by equation (A.4), possesses a constant scalar curvature:

$$\mathcal{R} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = 4\Lambda \tag{1.48}$$

Now, on the other hand, if we consider the vacuum solution with negative spatial

[9] Willem de Sitter (1872-1934)

curvature, i.e, $k = -1$ we have that the Friedmann's equation is given by:

$$H^2 = \frac{\Lambda}{3} + \frac{1}{a^2},$$
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{\Lambda}{3}a^2 + 1} \qquad \Rightarrow \qquad a(t) = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right), \tag{1.49}$$

and the line element is

$$\mathrm{d}s^2_{\text{open}} = -\mathrm{d}t^2 + \frac{3}{\Lambda}\sinh^2\left(\sqrt{\frac{\Lambda}{3}}t\right)[\mathrm{d}\chi^2 + \sinh^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)],$$
$$t \in (0, \infty), \quad \chi \in [0, \infty), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi). \tag{1.50}$$

Note that the time coordinate is defined over the positive branch of $\mathbb{R}$, since $\lim_{t\to 0} a = 0$. This, a *priori*, represents no physical problem, as the curvature is finite over the entire chart:

$$\mathcal{R} = 6\left[\frac{\Lambda}{3} + \frac{\Lambda}{3}\left(\frac{\cosh^2(\Lambda t/3) - 1}{\sinh^2(\Lambda t/3)}\right)\right] = 4\Lambda. \tag{1.51}$$

Last, but certainly not least, we have the positive spatial curvature case. As we shall see, out of all 3 possibilities for k, this chart in which the spacetime is foliated through 3-spheres $\mathbb{S}^3$ spatial sections is the only one that manifests the global structure of the de Sitter geometry, rather than merely a sub-manifold. Consider $k = +1$ in equation (1.33):

$$H^2 = \frac{\Lambda}{3} - \frac{1}{a^2},$$
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{\Lambda}{3}a^2 - 1} \qquad \Rightarrow \qquad a(t) = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right). \tag{1.52}$$

Thence, the line element of spatially closed sections is

$$\mathrm{d}s^2_{\text{closed}} = -\mathrm{d}t^2 + \frac{3}{\Lambda}\cosh^2(\sqrt{\Lambda t/3})[\mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)],$$
$$t \in \mathbb{R}, \quad \chi \in [0, \pi), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi), \tag{1.53}$$

and the Ricci scalar is

$$\mathcal{R} = 6\left[\frac{\Lambda}{3} + \frac{\Lambda}{3}\left(\frac{\sinh^2(\Lambda t/3) + 1}{\cosh^2(\Lambda t/3)}\right)\right] = 4\Lambda. \tag{1.54}$$

As we can see, additionally to being constant everywhere, the Ricci scalar $\mathcal{R}$ is the same throughout each chart here investigated. This suggests that, perhaps, each of the solutions for the vacuum with cosmological constant is just a different metric covering different regions of the same 4-dimensional structure. Indeed, these 3 solutions can be obtained by considering the 4-dimensional space-time to be a level surface with constant curvature embedded in a 5-dimensional space. Consider the Euclidean 5 dimensional space. A 4-dimensional hyperboloid is described by

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda}. \tag{1.55}$$

We can perform the coordinate transformation $(v, w, x, y, z) \to (\hat{t}, \hat{x}, \hat{y}, \hat{z})$:

$$\hat{t} = \sqrt{\frac{3}{\Lambda}} \ln \left[\sqrt{\frac{\Lambda}{3}} (w + v) \right], \quad \hat{x} = \sqrt{\frac{3}{\Lambda}} \frac{x}{w+v}, \quad \hat{y} = \sqrt{\frac{3}{\Lambda}} \frac{y}{w+v}, \quad \hat{z} = \sqrt{\frac{3}{\Lambda}} \frac{z}{w+v} \tag{1.56}$$

By doing so, one recovers the line element (1.47) in Cartesian coordinates, and we can identify the coordinate $\hat{t}$ as the co-moving time. Therefore, the surfaces of constant time is equivalent to the intersection of the hyperboloid with the planes of constant time, $w + v = \text{const}$. The Figure 1.2 illustrates this chart for a 2-dimensional hyperboloid. However, this chart only covers half of the hyperboloid, as this transformation in only defined for points above the plane $w + v = 0$.

Alternatively, we can introduce the coordinates (t, χ, θ, ϕ) over the hyperboloid through the coordinate change:

$$\begin{gathered} w = \sqrt{\frac{3}{\Lambda}} \sinh(\Lambda t/3), \quad v = \sqrt{\frac{3}{\Lambda}} \cosh(\Lambda t/3) \cos\chi, \quad x = \sqrt{\frac{3}{\Lambda}} \cosh(\Lambda t/3) \sin\chi \cos\theta, \\ y = \sqrt{\frac{3}{\Lambda}} \cosh(\Lambda t/3) \sin\chi \sin\theta \cos\phi, \quad z = \sqrt{\frac{3}{\Lambda}} \cosh(\Lambda t/3) \sin\chi \sin\theta \sin\phi. \end{gathered} \tag{1.57}$$

The line element in these coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{3}{\Lambda} \cosh^2(\Lambda t/3) [\mathrm{d}\chi^2 + \sin^2\chi (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)]. \tag{1.58}$$

Note that the line element above is the same as (1.53), which is the only one that covers the entire hyperboloid among the three possibilities. The spatially closed foliation for the 2-dimensional hyperboloid is illustrated below:

(a) Flat de Sitter Foliation (b) Closed de Sitter Foliation

Figure 1.2: Surfaces of constant time displayed for: (a) $k = 0$ case and (b) $k = 1$ cases.

These two cases ($k = 0, +1$) are the most relevant in the context of geodesic completeness: as we shall see, the time-like and null geodesics in the flat patch of the de Sitter manifold are incomplete. However, it is evident that this issue is only a matter of coordinate: the flat coordinates do not cover the entire geometry, which possesses no problem of curvature singularities, admitting extensions to a broader manifold, in this case, the entire hyperboloid. Nevertheless, a realistic beginning for the Universe cannot be an exact de Sitter space, but this particular case will motivate us to inquire into conditions under which geodesic incompleteness is not a physical problem for a general flat FLRW model in which the scale factor vanishes in the asymptotic past, allowing an extension.

1.5 The Inflationary Paradigm

The main importance of the de Sitter case in what concerns geodesic completeness is due to a phase in cosmic evolution called *inflation*. In the *Standard Model of Cosmology* ΛCDM, a series of problems arise when confronted with observations regarding the homogeneity of the large scale distribution of matter in the Universe since very early times. For instance, if the Universe had a beginning[10], causally disconnected regions of space have approximately the same energy density distribution despite being outside each other causal region, not having enough time to reach thermal equilibrium. Furthermore, given the set of energy

[10]It is relevant to emphasize that for alternative models in which the Universe had no beginning, the *horizon problem* is not posed, since any particle would have enough time to reach thermic equilibrium with the rest of the Universe.

density at each point in space at an initial time, we would still need to know the set of all the initial velocities in order to determine the complete evolution of the Universe. Notwithstanding, in order to remain homogeneous and isotropic at late times, the initial velocities necessary to explain the homogeneity and flatness would be exceedingly restrict. Evidently, to get a full description of the dynamics at very early times, one would need a theory of quantum gravity for such a high energy scale. In the absence of such a theory, we restrict our analysis of "initial conditions" to the Planck[11] time

$$t_p = \frac{1}{c^2}\sqrt{\frac{\hbar G}{c}} \sim 10^{-44} s. \tag{1.59}$$

Let us denote the size of the present observable Universe, which is homogeneous, by

$$d_h = ct_0, \tag{1.60}$$

where t_0 is the age of the Universe, $t_0 \sim 10^{17} s$. Since this region is highly homogeneous and isotropic, the region it originated from would need to be at least the size of the present horizon times the ratio of the respective scale factors

$$d_h^p = ct_0 \frac{a_p}{a_0}, \tag{1.61}$$

where the index p denotes measures at t_p. At such early times, the causal distance comprising every event inside the light cone from the Big Bang until t_p was $d_c^p = ct_p$. Comparing this length with the homogeneous region we obtain

$$\frac{d_h^p}{d_c^p} = \frac{ct_0}{ct_p}\left(\frac{a_p}{a_0}\right). \tag{1.62}$$

We can get an estimate of this ratio by noting that for a perfect fluid model

$$H \propto \frac{1}{t} \qquad \Rightarrow \qquad \dot{a} \sim \frac{a}{t}. \tag{1.63}$$

Thence

$$\frac{d_h^p}{d_c^p} \sim \frac{\dot{a}_p}{\dot{a}_0}, \tag{1.64}$$

and if gravity acts only as an attractive interaction, then the expansion at earlier times

[11] Max K. E. L. Planck (1858-1947)

is always greater than late times, $\dot{a}_p > \dot{a}_0$, and the homogeneous region has not been completely in causal contact. Furthermore, a great ratio of $\dot{a}_p/\dot{a}_0$ is also related to the initial curvature of the spatial sections. Recasting Friedmann equation (1.33) as

$$\Omega - 1 = \frac{k}{(aH)^2}, \tag{1.65}$$

where $\Omega \equiv 8\pi G\rho/3H^2 = \rho/\rho_{crit}$ (being ρ_{crit} the critical density for the Universe to be spatially flat), we can relate the equation above at the initial time and today using (1.63):

$$\Omega_p - 1 = (\Omega_0 - 1)\left(\frac{a_0 H_0}{a_p H_p}\right)^2 = (\Omega_0 - 1)\left(\frac{\dot{a}_0}{\dot{a}_p}\right)^2. \tag{1.66}$$

Thus, a large ratio of $\dot{a}_p/\dot{a}_0$ implies $\Omega_k^p \sim 0$. In fact, if we consider that at the initial time the Universe was dominated by radiation, $a \sim 1/T$, using the age of the Universe and the Planck temperature, $T_p = c^2\sqrt{\hbar c/G}/k_B \sim 10^{32}K$, in (1.62) we get an estimate of

$$\frac{\dot{a}_p}{\dot{a}_0} \sim \frac{d_h^p}{d_c^p} = \frac{t_0}{t_p}\frac{a_p}{a_0} \sim \frac{10^{17}}{10^{-44}}\frac{1}{T_p} \sim 10^{29} \tag{1.67}$$

which implies that the Universe must have started in a very homogeneous and isotropic state. Still, it could be argued that the set of initial conditions necessary to explain observations is a privileged one since it manifestly possesses many more symmetries when compared to other arbitrary possible sets. However, setting the initial conditions to the required specific values of very homogeneous energy density and highly isotropic velocity distribution does not explain all observation, as the space is not perfectly homogeneous, having inhomogeneities that exhibit correlations in distances outside the casual regions, implying that, regardless of the initial set, this regions must have been in causal contact at some moment of time. Surprisingly, a solution to all of the issues mentioned here relies in a epoch of the Universe history known as *inflation* [15, 16]: an epoch where gravity acted as a repulsive force in which the Universe underwent an accelerated expansion.

1.5.1 Accelerated Expansion

All of the initial condition problems presented so far are related to the expansion rate $\dot{a}_p/\dot{a}_0$. The larger this ratio, the more the homogeneous region exceed the causal region and the more flat the initial spatial section needs to be. Thereby, in order to solve both these issues, we need the Universe to undergo a phase of accelerated expansion, which is a necessary but not sufficient condition to solve all the issues previously mentioned. Through equation (1.32), we note that an acceleration in the expansion rate implies

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a > 0. \tag{1.68}$$

Thus, necessarily, $\rho + 3p < 0$[12]. An example of space-time that satisfies this condition is the flat de Sitter case. However, inflation can not be described by a de Sitter phase due to the following: to leave inflation into a regular non accelerating phase, we need $\dot{H}$ to become negative, for

$$\ddot{a} = a(H^2 + \dot{H}). \tag{1.69}$$

Additionally, the Hubble function must vary, which is never the case for the de Sitter case. Hence inflation must begin in a *quasi*-de Sitter scenario and, towards its end, we should have that $|\dot{H}|/H^2 \sim 1$. Nevertheless, the main point remains: the Universe needs a phase of accelerated expansion. Then, if the Universe is ever expanding, it must have been smaller and smaller in the past, to a point where all the matter content in the Universe must have been condensed in a infinitely dense point, entailing in a *singular* initial state, where General Relativity fails to give predictions. In such cases, given the fact that an event in the spacetime manifold is missing[13], we shall always have geodesic incompleteness, once any curve passing through such point would be incomplete, as for instance, in the Standard Model of Cosmology. However, this is not a necessary condition for a spacetime to be singular. In fact, even in the absence of an event with infinite curvature, it is possible that at least one non space-like geodesic have an endpoint given that a few physically reasonable conditions are satisfied. This is the content of the *singularity theorems*. An alternative to a model with an initial curvature singularity could be inflation: we know that inflation is not future eternal, but could it be eternal in the past? In this scenario, we should have $a(t \to -\infty) \to 0$, but as discussed, the infinite time as measured by a co-moving observer does not imply that non co-moving ones will observe the same. Therefore, since an inflationary epoch cannot be exactly de Sitter, it is possible that at least one of the non space-like geodesics are incomplete. In fact, as we will see in the

[12] In accelerated expansion, it is imperative that the Strong Energy Condition is violated.

[13] It is not appropriate to consider singular points as a part of the space-time since EFE's do not hold at these events

next section, this is the content of the Borde-Guth-Vilenkin[14] theorem, which states that inflationary space-times cannot be eternal in the past if not extensible.

[14] Arvind Borde, Alan Guth, Alexander Vilenkin

Chapter 2

The Borde-Guth-Vilenkin Theorem

The question of whether inflation might extend eternally in the past has been extensively discussed in the literature [3, 17, 18, 4, 19]. The singularity theorems [20, 21, 22, 23, 24, 25] guarantee the incompleteness of at least one non space-like geodesic under a few conditions regarding energy conditions satisfied by the matter fields, the causal structure of the space-time, and on topological assumptions on the space-like surfaces. Notwithstanding, once one of the hypothesis is evaded, the geodesic incompleteness is not assured neither dismissed. As previously discussed, since inflationary models require some energy condition violation, some of the hypothesis might not be satisfied by some models, which does not imply their geodesic completeness. For instance, inflationary models necessarily violate the strong energy condition, and therefore the theorems do not necessarily hold. Nevertheless, we cannot assume that the model is complete since, in these cases, the formation of singularity might be avoided. In this context, the Borde-Guth-Vilenkin (BGV)[1] theorem [3, 26] establishes another set of conditions for the geodesic incompleteness for non-static models, without the necessity of homogeneity or isotropy of the spatial sections of the spacetime. The only condition for incompleteness is that the average expansion along time-like and null geodesic congruences is positive. Let us start with a simple FLRW flat metric given by (1.15). For the sake of simplicity, we shall consider geodesics moving along only the radial direction. Let λ be an affine parameter that describes a null geodesic. Therefore, we can write:

[1]Originally proposed to show that inflationary spacetimes cannot be geodesically complete in past directions.

