Dissertação de

Mestrado

The photon and the granularity of space-time

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Centro Brasileiro de Pesquisas Físicas - CBPF

Rio de Janeiro - RJ

19 de Março de 2025

"THE PHOTON AND THE GRANULARITY OF SPACE-TIME"

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Dissertação de Mestrado em Física apresentada no Centro Brasileiro de Pesquisas Físicas do Ministério da Ciência Tecnologia e Inovação. Fazendo parte da banca examinadora os seguintes professores:

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Antônio Duarte Pereira Junior - UFF

Rudnei de Oliveira Ramos – UERJ

Rio de Janeiro, 19 de março de 2025.

Acknowledgements

I would like to first thank my mother, Cláudia Nascimento, for all her support and efforts, without whom none of this would be possible. I would also like to thank my wife Yanka, who has been by my side through the good times and the bad ones, and who has given me so much confidence, respect and joy. I would also like to thank my family, especially my stepfather Edson Nascimento, my grandmother Idalina and my grandfathers Marcel and Genuíno, for all the encouragement and trust they have given me.

I would also like to thank my advisor, José Abdalla Helayël-Neto, for accepting me as a Master student and taking me throughout this journey in a very beautiful partnership, which will continue into my doctorate. It is an honor to be part of the Diracstão group and to be able to work with one of the best human beings I have ever met and will ever meet in my life. The fantastic person that Prof. Helayel is has made everything easier, both in physics and in life, and his teachings are something that I will certainly carry with me forever.

I would like to thank my friends from college who are now friends for life: Patrick Belmont, Mariana Teixeira, Vitor Gozzi, Gabriel Moraes, Murilo Mano, Helena Ann and Diego Rigo, for all the times we spent together, on courses and in the bar. Thank you for all the laughs and support.

I can't forget to thank my cats, who have now grown to two: Solzinho, Branquinha, Gatão, Titito, Magnólia, Magrela, Feynman and Anne. The purity and companionship of cats has greatly influenced this journey.

I would like to thank Prof. Sebastião and Prof. Álvaro for all the conversations and courses we had. They both played a huge part not only in my development as a physicist but also as a person. The Fundamental Interactions and Theoretical Methods in High Energy Physics courses should become CBPF assets, the good you do for the students is surreal, and I was lucky enough to be part of it.

I couldn't leave out my thanks to all my friends at CBPF and in the Diracstão group. I'm very grateful to be part of such a solid and strong group as diracstão, in particular I'd like to thank Philipe de Fabritiis, João Paulo Melo, Mirian Reetz and Pedro Augusto for all their partnership, joint work and good laughs.

In addition, I would like to thank all the CBPF staff and people who have been part of this journey.

“Study hard what interests you the most in the most undisciplined, irreverent and original manner possible.”

- Richard P. Feynman

Resumo

Esta Dissertação pode ser lida em duas partes. Na primeira, iniciando com as equações (Abelianas) de Maxwell, apresentamos um procedimento, denominado método de Noether, que nos permite construir em alguns passos a versão não-Abeliana a partir de um modelo originalmente Abeliano. Dessa forma, obtivemos naturalmente uma formulação de Yang-Mills para partículas de spin-1 e massa nula a partir da Eletrodinâmica de Maxwell. Em seguida, dedicamos um capítulo inteiro à introdução de diferentes conceitos da Relatividade Geral, com o objetivo de preparar o leitor, de modo que, ao entrar na discussão sobre a Gravidade Quântica em Laços, já tenha familiaridade com determinados conceitos. Isso ocorre logo no capítulo seguinte, onde apresentamos uma teoria que busca unificar a Mecânica Quântica e a Relatividade Geral. Ao longo do texto, elencamos diferentes pontos que visam proporcionar uma melhor compreensão da teoria, culminando com o cerne da Dissertação: os principais observáveis de uma teoria eletromagnética não-linear incorporando efeitos da LQG e a posterior versão de Yang-Mills acrescida das correções da LQG. Essa abordagem é descrita em um cenário que permite conectar uma teoria de gravidade quântica a teorias não-Abelianas, como a Cromodinâmica Quântica e a Teoria Eletrofraca.

Palavras-chave: Teorias de Yang-Mills, Eletromagnetismo não-linear, Gravitação Quântica, Gravitação Quântica de Laços.

Abstract

This Dissertation can be read in two parts. In the first part, starting off from the (Abelian) Maxwell equations, we have adopted a procedure referred to as the Noether method, which allows us to build up the non-Abelian version of an Abelian model we start from. In this way, we have worked out a non-Abelian formulation for self-interacting massless spin-1 particles. This is the Yang-Mills theory. Next, we have dedicated an entire chapter to introducing different concepts of General Relativity, with the aim of preparing the readers so that, when entering the discussion on Loop Quantum Gravity, they may feel already familiar with the main concepts. This is the topic of the following chapter, where we present a theory that seeks to unify Quantum Mechanics and General Relativity. Throughout the text, we present various points to further provide a better understanding of the theory, getting to the core of the Dissertation: the main quantities of a nonlinear electromagnetic theory and extending it to finally arrive at a set of Loop-Quantum-Gravity-corrected Yang-Mills field equations . This approach is presented in a framework that allows to connect a quantum gravity model to non-Abelian theories, such as Quantum Chromodynamics and the Electroweak Theory.

Keywords: Yang-Mills theory, Non-linear Electromagnetism, Quantum Gravity, Loop Quantum Gravity.

Contents

Chapter 1

Introduction

Physics is, after all, an experimental natural science. Since the establishment of the scientific method, it has been understood that for any discovery regarding a subject to be valid, it must pass through all the stages of this method. However, the greatest discoveries and/or advancements in Physics have arisen from two entirely different sources from experimentation: contradiction and empiricism [1].

Contradiction, in turn, could almost be defined as the antithesis of Physics, as it represents the lack of logic concerning something. The reader may think this statement is contradictory, but there is a deeper connection. Einstein may have been the physicist who best utilized contradictions, as two of them led to significant discoveries. The first arose from the contradiction between classical mechanics and the photoelectric effect. The second came from the formulation of Special Relativity, which resolved the inconsistency between classical mechanics and electrodynamics. Of course, there are other examples of contradictions leading to great leaps in understanding. It is noteworthy that these logical inconsistencies between two previously successful theories paved the way for monumental theoretical developments.

Empiricism, on the other hand, plays a more subtle role. The major discoveries stemming from contradictions were achieved without new empirical data. These models were developed almost blindly. Undoubtedly, the most notable physicist to use this concept was Isaac Newton, one of the greatest scientists in history. Another major example of empiricism is found at the dawn of astronomy, with thinkers like Galileo, Kepler, and Copernicus working under such frameworks.

Physics, which describes fundamental interactions, benefited from the concepts of contradiction and empiricism throughout its development [2]. It began with the creation of Quantum Mechanics by W. Heisenberg, P. Dirac, and E. Schrödinger [3, 4]. However, the electron, the main protagonist, lacked a relativistic description. P. Dirac [5] resolved this issue, introducing gauge symmetry $U(1)$ [6] and unifying Quantum Mechanics and Special Relativity. Dirac, based on purely empirical reasoning, also predicted the existence of antiparticles, discovered in 1932. Around this time, the atom was undergoing theoretical development. I. Tamm and H. Yukawa began studying the interaction between protons and neutrons in the atomic nucleus [8], using Maxwell's electromagnetism as an analogy to develop their nuclear theory. Since vector bosons mediating interactions should be massless in electromagnetism, Yukawa formulated a theory of massive scalar bosons to account for the short range of strong nuclear interactions.

After Quantum Mechanics, Heisenberg investigated the origin of isospin (or isotopic spin). He addressed this by introducing the $SU(2)$ symmetry group as the basis for isospin, associating protons and neutrons, which have nearly identical masses, as members of a doublet [12]. Later, E. Wigner [13] demonstrated that this new quantum number should be conserved, invalidating Yukawa's scalar boson theory.

Efforts shifted towards the formulation of a gauge theory with spin-1 bosons, in which, self-interaction among the gauge fields was allowed. Initially, these bosons would be massless, but would acquire mass through spontaneous symmetry breaking. Massless mediators would imply that strong interactions have infinite range, contradicting the short-range nature of nuclear forces. Two groups independently formulated this theory for nuclear interactions: the first, more renowned, by C. N. Yang and R. Mills in 1954 (known as the Yang-Mills theory) [16], and the second, by R. Shaw [15], whose work remained unpublished but was recorded in his doctoral thesis at Cambridge under A. Salam's supervision. Both formulations addressed prior issues, providing a local $SU(2)$ gauge theory that conserved total isospin and it was anticipated (though not demonstrated at the time) to be renormalizable [16, 18, 36].

It is worth noting that by 1954, the Higgs mechanism had not been formulated. Consequently, the Yang-Mills theory faced significant challenges concerning mediator masses. Empiricism had also been set aside, as fundamental interaction theories failed to align with existing experimental data. In 1956, C. N. Yang and T. D. Lee published a study showing that weak interactions violate parity symmetry to comply with special relativity [20]. This work inspired Salam to develop a Yang-Mills theory for weak interactions, emphasizing the importance of such formulations and introducing the concept of chiral symmetry [24]. Other physicists, including S. Glashow [21] and S. Weinberg [22], also contributed to this effort. Salam had previously integrated weak interactions into the $SU(2)$ Yang-Mills framework [23].

The problem of the masses of mediators persisted until the spontaneous symmetry-breaking mechanism (the Higgs mechanism), which earned the 2013 Nobel Prize, was developed. These advancements solidified the Yang-Mills theory as the most comprehensive framework for describing fundamental interactions [25, 26].

Once again, the pillars of contradiction and empiricism have influenced physics, particularly in Quantum Gravity research since the 1930s. Developing a theory of Quantum Gravity faces challenges due to a lack of guiding empirical data and the contradiction between General Relativity and Quantum Mechanics. Achieving this theory is crucial for addressing significant open problems in contemporary physics, such as unifying General Relativity and Quantum Mechanics, resolving singularities in black holes and the Big Bang, understanding spacetime at Planck scales.

For years, efforts to quantize gravity have faced difficulties due to the foundational differences between the theories being unified. Einstein's General Relativity introduced a groundbreaking view of spacetime, merging space and time into a gravitational field. Consequently, quantizing the gravitational field equates to quantizing spacetime itself.

As with any minimally functional theory, a ground state for the quantized gravitational field is required. This state involves quantum metric fluctuations relevant at scales where a quantum particle's localization avoids being concealed by its horizon (the Planck length). Above these scales, such corrections can be disregarded. Quantizing the gravitational field inherently involves quantizing geometry, fundamentally challenging traditional perspectives. A Quantum Gravity framework must describe quantum states before spacetime itself emerges.

The Planck scale introduces the discrete, finite nature of quantum spacetime, establishing a fundamental constant l_0 akin to the speed of light (c) in special relativity and $\hbar$ in Quantum Mechanics. These concepts necessitate revisiting basic physical notions, suggesting that all phenomena in nature might be described by general-covariant quantum fields. This l_0 is a fundamental and discreteness constant of the world.

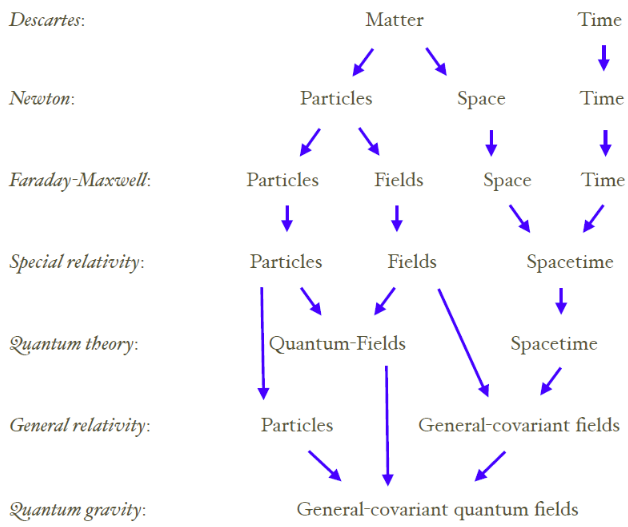

Figure 1 – Evolution of the Concept of Spacetime in the Development of Physical Theories [1].

The ultraviolet energy implies finiteness on a Planck scale. Two theories that successfully incorporate this consideration are String theory and Loop Quantum Gravity (LQG). String theory incorporates and addresses Quantum Gravity, aiming at the grand unification of physics. It originates from a different perspective than LQG, inheriting concepts from Supergravity theories, which, in turn, stem from Supersymmetry and Kaluza-Klein theories. These theories themselves are based on earlier ideas.

LQG, on the other hand, is solely a theory of Quantum Gravity, not directly concerned with grand unification, though it does not exclude it. Moreover, LQG presents two distinct versions, analogous to twin siblings raised differently. Both versions originate from a quantized metric. However, the canonical version, utilized in this work, stems from the loop solutions of the Wheeler-DeWitt equation and later redefined using Ashtekar variables. The other formulation, known as Spinfoams, resembles a "sum over geometries," inspired by the Euclidean functional integral developed by Hawking and his group in the 1970s. The Spinfoam theory has merged with the canonical formulation's kinematics, leading to the Covariant Loop theory, which can satisfactorily describes LQG. The figure below outlines the historical evolution described earlier.

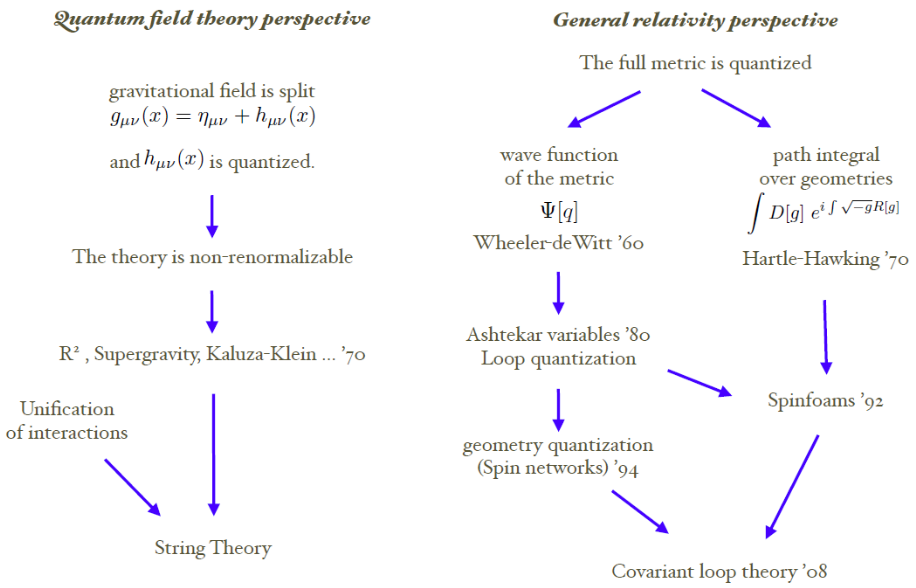

Figure 2 – The Development Stages of the Two Main Quantum Gravity Theories. On the left, we have String Theory. On the right, we have LQG [1].

After this brief introduction, this work is outlined as follows. In Chapter 2, we develop Yang-Mills theory as a self-interacting gauge boson theory. More precisely, starting from Maxwell's electromagnetism and its $U(1)$ group, we extend it to obtain a non-Abelian version of Maxwell's equations with $SU(N)$ symmetry. We achieve this using Noether's method to introduce self-interactions in a free theory. After deriving these equations, we apply them to fundamental quantities in electromagnetic theory, such as the Poynting vector and the energy-momentum tensor. Expanding beyond spin-1 vector bosons, we apply the method to fermionic matter, observing that minimal coupling of fermions to gauge bosons (Yang-Mills bosons) arises spontaneously [2].

In Chapter 3, we introduce the reader to the fundamentals of General Relativity (GR), which are essential for understanding the basics of LQG. We begin by presenting arguments framing GR as a gauge theory. Subsequently, we develop the tetrad formalism and its impact on the traditional description of GR. Here, we introduce the main quantities in this formalism and how the gravitational field action is described. Finally, we present the Hamiltonian formulation of GR, known in the literature as the ADM formalism, providing a pedagogical introduction to its key concepts, including the Hamiltonian and diffeomorphism constraints.

In Chapter 4, we delve into Loop Quantum Gravity, connecting the ADM formalism from the previous chapter to the Wheeler-DeWitt equation. We then explore LQG comprehensively, introducing its foundational concepts, such as area and volume operators. Additionally, we dedicate attention to Ashtekar variables, which are fundamental to LQG, and their connection to the tetrad formalism. To foster familiarity, we include an introduction to Wilson loops in the context of LQG. Finally, we discuss some applications of LQG in cosmology and black hole physics.

In Chapter 5, we extend LQG applications to the electromagnetic sector, presenting the main results of this work. We calculate key quantities in electromagnetic theory under LQG effects, such as the Poynting vector, stress tensor, wave equations, dispersion relations, group velocity, and refractive index. These expressions provide an overarching understanding of LQG's influence on classical electromagnetism. Building on these concepts, we further develop Yang-Mills theories incorporating LQG effects in the electromagnetic sector, following a similar approach to the non-Abelianization of the theory in Chapter 2 but with additional complexities. This led to intriguing results that may aid future understanding of Quantum Gravity.

Finally, we conclude the work with a chapter summarizing the findings of all chapters, emphasizing the novel results from Chapter 5. We also outline future perspectives emerging from this text.