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}=0. \tag{2.1}$$

For the radial trajectory, the geodesic equation and the null normalization yield

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[a^2\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right]=0 \qquad \Rightarrow \qquad a^2\frac{\mathrm{d}r}{\mathrm{d}\lambda}\equiv v_0, \tag{2.2}$$

$$-\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2+a^2\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2=0. \tag{2.3}$$

Then

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda}=\pm\frac{v_0}{a} \qquad \Rightarrow \qquad \mathrm{d}\lambda\propto a\mathrm{d}t. \tag{2.4}$$

Thus, the affine parameter grows with the scale factor. This result can be understood in terms of redshift: if we choose an affine parameter such that the tangent vector is proportional to the wave vector, $k^\mu\propto \mathrm{d}x^\mu/\mathrm{d}\lambda$, then, for the zeroth component, $\mathrm{d}\lambda\propto\mathrm{d}t/\omega$. So the frequency is redshifted in an expanding spacetime, as usual. To relate the expansion rate $H\equiv\dot{a}/a$ along the null geodesic over an interval (λ_i,λ_f), without loss of generality, normalize the tangent by the final value of the scale factor $a(t_f)$:

$$\mathrm{d}\lambda=\frac{a(t)}{a(t_f)}\mathrm{d}t. \tag{2.5}$$

Let us then integrate the expansion rate along a null geodesic. Since $H\equiv\dot{a}/a$, we can write $H\mathrm{d}\lambda$ as

$$\begin{aligned}\int_{\lambda_i}^{\lambda_f}H(\lambda(t))\mathrm{d}\lambda&=\int_{t(\lambda_i)}^{t(\lambda_f)}\frac{\dot{a}(\lambda(t))}{a(\lambda(t))}\frac{a(\lambda(t))}{a(\lambda(t_f))}\mathrm{d}t\\ =\frac{1}{a(\lambda(t_f))}\int_{t(\lambda_i)}^{t(\lambda_f)}\frac{\mathrm{d}a}{\mathrm{d}t}\mathrm{d}t&=\frac{1}{a(\lambda_f)}(a(\lambda_f)-a(\lambda_i)).\end{aligned} \tag{2.6}$$

Since $a(\lambda)$ is always a positive function

$$\int_{\lambda_i}^{\lambda_f}H(\lambda)\mathrm{d}\lambda=1-\frac{a(\lambda_i)}{a(\lambda_f)}\leq 1, \tag{2.7}$$

where the equality is true only when the initial scale factor $a(\lambda_i)$ vanishes. In any other case the integral of the expansion rate will be less than unity. Thus, if we define the *average* expansion $\mathcal{H}_{av}$ as the expansion over the interval divided by the interval, we conclude that

$$\mathcal{H}_{av} \equiv \frac{1}{\lambda_f - \lambda_i} \int_{\lambda_i}^{\lambda_f} H(\lambda) \mathrm{d}\lambda \leq \frac{1}{\lambda_f - \lambda_i}. \tag{2.8}$$

Since λ_f is arbitrary but fixed, if the average expansion is positive, $\mathcal{H}_{av} > 0$, then we can write the inequality

$$0 < \mathcal{H}_{av} \leq \frac{1}{\lambda_f - \lambda_i}. \tag{2.9}$$

Thereby, we get a contradiction if the initial parameter is unbounded as $t \to t_i$, since if this is the case, $lim_{\lambda_i \to -\infty} 1/(\lambda_f - \lambda_i) \to 0$. Therefore, if there is a null geodesic with average positive expansion, the spacetime given by metric (1.15) cannot be null geodesically complete, since there is a geodesic with finite parameter. A similar derivation can be obtained for the case of timelike congruences: now we shall parametrize the geodesic with the proper time τ along the curve. As shown in equation (1.25), the proper time of any non co-moving observer along a time-like geodesic can be written as $\mathrm{d}\tau = \mathrm{d}t/\sqrt{1 + v_0^2/a^2}$, where each v_0 defines a congruence and, the case $v_0 = 0$ describes the co-moving observer itself. Thus, along the geodesic between the two arbitrary events at the boundary of an interval (τ_i, τ_f), we can integrate the Hubble parameter:

$$\int_{\tau_i}^{\tau_f} H \mathrm{d}\tau = \int_{\tau_i}^{\tau} \frac{\dot{a}}{a} \frac{\mathrm{d}t}{\sqrt{1 + \frac{v_0^2}{a^2}}} = \int_{a_i}^{a_f} \frac{\mathrm{d}a}{\sqrt{a^2 + v_0^2}} = \ln \left[a + \sqrt{a^2 + v_0^2} \right] \Bigg|_{a_i}^{a_f}, \tag{2.10}$$

where $a_i \equiv a(\tau_i)$ and $a_f = a(\tau_f)$. Therefore, defining the average expansion over the interval as

$$\mathcal{H}_{av} \equiv \frac{1}{\Delta\tau} \int_{\tau_i}^{\tau_f} H(\tau) \mathrm{d}\tau, \tag{2.11}$$

we find that:

$$\mathcal{H}_{avg} = \frac{1}{\tau_f - \tau_i} \int_{\tau_i}^{\tau_f} H(\tau) \mathrm{d}\tau = \frac{1}{\tau_f - \tau_i} \ln \left[\frac{a_f + \sqrt{a_f^2 + v_0^2}}{a_i + \sqrt{a_i^2 + v_0^2}} \right]. \tag{2.12}$$

Thus, for a space that has expanded, on average, $a_f > a_i$:

$$\ln \left[\frac{a_f + \sqrt{a_f^2 + v_0^2}}{a_i + \sqrt{a_i^2 + v_0^2}} \right] \leq \ln \left[a_f + \sqrt{a_f^2 + v_0^2} \right]. \tag{2.13}$$

Thence, for a geodesic with positive average expansion, $\mathcal{H}_{av} > 0$, we have:

$$0 < \mathcal{H}_{av} \leq \frac{1}{\tau_f - \tau_i} \ln \left[a(\tau_f) + \sqrt{a(\tau_f)^2 + v_0^2} \right]. \tag{2.14}$$

Once again we see that, since $a(\tau_f)$ is an arbitrary positive value, in order to avoid a contradiction, $\tau_i > -\infty$. Despite its simplicity for being a kinematic theorem making no assumption on energy conditions, the BGV theorem has recently been target to criticism [27] regarding a possible loophole, which shall be discussed later on.

2.1 The Average Expansion of More General Space-times

Despite having a very direct evaluation for cosmological models where there is isotropy and homogeneity of the spatial sections – which is very restrictive, not only for the model but also for the choice of reference frame – the results of the BGV theorem can be applied to more general expanding space-times. For that, we need a more general definition of the average expansion, which we shall use for anisotropic foliations, but that should recover the usual definition of $H \equiv \dot{a}/a$ for homogeneous and isotropic spaces. Consider an observer[2] $\mathcal{O}$, with four-velocity v^μ crossing two test particles in a co-moving congruence with four-velocity given by u^μ. At an instant τ_i measured by the observer it crosses the trajectory of particle 1 and measures its four-velocity to be $u^\mu(\tau_i)$. Later, at instant $\tau_f = \tau_i + \Delta\tau$, it crosses the trajectory of test particle 2, where it measures the four-velocity to be $u^\mu(\tau_f)$ as illustrated in Figure 2.1:

[2]We call this geodesic "the observer" for simplicity, but v^μ could be a null geodesic.

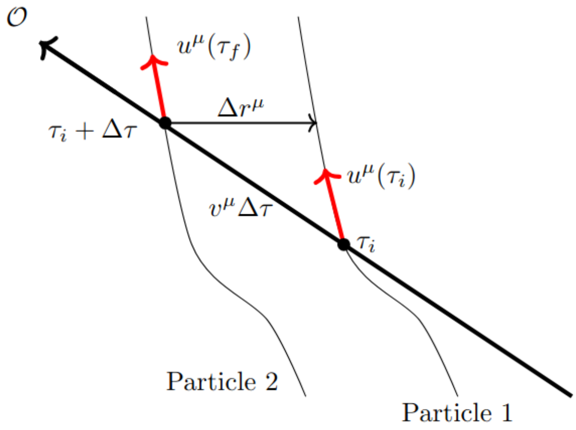

Figure 2.1: Worldline of the observer crossing two test particles of a co-moving congruence

Consider that the observer $\mathcal{O}$ makes the two observations at very close events, i.e, with $\Delta\tau$ being infinitesimal. We can calculate the separation of the two test particles at equal times in their frame. To compute this vector Δr^μ, which is perpendicular to the test particle worldlines u_μ, we project the vector $-v^\mu \Delta\tau$ onto u^μ to subtract the projection along the movement.[3] Thus:

$$\Delta r^\mu = -v^\mu \Delta\tau + \sigma u^\mu \Delta\tau, \tag{2.15}$$

where $\sigma \equiv -u_\nu v^\nu$. The norm squared of this vector is given by

$$\begin{aligned} |\Delta r^\mu|^2 &= g_{\mu\alpha}(v^\mu - \sigma u^\mu)(v^\alpha - \sigma u^\alpha)\Delta\tau^2 \\ &= [g_{\mu\alpha} v^\mu v^\alpha + 2\sigma^2 + \sigma^2 g_{\mu\alpha} u^\mu u^\alpha]\Delta\tau^2 \\ &= [\kappa + \sigma^2]\Delta\tau^2, \end{aligned} \tag{2.16}$$

where the metric $g_{\mu\nu}$ need not be homogeneous nor isotropic and κ is -1 for timelike or 0 for null geodesics. We can verify that the vector Δr^μ is indeed perpendicular to u^μ by calculating $\Delta r^\mu u_\mu$:

$$\begin{aligned} \Delta r^\mu u_\mu &= g_{\mu\nu} \Delta r^\mu u^\nu \\ &= \underbrace{-g_{\mu\nu} v^\mu u^\nu}_{\sigma} \Delta\tau + \sigma [\underbrace{g_{\mu\nu} u^\mu u^\nu}_{-1}] \Delta\tau \\ &= \Delta\tau(\sigma - \sigma) = 0, \end{aligned} \tag{2.17}$$

[3]Note by Figure 2.1 that the negative sign is so that $v^\mu \Delta\tau$ has the same orientation as Δr^μ.

The norm of the displacement measured at equal times by the co-moving test particles is $\Delta r^\mu = \sqrt{\kappa + \sigma^2}\Delta\tau$. So, the observer $\mathcal{O}$ can measure the variation in the four-velocity of the test particles Δu^μ by parallelly propagating the vector u^μ along its geodesic from τ_i to τ_f, so $\Delta u^\mu = (Du^\mu/\mathrm{d}\tau)\Delta\tau$. Then, locally, what $\mathcal{O}$ will use to define the expansion is the radial component of the variation of the four-velocity Δu_r, defined by projecting Δu^μ along the normalized spatial separation Δr^μ:

$$\Delta u_r \equiv \frac{\Delta u^\mu \Delta r_\mu}{\Delta r}. \tag{2.18}$$

We can now define the generalized Hubble parameter H_{BGV} as the rate of the radial velocity variation with respect to the radial distance to evaluate how the velocity of the congruence changes with the distance:

$$\begin{aligned} H_{BGV} &\equiv \frac{\Delta u_r}{\Delta r} = \frac{\Delta u^\mu \Delta r_\mu}{|\Delta r|^2} \\ &= \frac{\Delta\tau^2 (Du^\mu/\mathrm{d}\tau)(-v_\mu + \sigma u_\mu)}{\Delta\tau^2(\kappa+\sigma^2)} = \frac{-v_\mu (Du^\mu/\mathrm{d}\tau)}{\kappa+\sigma^2}, \end{aligned} \tag{2.19}$$

where in the last equality we used that

$$\begin{aligned} \frac{D(-\sigma)}{\mathrm{d}\tau} &= \frac{D}{\mathrm{d}\tau}(v_\nu u^\nu u_\mu u^\mu) = -\frac{D}{\mathrm{d}\tau}(v_\nu u^\nu) = -\sigma u_\mu \frac{Du^\mu}{\mathrm{d}\tau} + u^\mu \frac{D}{\mathrm{d}\tau}(v_\nu u^\nu u_\mu) \\ &= -\sigma u_\mu \frac{Du^\mu}{\mathrm{d}\tau} + u^\mu \cancelto{0}{\frac{Dv_\nu}{\mathrm{d}\tau}} u^\nu u_\mu - v_\nu \frac{Du^\nu}{\mathrm{d}\tau} - \sigma u^\mu \frac{Du_\mu}{\mathrm{d}\tau} \\ &= -2\sigma u_\mu \frac{Du^\mu}{\mathrm{d}\tau} - v_\nu \frac{Du^\nu}{\mathrm{d}\tau}, \end{aligned} \tag{2.20}$$

Since $v^\mu(\tau)$ is a tangent vector to the observer's $\mathcal{O}$ geodesic, we can write the last term in the last equality as $-D(v_\nu u^\nu)/\mathrm{d}\tau$. Thus, we conclude that

$$\begin{aligned} -\frac{D}{\mathrm{d}\tau}(v_\nu u^\nu) &= -2\sigma u_\mu \frac{Du^\mu}{\mathrm{d}\tau} - \frac{D}{\mathrm{d}\tau}(v_\nu u^\nu) \\ \Rightarrow \sigma u_\mu \frac{Du^\mu}{\mathrm{d}\tau} &= 0. \end{aligned} \tag{2.21}$$

Therefore, the generalized Hubble parameter in equation (2.19) does not have contributions from this term. Thence, using again that $v^\mu(\tau)$ is a geodesic, we can write the generalized Hubble parameter H_{BGV} as

$$H_{BGV} = -\frac{\mathrm{d}\sigma/\mathrm{d}\tau}{\kappa + \sigma^2}. \tag{2.22}$$

If $\mathcal{O}$ corresponds to a light geodesic ($\kappa = 0$), we can write the expression above as a total derivative

$$H_{BGV} = -\frac{1}{\sigma^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\sigma^{-1}. \tag{2.23}$$

By integrating the generalized Hubble parameter along the null geodesic from λ_i to λ_f, we find that

$$\int_{\lambda_i}^{\lambda_f} H_{BGV}\mathrm{d}\lambda = \int_{\lambda_i}^{\lambda_f}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\sigma^{-1}\right)\mathrm{d}\lambda = \sigma^{-1}(\lambda_f) - \sigma^{-1}(\lambda_i) \leq \sigma^{-1}(\lambda_f), \tag{2.24}$$

where the last inequality comes from the fact that, for the null case, $\sigma > 0$. Thence, if the average expansion is positive, $\mathcal{H}_{av} > 0$, then:

$$0 < \mathcal{H}_{av} \leq \frac{\sigma^{-1}(\lambda_f)}{\lambda_f - \lambda_i}, \tag{2.25}$$

where $\mathcal{H}_{av} \equiv (\Delta\lambda)^{-1}\int H_{BGV}\mathrm{d}\lambda$. Therefore, $\lambda_i > -\infty$ and the geodesic is incomplete.

In the case where the curve $\mathcal{O}$ is a timelike geodesic we have that, by equation (2.22), the generalized Hubble parameter can also be written as a total derivative, this time given by:

$$H_{BGV} = -\frac{1}{\sigma^2 - 1}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{2}\ln\left(\frac{\sigma+1}{\sigma-1}\right)\right]. \tag{2.26}$$

Then

$$\int_{\tau_i}^{\tau_f} H_{BGV}\mathrm{d}\tau = \int_{\tau_i}^{\tau_f}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{2}\ln\left(\frac{\sigma+1}{\sigma-1}\right)\right]\mathrm{d}\tau = \frac{1}{2}\ln\left(\frac{\sigma+1}{\sigma-1}\right)\Bigg|_{\tau_i}^{\tau_f}. \tag{2.27}$$

Since σ is the relative Lorentz factor, $\sigma = v_\nu u^\nu = 1/\sqrt{1 - v_{rel}^2}$, which is always $\sigma \geq 1$:

$$\int_{\tau_i}^{\tau_f} H_{BGV}\mathrm{d}\tau = \frac{1}{2}\ln\left(\frac{\sigma+1}{\sigma-1}\right)\Bigg|_{\tau_i}^{\tau_f} \leq \frac{1}{2}\ln\left(\frac{\sigma_f+1}{\sigma_f-1}\right). \tag{2.28}$$

Therefore, if the average is positive we get once again that the initial parameter cannot be arbitrary close to $-\infty$, given the inequality

$$0 < \mathcal{H}_{avg} \leq \frac{1}{2(\tau_f - \tau_i)} \ln\left(\frac{\sigma+1}{\sigma-1}\right). \tag{2.29}$$

Thence, the kinematic theorem shows that causal curves which possess an average positive expansion as defined by (2.22) are necessarily incomplete, regardless of any energy condition.

2.2 The Loophole in the BGV theorem

As previously stated, the BGV Theorem contains a loophole that stems from the fact that we cannot take the parameter value to be exactly $-\infty$ since, if it could be done, then the geodesic parameter can be made arbitrarily close to $-\infty$ and the geodesic is, therefore, complete. Recently, an amendment for the theorem was proposed in [27] in order to solve this issue. Let a causal geodesic be parametrized by $\lambda \in (\lambda_i, \lambda_f)$. If there exists a $\Delta > 0$ such that

$$\mathcal{H}_{av} \equiv \frac{1}{\lambda_f - \lambda_0} \int_{\lambda_0}^{\lambda_f} H(\lambda) \mathrm{d}\lambda \geq \Delta, \quad \forall \lambda_0 \in (\lambda_i, \lambda_f), \tag{2.30}$$

then the geodesic is incomplete in the past. The difference between the theorems is subtle: in the original theorem, the average only needed to be positive, but it could be arbitrarily close to zero. Therefore, as we approach $\lambda_i \to -\infty$, the inequality is not necessarily contradicted because $\mathcal{H}_{av}$ might go to zero always being less than $1/\Delta\lambda$. In the amended theorem, on the other hand, $\mathcal{H}_{av}$ is bounded from below, which means that as we approach $\lambda_i \to -\infty$ the average does not goes to zero as well, since $\Delta > 0$. In that case, the inequality

$$0 < \Delta \leq \mathcal{H}_{av} \leq \frac{1}{\lambda_f - \lambda_i}, \tag{2.31}$$

will unavoidably reach a contradiction if $\mathcal{H}_{av}$ is bounded from below for any parameter in (λ_i, λ_f).

Either way, whether applying the original or the amended version, the issue of geodesic incompleteness by the BGV theorem is restricted solely to the question of whether the spacetime can be extended, which, in its original proposal, spaces with a contraction phase are discarded using physical arguments instead of definitions. However, any method of geodesic incompleteness verification that aims to act as a singularity theorem must be applied only to maximal space-times, otherwise, the incompleteness might be merely a chart limitation. In the next section, we implement a local extension to the flat patch of de Sitter spacetime as a pivotal case to derive a more general procedure of extension for arbitrary space-times that could, in principle, be diagnosed as incomplete by the BGV theorem.

Chapter 3

Geodesically Complete Extension of the Spatially Flat de Sitter Spacetime

Despite using the original BGV theorem or its proposed amendment, the problem of mis-diagnosing space-times as incomplete still remains, as long as the question of extensibility is not addressed. For instance, by both theorems, the flat patch de Sitter spacetime, whose line element in spherical coordinates is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \qquad a(t) \equiv e^{\alpha t}, \quad t \in (-\infty, \infty), \tag{3.1}$$

should be incomplete. In the original proposal, we can calculate the average expansion along a non co-moving geodesic:

$$\mathcal{H}_{av} = \frac{1}{\tau_f - \tau_i}\int_{\tau_i}^{\tau_f} H \mathrm{d}\tau = \alpha > 0, \qquad \forall \tau_i \in (-\infty, \tau_f). \tag{3.2}$$

For the case of the original proposal of the theorem, the average expansion over any interval is positive and, hence, the space-time must be geodesically incomplete. Moreover, for the amended theorem, the criteria for incompleteness is also satisfied, if we choose trivially $\Delta = \alpha$, we see that for any interval (τ_i, τ_f), the average is always greater or equal to the positive value α so the parameter τ along the geodesic cannot be extended arbitrarily close to minus infinity. However, as discussed in Chapter 1, it is known that metric (3.1) only covers half of the de Sitter full manifold. Furthermore, the de Sitter space-time possesses well behaved and constant curvature scalars, such as the Ricci and Kretschmann scalars

$$\begin{aligned} \mathcal{R} &= R^{\mu\nu}R_{\mu\nu} = 4\Lambda, \\ \mathcal{K} &= R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{8\Lambda^2}{3}. \end{aligned} \tag{3.3}$$

Hence, the incompleteness of the geodesics diagnosed by the BGV theorem must be merely a coordinate problem rather than a physical pathology of the spacetime manifold when applied to a spacetime that is not maximal. Indeed, one of the assumptions of all the singularity theorems is that the space-time $(\mathcal{M}, g_{\mu\nu})$ is $\mathcal{C}^k$-inextensible, i.e, that there does not exist a space-time $(\mathcal{M}', g')$ such that there is an isometric $\mathcal{C}^k$ embedding:

$$\Phi : \mathcal{M} \to \mathcal{M}',$$

where $\mathcal{M} \subset \mathcal{M}'$. Therefore, through the BGV theorem, the issue of geodesic completeness is reduced to whether or not the space-time under consideration admits an extension, i.e, given a manifold $\mathcal{M}$ and a metric $g_{\mu\nu}$, what are the necessary conditions for it to be possible to find a metric extension. With this in mind, we turn our attention to the de Sitter space-time as a pivotal example. The spatially closed de Sitter foliation ($k = 1$) is a patch in which all geodesics are complete, since the foliation with topology $\mathbb{R} \times \mathbb{S}^3$ covers the entire regular manifold, recovering its global structure. However, in order to find a more general procedure to extend the geodesics for arbitrary cases, we turn to a local transformation physically motivated by the non co-moving observers, where their worldlines shall be considered as a coordinate chart to see their fate by their own point of view. In order to inquire about the fate of the incomplete non co-moving observers, we shall now construct a radial congruence of non co-moving geodesics and analyze its behavior in the past asymptotic limit $a \to 0$. First, we start by constructing a time-like vector field tangent to the curves on the congruence. By equation (2.2), we can write the time component of the tangent vector using the constant of motion v_0 that characterizes a congruence and the normalization condition:

$$-\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + a^2\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -1, \tag{3.4}$$

to obtain:

$$u^t \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1 + \frac{v_0^2}{a^2}} \equiv \gamma, \qquad u^r \equiv \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{v_0}{a^2}. \tag{3.5}$$

By direct integration of the time component u^t in (3.5), given that the scale factor of the flat patch is $a_{dS} = e^{\alpha t}$, we obtain

$$\tau - \tau_0 = \int_{\tau_0}^{\tau} \mathrm{d}\tau = \int_{t_0}^{t} \frac{\mathrm{d}t}{\sqrt{1 + v_0^2 e^{-2\alpha t}}} = \frac{1}{\alpha} \int_{a_0}^{a(t)} \frac{\mathrm{d}a}{\sqrt{a^2 + v_0^2}} = \frac{1}{\alpha} \ln\left(a_{dS} + \sqrt{a_{dS}^2 + v_0^2}\right)\Big|_{a_0}^{a}. \tag{3.6}$$

From the above equation, we see that in the limit $v_0 \to 0$, $a_{dS} = a_0 e^{\alpha(\tau - \tau_0)}$, which is the expected scale factor for the co-moving observer. Without loss of generality, we choose $\tau_0 = 0$ so that, we relate the scale factor with the proper time of the observer:

$$a_{dS} + \sqrt{a_{dS}^2 + v_0^2} = A_0 g(l) e^{\alpha\tau}, \tag{3.7}$$

where $A_0 \equiv a(\tau_0) + \sqrt{a(\tau_0)^2 + v_0^2}$ and $g(l)$ is a function of a parameter l which will label each different geodesic in the congruence. We see that, when $a = 0$, the proper time is a finite value, which explicits the incompleteness of non co-moving geodesics, since

$$\lim_{a \to 0} \tau = \frac{1}{\alpha} \ln\left(\frac{|v_0|}{A_0 g(l)}\right). \tag{3.8}$$

As shown in Appendix B, for a cosmological model all geodesic congruences have null vorticity and so we can define the vector tangent to a curve as hypersurface orthogonal, such that the covector of u^μ defined in (3.5) is the gradient of some function $\phi(t, r, v_0)$:

$$u_\mu = \partial_\mu \phi(t, r, v_0). \tag{3.9}$$

Since $u_t = -\gamma$ and $u_r = v_0$, we have that:

$$\begin{aligned} u_r = \partial_r \phi \qquad &\Rightarrow \qquad \phi = \int v_0 \mathrm{d}r = v_0 r + c_1(t), \\ u_t = \partial_t \phi \qquad &\Rightarrow \qquad \phi = -\int \sqrt{1 + \frac{v_0^2}{a_{dS}^2}} \mathrm{d}t + c_2(r). \end{aligned} \tag{3.10}$$

Comparing both equations, we obtain that the function $\phi(t, r, v_0)$ is given by:

$$\phi(t,r,v_0) = \frac{1}{\alpha}\left[\sqrt{1+\frac{v_0^2}{a_{dS}^2}} - \ln\left[a_{dS} + \sqrt{a_{dS}^2 + v_0^2}\right]\right]\Bigg|_{a_0}^{a_{dS}} + v_0 r. \tag{3.11}$$

A convenient parametrization of the function of the coordinate r is:

$$r = l - \frac{1}{\alpha v_0}\left(\sqrt{1+\frac{v_0^2}{a_{dS}^2}} - \sqrt{1+\frac{v_0^2}{a_0^2}}\right), \tag{3.12}$$

So that as $v_0 \to 0$ the parameter l can be identified with the radius coordinate of the co-moving observer. In order to see this, we can write the binomial expansion for the square root term to find that

$$\sqrt{1+\frac{v_0^2}{a^2}} = 1 + \sum_{n=1}^{\infty}\binom{1/2}{n}\left(\frac{v_0^2}{a^2}\right)^n. \tag{3.13}$$

Applying the same expansion for the square root term evaluated at a_0 leads to

$$\sqrt{1+\frac{v_0^2}{a^2}} - \sqrt{1+\frac{v_0^2}{a_0^2}} = \sum_{n=1}^{\infty}\binom{1/2}{n}\left(\frac{v_0}{a}\right)^{2n} - \sum_{n=1}^{\infty}\binom{1/2}{n}\left(\frac{v_0}{a_0}\right)^{2n}. \tag{3.14}$$

Since the lowest power in the expansion (3.14) is of order v_0^2, both terms go to zero faster than v_0 in the denominator of equation (3.12), so $\lim_{v_0\to 0} r = l$, and the parameter is identified with the radial co-moving coordinate, as intended. Now, substituting the parametrization of r in equation (3.11) and making $\phi = k = \text{const.}$, we get

$$-\alpha k + \alpha v_0 l = \ln\left[\frac{1}{A_0}\left(a_{dS} + \sqrt{a_{dS}^2 + v_0^2}\right)\right], \tag{3.15}$$

and

$$a_{dS} + \sqrt{a_{dS}^2 + v_0^2} = A_0 f(\tau) e^{\alpha v_0 l}, \tag{3.16}$$

where $f(\tau)$ is a function of τ, but which is constant over the non co-moving observer's hypersurface. Comparing the equations (3.7) and (3.16) which display the scale factor dependence on τ and l, we find that $f(\tau) = e^{\alpha\tau}$ and $g(l) = e^{\alpha v_0 l}$. Thus we have an equation for the scale factor as a function of both the proper time of the non co-moving observer and its spatial parameter

$$a_{dS} + \sqrt{a_{dS}^2 + v_0^2} = A_0 e^{\alpha(\tau + v_0 l)}. \tag{3.17}$$

Manipulating the equation above to obtain a scale factor in terms of τ and l,

$$\begin{aligned} 2a_{dS}^2 + 2a_{dS}\sqrt{a_{dS}^2+v_0^2} + v_0^2 &= A_0^2 e^{2\alpha(\tau+v_0 l)}, \\ 2a_{dS}^2 + \underbrace{v_0^2 - A_0^2 e^{2\alpha(\tau+v_0 l)}}_{\equiv C} &= -2a_{dS}\sqrt{a_{dS}^2+v_0^2}, \\ 4a_{dS}^4 + 4a_{dS}^2 C + C^2 &= 4a_{dS}^2(a_{dS}^2+v_0^2), \\ 4a_{dS}^2(v_0^2 - C) = C^2 \quad \Rightarrow \quad a_{dS} &= -\frac{C}{2\sqrt{v_0^2 - C}} \end{aligned}, \tag{3.18}$$

then

$$a_{dS} = \frac{A_0}{2}\left(e^{2\alpha(\tau+v_0 l)} - \frac{v_0^2}{A_0^2}\right) e^{-\alpha(\tau+v_0 l)} = |v_0| \sinh(\Theta - \Theta_0), \tag{3.19}$$

where we define $\Theta \equiv \alpha(\tau + v_0 l)$ and $\Theta_0 \equiv \ln(|v_0|/A_0)$. Once again, as $v_0 \to 0$ we recover the co-moving observer scale factor,

$$\lim_{v_0\to 0} |v_0| \sinh\left[\alpha(\tau+v_0 l) - \ln\left(\frac{|v_0|}{A_0}\right)\right] = \lim_{v_0\to 0} \frac{|v_0|}{2}\left(\frac{A_0}{|v_0|} e^{\alpha(\tau+v_0 l)} - \frac{|v_0|}{A_0} e^{-\alpha(\tau+v_0 l)}\right) = \frac{A_0}{2} e^{\alpha\tau}. \tag{3.20}$$

Now that we have obtained the dependency of the scale factor on the time and spatial parameters of the non co-moving observer, we can proceed to construct a coordinate basis in its local coordinates in order to fully determine its geometry.