At last, this work includes three Appendices that elucidate key equations and technical concepts regarding LQG that, while important, are too detailed for the main text. Following these Appendices, we present the references that support this Dissertation.

Chapter 2

From Maxwell's equations to Yang-Mills Theory

We begin this chapter with the aim of developing Yang-Mills (Y-M) theory as a theory of self-interaction for gauge bosons [2, 16]. More precisely, starting from the construction of Maxwell's electromagnetism using its $U(1)$ group, we will extend it to obtain a non-Abelian version of Maxwell's equations with the symmetry group $SU(N)$. We will achieve this by using a method to create fields self-interactions through a free theory, known as Noether's method*. With these equations in hand, we proceed to apply them to other key quantities in electromagnetic theory, such as the Poynting vector and the energy-momentum tensor. To extend beyond spin-1 vector bosons, we also apply the self-interaction framework to fermionic matter, where we find that minimal coupling between fermionic matter to mediating bosons (Y-M bosons) arises naturally.

The light particle was identified as the photon, which is massless, allowing electric and magnetic phenomena to be described by equations known as Maxwells equations in the vacuum [27] of the form:

$$\nabla \cdot \mathbf{E} = 0, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}, \tag{2.4}$$

the bold parameters are vectors. The electric field can be write as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{2.5}$$

and the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.6}$$

*Named after Emmy Noether, whose profound contributions, especially Noether's theorem, reveal the link between symmetries and conserved quantities in physics [28, 29].

The field ϕ is called the scalar potential and $\mathbf{A}$ the vector potential. Maxwell's equations enjoy gauge freedom, meaning that when expressing the electric and magnetic fields as in equations (2.5) and (2.6), the gauge symmetry associated with the potentials in Maxwell's equations must be preserved. Consequently, the potentials should remain invariant under the transformation form

$$\{\phi, \mathbf{A}\} \to \{\phi', \mathbf{A}'\}, \tag{2.7}$$

$$\phi' = \phi + \frac{\partial \alpha(t, \vec{x})}{\partial t} \qquad \mathbf{A}' = \mathbf{A} - \nabla \cdot \alpha(t, \vec{x}). \tag{2.8}$$

The following step involves applying Noether's method, a widely recognized approach in supersymmetry and supergravity due to its extensive range of applications [30–33], to construct a non-Abelian Maxwell theory. For simplicity, in this dissertation we have opted to use Euclidean signature.

2.1 Construction of Self-Interacting Theory for Spin-1 Fields

The quest for a non-Abelian framework fir spin-1, massless particles with self-interaction begins by the introduction of a multiplet of fields that are massless and own have spin-1, as described below:

$$A_i,\ i = 1, 2, ..., N \qquad \phi_i,\ i = 1, 2, ..., N \tag{2.9}$$

as a consequence we have the fields grouped into an N-plet. Yang-Mills theory is built upon Lie groups, so we assume a Lie-type symmetry for the potentials in an arbitrary N-dimensional representation, which transform as follows:

$$\phi'_i = R_{ij}\phi_j, \tag{2.10}$$

$$A'_i = R_{ij}A_j, \tag{2.11}$$

where R_{ij} is a element of the Lie group, which, due to the exponential structure of the group algebra [34], can be written as

$$R_{ij} = (e^{iw_a G_a})_{ij} \sim \delta_{ij} + iw_a(G_a)_{ij} + O(w^2), \tag{2.12}$$

whew w_a is the parameter of the group and the fields transform in infinitesimal form as

$$\delta A_i = iw_a(G_a)_{ij}A_j, \tag{2.13}$$

$$\delta \phi_i = iw_a(G_a)_{ij}\phi_j. \tag{2.14}$$

Where $(G_a)_{ij}$ are the generators of the Lie group in a given representation, forming the core of a Lie group's structure; they satisfy the following commutation relation:

$$[G_a, G_b] = if_{abc}G_c. \tag{2.15}$$

The f_{abc} is the structure constant of the group $SU(N)$. So, we use this representation, represented by complex, $n \times n$ unitary matrices with determinant equals to 1. Additionally, w_a are arbitrary functions of x. Let us begin with the free Lagrangian of electromagnetic theory in vacuum

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2). \tag{2.16}$$

We want to generate interactions between the fields; to achieve this, we will use the variational principle with respect to the potentials

$$\begin{aligned} \delta S = 0 &= \int d^4x \; \delta \cdot \left[\frac{E^2}{2} - \frac{B^2}{2}\right] \\ &= \frac{1}{2}\int d^4x \, (2\mathbf{E}_i \cdot \delta\mathbf{E}_i - 2\mathbf{B}_i \cdot \delta\mathbf{B}_i) \\ &= \int d^4x \left[\mathbf{E}_i \cdot \left(\nabla\delta\phi_i + \frac{\partial \delta\mathbf{A}_i}{\partial t}\right) - \mathbf{B}_i \cdot (\nabla \times \delta\mathbf{A}_i)\right], \end{aligned} \tag{2.17}$$

from this, we obtain

$$\nabla \cdot (\mathbf{B}_i \times \delta\mathbf{A}_i) - \underbrace{(\nabla \times \mathbf{B}_i) \cdot \delta\mathbf{A}_i}_{Eq.\ (2.4)} - \nabla \cdot (\mathbf{E}_i \cdot \delta\phi_i) + \underbrace{(\nabla\mathbf{E}_i) \cdot \phi_i}_{Eq.\ (2.1)} - \frac{\partial}{\partial t}(\mathbf{E}_i \cdot \delta\mathbf{A}_i) + \underbrace{\frac{\partial \mathbf{E}_i}{\partial t} \cdot \delta\mathbf{A}_i}_{Eq.\ (2.4)} = 0. \tag{2.18}$$

The terms selected as Eq. (2.1) are zero, as is the term Eq. (2.4), due to Maxwell's equations. The remaining terms are total spatial and temporal derivatives. If these terms involve the variation $\delta\mathbf{A}_i$, we substitute them using Eq. (2.13); if they involve $\delta\phi_i$, we substitute them using Eq. (2.14). These can be separated into spatial and temporal currents as follows

$$\mathbf{j}_a = -iw_a(G_a)_{ij}[(\mathbf{E}_i \cdot \phi_j) - (\mathbf{B}_i \times \mathbf{A}_j)], \tag{2.19}$$

$$j_a^0 = -iw_a(G_a)_{ij}(\mathbf{E}_i \cdot \mathbf{A}_j). \tag{2.20}$$

With the construction of these currents, we move from a theory of free particles to one where massless spin-1 vector bosons exhibit self-interactions. Observe that the currents carry the index (a) associated with the symmetry group's generators, while the fields are indexed by the representation label (i). Consequently, self-interaction is possible only if these indices coincide, meaning $i = a$. To enable self-interaction between the fields and currents, the fields initially assigned to an arbitrary representation must transform in the adjoint representation, aligning with the current's representation. Once this realignment is achieved, the subsequent step is to incorporate these currents into the standard Lagrangian, yielding a self-interaction Lagrangian for the photons as follows

$$\mathcal{L}_1 = \frac{1}{2}(E^2 - B^2) - l\phi_a j_a^0 - l\mathbf{A}_a\mathbf{j}_a, \tag{2.21}$$

we introduce l as the coupling constant for the self-interaction. An essential aspect of the adjoint representation is that the number of fields matches the number of generators. In this representation, the generators are expressed as

$$(G_a)_{bc} = -if_{abc}, \tag{2.22}$$

where $a, b, \cdots = 1, \ldots, N^2 - 1$. In the adjoint representation, as in the case of $SU(2)$, the generators of the representation are given by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.23}$$

and, in the case of $SU(3)$, the representation is the octet of Gell-Mann matrices, which are

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \tag{2.24}$$

Thus, it is possible for the previously obtained currents to be expressed in terms of these structure constants of the adjoint representation in $SU(N)$

$$\begin{aligned} \mathbf{j}_a &= -iw_a(G_a)_{ij}[(\mathbf{E}_i \cdot \phi_j) - (\mathbf{B}_i \times \mathbf{A}_j)] \\ &= -f_{abc}[(\mathbf{E}_b \cdot \phi_c) - (\mathbf{B}_b \times \mathbf{A}_c)], \end{aligned} \tag{2.25}$$

$$\begin{aligned} j_a^0 &= -iw_a(G_a)_{ij}(\mathbf{E}_i \cdot \mathbf{A}_j) \\ &= -f_{abc}(\mathbf{E}_b \cdot \mathbf{A}_c). \end{aligned} \tag{2.26}$$

The parameter w is no longer necessary. Using relation (2.25) and (2.26) in Eq. (2.21), we obtain the self-interaction Lagrangian for massless, spin-1 vector bosons in the form

$$\mathcal{L}_1 = \frac{1}{2}(E^2 - B^2) + l\phi_a f_{abc}\mathbf{E}_b \cdot \mathbf{A}_c + l\mathbf{A}_a f_{abc}[(\mathbf{E}_b \cdot \phi_c) - (\mathbf{B}_b \times \mathbf{A}_c)], \tag{2.27}$$

the interaction Lagrangian modifies the original one and, consequently, the original field equations given by (2.1) and (2.4). We will obtain new modified Maxwell equations derived from (2.27). Thus, we obtain expressions for Gauss's law and Ampère-Maxwell as

$$\nabla \cdot \mathbf{E}_a + 2lf_{abc}\nabla \cdot \mathbf{A}_b\phi_c + 2lf_{abc}\mathbf{A}_b\nabla \cdot \phi_c + 2l\mathbf{A}_b f_{abc}\mathbf{E}_c = 0, \tag{2.28}$$

$$(\nabla \times \mathbf{B}_a) - 2l\frac{\partial \mathbf{A}_b}{\partial t}f_{abc}\phi_c - 2lf_{abc}[(\mathbf{E}_b \cdot \phi_c) - (\mathbf{B}_b \times \mathbf{A}_c)] + lf_{abc}\nabla \times [(\mathbf{A}_b \times \mathbf{A}_c)] - \frac{\partial \mathbf{E}_a}{\partial t} = 0. \tag{2.29}$$

Notice how self-interaction modifies the standard equations. To eliminate any dependence on potential terms within the adjoint representation in the derivatives, we must repeat the same procedures. This involves redefining the fields in the form of

$$\delta A_a = iw_d(G_d)_{ae}\mathbf{A}_e = w_d f_{dae}\mathbf{A}_e, \tag{2.30}$$

$$\delta\phi_a = iw_d(G_d)_{ae}\phi_e = w_d f_{dae}\phi_e. \tag{2.31}$$

To obtain the new currents, it is necessary to use the principle of least action in the Lagrangian (2.27)

$$\begin{aligned}
\delta S = 0 = &\int d^4x\ \delta\left[\frac{1}{2}(E^2 - B^2) + l\phi_a f_{abc}\mathbf{E}_b \cdot \mathbf{A}_c + l\mathbf{A}_a f_{abc}[(\mathbf{E}_b \cdot \phi_c) - (\mathbf{B}_b \times \mathbf{A}_c)]\right] = 0,\\
= &\int d^4x \frac{1}{2}\left(2\mathbf{E}_i \cdot \delta\mathbf{E}_i - 2\mathbf{B}_i \cdot \delta\mathbf{B}_i\right) + l\delta\phi_a f_{abc}\mathbf{E}_b \cdot \mathbf{A}_c + l\phi_a f_{abc}\delta\mathbf{E}_b \cdot \mathbf{A}_c + l\phi_a f_{abc}\mathbf{E}_b \cdot \delta\mathbf{A}_c +\\
&l\delta\mathbf{A}_a f_{abc}[(\mathbf{E}_b \cdot \phi_c) - (\mathbf{B}_b \times \mathbf{A}_c)] + l\mathbf{A}_a f_{abc}[(\delta\mathbf{E}_b \cdot \phi_c) + (\mathbf{E}_b \cdot \delta\phi_c) - (\delta\mathbf{B}_b \times \mathbf{A}_c) -\\
&(\mathbf{B}_b \times \delta\mathbf{A}_c)]\\
= &\int d^4x \left[\mathbf{E}_i \cdot \left(\nabla\delta\phi_i + \frac{\partial\delta\mathbf{A}_i}{\partial t}\right) - \mathbf{B}_i \cdot (\nabla \times \delta\mathbf{A}_i) + l\delta\phi_a f_{abc}\mathbf{E}_b \cdot \mathbf{A}_c + l\phi_a f_{abc}\mathbf{E}_b \cdot \delta\mathbf{A}_c +\right.\\
&l\phi_a f_{abc}\left(\nabla\delta\phi_b + \frac{\partial\delta\mathbf{A}_b}{\partial t}\right) \cdot \mathbf{A}_c + l\delta\mathbf{A}_a f_{abc}[(\mathbf{E}_b \cdot \phi_c) - (\mathbf{B}_b \times \mathbf{A}_c)] + l\mathbf{A}_a f_{abc}[(\mathbf{E}_b \cdot \delta\phi_c) +\\
&\left.\left(\left(\nabla\delta\phi_b + \frac{\partial\delta\mathbf{A}_b}{\partial t}\right) \cdot \phi_c\right) - ((\nabla \times \delta\mathbf{A}_b) \times \mathbf{A}_c) - (\mathbf{B}_b \times \delta\mathbf{A}_c)]\right]
\end{aligned} \tag{2.32}$$

As previously done, it will be necessary to solve each term separately. Some terms will vanish as they correspond exactly to Eqs. (2.28) and (2.29). Other terms will remain, all of which are total spatial or temporal derivatives. These can be assigned to the spatial and temporal currents, as shown below

$$\mathbf{j}_d^{(1)} = -f_{dae}[(\mathbf{E}_A \cdot \phi_e) - (\mathbf{B}_a \times \mathbf{A}_e) - 2l f_{abc}\mathbf{A}_b\phi_e\phi_c], \tag{2.33}$$

$$j_d^{0(1)} = -f_{dae}(\mathbf{E}_A \cdot \mathbf{A}_e - 2l f_{abc}\mathbf{A}_b\mathbf{A}_e\phi_c). \tag{2.34}$$

Finally, the self-interaction Lagrangian is

$$\begin{aligned}
\mathcal{L}_2 &= \mathcal{L} - l'\phi_d j_d^{0(1)} - l'\mathbf{A}_d\mathbf{j}_d^{(1)}\\
&= \frac{1}{2}(E^2 - B^2) - l'\mathbf{A}_a f_{abc}(\mathbf{B}_b \times \mathbf{A}_c) - 4ll' f_{abc}f_{ade}\mathbf{A}_b\mathbf{A}_d\phi_c\phi_e + 2l' f_{abc}\phi_a\mathbf{A}_b\mathbf{E}_c,
\end{aligned} \tag{2.35}$$

observe that once expression (2.35) is obtained, deriving a new current is no longer necessary. This occur because the expression is now independent of the potential terms in the adjoint representation associated with the field. Therefore, if the same procedures are repeated n additional times, we will obtain n currents identical to the previous ones, as in (2.33) and (2.34). Consequently, there is no further need to derive self-interaction currents, and we can use the values $l = \frac{1}{4}g$ and $l' = \frac{1}{2}g$ for the coupling constants. We set these values by invoking the principle of universality, we can choose them in such a way that there is only a gauge coupling constant, exactly as it is the case for Yang-Mills theories [35], to write the Lagrangian as

$$\mathcal{L}_2 = \frac{1}{2}(E^2 - B^2) - \frac{1}{2}g\mathbf{A}_a f_{abc}(\mathbf{B}_b \times \mathbf{A}_c) + g f_{abc}\phi_a\mathbf{A}_b\mathbf{E}_c - \frac{1}{2}g^2 f_{abc}f_{ade}\mathbf{A}_b\mathbf{A}_d\phi_c\phi_e, \tag{2.36}$$

from equation (2.36), we find the following equations for the electric and magnetic fields of Yang-Mills:

$$\mathbf{E}_a = -\nabla \cdot \phi_a - \frac{\partial\mathbf{A}_a}{\partial t} + g f_{abc}\mathbf{A}_b\phi_c, \tag{2.37}$$

$$\mathbf{B}_a = (\nabla \times \mathbf{A}_a) + \frac{1}{2} g f_{abc} (\mathbf{A}_b \times \mathbf{A}_c). \tag{2.38}$$

Where, from the Lagrangian (2.36), it is possible to obtain the following field equations

$$\nabla \cdot \mathbf{E}_a + g f_{abc} \mathbf{A}_b \mathbf{E}_c = 0, \tag{2.39}$$

$$\nabla \times \mathbf{B}_a + g f_{abc} (\mathbf{A}_b \times \mathbf{B}_c) + g f_{abc} \phi_b \mathbf{E}_c = \frac{\partial \mathbf{E}_a}{\partial t}. \tag{2.40}$$

Note that equations (2.39) and (2.40) have the same structure as equations (2.28) and (2.29) when their electric and magnetic fields are defined as in (2.37) and (2.38). Furthermore, two more field equations are still needed to complete the quartet of Maxwell's equations. Thus, from the new formulations (2.37) and (2.38), it is possible to obtain the last two Maxwell equations (Gauss's law for magnetism and Faraday-Lenz law) in the non-Abelian formulation:

$$\nabla \times \mathbf{E}_a = -\frac{\partial \mathbf{B}_a}{\partial t} + g f_{abc} \phi_b \mathbf{B}_c - g f_{abc} (\mathbf{A}_b \times \mathbf{E}_c), \tag{2.41}$$

$$\nabla \cdot \mathbf{B}_a + g f_{abc} \mathbf{A}_b \mathbf{B}_c = 0. \tag{2.42}$$

In a compact form, we can write the new non-Abelian Maxwell equations, that is, expressed in the Yang-Mills formalism for the group $SU(N)$, as:

$$\nabla \cdot \mathbf{E}_a + g f_{abc} \mathbf{A}_b \mathbf{E}_c = 0, \tag{2.43}$$

$$\nabla \cdot \mathbf{B}_a + g f_{abc} \mathbf{A}_b \mathbf{B}_c = 0, \tag{2.44}$$

$$\nabla \times \mathbf{E}_a + g f_{abc} (\mathbf{A}_b \times \mathbf{E}_c) = -\frac{\partial \mathbf{B}_a}{\partial t} + g f_{abc} \phi_b \mathbf{B}_c, \tag{2.45}$$

$$\nabla \times \mathbf{B}_a + g f_{abc} (\mathbf{A}_b \times \mathbf{B}_c) = \frac{\partial \mathbf{E}_a}{\partial t} - g f_{abc} \phi_a \mathbf{E}_b. \tag{2.46}$$

From this newly derived set of non-Abelian equations, we can observe, when compared to equations (2.1), (2.2), (2.3), and (2.4), that Gauss's law for magnetism (2.44) undergoes significant structural changes. These changes suggest that the self-interacting spin-1, massless vector fields can generate magnetic monopoles even in the absence of fermionic matter, as demonstrated in equation (2.44). This distinction is a key difference between the Abelian and non-Abelian cases, as derived from Maxwell's equations. It is also essential to note that the electric and magnetic fields here lack physical meaning, as they are not gauge-invariant.