3.1 The geometry of non co-moving observers

Replacing the scale factor (3.19) in the parametrization $r(\tau, l)$ and deriving it w.r.t. l, we obtain:

$$\frac{\partial r}{\partial l} = 1 - \frac{\partial}{\partial l} \coth(\Theta - \Theta_0) = 1 + \operatorname{csch}^2(\Theta - \Theta_0) = 1 + \frac{v_0^2}{a^2}. \tag{3.21}$$

The derivative of t with respect to l reads:

$$\frac{\partial t}{\partial l} = \frac{\partial}{\partial l}\left[\frac{1}{\alpha}\ln(a(\tau, l))\right] = v_0\sqrt{1 + \frac{v_0^2}{a^2}}. \tag{3.22}$$

Thus, we can define a vector v^μ tangent to the spatial sections of the observer, which components are given by

$$v^t \equiv \frac{\partial t}{\partial l} = v_0\sqrt{1+\frac{v_0^2}{a^2}}, \quad v^r \equiv \frac{\partial r}{\partial l} = 1+\frac{v_0^2}{a^2}. \tag{3.23}$$

Note that the hypersurface here is defined by a curve because we are only working in the (t,r)-plane. The 3 dimensional nature of the hypersurface is only manifested when we include the angular coordinates. Since this vector is tangent to the spatial hypersurface of the observer, it is evidently orthogonal to the vector u^μ tangent to the non co-moving geodesic:

$$g_{\mu\nu}u^\mu v^\nu = -u^t v^t + a^2 u^r v^r = -v_0\gamma^2 + a^2\frac{v_0^2}{a^2}\gamma^2 = 0. \tag{3.24}$$

Furthermore, the norm of the vector v^μ is

$$g_{\mu\nu}v^\mu v^\nu = -v_0^2\gamma^2 + a^2\gamma^4 = -v_0^2\left(1+\frac{v_0^2}{a^2}\right) + a^2\left(1+2\frac{v_0^2}{a^2}+\frac{v_0^4}{a^4}\right) = a^2 + v_0^2. \tag{3.25}$$

In the limit $a \to 0$, the norm of the vector neither diverges nor vanishes. Therefore, we choose the vectors u^μ, v^ν as a basis for the non co-moving observers. We aim to describe the metric of de Sitter from (t,r)-coordinates to (τ,l)-coordinates. For that, we shall use the matrix transformation

$$\left[\frac{\partial x}{\partial x'}\right] = \begin{bmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial r}{\partial \tau} & \frac{\partial \theta}{\partial \tau} & \frac{\partial \phi}{\partial \tau} \\ \frac{\partial t}{\partial l} & \frac{\partial r}{\partial l} & \frac{\partial \theta}{\partial l} & \frac{\partial \phi}{\partial l} \\ \frac{\partial t}{\partial \theta'} & \frac{\partial r}{\partial \theta'} & \frac{\partial \theta}{\partial \theta'} & \frac{\partial \phi}{\partial \theta'} \\ \frac{\partial t}{\partial \phi'} & \frac{\partial r}{\partial \phi'} & \frac{\partial \theta}{\partial \phi'} & \frac{\partial \phi}{\partial \phi'} \end{bmatrix} = \begin{bmatrix} \gamma & \frac{v_0}{a^2} & 0 & 0 \\ v_0\gamma & \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.26}$$

and its inverse

$$\left[\frac{\partial x'}{\partial x}\right] = \begin{bmatrix} \frac{\partial \tau}{\partial t} & \frac{\partial l}{\partial t} & \frac{\partial \theta'}{\partial t} & \frac{\partial \phi'}{\partial t} \\ \frac{\partial \tau}{\partial r} & \frac{\partial l}{\partial r} & \frac{\partial \theta'}{\partial r} & \frac{\partial \phi'}{\partial r} \\ \frac{\partial \tau}{\partial \theta} & \frac{\partial l}{\partial \theta} & \frac{\partial \theta'}{\partial \theta} & \frac{\partial \phi'}{\partial \theta} \\ \frac{\partial \tau}{\partial \phi} & \frac{\partial l}{\partial \phi} & \frac{\partial \theta'}{\partial \phi} & \frac{\partial \phi'}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \gamma & -\frac{v_0}{a^2\gamma} & 0 & 0 \\ -v_0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.27}$$

Then, the metric components in the non co-moving basis is

$$
\begin{aligned}
g'_{00} &= \frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau} g_{\alpha\beta} = -\left(1+\frac{v_0^2}{a^2}\right) + a^2\frac{v_0^2}{a^4} = -1,\\
g'_{11} &= \frac{\partial x^\alpha}{\partial l}\frac{\partial x^\beta}{\partial l} g_{\alpha\beta} = -v_0^2\left(1+\frac{v_0^2}{a^2}\right) + a^2\left(1+\frac{v_0^2}{a^2}\right)^2 = a^2+v_0^2 = v_0^2\cosh^2(\Theta-\Theta_0),\\
g'_{22} &= \frac{\partial x^\alpha}{\partial x'^2}\frac{\partial x^\beta}{\partial x'^2} g_{\alpha\beta} = a^2 r^2(\tau,l),\\
g'_{33} &= \frac{\partial x^\alpha}{\partial x'^3}\frac{\partial x^\beta}{\partial x'^3} g_{\alpha\beta} = a^2 r^2(\tau,l)\sin^2\theta,
\end{aligned}
$$

and the line element can be written as

$$
\mathrm{d}s^2 = -\mathrm{d}\tau^2 + v_0^2\cosh^2(\Theta-\Theta_0)\mathrm{d}l^2 + v_0^2\sinh^2(\Theta-\Theta_0) r^2(\tau,l)(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2). \tag{3.28}
$$

Note that, the radial function is now a function of both τ and l. Thus, from the line element above, we see that in the non co-moving frame the metric is anisotropic, as the scale factor a and the angular component (ar) no longer depend solely on the time coordinate, but also on the observer's spatial coordinate. Another important observation is that, in the past asymptotic limit, as $a(t\to-\infty)\to 0$, which in the non co-moving frame is equivalent to $\Theta\to\Theta_0$, the metric (3.28) is non-degenerate, as all the components are non null. Despite the hyperbolic sine dependency on the angular component, from equation (3.12) we have that the radial coordinate is given by

$$
r = l - \frac{1}{\alpha v_0}[\coth(\Theta-\Theta_0)+\coth(\Theta_0)]. \tag{3.29}
$$

In the asymptotic limit, the angular component $(ar)^2$ goes as

$$
\begin{gathered}
ar = |v_0| l\sinh(\Theta-\Theta_0) - \frac{|v_0|}{\alpha v_0}[\cosh(\Theta-\Theta_0)+\sinh(\Theta-\Theta_0)\coth(\Theta_0)],\\
\lim_{\Theta\to\Theta_0} ar = -\frac{|v_0|}{\alpha v_0} \qquad \Rightarrow \qquad \lim_{\Theta\to\Theta_0}(ar)^2 = \frac{1}{\alpha^2},
\end{gathered} \tag{3.30}
$$

and the metric as seen by the non co-moving observer at the boundary $(a\to 0)$ reads

$$
\mathrm{d}s^2 = -\mathrm{d}\tau^2 + v_0^2\mathrm{d}l^2 + \frac{1}{\alpha^2}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2), \tag{3.31}
$$

which is not singular. Furthermore, the dependency of the metric determinant near the $\Theta = \Theta_0$ surface can be evaluated from the metric (3.28):

$$\sqrt{-g} = |v_0| \cosh(\Theta - \Theta_0)(ar)^2 \sin\theta. \tag{3.32}$$

Thus, at the past boundary, we have that the limit of the metric determinant is given by:

$$\lim_{\Theta\to\Theta_0} \sqrt{-g} = |v_0| \frac{\sin(\theta)}{\alpha^2}, \tag{3.33}$$

and the metric determinant reaches a velocity dependent minimum, that occurs at $\Theta = \Theta_0$. Since the determinant is related to the four-divergence of the congruence $\nabla_\mu u^\mu$, we see that no caustics are formed in the asymptotic limit, for

$$\begin{aligned} \nabla_\mu u^\mu &= \partial_\mu u^\mu + \Gamma^\mu_{\mu\beta} u^\beta \\ &= \partial_\mu u^\mu + u^\mu \partial_\mu \ln(\sqrt{-g}) = \frac{1}{\sqrt{-g}} \partial_\mu (u^\mu \sqrt{-g}) \\ &= \frac{1}{a^2\sqrt{a^2+v_0^2}} \partial_\tau \left(a^2 \sqrt{a^2+v_0^2} \right) + \frac{2}{r} \partial_\tau r \\ &= \frac{\dot{a}}{a} \left(2\frac{\sqrt{a^2+v_0^2}}{a} + \frac{a}{\sqrt{a^2+v_0^2}} \right) + 2\frac{v_0}{a^2 r}. \end{aligned} \tag{3.34}$$

For the de Sitter case under consideration:

$$\nabla_\mu u^\mu = \alpha \tanh(\Theta - \Theta_0) + 2\alpha \left[\frac{(\alpha v_0 l + \coth\Theta_0)\coth(\Theta - \Theta_0) - 1}{\alpha v_0 l - \coth(\Theta - \Theta_0) + \coth(\Theta_0)} \right]. \tag{3.35}$$

At $\Theta = \Theta_0$, the congruence divergence becomes $\nabla_\mu u^\mu = 2\alpha(\alpha v_0 l + \coth\Theta_0)$, which is never infinite. Moreover, given the hyperbolic cosine dependency of $\sqrt{-g}$, we already expect that, when attempting to extend this incomplete patch, a natural extension for the de Sitter manifold will be by allowing $\Theta < \Theta_0$, which, as we go further in the past, means that the congruence divergence grows again, and the hyperboloid can be covered by two disconnected flat patches [19]: the upper half expanding and the lower half contracting as defined by co-moving observers.

3.2 The de Sitter Extension

The non co-moving observer reaches the $a = 0$ boundary of the co-moving patch in a finite proper time, as shown by equation (3.8). However, since the de Sitter space is a regular solution with no divergences, a natural way to extend the metric (3.28) is by allowing $\Theta < \Theta_0$:

$$\alpha(\tau + v_0 l) < \ln\left(\frac{|v_0|}{A_0}\right). \tag{3.36}$$

Formally, we can define a new time t' and a new radial coordinate r' for the contracting sheet with metric

$$\mathrm{d}s^2 = -\mathrm{d}t'^2 + e^{-\alpha t'}[\mathrm{d}r'^2 + r'(\tau, l)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \tag{3.37}$$

and perform transformations (3.5) and (3.23) to find the non co-moving metric on the lower half. By implementing this procedure, we find[1]:

$$e^{-\alpha t'} = -|v_0| \sinh[\alpha(\tau + v_0 l) - \alpha(\tau_i + v_0 l_i) + \Theta_0], \tag{3.38}$$

$$r' = l - l_i - \frac{1}{\alpha v_0}[\coth(\Theta - \Theta_0) - \coth(\Theta_0)], \tag{3.39}$$

where we deliberately display the initial conditions τ_i, l_i in order to match the coordinates (t', r') with (t, r) at the boundary. Comparing (t', r') with (3.19) and (3.29) we find that they need to be such that:

$$\begin{aligned} \alpha(\tau_i + v_0 l_i) &= 2\Theta_0, \\ l_i &= 2\coth(\Theta_0)/\alpha v_0. \end{aligned} \tag{3.40}$$

Notice, however, that this frame is only defined for the lower half, which means that, a co-moving observer in this patch never crosses to the expanding sheet. Similarly, a co-moving observer in the expanding flat de Sitter never observes a contracting phase of the Universe. In the meantime, the non co-moving observer is covered by both charts along its worldline. The scenario, as perceived by the non co-moving observer is as follows: along its geodesic, it has a fixed value of l, so the changes in Θ are due to its proper time τ. As

[1] See Appendix D for a detailed calculation of the flat patch de Sitter with $a = e^{-\alpha t}$.

$\tau \to -\infty$, $\Theta \to -\infty$, and it observes an infinitely large space. As τ increases, the space contracts, until it reaches the $a = 0$ surface, where it enters the expanding de Sitter phase. This contracting behavior followed by an expanding phase can be explicitly seen by the average geodesic expansion H_{BGV}, as defined in (2.19). Let us consider the observer $\mathcal{O}$ in the BGV's construction to be the co-moving observer in the expanding sheet with tangent vector given by components $(1, 0, 0, 0)$. The generalized Hubble parameter reads

$$H_{BGV} = -\frac{v_\mu (Du^\mu/\mathrm{d}t)}{\sigma^2 - 1}. \tag{3.41}$$

Since $\sigma = u_\nu v^\mu = \gamma$

$$\begin{aligned} H_{BGV} &= \frac{a^2}{v_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\gamma = \frac{a^2}{v_0^2}\frac{v_0^2 \dot{a}}{a^3\sqrt{1 + v_0^2/a^2}} \\ &= \frac{a\alpha}{\sqrt{a^2 + v_0^2}} = \alpha \tanh(\Theta - \Theta_0). \end{aligned} \tag{3.42}$$

Thence, we see that for $\Theta > \Theta_0$ the congruence is expanding, vanishing when $\Theta = \Theta_0$ (which is the $a = 0$ surface) and contracting for $\Theta < \Theta_0$. Moreover, as the observer tends to the asymptotic past (future), where $\Theta - \Theta_0 << 1$ ($\Theta - \Theta_0 >> 1$), we recover the usual de Sitter expansion H_{BGV} = -α ($H_{BGV} = \alpha$). This is expected, as in the asymptotic limits co-moving and non co-moving observers coincide. Furthermore, since Θ is a function of both τ and l, for a surface of simultaneity in the non co-moving frame, i.e, $\tau =$ const., the local expansion depends on the spatial coordinate l, meaning that the hypersurface has regions of contraction and regions of expansion. Nevertheless, despite the hypersurface of simultaneity not being homogeneously expanding or contracting like in usual bounces, the structure of a contracting extension is manifested through the local expansion of geodesic congruences, as seen through equation (3.42). This result shall be used when we attempt to generalize the extension procedure for an arbitrary space-time.

Chapter 4

Conditions For the Extension of a General Spatially Flat FLRW Metric

So far, we have only worked with the spatially flat de Sitter spacetime. We can, however, investigate what conditions are necessary and/or sufficient to generalize the construction of an orthogonal space-like hypersurface for the case of a non co-moving observer in a general spatially flat FLRW spacetime, with metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)]. \tag{4.1}$$

As shown in Appendix B, in such spaces congruences of time-like geodesics have null vorticity, and hence, can always be made orthogonal to a hypersurface. More precisely, in an attempt to evade the BGV theorem, we focus on FLRW space-times that are geodesically incomplete in the past and inquire into whether they admit an extension or not. For that, we exclude models in which the domain of the cosmological time t does not go to minus infinity, i.e, models such that $t \in (t_i, t_f), \quad t_i > -\infty$, for either the scale factor is null at the boundary ($a(t_i) = 0$), (and hence we have a scalar singularity avoiding a spacetime completion of class $\mathcal{C}^2$ of differentiability, which is not in accordance with General Relativity), or the scale factor is non-null at a finite time ($a(t \to t_i) = \text{const.} \neq 0$), in which case the extension is trivial [28]. Furthermore, in the cases where the cosmic time goes up to minus infinity, if the scale factor does not vanish at the boundary, then either the spacetime is already complete ($a(t \to -\infty) = \text{const.} \neq 0$) and the proper time measured along non co-moving geodesics is divergent, or the scale factor vanishes as $t \to -\infty$, $a(t \to -\infty) = 0$. The latter case shall be our first assumption on the models to be extended:

Assumption 1. *The cosmological time t of the spatially flat FLRW spacetime, as seen by the co-moving observer, is defined up to $t \to -\infty$, with a vanishing scale factor at the past boundary, i.e,* $\lim_{t\to-\infty} a(t) = 0$.

Besides, although the absence of scalar curvature singularities does not imply geodesic completeness, the existence of scalar curvatures singularities is a sufficient condition for the nonexistence of a $\mathcal{C}^2$ extension. As the goal here is to find an extension for an arbitrary model that is compatible with General Relativity, from equations (A.6) and (A.7) for the Ricci and Kretschmann scalars of a spatially flat FLRW metric, we enunciate Assumption 2:

Assumption 2. *We consider spatially flat FLRW space-times, with no scalar curvature singularity, i.e,* $\lim_{t\to-\infty} H(t) = c_1$, $\lim_{t\to-\infty} \dot{H}(t) = c_2$, *where* $|c_1|, |c_2| < \infty$.

Even though the cosmic time of the co-moving observer is defined up to minus infinity, the past-incompleteness of apparent eternal spaces satisfying Assumptions 1 and 2 is manifested in the convergence of the invariant interval in the case of non co-moving time-like geodesics

$$\Delta\tau(t) = \int_{-\infty}^{t_0} \frac{\mathrm{d}t}{\sqrt{1+\frac{v_0^2}{a^2}}} < \infty, \qquad \forall t_0 \in \mathbb{R}, \tag{4.2}$$

or convergence of the affine parameter of null geodesics

$$\Delta\lambda(t) = \int_{-\infty}^{t_0} a\mathrm{d}t < \infty, \qquad \forall t_0 \in \mathbb{R}. \tag{4.3}$$

However, the notion of null and time-like incompleteness in flat FLRW models are equivalent, as shown in the following Lemma:

Lemma 1. *A space-time satisfying Assumptions 1 and 2 is time-like incomplete if and only if it is null incomplete.*

Proof. From assumption 1, we have that $a(t \to -\infty) = 0$. Therefore, since t_0 is arbitrary, we can choose it such that $a^2(t_0) << v_0^2$, so that $\sqrt{1+\frac{v_0^2}{a^2}} \approx \frac{|v_0|}{a}$. Thus

$$\Delta\tau(t) = \int_{-\infty}^{t_0} \frac{\mathrm{d}t}{\sqrt{1+\frac{v_0^2}{a^2}}} \approx \int \frac{a\mathrm{d}t}{|v_0|} = \frac{\Delta\lambda(t)}{|v_0|}. \tag{4.4}$$

If the space-time is null incomplete, then $\Delta\lambda(t) < \infty$.

$\square$

The converse is also true: if the spacetime is time-like complete, then it is null complete. This result is expected as time-like observers tend to null geodesics in the asymptotic limit.

From here onward, we attempt to construct a basis for non co-moving time-like geodesics in order to find conditions for a $\mathcal{C}^2$ extension. For a flat FLRW spacetime, any congruence of non co-moving observers has tangent vector field given by the vectors u^μ defined in (3.5). The vector field perpendicular to the congruence at each point which will be used to define a local orthogonal hypersurface reads

$$v^\mu = \left(v_0\sqrt{1+\frac{v_0^2}{a^2}}, 1+\frac{v_0^2}{a^2}, 0, 0 \right). \tag{4.5}$$

This tangent vector defines a parameter l such that

$$v^\mu \equiv \left(\frac{\mathrm{d}t}{\mathrm{d}l}, \frac{\mathrm{d}r}{\mathrm{d}l} \right). \tag{4.6}$$

Thereby, we have the elements of the transformation matrix to obtain the metric in the non co-moving observer coordinates τ, l for an arbitrary space:

$$\left[\frac{\partial x^\alpha}{\partial x'^\beta} \right] = \begin{bmatrix} \sqrt{1+\frac{v_0^2}{a^2}} & v_0\sqrt{1+\frac{v_0^2}{a^2}} \\ \frac{v_0}{a^2} & 1+\frac{v_0^2}{a^2} \end{bmatrix}. \tag{4.7}$$

Therefore

$$\begin{aligned} g'_{00} &= \frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau} g_{\alpha\beta} = -\left(1+\frac{v_0^2}{a^2}\right) + a^2\frac{v_0^2}{a^4} = -1, \\ g'_{11} &= \frac{\partial x^\alpha}{\partial l}\frac{\partial x^\beta}{\partial l} g_{\alpha\beta} = -v_0^2\left(1+\frac{v_0^2}{a^2}\right) + a^2\left(1+\frac{v_0^2}{a^2}\right)^2 = a^2 + v_0^2, \\ g'_{22} &= \frac{\partial x^\alpha}{\partial x'^2}\frac{\partial x^\beta}{\partial x'^2} g_{\alpha\beta} = a^2 r^2, \\ g'_{33} &= \frac{\partial x^\alpha}{\partial x'^3}\frac{\partial x^\beta}{\partial x'^3} g_{\alpha\beta} = a^2 r^2 \sin^2\theta. \end{aligned} \tag{4.8}$$

Thus, in the non co-moving observer frame, the metric reads

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + (a^2+v_0^2)\mathrm{d}l^2 + a^2 r^2(\tau, l)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{4.9}$$

From this metric, we get that the metric determinant is

$$\sqrt{-g} = \sqrt{a^2 + v_0^2}(ar)^2 \sin\theta. \tag{4.10}$$

Since u^μ, v^ν form an integrable basis, we can integrate $\partial r/\partial l = 1 + v_0^2/a^2$ and $\partial r/\partial \tau = v_0/a^2$, to find for the general case that

$$r = l + v_0^2 \int \frac{\mathrm{d}l'}{a^2}. \tag{4.11}$$

In the limit $a \to 0$, we want the metric determinant,

$$\lim_{a\to 0} \sqrt{-g} = |v_0| \sin\theta \lim_{a\to 0} \left(al + v_0^2 a \int_0^l \frac{\mathrm{d}l'}{a^2} \right)^2, \tag{4.12}$$

to be well defined at the boundary. Despite not knowing the dependence of the scale factor on the coordinates τ, l for an arbitrary model, we know that the cosmic time t of the co-moving observer is always a function of $\tau + v_0 l$:

$$\begin{aligned}
\frac{\partial t}{\partial \tau} &= \sqrt{1 + \frac{v_0^2}{a^2}} \qquad \Rightarrow \qquad \tau = \int \frac{\mathrm{d}t}{\sqrt{1 + v_0^2/a^2}} \equiv \mathcal{G}(t) \\
\frac{\partial t}{\partial l} &= v_0 \sqrt{1 + \frac{v_0^2}{a^2}} \qquad \Rightarrow \qquad v_0 l = \int \frac{\mathrm{d}t}{\sqrt{1 + v_0^2/a^2}} \equiv \mathcal{G}(t)
\end{aligned} \tag{4.13}$$

Hence, by adding both equations we see that $t = \mathcal{G}^{-1}(\tau + v_0 l)$. Therefore, from now on, we shall always treat any function of t as a function of τ and l in this particular combination. We can switch the integration variable by doing:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\partial a}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial a}{\partial l}\frac{\partial l}{\partial t} = \frac{\mathrm{d}a}{\mathrm{d}l}\left(\frac{\gamma}{v_0} - \frac{v_0}{\gamma a^2} \right) = \frac{1}{v_0 \gamma}\frac{\mathrm{d}a}{\mathrm{d}l}. \tag{4.14}$$

Thus

$$\mathrm{d}l = \frac{1}{v_0}\frac{\mathrm{d}a}{\gamma a H}, \tag{4.15}$$

or, alternatively

$$\mathrm{d}l = \frac{\mathrm{d}t}{v_0 \gamma}. \tag{4.16}$$

Thence, we can substitute the differential (4.15) in the determinant close to the boundary, such that $a^2(t_0) << v_0^2$

$$\begin{aligned}\lim_{a\to 0}\sqrt{-g} = |v_0|\sin\theta\lim_{a\to 0}\left(al + v_0 a\int\frac{\mathrm{d}t'}{a\sqrt{a^2+v_0^2}}\right)^2 \\ \approx |v_0|\sin\theta\lim_{a\to 0}\frac{v_0}{|v_0|}\left(\int_{t_0}^{t}\frac{\mathrm{d}t'}{a}\right)^2 = v_0\sin\theta\lim_{t\to-\infty}\left[a(t)\eta(t)\right]^2,\end{aligned} \tag{4.17}$$

where $\eta(t)$ is the primitive of $1/a(t)$. Notice, however, that in order for the determinant to be finite and non null, $\eta(t)$ cannot converge. This is in fact the case, given Assumption 2. The proof goes by contradiction: let us assume that $\lim_{a\to 0}\eta(t) < \infty$, then we can write

$$\eta(t) = \frac{a(t)\eta(t)}{a(t)} \qquad \Rightarrow \qquad \lim_{a\to 0}\eta(t) = \lim_{a\to 0}\frac{a(t)\eta(t)}{a(t)}. \tag{4.18}$$

Since we supposed $\eta(t)$ converges, we can apply L'Hôpital rule

$$\lim_{a\to 0}\eta(t) = \frac{1+\dot{a}\eta}{\dot{a}} = \lim_{a\to 0}\frac{1}{\dot{a}} + \lim_{a\to 0}\eta(t). \tag{4.19}$$

Therefore, we need that

$$\lim_{a\to 0}\frac{1}{\dot{a}} = 0, \tag{4.20}$$

and $\dot{a}$ must diverge. But this cannot be true since H is limited, by Assumption 2. If $\dot{a}$ were to diverge as $a\to 0$, then H would dirverge as well. Thence, we conclude that:

$$\lim_{a\to 0}\eta(t)\to\infty. \tag{4.21}$$

Then, knowing that $\eta(t)$ diverges, we can apply L'Hôpital rule to the term $a(t)\eta(t)$ in the metric determinant

$$\lim_{a\to 0}a(t)\eta(t) = \lim_{a\to 0}\frac{\eta}{1/a} = -\lim_{a\to 0}\frac{1}{H}, \tag{4.22}$$

hence

$$\lim_{a\to 0}\sqrt{-g} = v_0\sin\theta\lim_{t\to-\infty}\frac{1}{H^2}. \tag{4.23}$$

This gives us a third assumption for the construction of the metric in the non co-moving frame:

Assumption 3. *The Hubble parameter H, as measured by the co-moving observer does not vanish in the asymptotic limit, i.e,* $\lim_{t\to-\infty} H \neq 0$.

The cases where $\lim_{a\to 0} H \to 0$, despite having no scalar curvature singularity, are a curious case: either they are geodesically complete - and therefore, do not need an extension such as the one we are trying to build - or they have a *parallelly propagated curvature singularity*, as the one discussed in Appendix C, that prevents its completion. In the following Lemma we show that the cases $\lim_{a\to 0} H \to 0$ that are incomplete cannot be extended, neither by the method here presented or any other protocol, as they have a *parallelly propagated curvature singularity*:

Lemma 2. *Let a spatially flat FLRW with line element given by (4.1). If* $\lim_{t\to-\infty} H \to 0$ *and the space-time is geodesically past incomplete, it cannot be extended past the asymptotic boundary, as it possesses a parallelly propagated curvature singularity in the asymptotic limit.*

Proof. If the spacetime is incomplete (time-like and null incompleteness are equivalent, according to Lemma 1), then:

$$\Delta\lambda(t_0) = \int_{-\infty}^{t_0} a\mathrm{d}t \equiv F(t_0) - \lim_{t\to-\infty} F(t) < \infty. \tag{4.24}$$

The primitive of $a(t)$, $F(t)$, tends to a finite value $F^* < \infty$. However, assuming that H/a is finite as $t \to -\infty$, we get a contradiction

$$\begin{aligned}\lim_{t\to-\infty} F(t) &= \lim_{t\to-\infty} \frac{a(t)F(t)}{a(t)} = \lim_{t\to-\infty} \frac{a^2 + \dot{a}F(t)}{\dot{a}} \\ &= \lim_{t\to-\infty} \left(F(t) + \frac{a}{H}\right) = F^* + \frac{1}{\kappa} \neq F^*.\end{aligned} \tag{4.25}$$

Thus, we need that $\kappa \to \infty$, and therefore $\lim_{a\to 0} H/a \to \infty$. Because $H \to 0$, we can use L'Hôpital to finally obtain:

$$\lim_{a\to 0} \frac{H^2}{a^2} = \lim_{a\to 0} \frac{2H\dot{H}}{2a\dot{a}} = \lim_{a\to 0} \frac{\dot{H}}{a^2}. \tag{4.26}$$

Then, if $H/a \to \pm\infty$, so does $\dot{H}/a^2$, which represents a parallelly propagated curvature singularity [1], as proven in equation (C.18).

[1] For a detailed derivation, see Appendix C.

$\square$

As shown by contradiction, if a space-time is geodesically incomplete in the past, it is imperative that the limit of the ratio H/a as $a(t \to -\infty) \to 0$ diverges. Thus, as an immediate consequence, we have that, for incomplete space-times that are 'asymptotically Minkowski' ($\lim_{a\to 0} H = 0$), the incompleteness implies in a singularity parallelly propagated along light curves [29], since the ratio $\dot{H}/a^2$ diverges, and therefore, the spacetime is inextensible past the asymptotic boundary. Notice, however, that while the geodesic incompleteness of a spacetime implies in the divergence of H/a, the converse is not necessarily true, hence, a priori, a spacetime might be geodesically complete and have such divergence. The implications of this property for the extension of certain models are immediate: for the case of pre-Big-Bang (pBB) models, for instance, where

$$a(t) \propto t^{-\alpha}, \qquad \alpha > 1. \tag{4.27}$$

As discussed in Chapter 1, for a perfect fluid with state equation given by $p = \omega\rho$, the pBB models correspond to fluids with ω in the interval

$$\frac{2}{3(\omega+1)} < -1 \qquad \Rightarrow \qquad -\frac{5}{3} < \omega.$$

Adding the condition that the scale factor goes with an inverse power of t, we have that ω is limited in the interval

$$-\frac{5}{3} < \omega < -1, \tag{4.28}$$

and the Hubble factor for such a model is given by

$$H \equiv \frac{\dot{a}}{a} \propto \frac{1}{t}. \tag{4.29}$$

Thus, as $a(t \to -\infty) \to 0$, the Hubble parameter tends to zero, and hence, these models which were shown to be incomplete are also inextensible. Now, we manage to show that the pBB models are precisely the cases in which we have a curvature singularity parallelly propagated. The interval of ω for which Cosmological Models are incomplete is the interval which accepts $\mathcal{C}^2$ metric extension. Calculating the ratio $\dot{H}/a^2$ for scale factor of the form (4.27), we have that

$$\frac{\dot{H}}{a^2} \propto t^{2(\alpha-1)}. \tag{4.30}$$

We see that as $t \to -\infty$ this diverges for $\alpha > 1$, which yields condition (4.28). Thus, we conclude that the pBB models are inextensible due to a curvature singularity parallelly propagated. Furthermore, since $H(a \to 0) \to 0$, the presented continuous extension does not apply, since (ar) diverges.