2.1.1 The natural emergence of the covariant derivative

As an initial approach to developing a self-interaction theory for massless spin-1 vector bosons from first principles, we can naturally define the covariant gauge derivative with its spatial component as:

$$\mathbf{D} = \delta_a^c \nabla + g f_{abc} \mathbf{A}_b, \tag{2.47}$$

and the covariant gauge temporal derivative is:

$$D_t = \delta_a^c \frac{\partial}{\partial t} - g f_{abc} \phi_b. \tag{2.48}$$

From such definitions, it is possible to rewrite equations (2.43), (2.44), (2.45), and (2.46) with the same structure as the usual Maxwell equations in (2.1), (2.2), (2.3), and (2.4). They will be replaced by the derivative operators with a non-Abelian structure of the group $SU(N)$; thus, we obtain the following expressions:

$$\mathbf{D} \cdot \mathbf{E}_c = 0, \tag{2.49}$$

$$\mathbf{D} \cdot \mathbf{B}_c = 0, \tag{2.50}$$

$$\mathbf{D} \times \mathbf{E}_c = -D_t \cdot \mathbf{B}_c, \tag{2.51}$$

$$\mathbf{D} \times \mathbf{B}_c = D_t \cdot \mathbf{E}_c. \tag{2.52}$$

With these results, the analysis of the theory can be extended to the point of application to matter. This allows it to be used in the construction of the electroweak and quantum chromodynamic theories, since both are formulated within a non-Abelian framework. Moreover, these theories constitute fundamental components of the Standard Model of particle physics.

2.1.2 The energy and momentum of Yang-Mills waves

With the derivation of equations (2.49), (2.50), (2.51), and (2.52), which are analogous to the Maxwell equations, we can demonstrate that certain applications of these equations yield results similar to those in conventional electromagnetism. Moreover, the covariant gauge derivative behaves like a standard derivative when dealing with scalar terms in the Yang-Mills indices. A Yang-Mills scalar is defined as a singlet of the symmetry group, indicating that the generators, G_a, are trivial, with $G_a = 0$, when acting on Yang-Mills scalar quantities. Consequently, we have $D_t = \dfrac{\partial}{\partial t}$ and $\mathbf{D} = \nabla$. Thus, we can express:

$$(D_t \cdot \mathbf{B}_c) \cdot \mathbf{B}_c = \left(\frac{\partial \mathbf{B}_c}{\partial t} - g f_{cab} \phi_a \mathbf{B}_b \right) \cdot \mathbf{B}_c = \left(\frac{\partial \mathbf{B}_c}{\partial t} \right) \cdot \mathbf{B}_c - g f_{cab} \phi_a \mathbf{B}_b \mathbf{B}_c = \frac{\partial}{\partial t} \left(\frac{B^2}{2} \right). \tag{2.53}$$

Such a relationship occurs in the same way as the spatial covariant gauge derivative given in (2.47). Therefore, we will use this relation to find the directional energy flux, that is, the Poynting vector. The method of obtaining this quantity is similar to that used in undergraduate electromagnetism courses. Due to the property of the covariant gauge derivative shown previously, we have:

$$\nabla \cdot \mathbf{S} + \frac{\partial U_{em}}{\partial t} = 0, \tag{2.54}$$

this equation is associated with the conservation of energy since $\mathbf{S} = (\mathbf{E}_c \times \mathbf{B}_c)$ is the Poynting vector, a fundamental quantity in electromagnetic theory, as it represents the momentum flux, and $U_{em} = \dfrac{1}{2}(E^2 + B^2)$ provides the electromagnetic energy density. Taking the time derivative of the Poynting vector, we can naturally deduce the continuity equation relating the momentum of the non-Abelian wave to the corresponding stress tensor, as follows:

$$\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \mathbf{T} = 0, \tag{2.55}$$

where again **S** is the Poynting vector and T is the electromagnetic stress-energy tensor, which determines the fundamental properties of the electric and magnetic fields such as energy, momentum, pressure, and so on, and has components

$$T^{mn} = [\delta^{mn}(B_c^2 + E_c^2) - B_c^m B_c^n - E_c^m E_c^n]. \tag{2.56}$$

After having explored some applications through the equations describing spin-1 fields in vacuum, we can now consider the case where external sources are present. It is of highest importance to emphasize that the addition of sources is crucial for the equations, just as in the usual Maxwell equations. However, for the derivation of the non-Abelian spin-1 equations, they do not present a fundamental character. Therefore, now that the entire construction has been made, we can add them. This would lead to the equations (2.49) and (2.52) being rewritten as:

$$\mathbf{D} \cdot \mathbf{E}_c = \rho_c, \tag{2.57}$$

$$\mathbf{D} \times \mathbf{B}_c = D_t \cdot \mathbf{E}_c + \mathbf{J}_c. \tag{2.58}$$

An important observation when adding the sources is that the Yang-Mills indices of these sources must be the same as the indices of the fields.

2.1.3 Coupling with matter

In this work, we have developed a theory of self-interaction for massless, spin-1 vector bosons. Naturally, this leads to a question of how the theory behaves in the presence of matter, particularly in coupling with fermionic matter. Yang and Mills, in their foundational 1954 paper, initially explored the matter sector in a similar way. Such an analysis allows us to understand how gauge transformations act on the matter fields. We start from the Dirac Lagrangian

$$\mathcal{L}_D = i\psi^\dagger \gamma^0 \frac{\partial}{\partial t}\psi + i\psi^\dagger \gamma^\mu \nabla \psi, \tag{2.59}$$

where ψ represents the fermionic matter field, and its transpose conjugate $\psi^\dagger = (\psi^*)^T$ is used here. The spinorial indices are hidden and the γ-matrices, known as Dirac matrices, satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}, \tag{2.60}$$

where $\eta^{\mu\nu}$ denotes the Minkowski metric. Based on equation (2.59) a Lie transformation for the fermionic matter fields can be introduced similarly to equations (2.10) and (2.11) like

$$\psi'_A = R_{AB}\psi_B. \tag{2.61}$$

Here, we take these indices in the adjoint representation. For the group $SU(N)$, these indices range as $A, B = 1, \ldots, N^2 - 1$. Assuming a linear transformation for fermionic matter, we seek to identify the function R_{AB}. This group is characterized by unitary matrices, satisfying $R^\dagger R = 1$ in $U(N)$, and $|\det R|^2 = 1 \rightarrow \det R = 1$. Analysis shows that this group has N real constraints and $N^2 - 1$ degrees of freedom. To ensure the invariance of Lagrangian (2.59), we apply the covariant derivatives (2.47) and (2.48) on the N-plet of matter, with the following transformations:

$$(D_t\psi)'_A = R_{AB}(D_t\psi)_B, \tag{2.62}$$

$$(\mathbf{D}\psi)'_A = R_{AB}(\mathbf{D}\psi)_B. \tag{2.63}$$

Applying the gauge covariant derivative to both spatial and temporal components yields consistent results. For $SU(N)$, the covariant derivative transforms under similarity as:

$$D'_t = RD_tR^{-1} \to \frac{\partial}{\partial t} + ig = R\left(\frac{\partial}{\partial t} + ig\right)R^{-1}, \tag{2.64}$$

$$\mathbf{D}' = R\mathbf{D}R^{-1} \to \nabla + ig = R\left(\nabla + ig\right)R^{-1}. \tag{2.65}$$

These unitary matrices provide a unitary group representation. Thus, such transformations apply similarly to other fields. Using the field transformation in Eq. (2.8), we derive the transformation for the A field, covering both spatial and temporal components:

$$\mathbf{A}' = R\mathbf{A}R^{-1} + i(\nabla R)R^{-1}, \tag{2.66}$$

$$\Phi' = A'_0 = R\Phi R^{-1} + i\left(\frac{\partial}{\partial t}R\right)R^{-1}. \tag{2.67}$$

Choosing to work with spatial components, we set $A_0 = \Phi$ and substitute this into the following equation:

$$\nabla\psi' + i\mathbf{A}\psi' = R_{AB}\nabla\psi + iR_{AB}\mathbf{A}\psi, \tag{2.68}$$

which, upon resolution, gives

$$R_{AB} = (e^{iw_aG_a})_{AB}. \tag{2.69}$$

Using Noether's theorem, we obtain the fermionic self-interaction current for this system

$$\mathbf{j}_m = \psi^\dagger_A\gamma^\mu m(G_a)_{AB}\psi_B, \tag{2.70}$$

with the temporal component, also known as charge density, given by

$$\rho^0_m = \psi^\dagger_A\gamma^0(G_a)_{AB}\psi_B. \tag{2.71}$$

The Lagrangian for fermionic matter self-interaction is then written as

$$\mathcal{L}^1_D = \mathcal{L}_D - g\rho^0_m\Phi + g\mathbf{j}_m\mathbf{A}, \tag{2.72}$$

where g is the universal coupling constant of Yang-Mills theory. This universality implies that both the self-coupling of the mediating bosons and their coupling to matter are governed by the same parameter g, which typically matches the Yang-Mills coupling in the Dirac equation. Expanding all terms yields

$$\mathcal{L}^1_D = i\psi^\dagger_A\gamma^0\frac{\partial}{\partial t}\psi_B + i\psi^\dagger_A\gamma^\mu\nabla\psi_B - g\psi^\dagger_A\gamma^0(G_a)_{AB}\psi_B\Phi + g\psi^\dagger_A\gamma^\mu(G_a)_{AB}\psi_B\mathbf{A} \tag{2.73}$$

$$= \psi_A^\dagger i\gamma^0 \left(\delta_{AB}\frac{\partial}{\partial t} + ig(G_a)_{AB}\Phi\right)\psi_B + \psi_A^\dagger i\gamma^\mu(\delta_{AB}\nabla + ig(G_a)_{AB}\mathbf{A})\psi_B.$$

This expression can be reorganized using covariant derivatives defined in the adjoint representation as follows:

$$(\mathbf{D})_{AB} = \delta_{AB}\nabla + ig(G_a)_{AB}\mathbf{A}, \tag{2.74}$$

$$(D_t)_{AB} = \delta_{AB}\frac{\partial}{\partial t} + ig(G_a)_{AB}\Phi. \tag{2.75}$$

This formalism demonstrates that the gauge covariant derivative structure adapts to the representation proposed by the theory. Ultimately, this approach leads to the following expression for the Lagrangian:

$$\mathcal{L}_D^1 = i\psi_A^\dagger\gamma^0(D_t)_{AB}\psi_B + i\psi_A^\dagger\gamma^\mu(\mathbf{D})_{AB}\psi_B. \tag{2.76}$$

The central idea it becomes clear that, through Noether's procedure, we obtained the main goal — starting from a free Abelian theory of massless spin-1 vector bosons and arriving at a Yang–Mills theory, which retains these same bosonic properties, but now in a non-Abelian and self-interacting version.

We also showed that several developments originating from Maxwell's electromagnetism can be extended to the non-Abelian formalism. The illustrations we chose were: the energy conservation equation involving the Poynting vector and the computation of the electromagnetic energy-momentum tensor.

Finally, we chose to carry out the same analysis for the fermionic matter sector. This entire chapter serves as a way to prepare the reader for what is to come. We will once again employ Noether's procedure in this work, but now considering the effects of Loop Quantum Gravity (LQG) in the electromagnetic sector, which makes the procedure considerably more challenging.

Chapter 3

Fundamentals of General Relativity

General Relativity (GR), developed by Albert Einstein from 1907 to 1915, provides a comprehensive theory of gravitation. Before formulating GR, Einstein had established the theory of Special Relativity, which he later expanded upon to create GR. Unlike Newton's concept of gravitation as a force between masses, GR reinterprets it as an effect of spacetime geometry distortion around mass and energy. This theory has been supported by numerous physical phenomena, with black holes and gravitational waves being among the latest confirmations. We will start this chapter by examining the mathematical structure and physical consequences of General Relativity, focusing on advanced methods such as demonstrating that GR is a gauge theory [36, 37] and showing how the tetrad formalism [38–41] naturally arises. An alternative formulation of General Relativity can be achieved within the Hamiltonian framework, more specifically through the Arnowitt-Deser-Misner (ADM) foliation [1, 46–48]. These methods provide alternative perspectives on the theory's dynamics and serve as valuable tools for investigating gravitational phenomena, offering deeper insights into the structure of spacetime and its interactions with matter.

3.1 General Relativity is a Gauge Theory

The fundamental object of study in this work is Loop Quantum Gravity (LQG), with one of its applications being in the electromagnetic sector. Since one of its foundations is General Relativity (GR), we use this chapter to introduce the reader to the fundamentals of this theory, which will be employed later in the context of LQG. In turn, we make use of the ideas of [36, 37], can General Relativity, with all its characteristics, be considered a gauge theory? The formulate to this question in physical terms will be given briefly in this section, though we can begin to discuss it now. Let us try to outline some main points of both theories and see how they converge:

- Both Einstein's gravitation and Yang-Mills theories are highly nonlinear;
- In both theories, we have connections: the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ in GR and the gauge potential A_μ in Yang-Mills. It is possible to derive a curvature term $R_{\mu\nu}$ in gravitation and $F_{\mu\nu}$ in gauge theories;
- Both theories possess local symmetry;
- Einstein's gravitation includes diffeomorphisms, which are general transformations of local coordinates based on the covariant character of the metric. These transformations are analogous to those in Yang-Mills theory, and we will explore this more explicitly later.

Thus, there are significant similarities between these theories, forming essential foundations. However, the introduction of fermionic matter brings a crucial distinction between them. Fermionic matter comprises fields ψ_α carrying the index α, which is a spinorial index—not space-time based but in a complex space. This α index, for example, does not interact with the Jacobians present in diffeomorphisms, unlike bosonic matter, which, regardless of its vectorial nature, carries space-time indices and thus responds to diffeomorphism transformations.

The key lies in coupling fermionic fields in a way that enables them to interact with the curved geometry proposed by GR, whether it is a Riemannian or pseudo-Riemannian space. This coupling is achieved using tetrads.

3.2 Tetrads Formalism

The section we using the sign of the metric $g_{\mu\nu}$ will be taken as $(-,+,+,+)$. The vielbein is a formalism in General Relativity [38–41], offering an alternative to the traditional description of spacetime geometry via the metric tensor. In this approach, the metric is expressed through a set of local basis vectors, known as vielbeins (or tetrads in four dimensions), which establish a locally flat reference frame at each spacetime point. As already said, this formalism is particularly advantageous when coupling General Relativity to spinor fields, such as in fermionic theories, as it provides a natural way to incorporate spin in curved spacetime. Vielbeins also simplify certain calculations by providing a flexible framework for analyzing geometric properties and are frequently applied in Quantum Gravity. Through this formalism, the local Lorentz symmetry of spacetime becomes explicit, allowing for a clearer understanding of the relationship between local and global symmetries in gravitational systems. Let us consider the transformation from a curvilinear coordinate system to a flat one

$$x^\mu \to \xi^a, \tag{3.1}$$

where the Greek letters $\mu, \nu, \cdots = 1, \ldots, N$ denote indices in curved space-time, also referred to as world indices, while lowercase Latin letters $a, b, \cdots = 1, \ldots, N$ denote local or frame indices.