Thus far, our 3 assumptions alone are sufficient for metric (4.9) to be continuously extended through the past boundary. However, in order for the metric components to be of class $\mathcal{C}^2$ we need to compute its derivatives:

$$\begin{aligned} \partial_\tau g_{11} = \partial_\tau(a^2+v_0^2) = \frac{1}{v_0}\partial_l(a^2+v_0^2) = 2aH\sqrt{a^2+v_0^2}; \\ \partial_\tau^2(a^2+v_0^2) = \frac{1}{v_0^2}\partial_l^2(a^2+v_0^2) = \frac{1}{v_0}\partial_l\partial_\tau(a^2+v_0^2) = 2[(a^2+v_0^2)\dot{H} + (2a^2+v_0^2)H^2]; \end{aligned} \tag{4.31}$$

Denoting the limit $a \to 0$ as $\xi \equiv \tau + v_0 l \to \xi_*$ we get:

$$\begin{aligned} \lim_{\xi\to\xi_*} \partial_\tau g_{11} = \lim_{\xi\to\xi_*} \partial_\tau(a^2+v_0^2) = \frac{1}{v_0}\partial_l(a^2+v_0^2) = 0; \\ \lim_{\xi\to\xi_*} \partial_\tau^2 g_{11} = \lim_{\xi\to\xi_*} \partial_\tau^2(a^2+v_0^2) = \frac{1}{v_0^2}\lim_{\xi\to\xi_*} \partial_l^2(a^2+v_0^2) = 2v_0^2(\dot{H}+H^2). \end{aligned} \tag{4.32}$$

For the angular component $g_{22} = (ar)^2$ (the result is analogous for g_{33}, since θ and ϕ do not depend on τ, l):

$$\begin{aligned} \partial_\tau(ar) = \sqrt{a^2+v_0^2}\frac{H}{a}(ar) + \frac{v_0}{a}, \\ \partial_l(ar) = v_0\left[\sqrt{a^2+v_0^2}\frac{H}{a}(ar) + \frac{v_0}{a}\right] + a. \end{aligned} \tag{4.33}$$

Using (4.22), we find that

$$\lim_{\xi\to\xi_*} \sqrt{a^2+v_0^2}\frac{H}{a}(ar) = |v_0| l \lim_{\xi\to\xi_*} H - \lim_{\xi\to\xi_*} \frac{v_0}{a},$$

thus,

$$\lim_{\xi\to\xi_*} \partial_\tau(ar) = |v_0| l \lim_{\xi\to\xi_*} H = \lim_{\xi\to\xi_*} \frac{1}{v_0}\partial_l(ar). \tag{4.34}$$

Thence, the first derivative of the metric component $(ar)^2$:

$$\begin{aligned}\partial_\tau (ar)^2 &= 2(ar)\left[\sqrt{a^2+v_0^2}\frac{H}{a}(ar)+\frac{v_0}{a}\right],\\ \partial_l (ar)^2 &= 2v_0(ar)\left[\sqrt{a^2+v_0^2}\frac{H}{a}(ar)+\frac{v_0}{a}\right]+2a(ar).\end{aligned} \tag{4.35}$$

In the limit $\xi \to \xi_*$

$$\begin{aligned}\lim_{\xi\to\xi_*}\partial_\tau (ar)^2 &= \lim_{\xi\to\xi_*} 2(ar)\lim_{\xi\to\xi_*}\left[\sqrt{a^2+v_0^2}\frac{H}{a}(ar)+\frac{v_0}{a}\right] = -2|v_0|l,\\ \lim_{\xi\to\xi_*}\partial_l (ar)^2 &= \lim_{\xi\to\xi_*} 2v_0(ar)\lim_{\xi\to\xi_*}\left[\sqrt{a^2+v_0^2}\frac{H}{a}(ar)+\frac{v_0}{a}\right]+\lim_{\xi\to\xi_*}2a(ar) = -2v_0|v_0|l.\end{aligned} \tag{4.36}$$

Furthermore, the second order derivatives are given by

$$\partial_\tau^2 (ar)^2 = 2\left\{[\partial_\tau (ar)]^2+(ar)\partial_\tau^2(ar)\right\} = 2\left\{[\partial_\tau(ar)]^2+(ar)^2\left[H^2+(a^2+v_0^2)\frac{\dot{H}}{a^2}\right]\right\}, \tag{4.37}$$

$$\partial_l^2 (ar)^2 = 2\left\{[\partial_l(ar)]^2+v_0^2(ar)^2\left[(a^2+v_0^2)\frac{\dot{H}}{a^2}+H^2\right]+2v_0H(ar)\sqrt{a^2+v_0^2}\right\}, \tag{4.38}$$

$$\partial_l\partial_\tau (ar)^2 = 2(ar)\left\{[\partial_l(ar)[\partial_\tau(ar)]+(ar)^2[(a^2+v_0^2)\dot{H}+H^2]+H\sqrt{a^2+v_0^2}\right\}. \tag{4.39}$$

Taking the limit $\xi \to \xi_*$ of the second order derivatives:

$$\lim_{\xi\to\xi_*}\partial_\tau^2 (ar)^2 = 2v_0^2l^2+2\lim_{\xi\to\xi_*}\left[1+\frac{v_0^2}{H^2}\frac{\dot{H}}{a^2}+\frac{\dot{H}}{H^2}\right], \tag{4.40}$$

$$\lim_{\xi\to\xi_*}\partial_l^2 (ar)^2 = 2\lim_{\xi\to\xi_*}[\partial_l(ar)]^2+2\lim_{\xi\to\xi_*}v_0^2\left[1+\frac{\dot{H}}{H^2}+\frac{v_0^2}{H^2}\frac{\dot{H}}{a^2}\right]-4v_0|v_0|, \tag{4.41}$$

$$\lim_{\xi\to\xi_*}\partial_l\partial_\tau (ar)^2 = 2v_0\lim_{\xi\to\xi_*}\partial_\tau^2(ar)^2+2|v_0|H. \tag{4.42}$$

Through expressions (4.40), (4.41) and (4.42) we see that a last assumption needs to be made:

Assumption 4. $\lim_{a\to 0}\frac{\dot{H}}{a^2}=c$, *where* $c\in\mathbb{R}$.

Note that, as shown in Lemma 2, if the space-time is past incomplete and if $\lim_{a\to 0} H = 0$, then $\lim_{a\to 0}\dot{H}/a^2$ necessarily diverges. However, the fact that a space-time is incomplete, solely, does not guarantee this divergence without knowledge of the limit of H. That is why Assumption **(iv)** is needed in order to have a $\mathcal{C}^2$ metric extension. This assumption happens to be the condition for no parallelly propagated curvature singularity. Moreover, regarding the extension, an interesting result arises from the calculation of derivatives of a^2 with respect to the function ξ:

$$\frac{\partial a}{\partial \tau}=\frac{\partial a}{\partial \xi}\frac{\partial \xi}{\partial \tau}=\partial_\xi a = H\sqrt{a^2+v_0^2}. \tag{4.43}$$

Therefore, we have that

$$\begin{gathered}\partial_\xi a^2 = 2a\partial_\xi a = 2aH\sqrt{a^2+v_0^2},\\ \lim_{\xi\to\xi_*}\partial_\xi a^2 = 0,\end{gathered} \tag{4.44}$$

which signalizes a point of either maximum or minimum of a^2 at the boundary. Additionally, the second derivative is positive in the asymptotic limit:

$$\begin{gathered}\partial_\xi^2 a^2 = 2(\partial_\xi a)^2 + 2a\partial_\xi^2 a = 2H^2(a^2+v_0^2) + 2a\left(\dot{H}\sqrt{a^2+v_0^2}+2aH^2\right),\\ \lim_{\xi\to\xi_*}\partial_\xi^2 a^2 = 2v_0^2H^2 > 0.\end{gathered} \tag{4.45}$$

Thus, we have that $\xi = \xi_*$ is a local minimum of a^2, and the extension, locally, must have a bounce. This can be seen through the local expansion of geodesics: let us compute the H_{BGV} between a co-moving $v^\mu = (1,0,0,0)$ and a non co-moving observer $u^\mu = (\gamma, v_0/a^2, 0, 0)$:

$$H_{BGV} = -\frac{v_\mu Du^\mu/\mathrm{d}t}{(u^\mu v_\mu)^2-1} = \frac{aH}{\sqrt{a^2+v_0^2}}. \tag{4.46}$$

Note that this goes to zero as $a\to 0$. Furthermore, given that $\partial_\tau a^2 = \partial_\xi a^2 = 2a\partial_\tau a$, using equation (4.43) the BGV expansion (4.46) can be recast in the more convenient manner

$$H_{BGV} = \frac{aH\sqrt{a^2+v_0^2}}{(a^2+v_0^2)} = \frac{a\partial_\tau a}{(a^2+v_0^2)} = \frac{1}{2}\frac{\partial_\xi a^2}{(a^2+v_0^2)}. \tag{4.47}$$

We can picture $\sqrt{a^2+v_0^2}H_{BGV}$ as a generalization of the local expansion between the geodesics without the normalization factor. For instance, if we take the co-moving limit, we see that as $v_0 \to 0$, $\sqrt{a^2+v_0^2}H_{BGV} \to \dot{a}$. Thus, we can write the generalized expansion as

$$\sqrt{a^2+v_0^2}H_{BGV} = \frac{1}{2}\frac{\partial_\xi a^2}{\sqrt{a^2+v_0^2}}, \tag{4.48}$$

which is null at $a=0$ and with positive derivative since $\partial_\xi^2 a^2 = 2v_0^2 H^2 > 0$. Even though the bounce is not homogeneous nor isotropic, the nature of the contracting phase needed in whatever extension one has in mind is explicit through the second derivative $\partial_\xi^2 a^2$ being positive, which implies a point of local minimum in the expansion. Additionally, a novelty of this extension procedure is that there is no need of the *null energy condition*[2] (NEC) violation, as for example, the de Sitter extension here presented. Since for de Sitter case $p=-\rho$, we have that $\rho+p=0$ and the NEC is intact. Another example with no NEC violation that could be extended through the protocol here presented is the toy model $a(t) \propto \operatorname{sech} t, \quad \forall t \in (-\infty, t_0)$. Note that such model satisfies all the four assumptions required

$$\begin{gathered}
\textbf{(i)} \lim_{t\to-\infty} a(t) = \lim_{t\to-\infty} \operatorname{sech}(t) = 0, \\
\textbf{(ii)} \qquad \lim_{t\to-\infty} H = \lim_{t\to-\infty} -\tanh(t) = 1 < \infty, \\
\lim_{t\to-\infty} \dot{H} = \lim_{t\to-\infty} -\operatorname{sech}^2(t) = 0, \\
\textbf{(iii)} \qquad \lim_{t\to-\infty} H \neq 0, \\
\textbf{(iv)} \qquad \lim_{t\to\infty} \frac{\dot{H}}{a^2} = -1 < \infty.
\end{gathered} \tag{4.49}$$

Thus, this model satisfies all the conditions for a C^2 extension. Furthermore, from Friedmann's equation:

$$\rho = \frac{3}{8\pi G}H^2 = \frac{3}{8\pi G}(1-a^2), \tag{4.50}$$

[2]For the case of a perfect fluid, the NEC can be expressed as $\rho + p \geq 0$

$$\begin{aligned}\dot{H} &= -4\pi G(\rho + p),\\ a^2 &= -4\pi G\left[\frac{3}{8\pi G}(1-a^2) + p\right],\\ \Rightarrow \frac{p}{\rho} &= \omega(a) = \frac{\left[\frac{5}{3}a^2 - 1\right]}{1-a^2}.\end{aligned} \tag{4.51}$$

We see that our toy model tends to a space with cosmological constant in the asymptotic past, and

$$p + \rho = \frac{a^2}{4\pi G} > 0, \qquad \forall t \in (-\infty, t_0). \tag{4.52}$$

Chapter 5

Cyclic Cosmological Models and Geodesic Completeness

We have exhausted all the possibilities for the asymptotic behavior of the Hubble parameter. However, one last interesting case concerning geodesic completeness of cosmological model is the *cyclic model*. The question of whether the Universe could exhibit a strictly periodic behavior has long been sorted out in literature. Since the works of Richard Tolman[1], it is known that a strictly cyclic cosmological model, in which the co-moving observer sees its isotropic and homogeneous spatial sections oscillating between a minimum scale factor, $a_{\rm min}$, and a maximum, $a_{\rm max}$, within a periodic time interval $\Delta t = T$

$$
\begin{aligned}
a(t) = a(t+T), \qquad \forall t \in \mathbb{R}, \\
0 < a_{\rm min} \leq a(t) \leq a_{\rm max} < \infty,
\end{aligned}
\tag{5.1}
$$

is not thermodynamically allowed. However, it is trivial to show that, if it was not for thermodynamical considerations, for a reasonable strictly periodic eternal Universe, any observer, regardless of his state of movement w.r.t to the background, would measure an infinite proper time from the asymptotic past, once that, if $a(t)$ satisfies (5.1), so does $\sqrt{1 + v_0^2/a^2(t)}$. Thus, let C be the integral of one period. Assuming $a(t)$ has a finite maximum, we have

$$
\int_{t-T}^{t} \frac{\mathrm{d}t}{\sqrt{1 + \frac{v_0^2}{a^2}}} = C < \infty. \tag{5.2}
$$

[1]Richard C. Tolman (1881 - 1948)

If we change the integration limit to cover two previous periods, we have

$$\int_{t-2T}^{t} \frac{\mathrm{d}t}{\sqrt{1+\frac{v_0^2}{a^2}}} = 2C. \tag{5.3}$$

By mathematical induction, if we integrate over N periods:

$$\int_{t-NT}^{t} \frac{\mathrm{d}t}{\sqrt{1+\frac{v_0^2}{a^2}}} = NC. \tag{5.4}$$

Therefore, to evaluate the limit of this integral in the asymptotic past, we must integrate over an infinite number of past periods $N \to \infty$:

$$\int_{t\to-\infty}^{t} \mathrm{d}\tau = \lim_{N\to\infty} \int_{t-NT}^{t} \frac{\mathrm{d}t}{\sqrt{1+\frac{v_0^2}{a^2}}} = \lim_{N\to\infty} NC \to \infty, \tag{5.5}$$

and hence all time-like geodesics should have infinite parameter. A similar reasoning can be carried out for the null case since $\Delta\lambda \propto \int_{t-T}^{t} a\mathrm{d}t$.