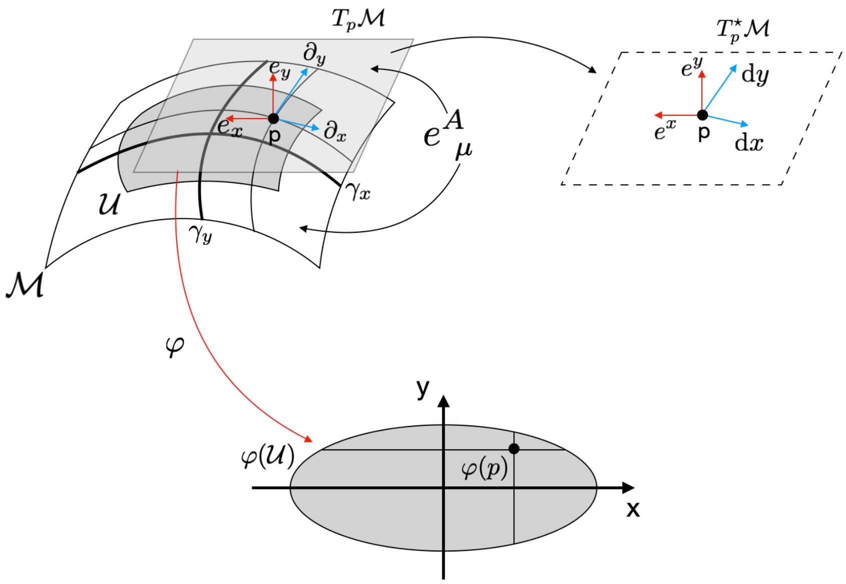

Figure 3 – This figure illustrates the action of the vielbeins [43]. At the bottom, we have an illustration of a curved manifold denoted by $\mathcal{U}$. If we expand the view, this manifold $\mathcal{U}$ is embedded in the manifold $\mathcal{M}$, we can take a tangent plane over this manifold in the form $T_p\mathcal{M}$. There, the vielbeins provide the connection between the curvilinear coordinates in the form dx and dy and the coordinates of the tangent plane, which are flat.

Consequently,

$$dx^\mu = \frac{\partial \xi^a}{\partial x^\mu} d\xi^a = e^a_\mu d\xi^a, \tag{3.2}$$

where the $\dfrac{\partial \xi^a}{\partial x^\mu}$ is the Jacobian associated with the transformation, establishing the transition from curvilinear to flat coordinates. This Jacobian is the vielbein e^a_μ. Thus, the line element given by the flat metric η_{ab} can be rewritten as

$$\begin{aligned} ds^2 &= \eta_{ab} d\xi^a d\xi^b \\ &= \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} dx^\mu \frac{\partial \xi^b}{\partial x^\nu} dx^\nu \\ &= \frac{\partial \xi^a}{\partial x^\mu} \eta_{ab} \frac{\partial \xi^b}{\partial x^\nu} dx^\mu dx^\nu \\ &= g_{\mu\nu}(x) dx^\mu dx^\nu. \end{aligned} \tag{3.3}$$

Note that we could have started from the curvilinear metric and arrived at the equation for the flat metric. From Eq. (3.3), we obtain a relation between the curved metric and the flat metric in terms of the vielbein

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu, \tag{3.4}$$

the tangent space, as well as its quantities, possesses a local symmetry, endowed with the Lorentz group $SO(1,3)$. This is already an indication that the coupling of fermions is possible, as the present group allows coupling with spinorial indices. The connection between the rotation group and spinors in the Lie algebra will be given by a vector transformation of the form

$$A'^{a} = \left(e^{\frac{i}{2}\alpha^{cd}\Sigma_{cd}}\right)^{a}_{\ b} A^{b}, \tag{3.5}$$

the α^{cd} is the parameter of the transformation, i.e., the angles, and Σ_{cd} are the rotation generators in the spinorial space of $SO(1,3)$, given by

$$\Sigma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b], \tag{3.6}$$

where γ_a are the Dirac gamma matrices, which are related to the flat metric through the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{3.7}$$

Note that if we were dealing with scalars, Eq. (3.5) would have the generators $\Sigma_{cd} = 0$. Remember, a typical Yang-Mills transformation for any given field is given from Eqs. (2.10) and (2.11) as follows

$$\Phi'(x) = R\Phi(x). \tag{3.8}$$

If we apply a derivative $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$, we obtain

$$\begin{aligned} \partial_\mu \Phi'(x) &= \partial_\mu (R\Phi(x)) \\ &= R\partial_\mu \Phi + (\partial_\mu R)\Phi(x). \end{aligned} \tag{3.9}$$

As we know, the last term in this expression breaks gauge covariance. The solution to this problem is the introduction of the covariant derivative D_μ, such that it does not act on the Lie group parameter R which can be describe by the Eq. (2.12) and has the following form

$$D_\mu = \partial_\mu + igw_\mu, \tag{3.10}$$

where g is the coupling constant and w_μ is the gauge field. Here, for a gravitational theory, is called the spin connection. This gauge field is written in contraction with the group generator, a notation commonly used in Yang-Mills theory, as follows

$$w_\mu = \frac{1}{2} w^{ab} \Sigma_{ab}, \tag{3.11}$$

the term 1/2 is a convention due to the antisymmetry of the indices ab in both quantities. The spin connection field typically transforms as a Yang-Mills field in the following form:

$$\begin{aligned} D'_\mu &= RD_\mu R^{-1} \\ \partial_\mu + igw'_\mu &= R(\partial_\mu + igw_\mu)R^{-1} \\ w'_\mu &= Rw_\mu R^{-1} - \frac{i}{g} R(\partial_\mu R^{-1}). \end{aligned} \tag{3.12}$$

The same analysis can be applied to the vielbein, showing that it is a gauge field

$$\begin{aligned} e^a_\mu(x+\delta x) &= \frac{\partial x^\nu}{\partial x^{\mu'}} e^a_\nu \\ e^a_\mu + \delta x^\nu \partial_\nu e^\mu_a &= e^a_\mu - (\partial_\mu \delta x^\nu) e^a_\nu \\ \delta e^a_\mu &= \delta x^\nu \partial_\nu e^\mu_a - (\partial_\mu \delta x^\nu) e^a_\nu. \end{aligned} \tag{3.13}$$

As we have just shown in Eqs. (3.12) and (3.13), our theory will have two fundamental gauge fields instead of just one. It should be noted that these fields transform in the same manner as the fields in Eqs. (2.66) and (2.67). Let us consider a field with the following transformation

$$V^\mu = e^\mu_a V^a, \tag{3.14}$$

which is invariant under local Lorentz transformations. As we already know the action of the flat covariant derivative on this field, we can define the covariant derivative in curved space as

$$\begin{aligned} \nabla_\mu V^\nu &= e^\mu_a D_\mu (e^a_\rho V^\rho) \\ &= e^\nu_a (D_\mu e^a_\rho) V^\rho + e^\nu_a e^a_\rho D_\mu V^\rho \\ &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \end{aligned} \tag{3.15}$$

where $\Gamma^\nu_{\mu\rho}$ is the affine connection, also known as the Christoffel symbol (when torsion is equal to 0). This new derivative does not account for the indices of the vielbeins, thus

$$\nabla_\mu e^A_\nu = D_\mu e^A_\nu - \Gamma^\rho_{\mu\nu} e^A_\rho = 0. \tag{3.16}$$

We previously mentioned that when describing a theory of gravitation through gauge components, the gauge field is called the spin connection. But how does it relate to gravity? In Yang-Mills theories, we take $[D_\mu, D_\nu] = igF_{\mu\nu}$, where $F_{\mu\nu}$ is the field strength. As in the previous chapter we performed the non-Abelianization using the Euclidean formalism, the field strength tensor $F_{\mu\nu}$ did not play a significant role. However, this quantity is fundamental both for the Abelian Maxwell theory and for Yang-Mills theories. From the relation commutator of covariant derivatives, we can define this quantity as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \tag{3.17}$$

the equation above is for the non-Abelian field strength. Let us do the same here, considering the different types of derivatives and analyzing their contributions

$$\begin{aligned} [\partial_a, \partial_b] &= [e^\mu_a \partial_\mu, e^\nu_b \partial_\nu] \\ &= (e^\mu_a \partial_\mu e^\nu_b - e^\mu_b \partial_\mu e^\nu_a) \partial_\nu \\ &= (\partial_a e^\nu_b - \partial_b e^\nu_a) \partial_\nu = T^\nu_{ab} \partial_\nu, \end{aligned} \tag{3.18}$$

where T^ν_{ab} is the torsion. This is the field strength in the tangent plane. In the Einstein-Hilbert theory, this quantity is zero due to the field equations. At this point, let us consider it as a non-zero parameter. Torsion is a quantity that arises naturally when fermions are introduced into our theory. The torsion and the affine connection are related as

$$\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = \Gamma^{\rho}_{[\mu,\nu]} = e^{\rho}_{a} T^{a}_{\mu\nu}. \tag{3.19}$$

Let us examine other quantities obtained when computing the commutator of gauge covariant derivatives, which are

$$\begin{aligned} [D_A, D_B] &= [e^{\mu}_{a} D_{\mu}, e^{\nu}_{b} D_{\nu}] = (e^{\mu}_{a} D_{\mu} e^{\nu}_{b} - e^{\nu}_{b} D_{\nu} e^{\nu}_{a}) D_{\mu} + e^{\mu}_{a} e^{\nu}_{b} [D_{\mu}, D_{\nu}] \\ &= (e^{\mu}_{a} D_{\mu} e^{\nu}_{b} - e^{\mu}_{b} D_{\mu} e^{\nu}_{A}) e^{c}_{\nu} D_c + \frac{1}{2} e^{\mu}_{a} e^{\nu}_{B} R^{cd}_{\mu\nu} \Sigma_{cd} \\ &= -T^{c}_{ab} D_c + \frac{1}{2} R^{cd}_{ab} \Sigma_{cd}, \end{aligned} \tag{3.20}$$

here R^{cd}_{ab} is the Ricci curvature tensor. From Eq. (3.20), it is possible to obtain the two Maurer-Cartan equations. The first one is

$$\begin{aligned} T^{a}_{\mu\nu} &= D_{\mu} e^{a}_{\nu} - D_{\nu} e^{a}_{\mu} \\ &= \partial_{\mu} e^{a}_{\nu} - \partial_{\nu} e^{a}_{\mu} + w^{a}_{\mu b} e^{b}_{\nu} - w^{a}_{\nu b} e^{b}_{\mu}. \end{aligned} \tag{3.21}$$

The second Maurer-Cartan equation is obtained through the Ricci curvature tensor

$$R^{ab}_{\mu\nu} = \partial_{\mu} w^{ab}_{\nu} - \partial_{\nu} w^{ab}_{\mu} + w^{a}_{\mu c} w^{cb}_{\nu} - w^{a}_{\nu c} w^{cb}_{\mu} \tag{3.22}$$

Here we define that the Ricci scalar is

$$R = R^{a}_{\mu} e^{\mu}_{a}. \tag{3.23}$$

With all the ingredients present in the new formalism, naturally, the next step is to use this mathematical formalism of tetrads to write the action of fields.

3.2.1 Light, Mechanics and Action

The action is a fundamental quantity to describe classical or Quantum Mechanics. Using the previous notation introduced in the last section, we can exemplify some important actions in physics. First, we can write the Einstein-Hilbert action in metric formalism, this action is very important, as it is the fundamental action of General Relativity, and therefore the starting point for obtaining Einstein's equations by varying it in relation to the metric. This formulation highlights the geometrization of gravity: all the dynamic properties of the gravitational field are contained in the curvature of space-time and are of the form

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \tag{3.24}$$

where $\sqrt{-g}$ is the invariant volume element, with the minus sign for the signature of metric. The parameter $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar as already know, G is the Newton's gravitational constant and there could also be a term Λ related to the cosmological constant, where in our case we take $\Lambda = 0$. How can we write this action in the tetrad formalism? First, we can see that $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu \to g = -e^2$, and we can relate the determinant as $e = \det e^a_\mu$. Using Eq. (3.23), we can rewrite $R = e^\mu_a e^\nu_b R^{ab}_{\mu\nu}$. Thus, the action in Eq. (3.24) can be rewritten as

$$S[e] = \frac{1}{2\kappa} \int d^4x \left(e e^\mu_a e^\nu_b R^{ab}_{\mu\nu} \right), \tag{3.25}$$

where $\kappa = 1/\sqrt{G}$ is the coupling constant with the dimension of energy, since G is the Newtonian gravitational constant. We can apply the variational method to this action to obtain the equations of motion. If we do this for the vielbein field δe^a_μ, we obtain

$$\begin{aligned} 0 &= (\delta e) e^\mu_a e^\nu_b R^{ab}_{\mu\nu} + e(\delta e^\mu_a) e^\nu_b R^{ab}_{\mu\nu} + e e^\mu_a (\delta e^\nu_b) R^{ab}_{\mu\nu} \\ &= -e e^c_\rho (\delta e^\rho_c) R + e(\delta e^\mu_a) e^\nu_b R^{ab}_{\mu\nu} + e e^\mu_a (\delta e^\nu_b) R^{ab}_{\mu\nu} \\ &= (\delta e^\mu_a) e [2 e^\nu_b R^{ab}_{\mu\nu} - e^a_\mu R] \\ &= R_{\mu\nu} - \frac{1}{2} e^a_\mu R. \end{aligned} \tag{3.26}$$

This is the famous Einstein equation in vacuum. Just as we applied the principle of least action to one of the gauge fields, we can apply it to the other, that is, to the spin connection, in the following way

$$e e^\mu_a e^\nu_b (\delta R^{ab}_{\mu\nu}) = 0, \tag{3.27}$$

after some algebraic, we manipulations

$$-e \left[e^\mu_d e^\nu_{[b} e^\lambda_{a]} + \frac{1}{2} e^\mu_{[a} e^\lambda_{b]} e^\nu_d \right] T^d_{\mu\lambda} = 0. \tag{3.28}$$

The Eq. (3.28) allows, in the Einstein-Hilbert formalism, the torsion to be set to zero, since all the terms in the equation are non-zero. One question that the reader may have asked some looking at Eq. (3.26) is where the energy-momentum tensor is on the right-hand side of the equality. This quantity arises when coupling matter to gravity, such as a scalar field

$$S_{full} = S_{EH} + \int d^4x e \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 \right). \tag{3.29}$$

Let us now write the action for electromagnetism. The reader may think that the first thing to do, as in a non-Abelian formulation, is to promote the usual derivatives to covariant derivatives of the field strength $F_{\mu\nu}$. However, this strategy is invalid here, and the reason for this is the following

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \nabla_\mu A_\nu - \nabla_\nu A_\mu + \Gamma^\lambda_{\mu\nu} A_\lambda - \Gamma^\lambda_{\nu\mu} A_\lambda \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu) + T^\lambda_{\mu\nu} A_\lambda, \end{aligned} \tag{3.30}$$

this last term breaks gauge invariance, which shows that covariantizing the derivative is not a good approach. In fact, we will leave $F_{\mu\nu}$ in its usual form and couple the electromagnetic sector to gravity through

$$S_{full} = S_{EH} + \int d^4x e g^{\mu\kappa} g^{\nu\lambda} F_{\mu\nu} F_{\kappa\lambda} = S_{EH} + \int d^4x e (e^{\mu}_{a} e^{a\kappa} e^{\nu}_{b} e^{b\lambda} F_{\mu\nu} F_{\kappa\lambda}), \tag{3.31}$$

and we can obtain the equation of motion for the vielbein δe^{a}_{μ}, which gives

$$R_{\mu\nu} - \frac{1}{2} e^{a}_{\mu} R = \kappa^2 \left(F_{\mu\lambda} F^{\lambda}_{\nu} - \frac{1}{4} \eta_{\mu\nu} F^2 \right). \tag{3.32}$$

Finally, we already have shown that the vielbein formalism can help describing the fermionic matter. What happens, when we want to describe the Dirac action? Let us start from the usual Lagrangian, where the spinorial indices of the fermionic fields will not be written explicitly

$$S_D = \int d^4x \ \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi. \tag{3.33}$$

For simplicity, let us consider $m = 0$ and add the determinant term e, as well as covariantize the derivative, in the following way

$$\begin{aligned} S_D &= \int d^4x \ \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi \\ &= \int d^4x \ e e^{\mu}_{a} \bar{\psi} i \gamma^{a} D_{\mu} \psi \\ &= \int d^4x \ e e^{\mu}_{a} \bar{\psi} i \gamma^{a} \left(\partial_{\mu} \psi - \frac{1}{8} w_{\mu cd} [\gamma^c, \gamma^d] \psi \right), \end{aligned} \tag{3.34}$$

from here, it is possible to couple with the Einstein-Hilbert action, obtaining

$$S_{full} = S_{EH} + \int d^4x \ e e^{\mu}_{a} \bar{\psi} i \gamma^{a} \left(\partial_{\mu} \psi - \frac{1}{8} w_{\mu cd} [\gamma^c, \gamma^d] \psi \right), \tag{3.35}$$

and obtain the equations of motion for both gauge fields. Note that, as mentioned earlier, torsion is a quantity that arises naturally in the presence of fermions, and this becomes evident when we extremize this action with respect to the spin connection, yielding

$$-e \left[e^{\mu}_{d} e^{\nu}_{[b} e^{\lambda}_{a]} + \frac{1}{2} e^{\mu}_{[a} e^{\lambda}_{b]} e^{\nu}_{d} \right] T^{d}_{\mu\lambda} + e e^{\mu}_{a} \bar{\psi} i \gamma^{a} [\gamma^c, \gamma^d] \psi = 0. \tag{3.36}$$

The first term is the standard one obtained earlier in Eq. (3.28), so the torsion in this case is no longer zero as seen before. This occurs due to this new term that appears, which is a fermionic bilinear. The fermionic matter induces torsion in the system. We could go on to describe the action and consequently the dynamics of other fields, as can be seen in [1] and [44].

3.3 Hamiltonian and ADM Variables

Throughout this dissertation, we have mentioned one of Dirac's contributions to physics with the publication of his 1928 paper. As we know, he ventured into other areas of study, one of which was the problem of incompatibility between General Relativity and Quantum

Mechanics. Up until that time, no one had formally worked on this topic, making Dirac the pioneer in attempting to find a Hamiltonian for General Relativity and thereby quantize the theory [45]. This was a very challenging task due to two main factors. First, because the theory has constraints, it was necessary to develop Hamiltonians for constrained systems. Second, as Carlo Rovelli and Francesca Vidotto notes in [1], "the horrendous complexity of the algebra in the canonical analysis of the theory, when using the metric variable."

Dirac managed to overcome both problems, although the second was particularly difficult. However, in an effort to improve and simplify Dirac's contributions, Richard Arnowitt, Stanley Deser, and Charles W. Misner published the ADM formulation in 1959. They proposed variables that greatly simplified the canonical algebra and provided a clearer interpretation of the geometry, because the foliation of space-time. This section will be based on the references [1, 46–54] and the metric $g_{\mu\nu}$ is $(+,-,-,-)$.

3.3.1 Differential Geometry and the foliation of Space-time

The ADM formalism has the clear goal of describing General Relativity using Hamiltonians. The approach taken was to propose something rather bold in a covariant theory*, namely, to separate the spacetime manifold $(M)^\dagger$ described by the metric $g_{\mu\nu}$ into space and time through hypersurfaces $(\Sigma_t)^\ddagger$, where the temporal coordinate becomes a label in the form $M = \sigma \times \mathbb{R}$ where σ is a fixed 3-dimensional manifold with arbitrary topology. The label t will represent the constant spatial slices σ. This was conceived because, in a canonical approach, it is not possible to define velocities and conjugate momenta without breaking diffeomorphism invariance. The foliation into hypersurfaces generates new terms, as can be seen in figure 4. These are vectors that we denote as n^μ, the normal vector to the hypersurface Σ_t, where the coefficient N, known as the Lapse function, is related to the motion between spatial slices, or in a sense, determines the rate at which physical time progresses in the chosen coordinates. Finally, we also have the Shift vector N_a, which measures the displacement of the spatial coordinates from one constant surface to the next. With the proposal of foliating spacetime and the addition of these new vectors, the metric of curved spaces $g_{\mu\nu}$ is directly related to them, in the following way

$$q_{ab} = g_{ab}, \tag{3.37}$$

$$N_a = g_{a0}, \tag{3.38}$$

$$N = \frac{1}{\sqrt{-g^{00}}}, \tag{3.39}$$

$$g_{00} = -N^2 + N_a N^a \to N^2 \det\, q = \det\, g. \tag{3.40}$$

*This is a theory independent of the coordinate system. More fundamentally, it is centered on Einstein's Equivalence Principle.

†This is a topological space that can be locally mapped to $\mathbb{R}^n$. Locally, near each point, it resembles Euclidean space. However, globally, it may have a different topology. It can consist of several dimensions and have various shapes and sizes.

‡These are $(n-1)$-dimensional manifolds embedded in an n-dimensional space.

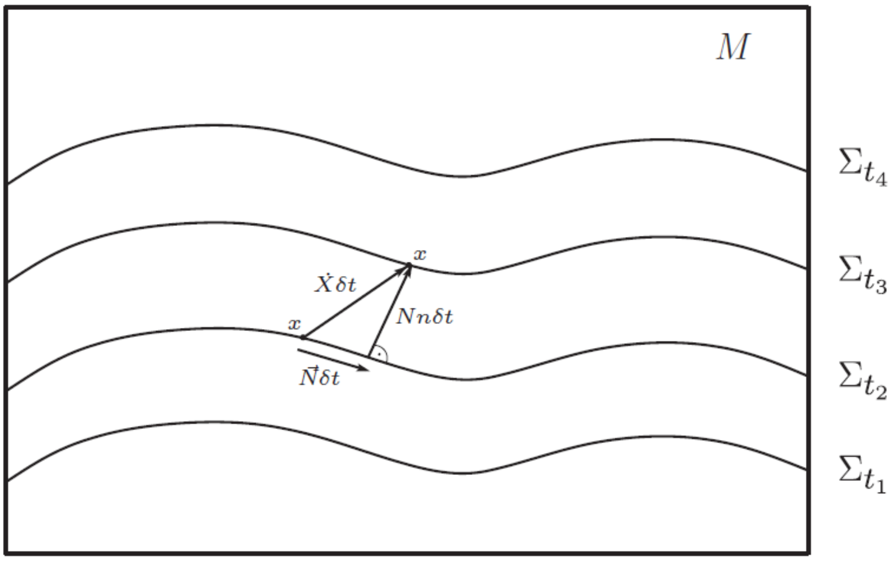

Figure 4 – The foliation of spacetime into hypersurfaces and into Lapse and Shift functions [48].

Here $a, b = 1, 2, 3$. Notice that in the last equation N is defined by the time-time component of $g_{\mu\nu}$ with upper indices. As already said, the N and N_a are called Lapse and Shift functions and q_{ab} is called the three-metric, where it will now be used to raise and lower the indices.

What is the consequence of writing the system in terms of these new functions? Simplicity is the key point. Later, the reader will see this in practice, but using the notation as in Eqs. (3.37), (3.38), (3.39), and (3.40) allows the Einstein-Hilbert Lagrangian to be written in terms of these quantities, and since the time parameter t has become a label for our system, the terms $\dot{N}$ and $\dot{N}_a$ vanish. Consequently, the Lagrangian no longer has temporal dependence, which leads to tremendous simplifications.

Note that if $-N^2 + g_{\mu\nu}N^\mu N^\nu < 0$, the Lapse function would vanish. The line element in this new formalism, using the previous equations, is

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= g_{00}dx^0dx^0 + g_{0a}dx^0dx^a + g_{a0}dx^adx^0 + g_{ab}dx^adx^b \\ &= [-N^2 + N^aN^a]dt^2 + 2N_adtdx^a + q_{ab}dx^adx^b. \end{aligned} \tag{3.41}$$

This quantity can be interpreted as the Lorentzian version of the Pythagorean theorem. Let us find the components of the metric tensor in terms of the new Lapse and Shift functions

$$g_{\mu\nu} = \left(\begin{array}{c|c} A & B_b \\ \hline B_a & q_{ab} \end{array}\right), \qquad g^{\mu\nu} = \frac{1}{N^2}\left(\begin{array}{c|c} -1 & N^b \\ \hline N^a & C^{ab} \end{array}\right).$$

We need to determine what A, B_i, and C^{ab}. The determination of these components is done through the identity $g_{\mu\sigma}g^{\sigma\nu} = \delta^\nu_\mu$

$$g_{a\sigma}g^{\sigma 0} = \frac{1}{N^2}(-B_a + q_{ab}N^b) = 0 \quad \rightarrow \quad B_a = q_{ac}N^c, \tag{3.42}$$

$$g_{0\sigma}g^{\sigma 0} = \frac{1}{N^2}(-A + q_{ac}N^cN^a) = 1 \quad \rightarrow \quad A = q_{ac}N^cN^a - N^2, \tag{3.43}$$

$$g_{a\sigma}g^{\sigma b} = \frac{1}{N^2}q_{ac}(N^cN^b + C^{cb}) = \delta^b_a \quad \rightarrow \quad C^{ab} = N^2q^{ab} - N^aN^b. \tag{3.44}$$

Using the notation $N_a = q_{ab}N^b$, the metric tensor in terms of its covariant and contravariant components are

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_aN^a & N_b \\ N_a & q_{ab} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^a}{N^2} & q^{ab} - \frac{N^aN^b}{N^2}. \end{pmatrix} \tag{3.45}$$

As shown in Eq. (3.40), the determinants of both metrics are related, and this allows us to obtain the relation for the 4-dimensional volume element as

$$\sqrt{-g}\ d^4x = N\sqrt{q}\ d^3xdt, \tag{3.46}$$

and the inverse of the metric is

$$g^{00} = \frac{\det\ q}{\det\ g} = \frac{q}{g} = -\frac{1}{N^2}. \tag{3.47}$$

We can also relate the normal vector n to the ADM formalism. Commonly in the literature, the normal vector is defined as

$$\begin{aligned} n_\mu &= -\frac{\delta^0_\mu}{\sqrt{-g^{00}}}, \\ n^\mu &= -\frac{g^{0\mu}}{\sqrt{-g^{00}}}. \end{aligned} \tag{3.48}$$

Then, using Eq. (3.39), we have

$$\begin{aligned} n_\mu &= (-N, 0, 0, 0), \\ n^\mu &= \left(\frac{1}{N}, -\frac{N^a}{N}\right). \end{aligned} \tag{3.49}$$

Consequently, we have that $n^\mu n_\mu = -1$, that is, a constant. From these equations, we can relate the $3+1$ metric to the metric of curved spaces and the normal vector as

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \iff q^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu, \tag{3.50}$$

or in matrix form

$$q^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu = \left(\begin{array}{c|c} 0 & -\frac{g^{0b}}{g^{00}} \\ \hline 0 & \delta^a_b \end{array}\right). \tag{3.51}$$

We use Greek indices due to the fact that q_{ab} is an object that is embedded in spacetime and can be easily converted to Latin indices when considering a 3-dimensional structure. The foundation of the ADM formalism is composed of two pillars: the metric separated into $3+1$ components with time becoming a label, and the other pillar is a fundamental quantity called the extrinsic curvature. Before we properly examining this quantity, let's briefly discuss some well-known and fundamental quantities in $3+1$ General Relativity. The 3-Christoffel symbol is

$$^{(3)}\Gamma^{\mu}_{\alpha\nu} = \frac{1}{2}q^{\mu\sigma}(\partial_\alpha q_{\nu\sigma} + \partial_\nu q_{\sigma\alpha} - \partial_\sigma q_{\alpha\nu}). \tag{3.52}$$

As already mentioned earlier in this dissertation (3.20), the commutator of the covariant derivatives gives the 3-curvature (Ricci tensor), which is intrinsic and refers to each hypersurface Σ_t, as

$$[D_\mu, D_\nu]V^\alpha = {}^{(3)}R_{\mu\nu}V^\beta. \tag{3.53}$$

Here, we use the test vector V^α present on the hypersurface to show the relationship. The curvature has the same properties as in the usual Einstein gravitational theory, with the Ricci scalar given by

$$^{(3)}R = {}^{(3)}R_{\alpha\beta}q^{\alpha\beta}. \tag{3.54}$$

Unlike in General Relativity, the intrinsic curvature is not sufficient to provide all the geometric information of the space in question, as only one hypersurface will be selected. Therefore, we will define the extrinsic curvature, which will contain the information of all Σ_t, and is given by

$$K_{\mu\nu} = -q^{\alpha}_{\mu}q^{\beta}_{\nu}\nabla_\alpha n_\beta. \tag{3.55}$$

This term is symmetric in μ and ν, and due to its expression, it measures how the normal vector changes point by point. Moreover, $K_{\mu\nu}$ is a purely spatial quantity because $n^\mu K_{\mu\nu} = 0$. This allows us to measure the rate at which the hypersurface deforms as it is carried along the normal. The change of this normal vector leads to the appearance of a type of foliation acceleration (or proper acceleration), a quantity that measures how quickly the curvature of one hypersurface changes to the next. This quantity is defined as a_μ with the expression

$$a_\mu = n^\nu \nabla_\nu n_\mu, \tag{3.56}$$

alternatively, let f be a scalar, we can write $n_\mu = -N\nabla_\mu f$, assuming we are considering a system without torsion. Thus, when $[\nabla_\mu, \nabla_\nu]f = 0$, to recover Eq. (3.49), we have $\nabla_\mu f = -\dfrac{n_\mu}{N}$, so

$$\begin{aligned}
a_\mu &= n^\nu \nabla_\nu n_\mu \\
&= -n^\sigma \nabla_\sigma (N\nabla_\mu t) \\
&= -n^\sigma (\nabla_\sigma N)(\nabla_\mu t) - n^\sigma N \nabla_\sigma \nabla_\mu t \\
&= \frac{1}{N} n^\sigma n_\mu (\nabla_\sigma N) + n^\sigma N \nabla_\mu \left(\frac{n_\mu}{N}\right) \\
&= n_\mu n^\sigma \nabla_\sigma \ln N - n_\sigma \nabla_\mu n^\sigma + n^\sigma N n_\sigma \left(\frac{-1}{N^2}\right) \nabla_\mu N \\
&= n_\mu n^\sigma \nabla_\sigma \ln N - n_\sigma n^\sigma \nabla_\mu \ln N \\
&= (n^\sigma n_\mu + \delta^\sigma_\mu) \nabla_\sigma \ln N = q^\sigma_\mu \nabla_\sigma \ln N = D_\mu \ln N.
\end{aligned} \tag{3.57}$$

Substituting into Eq. (3.55), we obtain

$$\begin{aligned} K_{\mu\nu} &= -q_\mu^\alpha q_\nu^\beta \nabla_\alpha n_\beta \\ &= -(\delta_\mu^\alpha + n_\mu n^\alpha)(\delta_\nu^\beta + n_\nu n^\beta)\nabla_\alpha n_\beta \\ &= -(\delta_\mu^\alpha + n_\mu n^\alpha)(\delta_\nu^\beta)\nabla_\alpha n_\beta \\ &= -\nabla_\mu n_\nu - n_\mu a_\nu. \end{aligned} \tag{3.58}$$

Where we used that $n^\mu \nabla_\nu n_\mu = 1/2\nabla_\nu(n^\mu n_\mu) = 0$. It is still possible to express the extrinsic curvature in another form convenient for future discussions. Let $\mathcal{L}_n$ be the Lie derivative along the flow of the normal vector n of the spatial metric $q_{\mu\nu}$, we have

$$\begin{aligned} \mathcal{L}_n q_{\mu\nu} &= \mathcal{L}_n(g_{\mu\nu} + n_\mu n_\nu) \\ &= n^\sigma \nabla_\sigma q_{\mu\nu} + q_{\mu\sigma}\nabla_\nu n^\sigma + q_{\sigma\nu}\nabla_\mu n^\sigma \\ &= -2K_{\mu\nu} \rightarrow K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n q_{\mu\nu}. \end{aligned} \tag{3.59}$$

As with the intrinsic curvature, the extrinsic curvature can be contracted and it gives rise to a scalar denoted as

$$K \equiv q^{\mu\nu} K_{\mu\nu}. \tag{3.60}$$

The time vector is a constant of the form $t^\mu = (1, 0, 0, 0)$ and can be written in terms of the Lapse and Shift functions as

$$t^\mu = N n^\mu + N^\mu, \tag{3.61}$$

an important contribution comes to light when we consider an adapted coordinate system, that is, what happens to the Lie derivative when we use the time vector t instead of the normal vector

$$\begin{aligned} \mathcal{L}_t q_{\mu\nu} &= t^\sigma \partial_\sigma q_{\mu\nu} + q_{\mu\sigma}\partial_\nu t^\sigma + q_{\sigma\nu}\partial_\mu t^\sigma \\ &= \partial_0 q_{\mu\nu} = \dot{q}_{\mu\nu}. \end{aligned} \tag{3.62}$$

We can do the same exercise for the Shift vector, as follows:

$$\begin{aligned} \mathcal{L}_N q_{\mu\nu} &= N^\sigma \partial_\sigma q_{\mu\nu} + q_{\mu\sigma}\partial_\nu N^\sigma + q_{\sigma\nu}\partial_\mu N^\sigma \\ &= D_\mu N_\nu + D_\nu N_\mu. \end{aligned} \tag{3.63}$$

Note that if we multiply by the metric $q_{\mu\nu}$ and take the Lie derivative of the entire Eq. (3.61), and use Eqs. (3.59), (3.63), and (3.62), we obtain

$$\begin{aligned} &\mathcal{L} q_{\mu\nu}[t^\mu = N n^\mu + N^\mu] \\ &\mathcal{L}_t q_{\mu\nu} = N \mathcal{L}_n q_{\mu\nu} + \mathcal{L}_N q_{\mu\nu} \\ &\dot{q}_{\mu\nu} = -2N K_{\mu\nu} + D_\mu N_\nu + D_\nu N_\mu \\ &K_{\mu\nu} = \frac{1}{2N}\left(D_\mu N_\nu + D_\nu N_\mu - \dot{q}_{\mu\nu}\right), \end{aligned} \tag{3.64}$$

and these four equations above lead to the evolution of the system. There are six relations regarding the Shift and Lapse vectors, and they directly depend on the choice of gauge. Before we reach the final expressions, we will present the reader with the Gauss relation and Codazzi Mainardi identity. These are fundamental for the deduction of the Ricci tensor in the ADM formalism and for the Hamiltonian constraints, which will be shown in the future. More details can be found in Appendix A. The Gauss relation is

$$q^{\mu}_{\alpha} q^{\nu}_{\beta} q^{q}_{\rho} q^{\sigma}_{\delta} {}^{(4)}R^{\rho}_{\sigma\mu\nu} = {}^{(3)}R^{q}_{\delta\alpha\beta} + K^{q}_{\alpha} K_{\delta\beta} - K^{q}_{\beta} K_{\alpha\delta} \,, \tag{3.65}$$

and the Codazzi-Mainardi identity is

$$q^{q}_{\rho} n^{\sigma} q^{\mu}_{\alpha} q^{\nu}_{\beta} {}^{(4)}R^{\rho}_{\sigma\mu\nu} = D_{\beta} K^{q}_{\alpha} - D_{\alpha} K^{q}_{\beta} \,. \tag{3.66}$$

Finally, we can express the intrinsic curvature in terms of the extrinsic curvature and the Lapse and Shift vectors. Similarly to Eq. (3.53), in 4-dimensions, we have

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\alpha} = {}^{(4)}R_{\mu\nu} V^{\beta} \tag{3.67}$$

Thus, it is possible to derive three fundamental equations

$$q_{\rho\alpha} q^{\mu}_{\beta} {}^{(4)}R^{\rho}_{\sigma\mu\nu} n^{\sigma} n^{\nu} = -K_{\alpha\lambda} K^{\lambda}_{\beta} + q^{\mu}_{\alpha} q^{\nu}_{\beta} \nabla_{n} K_{\mu\nu} + \frac{1}{N} D_{\alpha} D_{\beta} N \,, \tag{3.68}$$

$$q^{\alpha}_{\mu} q^{\beta}_{\nu} {}^{(4)}R_{\alpha\beta} = {}^{(3)}R_{\mu\nu} + K K_{\mu\nu} - q^{\alpha}_{\mu} q^{\beta}{}_{\nu} \nabla_{n} K_{\alpha\beta} - \frac{1}{N} D_{\mu} D_{\nu} N \tag{3.69}$$

and

$$ {}^{(4)}R = {}^{(3)}R + K^2 + K^{ab} K_{ab} - 2\nabla_{n} K - \frac{2}{N} D^{a} D_{a} N \,. \tag{3.70}$$

These Eqs. (3.68), (3.69), and (3.70) relate the Ricci tensor or its scalar with the new quantities from the ADM formalism, extrinsic curvature, Lapse and Shift vectors in $3+1$ components of spacetime. Details of these expressions can be found in the Appendix B. From here, we have all the ingredients necessary to write the action of a field or Hamiltonians in terms of this new formalism. Recall that in the Einstein-Hilbert action (3.24), the term R appears, which can now be written as (3.70).