5.1 A Model with Entropy Dissipation

Despite being geodesic complete, it is not physically reasonable for our Universe to be strictly cyclic, as these models can not satisfy both the conditions for periodicity and conditions for thermodynamical reversible process that would need to take place in order to have a cyclic cosmos [30, 31].

However, numerous models with somewhat cyclic properties have been proposed ever since, as for instance a model with a scale factor growing from one cycle to the other but with strictly periodic Hubble parameter [32].

To solve the problem of incompatibility between the reversibility of thermodynamical process and a strictly periodic solution, a new kind of cyclic model was proposed [33], where the assumption of periodicity of $a(t)$ is put aside. Instead, the Hubble parameter is periodic in time

$$H(t) = H(t+T), \qquad \forall t \in \mathbb{R}. \tag{5.6}$$

In this model, in order to dissipate entropy outside the Hubble horizon, such that, in the observable Universe the thermodynamic processes are reversible, the scale factor is allowed to grow exponentially from one cycle to the next:

$$a(t+T) = e^N a(t), \qquad \forall t \in \mathbb{R}. \tag{5.7}$$

This space satisfies assumption 1, that the scale factor vanishes at the past boundary. However, when we consider the scale factor of form (5.7), we have a Hubble function of the type

$$H(t) = \frac{N}{T} + \frac{\dot{P}(t)}{P(t)}, \tag{5.8}$$

where $P(t)$ is periodic in the cosmic time. Even if the periodic function is bounded by a maximum finite value, this does not guarantee that the extension is possible, as H does not approach a definite value in the asymptotic past. This means that, as the limit of the Hubble parameter does not exist and, since the curvature scalars are proportional to powers of H^2, no $\mathcal{C}^2$ extension for this space is possible, since the curvature scalars do not have a definite value, even in bounded cases. What happens is that, the co-moving reference observes evenly time intervals between each cycles. However, a non co-moving observer does not agree with the period of each phase, as can be seen through his proper time. Consider the time elapsed as measured by a non co-moving observer during on single cycle

$$\Delta T(t) = \int_t^{t+T} \frac{\mathrm{d}t}{\sqrt{1+\frac{v_0^2}{a^2}}}. \tag{5.9}$$

Since $a \to 0$, we can choose the starting time to measure the oscillation such that $a(t)^2 << v_0^2$, such that a period measured by the non co-moving observer is

$$\Delta T(t) \approx \frac{1}{|v_0|}\int_t^{t+T} a(t)\mathrm{d}t = \frac{1}{|v_0|}\int_t^{t+T} P(t)e^{Nt/T}\mathrm{d}t. \tag{5.10}$$

Since the function $P(t)$ is periodic oscillating between a maximum P_{max} and a minimum P_{min}

$$P_{\text{min}} \leq P(t) \leq P_{\text{max}}, \qquad \forall t \in \mathbb{R}, \tag{5.11}$$

we have that integral (5.10) will always be bounded by

$$\int_t^{t+T} P_{\rm min} e^{Nt/T} {\rm d}t \le \Delta T(t) \le \int_t^{t+T} P_{\rm max} e^{Nt/T} {\rm d}t,$$
$$\frac{T}{N} P_{\rm min} e^{Nt/T}(e^N - 1) \le \Delta T(t) \le \frac{T}{N} P_{\rm max} e^{Nt/T}(e^N - 1). \tag{5.12}$$

Thus the period is always bounded from above and below. However, as $t \to -\infty$:

$$\lim_{t\to-\infty} \frac{T}{N} P_{\rm min} e^{Nt/T}(e^N - 1) = \lim_{t\to-\infty} \frac{T}{N} P_{\rm max} e^{Nt/T}(e^N - 1) = 0, \tag{5.13}$$

and hence the bounds go to zero, forcing the period oscillation of the Universe as seen by the non co-moving observer to vanish and consequently, to see an infinite frequency of oscillation between the cycles.

Another interesting cyclic model is Penrose's Conformal Cyclic Cosmologies (CCC), proposed in Ref. [34]. In these scenarios the asymptotic future limit is matched to the asymptotic past through a conformal transformation, thus connecting different cycles (referred to as *aeons*) by a conformal re-escaling

$$g^F_{\mu\nu} = (\Omega^F)^2 g_{\mu\nu} = (\Omega^P)^2 g_{\mu\nu} = g^P_{\mu\nu}, \tag{5.14}$$

where the superscripts F and P correspond to the *future* and *past* conformal factors, respectively, and metric $g_{\mu\nu}$ is a metric used to connect the two asymptotic limits. One of the many realizations of such scenarios which is cosmologically relevant due to the late time cosmic expansion is the connection of subsequent phases of expansion followed by a conformal matching to a contracting phase. In this scenario, the asymptotic future behavior would be de Sitter, dominated by a cosmological constant. It was previously argued in Ref. [35] that non co-moving observers would be described by a co-moving frame in the de Sitter metric with spatially negative curvature ($k = -1$), and hence, would observe its geodesic to be finite in the past, due to coordinate singularity at $\tau = 0$ in $a = \sinh(\alpha\tau)$. Constructing a bounding de Sitter space on each *aeon* and matching its null boundaries $\mathscr{J}^\pm$ would map the *aeon* expanding in the past with the one contracting in the future. Thus, any non co-moving observer in the expanding *aeon* would see incomplete geodesics in the past due to the incompleteness of the open coordinates. As was shown in this work, this is not true since, despite having a hyperbolic sine dependency, the non co-moving frame is neither homogeneous nor isotropic, and hence, the open patch does not correspond to a non co-moving observer, which could allow a CCC extension.

Chapter 6

Concluding Remarks

In this dissertation we discussed the issue of geodesic completeness in Cosmological Models where the co-moving observer sees an eternal Universe in the past. However, we have found that the same is not necessarily true for an observer in movement with respect to the homogeneous and isotropic background. For a cosmological model filled by a perfect fluid with equation of state $p = \omega\rho$, we obtained that the interval $-5/3 < \omega < -1$ is incomplete and $\mathcal{C}^2$ inextensible. Moreover, in the context of the Standard Cosmological Model, which needs an inflationary phase where the scale factor undergoes an accelerated expansion, the BGV theorem ascertains its incompleteness (or any space with an average positive expansion), rendering past eternal inflationary models still incomplete regardless of the infinite co-moving interval. Notwithstanding, in order for the past incompleteness to portray a physical issue, in what concerns General Relativity, the model can admit no $\mathcal{C}^2$-extension in order to be truly singular rather than only coordinate incomplete. For that purpose, since the finitude of the parameter is always displayed through non co-moving observers, we have established a method through which any incomplete extensible flat FLRW metric can be C^2 extended past the asymptotic boundary and the criteria for the incomplete space to be extensible can be summed up by the following definition:

Definition 6.0.1. *A spatially flat FLRW space that satisfies assumptions **(i)**, **(ii)**, **(iii)**, and **(iv)** is said to be* asymptotically de Sitter*.*

Theorem 1. *Consider an incomplete spatially flat FLRW model with line element given by (4.1) in which $a \to 0$ in the asymptotic past, $t \to -\infty$. The space-time admits a $\mathcal{C}^2$ extension through the asymptotic boundary if and only if it is* asymptotically de Sitter*, as per Definition 6.0.1.*

Proof. Assumption (i) guarantees that the space-time is defined up to minus infinity, thus, the scale factor does not vanish for a finite value of t, preventing a Big Bang type of singularity [28]. If assumption (ii) is not fulfilled, then we have *scalar curvature singularities* since one of the curvature scalars, which are linearly independent polynomials of H and $\dot{H}$, will diverge, preventing a $\mathcal{C}^2$ extension. If assumption (iii) is not fulfilled, $\lim_{a\to 0} H = 0$, we have shown through Lemma 2, that it possess a *parallelly propagated curvature singularity*. Since these exaust all the possibilities for the limit of H, the only cases in which there is no type of singularity is the case $0 < \lim_{a\to 0} H < \infty$. Moreover, if assumption (iv) is not fulfilled, $\lim_{a\to 0} \dot{H}/a^2 \to \pm\infty$ and there is a parallelly propagated curvature singularity. Therefore, to be $\mathcal{C}^2$ extensible the space must satisfy all the assumptions, and hence, it must be *asymptotically de Sitter*. □

Furthermore, despite not being de Sitter, a space-time does not need to possess all the symmetries of the exact de Sitter space, which would consequently discard the possible completion of any realistic inflationary model. For a spatially flat FLRW model to have a $\mathcal{C}^2$ extension compatible with General Relativity, it is *sufficient* to be *asymptotically de Sitter*, satisfying assumptions $(i)-(iv)$. In fact, it is not only sufficient, but also *necessary.*

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Appendix A

Curvature Tensors in FLRW

Some useful computation throughout this dissertation are summed in this appendix. For a FLRW metric, given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}\chi^2 + r^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \tag{A.1}$$

where

$$r(\chi) = \begin{cases} \sin(\chi) & k = +1, \\ \chi & k = 0, \\ \sinh(\chi) & k = -1. \end{cases} \tag{A.2}$$

The non-null Christoffel symbols for this metric are:

$$\Gamma^t_{rr} = \frac{a\dot{a}}{(1-kr^2)}, \quad \Gamma^t_{\theta\theta} = a\dot{a}r^2, \quad \Gamma^t_{\phi\phi} = a\dot{a}r^2\sin^2\theta, \quad \Gamma^r_{rr} = \frac{kr}{1-kr^2};$$

$$\Gamma^t_{rt} = \Gamma^\theta_{t\theta} = \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a}, \quad \Gamma^r_{\theta\theta} = -r(1-kr^2), \quad \Gamma^t_{\phi\phi} = -r(1-kr^2)\sin^2\theta;$$

$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta.$$

The non-null components of the Ricci Tensor are:

$$
\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}; \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2}; \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k); \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta.
\end{aligned} \tag{A.3}
$$

Consequently, the Ricci scalar is given by

$$
\mathcal{R} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \tag{A.4}
$$

and the Kretschmann scalar

$$
\mathcal{K} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 12\left(\frac{\ddot{a}^2}{a^2} + \frac{(k + \dot{a}^2)^2}{a^4}\right). \tag{A.5}
$$

In particular, in the flat case:

$$
\mathcal{R} = 6(\dot{H} + 2H^2), \tag{A.6}
$$

$$
\mathcal{K} = \dot{H}^2 + 2H^2(\dot{H} + H^2). \tag{A.7}
$$

Appendix B

Geodesic Congruences and Orthogonal Hypersurfaces

An important analysis for the implementation of a general extension procedure is the inquiry of congruences, in particular, for the case of geodesic completeness as a criteria for singularity-free space-times, it is essential that we investigate the evolution of kinematic parameters of geodesic congruences. In this Appendix, we aim to show that, for a timelike congruence of geodesics in a spatially flat FRLW model, the vorticity $\omega_{\alpha\beta}$ is null, i.e, the vector field tangent to the geodesics is *hypersurface orthogonal*, and the spacetime can be foliated by a sequence of hypersurfaces. First, let us consider a time-like geodesic congruence defined by a vector field $u^{\mu}(\tau, l)$ such that at each event $\mathcal{P}$, there exists only one curve passing through $\mathcal{P}$. We can consider the separation vector between two geodesics in this congruence η^{α}:

$$u^{\mu}\nabla_{\mu}\eta^{\alpha} = \eta^{\mu}\nabla_{\mu}u^{\alpha}, \tag{B.1}$$

Since the congruence is compost by geodesics, with $u^{\mu}u_{\mu} = -1$, we have that $u^{\mu}\nabla_{\mu}u^{\alpha} = u_{\mu}\nabla_{\alpha}u^{\mu} = 0$. Thus, the tensor $D^{\mu}_{\alpha} \equiv \nabla_{\alpha}u^{\mu}$ is orthogonal to u^{μ}, so that in the rest frame of the congruence D^{μ}_{α} is purely spatial. We can decompose D^{μ}_{α} in its antisymmetric and symmetric part, and the latter can be decomposed in a traceless term and a term proportional to identity δ^{μ}_{α}, with the trace as proportionality constant. So, writing the projection of $\nabla_{\alpha}u^{\mu}$, we find

$$\nabla_\alpha u_\beta = P^\mu_\alpha \nabla_\mu u_\beta + \frac{1}{3}\theta P_{\beta\alpha} - \frac{1}{3}\theta P_{\beta\alpha}, \tag{B.2}$$

where we have added and subtracted $\theta \equiv \nabla_\mu u^\mu$, which is the expansion of the spatial sections. We can define the symmetric *shear* tensor, $\sigma_{\alpha\beta}$, and the antisymmetric vorticity tensor, $\omega_{\beta\alpha}$:

$$\sigma_{\alpha\beta} \equiv \frac{1}{2}(P^\mu_\alpha \nabla_\mu u_\beta + P^\mu_\beta \nabla_\mu u_\alpha) - \frac{1}{3}\theta P_{\beta\alpha}, \tag{B.3}$$

$$\omega_{\beta\alpha} \equiv \frac{1}{2}\left(P^\mu_\alpha \nabla_\mu u_\beta - P^\mu_\beta \nabla_\mu u_\alpha\right). \tag{B.4}$$

With these definitions, we can rewrite equation (B.2) as

$$\nabla_j u_i = \sigma_{ij} + \omega_{ij} + \frac{1}{3}\theta P_{ij}. \tag{B.5}$$

Note that the above equation only applies for the case of a geodesic congruence since for more general curves the acceleration term $a_i u_j$ will be non-vanishing.

This decomposition is particularly useful for the case when a geodesic congruence is hypersurface orthogonal, i.e, when the timelike tangent vector can be written as the gradient vector of a spacelike hypersurface

$$u_\mu = f(x)\partial_\mu g(x), \tag{B.6}$$

for two functions $f(x)$ and $g(x)$. In this case the covariant derivative is given by

$$\nabla_\alpha u_\mu = f(x)\nabla_\alpha \nabla_\mu g(x) + \partial_\alpha f(x)\partial_\mu g(x). \tag{B.7}$$

If this is the case, the vorticity of the geodesic congruence is null. Notwithstanding, for our purposes, it is more convenient to prove the converse is also true: if a geodesic congruence has null vorticity, its tangent vector can always be made hypersurface orthogonal. By the definition (B.4) we have that

$$\omega_{\beta\alpha} = \frac{1}{2}\left[(\delta^\mu_\alpha + u^\mu u_\alpha)\nabla_\mu u_\beta - (\delta^\mu_\beta + u^\mu u_\beta)\nabla_\mu u_\alpha\right]. \tag{B.8}$$

Since u^μ is tangent to a geodesic, $u^\mu \nabla_\mu u_\alpha = 0$. Thence, what remains is:

$$\omega_{\beta\alpha} = \frac{1}{2}[\nabla_\alpha u_\beta - \nabla_\beta u_\alpha]. \tag{B.9}$$

For a Riemannian geometry with no torsion, $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$:

$$\omega_{\beta\alpha} = \frac{1}{2}[\partial_\alpha u_\beta - \Gamma^\mu_{\alpha\beta} u_\mu - \partial_\beta u_\alpha + \Gamma^\mu_{\beta\alpha} u_\mu] = \frac{1}{2}(\partial_\alpha u_\beta - \partial_\beta u_\alpha). \tag{B.10}$$

Thence, if $\omega_{\beta\alpha} = 0$

$$\partial_\alpha u_\beta - \partial_\beta u_\alpha = 0, \tag{B.11}$$

which the gradient of any function $f(t, r)$ satisfies. Let u^μ be a vector field tangent to a non co-moving congruence of geodesics in a flat FLRW. Its components can always be written as

$$u^t = \sqrt{1 + \frac{v_0^2}{a^2}}, \quad u^r = \frac{v_0}{a^2}, \qquad v_0 = \text{const.} \tag{B.12}$$

Then the covariant components are given by

$$\begin{aligned} u_t = g_{r\mu} u^\mu &= -\sqrt{1 + \frac{v_0^2}{a^2}}, \\ u_r = g_{r\mu} u^\mu &= v_0, \end{aligned} \tag{B.13}$$

and the vorticity of any geodesic congruence results in

$$\omega_{\beta\alpha} = \frac{1}{2}(\partial_\beta u_\alpha - \partial_\alpha u_\beta) \tag{B.14}$$

Since the components only depend on time, $\omega_{r\alpha} = 0$. The only component that could be non vanishing is

$$\omega_{t\alpha} = \frac{1}{2}\partial_t u_\alpha, \qquad \alpha \neq t. \tag{B.15}$$

As the covariant components with index different from zero are either v_0 or 0, we conclude that

$$\omega_{\beta\alpha} = 0, \tag{B.16}$$

and therefore, as a consequence of vanishing vorticity, Frobenius Theorem guarantees that we can always make

$$u_\mu = \partial_\mu f(t, r), \tag{B.17}$$

for some function $f(t, r)$.

Appendix C

Curvature Singularity Paralelly Propagated Along Light Geodesics in flat FLRW

In this appendix we follow the Ref. [29] to show sufficient conditions for the presence of a *parallelly propagated curvature singularity* in a flat FLRW model with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]. \tag{C.1}$$

Let us construct a tetrad basis $\hat{\mathbf{e}}^A$

$$(\hat{\mathbf{e}}^0)_\nu = (a, 0, 0, 0) = a\mathbf{d}\eta; \tag{C.2}$$

$$(\hat{\mathbf{e}}^1)_\nu = (0, a, 0, 0) = a\mathbf{d}r; \tag{C.3}$$

$$(\hat{\mathbf{e}}^2)_\nu = (0, 0, ar, 0) = ar\mathbf{d}\theta; \tag{C.4}$$

$$(\hat{\mathbf{e}}^3)_\nu = (0, 0, 0, ar\sin\theta) = ar\sin\theta\mathbf{d}\phi. \tag{C.5}$$

Since this is a tetrad basis, the vectors satisfy $g_{\mu\nu}(\hat{\mathbf{e}}_A)^\mu(\hat{\mathbf{e}}_B)^\nu = \eta_{AB}$ where η_{AB} is the Minkowski metric in the tetrad indices. However, this tetrad basis is not parallelly propagated along a null geodesic with affine parameter given by $\lambda \propto a\mathrm{d}t$, since

$$k^\mu\nabla_\mu(\hat{e}^0)_\nu \propto H\mathbf{dr}, \tag{C.6}$$

$$k^{\mu}\nabla_{\mu}(\hat{e}^1)_{\nu} \propto H\mathbf{d}r, \tag{C.7}$$

and the vectors $(\hat{\mathbf{e}}^2)$ and $(\hat{\mathbf{e}}^3)$ are parallelly propagated along k^{μ}. The main point here is to construct a tetrad basis that is parallelly propagated along the null geodesic and evaluate the Ricci tensor $R_{\mu\nu}$ along such basis in order to verify what condition prevents singularity on the asymptotic limit. For that, we notice that two tetrad basis are related to each other through a Lorentz transformation. Thus, let us consider a new tetrad basis $\mathbf{e}^i$ (without the hat superscript), such that it is related to $\hat{\mathbf{e}}^A$ through the Lorentz transformation:

$$\mathbf{e}^0 = \cosh\zeta(\eta, r)\hat{\mathbf{e}}^0 + \sinh\zeta(\eta, r)\hat{\mathbf{e}}^1, \tag{C.8}$$

$$\mathbf{e}^1 = \sinh\zeta(\eta, r)\hat{\mathbf{e}}^0 + \cosh\zeta(\eta, r)\hat{\mathbf{e}}^1. \tag{C.9}$$

The rapidity ζ must be such that, at the initial point (η_0, r_0), $\zeta = 0$. Then, we can compute the covariant derivative of the new basis:

$$k^{\mu}\nabla_{\mu}\mathbf{e}^0 = \hat{\mathbf{e}}^0(k^{\mu}\nabla_{\mu}\cosh\zeta) + \cosh\zeta\,\underbrace{(k^{\mu}\nabla_{\mu}\hat{\mathbf{e}}^0)}_{(H/a)\hat{\mathbf{e}}^1} + \hat{\mathbf{e}}^1(k^{\mu}\nabla_{\mu}\sinh\zeta) + \sinh\zeta\,\underbrace{(k^{\mu}\nabla_{\mu}\hat{\mathbf{e}}^1)}_{(H/a)\hat{\mathbf{e}}^0}$$

$$= \frac{(\partial_{\eta}\zeta + \partial_r\zeta)}{a^2}\underbrace{[\hat{\mathbf{e}}^0\sinh\zeta + \hat{\mathbf{e}}^1\cosh\zeta]}_{\mathbf{e}^1} + \frac{H}{a}\underbrace{[\hat{\mathbf{e}}^1\cosh\zeta + \hat{\mathbf{e}}^0\sinh\zeta]}_{\mathbf{e}^1}.$$

Then

$$k^{\mu}\nabla_{\mu}\mathbf{e}^0 = \frac{1}{a^3}[\partial_{\eta}a + a(\partial_{\eta}\zeta - \partial_r\zeta)]\mathbf{e}^1. \tag{C.10}$$

Similarly, we have for the parallel transport of $\mathbf{e}^1$

$$k^{\mu}\nabla_{\mu}\mathbf{e}^1 = \frac{1}{a^3}[\partial_{\eta}a + a(\partial_{\eta}\zeta - \partial_r\zeta)]\mathbf{e}^0. \tag{C.11}$$

Therefore, to be parallelly transported, the right side of both equations (C.10) and (C.11) must be null. Hence, we get that the rapidity of the Lorentz transformation that will transform the tetrad basis e in a parallelly propagated one is

$$\zeta = -\ln\left(\frac{a}{a_0}\right). \tag{C.12}$$

Replacing solution (C.12) in the new basis defined in (C.8, C.9) we obtain:

$$\mathbf{e}^0 = \frac{a_0}{2a}\left(1+\frac{a^2}{a_0^2}\right)\hat{\mathbf{e}}^0 + \frac{a_0}{2a}\left(1-\frac{a^2}{a_0^2}\right)\hat{\mathbf{e}}^1, \tag{C.13}$$

$$\mathbf{e}^1 = \frac{a_0}{2a}\left(1-\frac{a^2}{a_0^2}\right)\hat{\mathbf{e}}^0 + \frac{a_0}{2a}\left(1+\frac{a^2}{a_0^2}\right)\hat{\mathbf{e}}^1, \tag{C.14}$$

or, by inverting the relations above,

$$\hat{\mathbf{e}}^0 = \frac{a_0}{2a}\left(1+\frac{a^2}{a_0^2}\right)\mathbf{e}^0 - \frac{a_0}{2a}\left(1-\frac{a^2}{a_0^2}\right)\mathbf{e}^1, \tag{C.15}$$

$$\hat{\mathbf{e}}^1 = -\frac{a_0}{2a}\left(1-\frac{a^2}{a_0^2}\right)\mathbf{e}^0 + \frac{a_0}{2a}\left(1+\frac{a^2}{a_0^2}\right)\mathbf{e}^1. \tag{C.16}$$

Thus, the Ricci tensor components in the parallelly propagated tetrad basis can be written as

$$\begin{aligned} R_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = \frac{\dot{H}a_0^2}{2a^2}\Bigg[-\left(1+\frac{a^2}{a_0^2}\right)^2 e^0\otimes e^0 + \left(1-\frac{a^4}{a_0^4}\right) e^0\otimes e^1 \\ + \left(1-\frac{a^4}{a_0^4}\right) e^1\otimes e^0 - \left(1-\frac{a^2}{a_0^2}\right)^2 e^1\otimes e^1 \Bigg] + (3H^2+\dot{H})\eta_{AB}\mathbf{e}^A\otimes\mathbf{e}^B \end{aligned} \tag{C.17}$$

Thence, in order for the Ricci tensor components not to diverge, we need, additionally to $H<\infty$ and $|\dot{H}|<\infty$ that

$$\lim_{a\to 0}\frac{\dot{H}}{a^2} \to c, \tag{C.18}$$

where c $\in \mathbb{R}$.

Appendix D

Contracting de Sitter Covering and Matching Conditions

In this appendix we proceed to apply the same coordinate transformations used in the expanding de Sitter patch for the contracting sheet with metric given by

$$\mathrm{d}s^2 = -\mathrm{d}t'^2 + e^{-2\alpha t'}[\mathrm{d}r'^2 + r'^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi)], \tag{D.1}$$

where we use the prime notation to differentiate between the cosmic time in the expanding flat patch, and the co-moving time in the contracting sheet. Since a tangent vector to a non co-moving geodesic parametrized by τ can always be written as

$$u^\mu \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1 + \frac{v_0^2}{a^2}}, \tag{D.2}$$

we can integrate with the scale factor $a = e^{-\alpha t}$ to find:

$$\tau - \tau_0 = \int \frac{\mathrm{d}t'}{\sqrt{1 + \frac{v_0^2}{a^2}}}. \tag{D.3}$$

By performing the transformation $\mathrm{d}t' = -\mathrm{d}a/\alpha a$ we obtain

$$\tau - \tau_0 = -\frac{1}{\alpha}\int_{a_0}^{a} \frac{\mathrm{d}a}{\sqrt{a^2 + v_0^2}} \qquad \Rightarrow \qquad -\alpha(\tau - \tau_0) = \ln\left[a + \sqrt{a^2 + v_0^2}\right]\Bigg|_{a_0}^{a}. \tag{D.4}$$

Then

$$e^{-\alpha(\tau-\tau_0)} = \frac{1}{A_0}\left[a + \sqrt{a^2 + v_0^2}\right], \tag{D.5}$$

where $A_0 \equiv a_0 + \sqrt{a_0^2 + v_0^2}$. Then, we write

$$a + \sqrt{a^2 + v_0^2} = A_0 g(l) e^{-\alpha(\tau-\tau_0)}, \tag{D.6}$$

where $g(l)$ carries the dependency of the scale factor on the spatial parameter. Now, let us consider u_μ to be hypersurface orthogonal such that $u_\mu = \partial\phi$ for some function ϕ to be determined. By integrating in the r coordinate we find

$$\phi(t', r, v_0) = \int v_0 \mathrm{d}r = v_0 r + \mathcal{F}(t'), \tag{D.7}$$

where $\mathcal{F}(t')$ is a function of the cosmic time t'. Integrating $\partial_{t'}\phi$ we obtain

$$\phi(t', r, v_0) = -\int \sqrt{1 + \frac{v_0^2}{a^2}} \mathrm{d}t = \mathcal{F}(t') + v_0 r. \tag{D.8}$$

The above integral returns

$$-\frac{1}{\alpha}\left[\sqrt{1 + \frac{v_0^2}{a^2}} - \frac{1}{\alpha}\ln\left(a + \sqrt{a^2 + v_0^2}\right)\right]\Bigg|_{t_0'}^{t'}. \tag{D.9}$$

Thence, the function $\phi(t', r, v_0)$ can be written as

$$\phi = -\frac{1}{\alpha}\left[\sqrt{1 + \frac{v_0^2}{a^2}} - \sqrt{1 + \frac{v_0^2}{a_0^2}}\right] + \frac{1}{\alpha}\ln\left[\frac{1}{A_0}\left(a + \sqrt{a^2 + v_0^2}\right)\right] + v_0 r. \tag{D.10}$$

Let us parametrize r by

$$r = (l - l_i) + \frac{1}{\alpha v_0}\left\{\sqrt{1 + \frac{v_0^2}{a^2}} - \sqrt{1 + \frac{v_0^2}{a_0^2}}\right\}. \tag{D.11}$$

Then, we can write ϕ as

$$\phi = \frac{1}{\alpha}\ln\left[\frac{1}{A_0}\left(a + \sqrt{a^2 + v_0^2}\right)\right] + v_0(l - l_i). \tag{D.12}$$

For surfaces of $\phi = \text{const.} \equiv k$ we have

$$
\begin{aligned}
\alpha k - \alpha v_0(l - l_i) &= \ln\left[\frac{1}{A_0}\left(a + \sqrt{a^2 + v_0^2}\right)\right] \\
&= A_0 f(\tau) e^{-\alpha v_0 (l - l_i)}.
\end{aligned} \tag{D.13}
$$

By comparing eq. (D.13) with (D.6) we see that

$$
\sqrt{a^2 + v_0^2} + a = A_0 e^{-\alpha[(\tau - \tau_0) + v_0(l - l_i)]}, \tag{D.14}
$$

which returns the scale factor

$$
\begin{aligned}
a &= -|v_0| \sinh\left\{\alpha[(\tau - \tau_0) + v_0(l - l_i)] + \ln\left(\frac{|v_0|}{A_0}\right)\right\} \\
&= -|v_0| \sinh\left\{\Theta - \alpha(\tau_0 + v_0 l_i) + \Theta_0\right\},
\end{aligned} \tag{D.15}
$$

where once again we define $\Theta \equiv \alpha(\tau + v_0 l)$ and $\Theta_0 \equiv \ln(|v_0|/A_0)$. Therefore, in order to match the expanding solution at $a = 0$, we need that $\Theta_i \equiv \alpha(\tau_0 + v_0 l_i) = 2\Theta_0$ and by replacing the scale factor (D.15) in (D.11) we get

$$
r = l - l_i - \frac{1}{\alpha v_0}[\coth(\Theta - \Theta_0) - \coth(\Theta_0)]. \tag{D.16}
$$

Then

$$
l_i = \frac{2}{\alpha v_0} \coth \Theta_0. \tag{D.17}
$$