3.3.2 ADM Action

After deducing all the important quantities in the ADM formalism, it is possible to write the Einstein-Hilbert action in this same formalism. The action can be written as in Eq. (3.24), where R is in the form ${}^{(4)}R$, which is given by Eq. (3.70), and the term $\sqrt{-g}$ comes from Eq. (3.46). Thus, we have

$$S_H = \frac{1}{2\kappa} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x N \sqrt{q} \left[{}^{(3)}R + K^2 + K^{ab} K_{ab} - 2\nabla_{n} K - \frac{2}{N} D^{a} D_{a} N \right] . \tag{3.71}$$

Note that the last two terms are divergences in terms of covariant derivatives. Let us work with these terms separately:

$$
\begin{aligned}
\sqrt{q} D_a D^a N &= D_a \left(\sqrt{q}\partial^a N\right) . \\
&= \partial_a \left(\sqrt{q}\partial^a N\right) \\
&= N\sqrt{q}\nabla_n K \\
&= N\sqrt{q} n^\alpha \nabla_\alpha K \\
&= \partial_\alpha(\sqrt{q} N K n^\alpha) + \sqrt{q} N K^2,
\end{aligned} \tag{3.72}
$$

substituting Eq. (3.72) into Eq. (3.71), we obtain

$$
S_H = \frac{1}{2\kappa}\int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x N\sqrt{q}\left({}^{(3)}R - K^2 + K^{ab}K_{ab}\right) . \tag{3.73}
$$

3.3.3 Hamiltonian Formalism in ADM

From Eq. (3.73), we know that the Lagrangian density is given by

$$
\mathcal{L}_H = N\sqrt{q}\left({}^{(3)}R - K^2 + K^{ab}K_{ab}\right) , \tag{3.74}
$$

in turn, $\mathcal{L}_H$ will depend on all these new quantities we have defined in the ADM formalism together with their derivatives, i.e., $\{q, \dot{q}_{ij}, N, \vec{N}, \partial_i N, \partial_i \vec{N}\}$. An important and fundamental observation about these quantities is that both the Lapse function and the Shift function have their time derivatives equal to zero. Later on, we will see that this condition greatly simplifies the equations of motion in the Hamiltonian formalism. Thus, N and $\vec{N}$ become Lagrange multipliers in the theory.

The equation for the conjugate momentum corresponding to the metric q_{ab} is

$$
\pi^{ab} = \frac{\partial \mathcal{L}_H}{\partial q_{ab}}, \tag{3.75}
$$

where π^{ab} is symmetric in the indices. As mentioned earlier, the conjugate momentum of the Lapse and Shift vectors is:

$$
\begin{aligned}
\pi_N &= \frac{\partial \mathcal{L}_H}{\partial \dot{N}} = 0 \, , \\
\pi_{N^i} &= \frac{\partial \mathcal{L}_H}{\partial \dot{N}^i} = 0 \, .
\end{aligned} \tag{3.76}
$$

Just as in classical mechanics, q and $\dot{q}$ are independent variables. Let us now look at how each term in (3.74) behaves when derived with respect to these terms:

- Since ${}^{(3)}R$ depends only on q_{ij}, we have

$$
\frac{\partial^{(3)}R}{\partial \dot{q}_{ij}} = 0. \tag{3.77}
$$

- For the extrinsic curvature, we use Eq. (3.64), so

$$\begin{aligned}\frac{\partial K_{ab}}{\partial \dot{q}_{ij}} &= \frac{\partial}{\partial \dot{q}_{ij}}\left(\frac{1}{2N}\left(D_a N_b + D_b N_a - \dot{q}_{ab}\right)\right)\\ &= -\frac{1}{2N}\delta^i_a\delta^j_b\,. \end{aligned} \tag{3.78}$$

- The scalar of the extrinsic curvature is

$$\begin{aligned}\frac{\partial K^2}{\partial \dot{q}_{ij}} &= -2K\frac{\partial K}{\partial \dot{q}_{ij}}\\ &= -2K\frac{\partial\left(q^{ab}K_{ab}\right)}{\partial \dot{q}_{ij}}\\ &= -2Kq^{ab}\frac{\partial\left(K_{ab}\right)}{\partial \dot{q}_{ij}}\\ &= -2Kq^{ab}\left(-\frac{1}{2N}\delta^i_a\delta^j_b\right). \end{aligned} \tag{3.79}$$

Now, we can combine the equations obtained to find the total expression for the conjugate momentum. Using Eqs. (3.77), (3.78), and (3.79) in (3.75), we obtain

$$\begin{aligned}\pi^{ij} = \frac{\partial \mathcal{L}_H}{\partial q_{ij}} &= \frac{\partial}{\partial \dot{q}_{ij}}\left[N\sqrt{q}\left({}^{(3)}R - K^2 + K^{ab}K_{ab}\right)\right]\\ &= N\sqrt{q}\left[-2Kq^{ab}\left(-\frac{1}{2N}\delta^i_a\delta^j_b\right) + 2K^{ab}\left(-\frac{1}{2N}\delta^i_a\delta^j_b\right)\right]\\ &= \sqrt{q}\left(Kq^{ij} - K^{ij}\right)\,. \end{aligned} \tag{3.80}$$

This is still not the final expression for the conjugate momentum. This is because we want to express this quantity in terms of $\{{}^{(3)}R, q, N, \vec{N}\}$.To do so, we will use Eq. (3.80) and take the trace of π^{ij}:

$$\pi = q_{ij}\pi^{ij} = 2\sqrt{q}K, \tag{3.81}$$

where we used $q^{ij}K_{ij} = K$, as well as $q^{ij}q_{ij} = 3$ and $q^{ij}\pi_{ij} = \pi$. Therefore, it is possible to write the extrinsic curvature in terms of the conjugate momentum:

$$K^{ij} = \frac{1}{2\sqrt{q}}\left(\pi q^{ij} - 2\pi^{ij}\right)\,. \tag{3.82}$$

We can now find the equation of motion for the metric in terms of the conjugate momentum. Again, we will use Eq. (3.64), where we can now substitute K_{ij} with Eq. (3.82) and thus obtain

$$\dot{q}_{ij} = D_i N_j + D_j N_i - \frac{N}{\sqrt{q}}\left(\pi q_{ij} - 2\pi_{ij}\right)\,. \tag{3.83}$$

Finally, we can express Eq. (3.74) in terms of the Ricci scalar, the conjugate momentum, and the Lapse and Shift functions. For this, we simply substitute Eqs. (3.82) and (3.81):

$$\begin{aligned}\mathcal{L}_H &= N\sqrt{q}\left({}^{(3)}R - K^2 + K^{ij}K_{ij}\right)\\ &= N\sqrt{q}{}^{(3)}R - N\sqrt{q}\frac{\pi^2}{4q} + N\sqrt{q}\left(\frac{1}{2\sqrt{q}}\right)^2\left(\pi q^{ij} - 2\pi^{ij}\right)\left(\pi q_{ij} - 2\pi_{ij}\right)\\ &= N\sqrt{q}{}^{(3)}R + \frac{N}{\sqrt{q}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right)\,. \end{aligned} \tag{3.84}$$

With the Lagrangian density calculated in Eq. (3.84), it is possible to obtain the Hamiltonian density via the Legendre transform. Recall that, as mentioned earlier, N and $\vec{N}$ are Lagrange multipliers of the theory, so we have the expression

$$\begin{aligned}\mathcal{H}_H &= \pi_N \dot{N} + \pi_{N^i}\dot{N}^i + \pi^{ij} q_{ij} - \mathcal{L}_H \\ &= \pi^{ij}\left[D_i N_j + D_j N_i - \frac{N}{\sqrt{q}}\left(\pi q_{ij} - 2\pi_{ij}\right)\right] - \left[N\sqrt{q}^{(3)}R + \frac{N}{\sqrt{q}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right)\right] \\ &= 2\pi^{ij} D_i N_j - N\sqrt{q}^{(3)}R + \frac{N}{\sqrt{q}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) .\end{aligned} \tag{3.85}$$

Thus, the Hamiltonian in the ADM formalism, $H_{\text{ADM}} = \int_{\Sigma_t} d^3x \mathcal{H}_H$, can be rewritten in a compact form as

$$H_{\text{ADM}} = \mathbb{H}[N] + \mathbb{D}[N^j] = N\mathbb{H} + N^j \mathbb{D}_j \,, \tag{3.86}$$

where $\mathbb{H}[N]$ is the Hamiltonian constraint:

$$\mathbb{H}[N] \equiv \int_{\Sigma_t} d^3x N \left[-\sqrt{q}^{(3)}R - \frac{1}{\sqrt{q}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right)\right] , \tag{3.87}$$

and $\mathbb{D}[N^i]$ are the 3 diffeomorphism constraints:

$$\mathbb{D}[N^i] \equiv \int_{\Sigma_t} d^3x N^i \left[-2D^j \pi_{ij}\right] . \tag{3.88}$$

These constraints only need to be satisfied on the hypersurfaces, not between them. We will not go into the motivations for these constraints here. Finally, it is possible to obtain the equations of motion in the Hamiltonian formalism. The conjugate momenta are of the form

$$\dot{q}_{ij} = \frac{\delta \mathcal{H}}{\delta \pi^{ij}} \,, \qquad \dot{\pi}^{ij} = -\frac{\delta \mathcal{H}}{\delta q_{ij}} \,. \tag{3.89}$$

In the ADM formalism, the action is

$$\begin{aligned} S_{\text{ADM}} &= \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \left(\pi^{ij}\dot{q}_{ij} - \mathcal{H}\right) \\ &= \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \left[\pi^{ij}\dot{q}_{ij} - \left(2\pi^{ij} D_i N_j - N\sqrt{q}^{(3)}R + \frac{N}{\sqrt{q}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right)\right)\right] .\end{aligned} \tag{3.90}$$

The necessary boundary conditions for the variables $\{N, \vec{N}, q_{ij}\}$ on the hypersurface Σ_t are zero:

$$\delta N|_{\partial\Sigma_t} = \delta N^i|_{\partial\Sigma_t} = \delta q_{ij}|_{\partial\Sigma_t} = 0 \,. \tag{3.91}$$

We will not calculate the equations of motion explicitly in this dissertation, as they are not the main focus of this work. The calculations can be found in the references cited at the beginning of the ADM formalism section. For the variable $\dot{q}_{ij}$, we have

$$\dot{q}_{ij} = \frac{\delta \mathcal{H}}{\delta \pi^{ij}} = D_i N_j + D_j N_i - 2NK_{ij} \,, \tag{3.92}$$

and for conjugate momentum $\dot{\pi}^{ij}$ the motion equation is

$$\dot{\pi}^{ij} = -\frac{\delta \mathcal{H}}{\delta q_{ij}} = - N\sqrt{q}\left(R^{ij} - \frac{1}{2}q^{ij}R\right) + \frac{N}{2\sqrt{q}}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)q^{ij} - \frac{2N}{\sqrt{q}}\left(\pi^{ic}\pi^{j}_{c} - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{q}\left(D^iD^jN - q^{ij}D_cD^cN\right) + D_c\left(\pi^{ij}N^c\right) - \pi^{ic}D_cN^j - \pi^{jc}D_cN^i\,. \tag{3.93}$$

In this chapter, we have developed the necessary mathematical and physical foundations of General Relativity to support the framework used throughout the rest of this dissertation. These tools prepare for the next step of this work. In the upcoming chapter, we will apply the formal machinery developed here to investigate the LQG and the effects of Loop Quantum Gravity in the electromagnetic sector. The techniques introduced in this chapter will serve as the foundational elements for such developments.

Chapter 4

Loop Quantum Gravity

When we try formulate the General Relativity alongside Quantum Mechanics, we see these two frameworks are incompatible in extreme conditions, so it is necessary develop the theory of Quantum Gravity. In the literature, there are several theories that can be classified as approaches to Quantum Gravity, such as String Theory, Loop Quantum Gravity, D-Branes, Asymptotic Safety, and others. In this chapter, we will motivate why a theory of Quantum Gravity is necessary and explore the formalism of one of Loop Quantum Gravity approach.

4.1 Why Quantum Gravity?

As previously mentioned, Quantum Gravity theory is a pathway to explaining intriguing physical phenomena that remain unresolved because the Standard Model of particle physics is unable to account for them, such as:

1. Black Holes. These objects are regions in spacetime where the gravitational field is extremely strong, and light cannot escape. They have been the subject of intense study and there was good evidence of their discovery in 2017. Quantum Gravity is a theory that can describe the microscopic constitution of black holes and, consequently, their entropy. The formulation of a Quantum Gravity theory can also help in understanding other black hole-related problems, such as supermassive black holes in the Universe [56, 57].

2. Singularity Resolution. General Relativity (GR) is a theory where singularities naturally arise. These singularities appear in different contexts, such as in black holes and at the Big Bang. A common approach to solving singularities in other fields of physics is through renormalization; however, gravity is not a renormalizable interaction. Therefore, another possible way to address this problem is through Quantum Gravity. In particular, the strong gravitational fields near singularities suggest that quantum effects may become significant, potentially modifying the classical trajectories.

3. Finite Hilbert Space. In Quantum Field Theory, it is possible to give infinite energy to the fields in the form of quanta. Consequently, the dimension of the Hilbert space is $\dim(\mathcal{H}) = \infty$. However, in regions with gravity, if we add a large number of quanta, or an infinite amount of quanta, a collapse occurs, resulting in the formation of a black hole. Thus, the Hilbert space dimension is not infinite and is proportional to $\dim(\mathcal{H}) = e^{A_{BH}/A_p} \neq \infty$, where A_{BH} is the black hole's event horizon area, and $A_p = 10^{-70}m^2$ is the Planck area.

4. Matter Quantization. The Einstein field equations, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$, can be interpreted as the geometry of spacetime being determined by the energy-momentum tensor. However, one of the pillars of Quantum Field Theory is the quantization of matter, which allows the energy-momentum tensor to be quantized and promoted to the operator $\widehat{T}_{\mu\nu}$ in the Hilbert space. So, can we now combine the fields $g_{\mu\nu}$ and $\widehat{T}_{\mu\nu}$ in the Einstein equations? There are several ways to approach this problem. In this dissertation, we present two of the most well-known approaches. The first is to quantize all terms in the Einstein equation and use a path integral approach, which is adopted in string theory and therefore we're not interested. The second approach is through the formalism of Quantum Field Theory in curved spacetime, using the expectation value $\langle\widehat{T}_{\mu\nu}\rangle$, which can lead to fruitful applications.

4.2 Formalism of Quantum Gravity

We will revisit the idea presented in section 3.3.1, but now, with Hamiltonians. Using the same procedure as in Quantum Mechanics, it is possible to propose the canonical quantization of the system and transform the GR quantities into operators.

4.2.1 Wheeler-DeWitt Equation

The ADM formalism presented in 3.3.1 is based on classical canonical formulation. The transition to a quantum description occurs by promoting the main quantities of the theory to operators. This idea was proposed by Wheeler [58] and DeWitt [59], who, inspired by the Hamilton-Jacobi approach [60] and consequently ADM formalism, obtained the following equation

$$\hat{H}\psi = 0. \tag{4.1}$$

It is possible to obtain the wave function of the Universe, which enables discussions about the properties of spacetime. Note that this is a Schrödinger equation without the time derivative term; this arises because the ADM formalism's foliation of spacetime lacks a temporal term. The absence of this term also leads to the so-called "problem of time", as discussed in [48].

The Wheeler-DeWitt operator, $\hat{H}$, is a constrained, quantized Hamiltonian in General Relativity like we show in 3.3.3. The solutions to this equation are expressed by an orthogonal basis of spin network states on the 3D hypersurface, which can be represented by Figure 6. We will explore the concept of spin networks in more detail later. The Hamiltonian will act only on the nodes of this spin network, so that the loop states in Loop Quantum Gravity (LQG) will be the solution to equation (4.1). Note also that, in transitioning from the ADM formalism to Wheeler-DeWitt, constraints are applied, which introduce restrictions. Hamiltonians of this type are linear combinations of spatial diffeomorphism constraints.

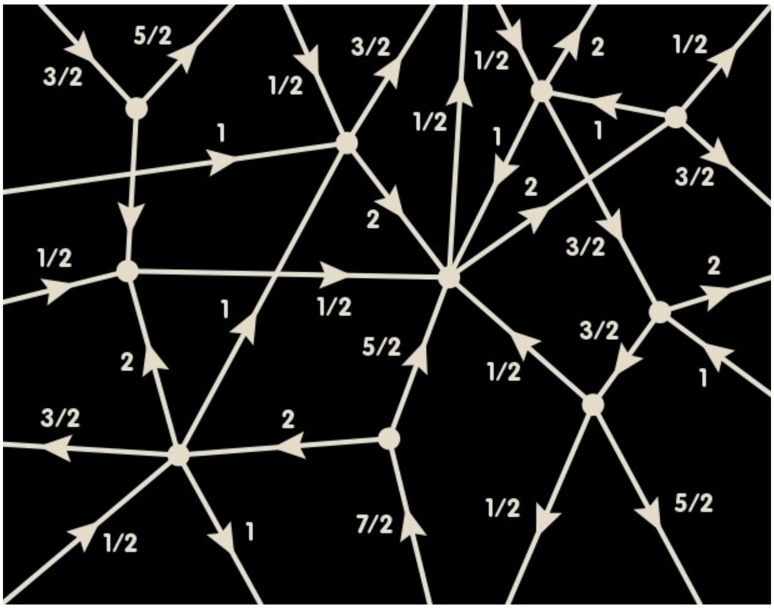

Figure 5 – A representation of a simple spin network. The nodes of this graph are the grains of space-time [61].

4.3 Loop Quantum Gravity

Loop Quantum Gravity or commonly know as LQG, offers a consistent theoretical framework which extends its exploration to include phenomena that contribute to the understanding of what gravity is at a fundamental quantum scale. Possible events that could detect traces of LQG are [62–64], the most notable ones are: noise present in gravitational wave detectors [65, 66], neutral kaon systems [67, 68], and the time-dependent energy of arrival of photons and neutrinos from distant sources [69, 70].

Loop Quantum Gravity is a theory of Quantum Gravity that aims at unifying General Relativity and Quantum Mechanics. The idea of proposing modifications to known physical theories in light of new insights dates back to the early 20th century. In the conclusion of his 1916 paper predicting the existence of gravitational waves, Einstein says: "it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation"*. LQG does not propose a grand unification of physics, unlike other Quantum Gravity theories, like String Theory. Instead, it simply quantizes gravity, assuming that the gravitational interaction exhibits quantum effects at the Planck scale. This leads to

*This is a translation of the article originally published in German, which can be found in [71].

the notion that spacetime is discrete, displaying a granular structure. In addition to quantizing gravity, LQG is a non-perturbative theory and is background-independent. This stems from General Relativity, which is founded on two major principles: diffeomorphism invariance and background independence, both of which are fundamental to LQG [1, 83, 85, 86].

In contrast to its major competitor String Theory, Loop Quantum Gravity operates in 4 dimensions and does not require supersymmetry (although it does not discard it, as seen in [72], [73] and [74]). To introduce the topic in a didactic manner, we will briefly contextualize the main points that form the foundation of LQG. The concept of background independence is a direct consequence of the theory being non-perturbative. Since gravity, or spacetime itself, is quantized, there are no background fields, and thus perturbation is not possible. For example, in quantum field theory, we have an electromagnetic field embedded in the geometry of spacetime, interacting with charges. Due to the complexity of describing this phenomenon, a Taylor series expansion is performed, i.e., a perturbation. This expansion generates Feynman diagrams in the form of loops. LQG, however, is a theory that does not depend on an external spacetime geometry. The background independence proposed by LQG means it cannot use the conventional methods of quantum field theory, which rely on background-dependent fields. Background independence manifests as diffeomorphism invariance of the action, meaning the action is invariant under coordinate transformations and there is no dynamical background field. Therefore, a different approach is used by employing the Hilbert space of states, operators, and amplitude transformations.

The use of the Hilbert space in this alternative approach leads to a modification of the algebra of fields into an algebra of parallel transport matrices along closed curves, what we call holonomies or Wilson loops. The concept of these loops is essentially that of phase factors in both Abelian and non-Abelian gauge theories. These loops can be observed in the Aharonov-Bohm effect [75] and are central to the formulation of gauge theories such as quantum chromodynamics. They are also extremely useful in solving matrix models. More detailed information will be provided later in the text, but an excellent reference on the subject is [76].

Loops are essential to the theory (as emphasized by its name), because when quantized, they become operators that create loop states. More precisely, loops serve as the quantized coordinates of the theory, and when these states undergo infinitesimal transformations, they become an equivalent representation of the same state. However, finite transformations alter the state, and this is due to one of LQG's most important considerations: the position of a loop relative to other loops matters. Since it is a theory based on Quantum Mechanics, Hamiltonians play a crucial role in constructing the type of problem addressed by LQG.

Let us define the quantum operator in the spin network basis, where the volume V of a physical region is given by

$$V = \int d^3x |\det\, e(x)|, \tag{4.2}$$

where $e(x)$ is the gravitational field in the tetrad formalism 3.2. The spectrum of V is discrete.

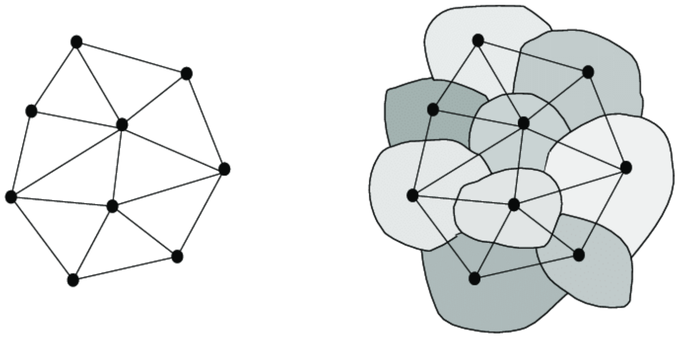

Figure 6 – Illustration of a quanta volume, which represents an abstract spinfoam. Note that the pieces are adjacent and connected by nodes [44].

The quanta of spacetime are granular, and it is necessary to understand which grains are adjacent to one another. Adjacency forms the basis of spatial relations: if two regions of spacetime are adjacent, then they touch each other and are separated by the surface S. Let A be the area of this surface. In this way, the grains of spacetime are separated by quanta of area. The eigenvalues of the area operator are given by

$$A = 8\pi\gamma\hbar G \sum_i \sqrt{j_i(j_i+1)}, \tag{4.3}$$

where j_i are labels on the spin network, determined by the local gauge group $SU(2)$, and γ is the Barbero-Immirzi parameter, that can arise from calculating the entropy of the black hole.

There are other equivalent ways to describe Quantum Mechanics. Naturally, one might think of describing LQG using these new equivalent ideas, for example, employing path integrals instead of Hamiltonians. This idea is relevant and does exist, being described by spin foams. A spin foam is the characterization of the surface of some quantity, such as the Universe, described by spin networks, which in turn evolve over time, forming a discrete geometry. Within LQG, there are strong research groups focused on this topic, with the main result being the recovery of the graviton propagator, i.e., Newton's law in the classical limit is rediscovered. More detailed information can be found in [77, 78].

The canonical formulation is described by considering spacetime as a manifold M whose topology is $\Sigma \times \mathbb{R}$, where Σ is a three-dimensional Riemannian manifold. The so-called Ashtekar variables are the formalism used for a non-compact gauge group, $SL(2, \mathbb{C})$. Due to its importance and the connection that can be made with the tetrad formalism in 3.2 and ADM in 3.3.1, we will describe this idea in more detail below.

4.3.1 Ashtekar Variables

In 1986, Abhay Ashtekar proposed a new formulation of General Relativity in his paper [79]. He expressed the theory in a set of variables that describes General Relativity using canonical quantization, inspired by gauge theories as already said of non-compact gauge group, $SL(2, \mathbb{C})$. In this formulation, the indices a, b, c stand for the space coordinates and i, j, k

are their indices assigned to the generators of $su(2)$. Due to the definition of the tetrads and subsequently the ADM formalism, it is possible to define triads where time is a constant surface of the form

$$q_{ab}(x) = e^i_a(x)e^j_b(x)\delta_{ij}, \tag{4.4}$$

these new variables, the spatial metric e^i_a can be written in terms of the so-called D-Bein, the dimension is linked to the dimension of the hypersurface. The introduction of these variables adds a local $SO(D)$ invariance where a fixed, constant time exists on the surface. We can define the triadic version of the extrinsic curvature by

$$K^a_i e^i_b = K_{ab}. \tag{4.5}$$

Let us consider that the canonically conjugate pair of the form $\{K^a_i, e^j_b\} \simeq \delta^j_i \delta^a_b$ leads to an important constraint

$$G_c = \epsilon_{cab} K^a_i e^b_i = 0, \tag{4.6}$$

the Eq. (4.6) allows us to interpret that the Poisson brackets lead precisely to the $SO(3)$ rotation. The connection A^i_a is related both to the spin connection $\Gamma^i_a = \Gamma_{ajk}\epsilon^{jki}$ and to the extrinsic curvature, given by:

$$A^i_a = \Gamma^i_a[e] + \beta K^i_a, \tag{4.7}$$

where $\Gamma^i_a[e]$ represents the torsion-free spin connection of the triad (it is the only possible solution for the Cartan equation in 3 dimensions) and β is an arbitrary parameter, which can be either complex (i) or real. In the case of a real value, there is no restriction for it to be γ, the Barbero-Immirzi parameter. We can define the covariant derivative D_a with respect to the connection A^i_k and also define the curvature F^i_{ab}, respectively, as:

$$D_a v_i = \partial_a v_i + \epsilon_{ijk} A^j_a v^k, \tag{4.8}$$

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b. \tag{4.9}$$

And note that the constraint present in Eq. (4.6) is the same as in Yang-Mills theory

$$G^i = D_a E^a_i. \tag{4.10}$$

The other variable that is part of the pair of Ashtekar variables is its electric field, given by

$$E^a_i(x) = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c \tag{4.11}$$

and it is known as the densitized triad, $\tilde{E}^a_i = \sqrt{q} E^a_i$. This pair of variables satisfy the following Poisson bracket algebra:

$$\{A^i_a(x), A^j_b(y)\} = 0, \quad \{E^a_i(x), E^b_j(y)\} = 0, \quad \{A^i_a(x), E^b_j(y)\} = 8\pi G\beta\delta^b_a\delta^i_j\delta^3(x-y). \tag{4.12}$$

In the last Poisson bracket, if $\beta = i$, we obtain what is called the complex Ashtekar connection or more commonly known in the literature simply as the Ashtekar connection. This leads to quite simple constraints for vectors and scalars of the form

$$C = \epsilon_{ijk} F^i_{ab} E^{aj} E^{bk}, \tag{4.13}$$

$$C_a = F^i_{ab} E^{bi}. \tag{4.14}$$

We can define the volume interns of the Ashtekar's Variables as

$$V = \int d^3x \sqrt{|\det\, E(x)|}, \tag{4.15}$$

a commutation between the volume and the connection for $\beta = i$ is

$$\{V, A^i_a(x)\} = (8\pi i G) \frac{E^b_j(x) E^c_k(x) \epsilon^{ijk}_{abc}}{4\sqrt{|\det\, E(x)|}}, \tag{4.16}$$

so, the equation related to the Hamiltonian constraints is

$$H(N) = \int d^3x N \frac{E^{ai} E^{bj} F^k_{ab} \epsilon_{ijk}}{\sqrt{q}}. \tag{4.17}$$

The fundamental idea we want to convey here is that, in Loop Quantum Gravity (LQG), the integral form of the Hamiltonian is given by $H(N) = \int_\sigma d^3x\, NH$. Note that the constraints mentioned here are implicitly constant over time. This leads to an important property, well-known in theories that employ Poisson brackets, which is that the Poisson bracket of the total Hamiltonian of the system with any quantity is zero. The equation related to the Hamiltonian constraint, described in (4.17), can still be rewritten in another form. This new form is clearer, easier to use, and well-defined with respect to the Wheeler-DeWitt equation (4.1). Let us set the constants in equation (4.16) to be equal to 1; substituting into equation (4.17), we obtain a new, quantum-restricted Hamiltonian equation in the form

$$H(N) = \int d^3x N \{V, A^k_c(x)\} F^k_{ab} \epsilon^{abc}. \tag{4.18}$$

4.3.2 Holonomies and Wilson Loops

Holonomies are important because we will consider one in the form $\tilde{W}(\tau)$, which represents parallel transport along the closed curve τ. Furthermore, the holonomy for any closed curve implies that the connection at any point is a gauge transformation.

Consider two closed curves. They are said to be equivalent if one can be continuously deformed into the other in such a way that the loops of one correspond to those of the other. This notion of equivalence defines a class of curves, and the set of all such equivalence classes forms a group, known as the loop group.

Holonomies can then be understood as a map from this loop group to a Lie group G, encoding how the gauge connection transforms along the loops. Functions defined on the elements of the loop group are called wave functions, and they constitute what is known as the loop representation.

Consider closed curves $l, m, \ldots$ such that they start and end at the same point, denoted by o. Let L_o denote the complete set of all such closed curves; this parameter L_o forms a semigroup under the composition law $(l, m) \to l \circ m$. The identity element, that is, the null curve, is defined as $I(s) = o$ for any s and any parametrization. The reason for this being a semigroup structure is that the inverse curve l^{-1} is not a group inverse, since $l \circ l^{-1} \neq I$. Parallel transport along a closed curve $l \in L_o$ is given by:

$$\tilde{W}_A(l) = P \exp\left(\int_l A_a(y) dy^a\right), \tag{4.19}$$

where A_a is the connection and $P(M, G)$ is the principal bundle[†] of the group G over M, which defines the holonomy map. Now, it is important to examine some properties of curves and holonomies. Let us choose a point $\hat{o}$ over o. If we use the connection A along the curve l that belongs to M, we obtain the curve $\hat{l}$ in P, such that the initial point is:

$$\hat{l}(0) = \hat{o}, \tag{4.20}$$

and the final point

$$\hat{l}(1) = \hat{l}(0)\tilde{W}_A(l). \tag{4.21}$$

The holonomy $\tilde{W}_A$ is an element of the group G, such that its main property is:

$$\tilde{W}_A(l \circ m) = \tilde{W}_A(l)\tilde{W}_A(m), \tag{4.22}$$

and if we choose to change the point o to $\hat{o}$ in the bundle, we will have $\hat{o}' = \hat{o}g$, leading to the transformation:

$$\tilde{W}'_A(l) = g^{-1}\tilde{W}_A(l)g. \tag{4.23}$$

Now, we want to transform L_o into a group. To make this possible, we introduce an equivalence relation that identifies all closed curves yielding the same holonomy for a smooth connection. This is essential because curves with the same holonomy carry the same physical information, which is important for building a theory. These equivalence classes are properly the loops. Further definitions regarding these quantities can be found in [115].

Since we have constructed a group, the loops (which we will represent with Greek letters) satisfy certain relations that have been previously shown. The inverse is defined as a loop in the opposite direction, τ^{-1}, such that $\tau \circ \tau^{-1} = I$, where I is the set of null curves. Furthermore, we still have that:

$$\tilde{W}(\tau_1 \circ \tau_2) = \tilde{W}(\tau_1)\tilde{W}(\tau_2), \tag{4.24}$$

$$\tilde{W}(\tau^{-1}) = (\tilde{W}(\tau))^{-1}. \tag{4.25}$$

[†]A bundle is a topological term where a space is locally similar to a certain product space but may have a different topological structure globally.

4.3.2.1 Wilson Loops

Gauge theories form the basis of a vast number of physical theories. In LQG, it is no different, as mentioned earlier; this is also a gauge theory. This implies that observable quantities must be gauge-invariant. Quantum mechanically, these quantities are wave functions, which must also be gauge-invariant. Now, let us introduce objects that involve the connection A_a and can be written as gauge invariants. What are these objects? They are the so-called Wilson Loops, which are constructed from the traces of holonomy

$$W_A(\tau) = \mathrm{Tr}\left[P\exp\left(i\oint_\tau dy^a A_a\right)\right]. \tag{4.26}$$

These objects are observables in the canonical sense, and their Poisson brackets vanish when the constraints of the theory are imposed. As stated in [115], Wilson loops have two fundamental properties that should be noted, which are:

1. Mandelstam Identity;

   The first to introduce this identity was Mandelstam for the group $O(3)$ [116]. Subsequently, Giles extended it to the groups $GL(N)$ [117], and later, Gambini and Trias extended it to special and unitary groups $SU(D)$ [118].

   The idea starts from considering gauge groups that have fundamental representations in terms of $N \times N$ matrices, for groups such as $GL(N), SL(N), U(N), SU(N)$. Mandelstam identified two identities for the traces of $N \times N$ matrices. The first is due to the cyclic property of traces, yielding an identity that holds for any gauge group in any dimension of the form:

   $$W(\tau_1 \circ \tau_2) = W(\tau_2 \circ \tau_1). \tag{4.27}$$

   Here, Wilson loops extend to the connection A in a general form, as these results do not depend on a particular choice of connections. The second type of identity that Mandelstam observed is that Wilson loops are the traces of $N \times N$ matrices.

   An important development in this area is the recognition by Rovelli and Smolin that spin networks can be used to characterize a complete set of independent products of Wilson loops [120].

2. Reconstruction Properties.

   As previously mentioned, the reconstruction property is one of the fundamental properties of Wilson loops. But what is it exactly? Furthermore, is it possible to reconstruct the holonomy from a given function (since, as we know, the information present in the holonomy can be reconstructed as a Wilson loop)?

   The proof that this can indeed be achieved, that is, that given a loop function satisfying the Mandelstam constraints, one can reconstruct the gauge-invariant information encoded within it, is the subject of the so-called "reconstruction theorems."

Starting from a function $W(\tau)$ that satisfies equation (4.27), it is possible to write a set of $N \times N$ matrices whose trace is precisely $W(\tau)$. Thus, this function can reconstruct the holonomy.

Another important step is how the wave functions are defined. They can be defined in terms of the loop bases as

$$\psi(\tau) = \int dA\, W_A^*(\tau)\psi[A]. \tag{4.28}$$

The holonomy constructed from Wilson loops is a representation of the loop group, and the traces of this representation satisfy the Mandelstam identities. Any gauge-invariant function can be expressed as a combination of products of Wilson loops.

4.4 Applications of LQG

As said before, Loop Quantum Gravity is a theory that aims to quantizing gravity from General Relativity and not proposed a grand unification. This theory has many applications; here we mention three of them:

1. Cosmology;
2. Black holes;
3. Electrodynamics.

We revisit these applications with a didactic purpose. Furthermore, the effects of Loop Quantum Gravity (LQG) in electrodynamics constitute a central topic of our study. For this reason, we devote more attention to the introduction of this theme.

4.4.1 Cosmology - Loop Quantum Cosmology

One of the applications of Loop Quantum Gravity is Loop Quantum Cosmology (LQC). The theory is obtained from symmetry-reduction of LQG, which enables one to obtain an approach which describes cosmologicals models [83, 97–99]. In this model the energy scales is high; consequently the Einstein equations do not describe space-time well, because the Universe has the quantum effects and the geometry of this space to undergo a kind of bounce. The bounce has some interesting properties like: solve the singularity problem of the Big Bang, when we analyze inflation and using an inflationary scalar field ϕ, called inflaton, the kinetic energies dominate the bounce.

The LQC split the evolution of the Universe in different phases, in each phase we have an independent comportment of the scalar field in function of the choice for the inflationary potential. We have many of these potentials and we can cite: Power-law monomial potentials

$$V = \frac{V_0}{2n}\left(\frac{\phi}{m_{Pl}}\right)^{2n}, \tag{4.29}$$

where n is some power, $m_{Pl} = 1/\sqrt{G} = 1,22 \times 10^{19}$ GeV is the Planck mass, with G is the Newton's gravitational constant. The Higgs-like symmetry breaking potential

$$V = \frac{V_0}{4\, m_{Pl}^4}(\phi^2 - v^2)^2, \tag{4.30}$$

where v denotes the vacuum expectation value (VEV) of the field. The modifications that LQC

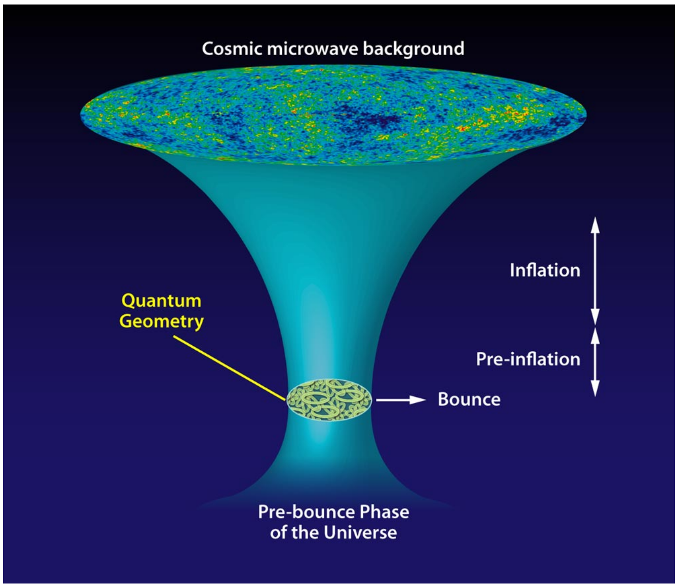

Figure 7 – This picture shows how the evolution of the Universe can be splited in differents phases. In the figure we have the pre bounce phase, bounce phase with quantum effects, pre-inflation and inflation phase for the scalar field.

introduces to the spacetime conjecture lead to changes in the cosmological equations through General Relativity. It is convenient to define the spatial geometry by the variable ν, which is proportional to the volume of a fixed cubic cell [87], instead of the scale factor $a(t)$. In this way, we have:

$$\nu = -\frac{V_0\, a^3\, m_{Pl}^2}{2\pi\gamma}, \tag{4.31}$$

the parameter V_0 is the comoving volume of fiducial cell. Moreover, γ is known the Barbero-Immirzi parameter, with constant value $\gamma \approx 0.2375$ [95] motivated of the black hole entropy. However, nowadays the validity of this constant value is commonly discussed in the literature. This happens because the calculation was based on Bekenstein-Hawking entropy, and part of the literature disagrees with how it was done. As a result, many papers treat the γ parameter as a free parameter of the theory. We dedicate the Appendix C to showing how this parameter is obtained within LQG and some implications regarding such a deduction. The Friedmann equation with LQC corrections is

$$\frac{1}{9}\left(\frac{\dot{\nu}}{\nu}\right)^2 \equiv H^2 = \frac{8\pi}{3m_{Pl}^2}\rho\left(1 - \frac{\rho}{\rho_{cr}}\right), \tag{4.32}$$

where ρ is energy density and the critical energy density is

$$\rho_{cr} = \frac{\sqrt{3}m_{Pl}^4}{32\pi^2\gamma^3}. \tag{4.33}$$

We can see the explicit modifications to the main equations of "traditional" cosmology, showing the quantum geometric effects that come from LQC. The main phenomenon that emerges is the replacement of the Big Bang singularity with a bounce. The bounce occurs when the densities are equal, i.e, $\rho = \rho_{cr}$. If $\rho \ll \rho_{cr}$, the quantum geometric effects vanish, and we recover the GR limit as expected. It is possible to work with multiple scalar fields, but in this work, we only consider a single scalar field with potential $V(\phi)$. In the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the equation of motion for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \tag{4.34}$$

where $V_{,\phi} = dV(\phi)/d\phi$. These equations showed before are the main equations of LQC, and it is possible analyses the dynamics of inflationary field. The reviews can be see in the Ref. [78,88,89].

4.4.2 Black Hole

Continuing with the applications of LQG, we dedicate this section to the exposition of black holes in the context of LQG. We will not present the subject in a detailed or complete manner, nor aim to provide an overview; the purpose is to briefly show how the study of black holes is modified by the introduction of effects generated by LQG.

The physics of black holes is currently a broad testing ground for Quantum Gravity theories, as scientists have made advances in observing gravitational waves from binary black holes and the shadow of supermassive black holes [90, 91].

An important contribution that LQG offers to the study of black hole physics is that it allows the resolution of the singularity problem for spherically symmetric black holes. It was found that, by considering black holes in LQG, there is no singularity at the center of the black hole. Furthermore, the black hole's event horizon is preserved. This solution is valid both for the full case of Loop Quantum Gravity theory, with the analytical form of the black hole metric under spherical symmetry conditions being obtained [92], and for the semi-classical case [93, 94], the metric has the general form

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + h(r)(d\theta^2 + sin^2\theta d\phi^2), \tag{4.35}$$

where the metric functions $f(r), g(r)$ and $h(r)$ are

$$f(r) = \frac{(r-r_+)(r-r_-)(r+r_*)^4}{r^4+a_0^2}, \tag{4.36}$$

$$g(r) = \frac{(r-r_+)(r-r_-)r^4}{(r+r_*)^2(r^4+a_0^2)}, \tag{4.37}$$

$$h(r) = r^2 + \frac{a_0^2}{r^2}, \tag{4.38}$$

$$(d\theta^2 + sin^2\theta d\phi^2) = d\Omega, \tag{4.39}$$

Here, the two horizon parameters are; $r_+ = 2M/(1+P)^2$ and $r_- = 2MP^2/(1+P)^2$. Another variable is $r_* = \sqrt{r_+ r_-} = 2MP/(1+P)^2$, where $M = (6.5 \pm 0.7) \times 10^9 M_\odot$ is the ADM mass and we can define the polymeric function[‡]

$$P = \frac{\sqrt{1+\epsilon^2} - 1}{\sqrt{1+\epsilon^2} + 1} \tag{4.40}$$

where $\epsilon = \gamma\delta \ll 1$, i.e the product of Barbero-Immirzi parameter and polymeric parameter. Finally, the $a_0 = A_{min}/8\pi$ is the minimum area gap of LQG and we reduces to Schwarzschild black hole solution when $a_0 = P = 0$. Now, from the LQG-corrected Schwarzschild metric we can work with many possibilities about Loop Quantum Black Hole. We won't go into this analysis here, but we can point to the following works:

- Overview about this Subject [132, 133]
- Charged Loop Quantum Black Hole [134]
- Rotation Loop Quantum Black Hole [135, 136]

In the next section, we turn our attention to a central topic in this work: the implications of LQG in the electromagnetic sector. We will present the motivations for this application and develop the complete mathematical framework necessary to describe the resulting quantum-corrected electrodynamics.

In the next section, our focus will be on the effects of Loop Quantum Gravity in the electromagnetic sector—an application that is more thoroughly developed in this work. We will derive the electrodynamics equations within the framework of the theory, as well as carry out the process of non-Abelianization.

[‡]The polymeric function in effective loop-quantum black-hole models is a bounded "replacement" for classical phase-space variables, and parameterizing holonomy corrections to regularize singularities [129–131].

Chapter 5

Loop Quantum Gravity Effects in the Electromagnetic Sector

Loop quantum gravity provides electromagnetic effects, valuable insights into the nature of gravity at the fundamental quantum scale, particularly through the study of time-dependent energy variations in the arrival of photons and neutrinos from distant sources [69, 70]. At the Planck scale, it is expected that (local) Lorentz invariance—a foundational principle of General Relativity could be violated. However, achieving the required energy scales to test both Lorentz invariance violation (LIV) and Planck-scale effects in terrestrial experiments is currently a big challenge [62, 70, 100–102]. An alternative approach is to examine the energy of Gamma-Ray Bursts (GRBs) and their immense energy emissions, which may allow investigation of the energy-dependent time of arrival of photons or neutrinos from distant sources. In this work, we focus on the photon sector. The speed of light in vacuum, as characterized by a granular spacetime, may take the form

$$v(E) \simeq c\left(1 - \frac{E}{E^{\gamma}_{LIV}}\right), \tag{5.1}$$

where $E^{\gamma}_{LIV} \approx 3.6 \times 10^{17}$ GeV represents the characteristic energy scale related to possible LIV effects in the photon sector and is independent of the photon's helicity. The high-energy scale of the E^{γ}_{LIV}-parameter supports the notion that GRBs are the most suitable candidates for investigating this parameter, more information can be seen at [103–110, 112–114]. The Hamiltonian formulation of Loop Quantum Gravity (LQG) effects on electromagnetic theory was explored in the work of Ref. [69].

$$H_{LQG} = \frac{1}{Q^2}\int d^3x \Bigg\{ \left[1+\theta_7\left(\frac{l_P}{\mathcal{L}}\right)^{2+2\Upsilon}\right]\frac{1}{2}(\underline{\vec{B}}^2+\underline{\vec{E}}^2)+\theta_3 l_P^2(\underline{B}^a\nabla^2\underline{B}_a+\underline{E}^a\nabla^2\underline{E}_a) \tag{5.2}$$
$$+\theta_2 l_P^2\underline{E}^a\partial_a\partial_b\underline{E}^b+\theta_8 l_P[\underline{\mathbf{B}}\cdot(\nabla\times\underline{\mathbf{B}})+\underline{\vec{E}}\cdot(\nabla\times\underline{\vec{E}})]+\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}(\underline{\vec{B}}^2)^2+\ldots\Bigg\},$$

where Q^2 is the coupling constant of electromagnetism, $l_p \approx 1.6\times10^{-35}m$ is the Planck length. The characteristic length, $\mathcal{L}$, satisfies the condition $l_P \ll \mathcal{L} \leq \lambda$. The parameter λ is the de Broglie wavelength, and the characteristic length, $\mathcal{L}$, has a maximum value at momentum k whenever $\mathcal{L}=k^{-1}$. This condition allows us to interpret such a Hamiltonian and theory as effective, due to its wide energy range. Another parameter appearing in the equation is Υ, which represents the order of the contribution of the gravitational connection to the expected value and may be determined through the phenomenological analysis of a specific event. This parameter may depend on the helicity of the particle under consideration [111, 121, 122]. The θ_i's are non-dimensional parameters of order one or are extremely close to zero [69, 123]; finally, a, b are spatial tensor indices. Hereafter, we replace the spatial indices using vector notation and omit the underline referring to canonical pairs in the electromagnetic sector for all quantities. From Eq. (5.2), we derive the field equations as follows:

$$\nabla\cdot\mathbf{E}=0, \tag{5.3}$$

$$A_\gamma(\nabla\times\mathbf{B})-\frac{\partial\mathbf{E}}{\partial t}+2l_P^2\theta_3\nabla^2(\nabla\times\mathbf{B})-2\theta_8 l_P\nabla^2\mathbf{B}+4\theta_4\mathcal{L}^2\left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon_\gamma}l_P^2\nabla\times(B^2\cdot\mathbf{B})=0, \tag{5.4}$$

$$\nabla\cdot\mathbf{B}=0, \tag{5.5}$$

$$A_\gamma(\nabla\times\mathbf{E})+\frac{\partial\mathbf{B}}{\partial t}+2l_P^2\theta_3\nabla^2(\nabla\times\mathbf{E})-2\theta_8 l_P\nabla^2\mathbf{E}=0, \tag{5.6}$$

with

$$A_\gamma = 1+\theta_7\left(\frac{\ell_p}{\mathcal{L}}\right)^{2+2\Upsilon}. \tag{5.7}$$

The equation (5.2) is an infinite expression. In the literature, it is common to truncate it at the l_p^2 order. When the field equations are computed, this non-linear term contributes to equation (5.4). The key quantities of electromagnetic theory have already been calculated and analyzed previously, as seen in [69, 112]. In this work, such quantities are not the primary focus; instead, the modified Maxwell equations in the context of Loop Quantum Gravity (LQG), along with the corresponding Hamiltonian, are the central equations for the development of this contribution.

5.1 Main quantities of LQG effects in the Electromagnetic Sector

In this dissertation, we keep the non linear term from the Eq.(5.4) to understand how a non linear term for magnetic field change the main quantities for a classical theory like electromagnetism. We assume the rotation angles contribute with a single mode in the equations, i.e

$$\bar{\theta}_i \cdot \bar{\theta}_j = \begin{cases} 0, \text{ if } i \neq j \\ \bar{\theta}_i^2, \text{ if } i = j. \end{cases} \tag{5.8}$$

For simplicity, we consider

$$\bar{\theta}_3 = 2l_p^2\theta_3,\ \bar{\theta}_8 = 2l_p\theta_8,\ \bar{\theta}_4 = 4\theta_4\mathcal{L}^2\left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon_\gamma} l_P^2. \tag{5.9}$$

The first quantity calculated is the Poynting vector. Let's multiply Eq. (5.4) by **E** and Eq. (5.6) by **B**, and then subtract one from the other until

$$A_\gamma[\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \mathbf{B}\cdot(\nabla\times\mathbf{E})] - \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} - \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} + \bar{\theta}_3\mathbf{E}\cdot[\nabla^2(\nabla\times\mathbf{B})] - \bar{\theta}_3\mathbf{B}\cdot[\nabla^2(\nabla\times\mathbf{E})] -$$
$$\bar{\theta}_8[\mathbf{E}\cdot(\nabla^2\mathbf{B})] + \bar{\theta}_8[\mathbf{B}\cdot(\nabla^2\mathbf{E})] + \bar{\theta}_4\mathbf{E}\cdot[\nabla\times(B^2\cdot\mathbf{B})] = 0$$
$$\Rightarrow A_\gamma\nabla\cdot(\mathbf{E}\times\mathbf{B}) + \frac{\partial}{\partial t}\left(\frac{E^2}{2}+\frac{B^2}{2}\right) = \bar{\theta}_3[\mathbf{E}\cdot\nabla^2(\nabla\times\mathbf{B}) - \mathbf{B}\cdot\nabla^2(\nabla\times\mathbf{E})] - \bar{\theta}_8[\mathbf{E}\cdot(\nabla^2\mathbf{B}) -$$
$$\mathbf{B}\cdot(\nabla^2\mathbf{E})] + \bar{\theta}_4\mathbf{E}\cdot[\nabla\times(B^2\cdot\mathbf{B})] \tag{5.10}$$

From Eq. (5.10), we can already identify the classical terms of the Poynting vector expression and the electromagnetic energy density, respectively. They are

$$\nabla\cdot\mathbf{S}_{Maxwell} = A_\gamma\nabla\cdot(\mathbf{E}\times\mathbf{B}), \tag{5.11}$$

and

$$\frac{\partial}{\partial t}u_{Maxwell} = \frac{\partial}{\partial t}\left(\frac{E^2}{2}+\frac{B^2}{2}\right). \tag{5.12}$$

However, there are new terms on the right-hand side of Eq. (5.10). Just like in the two equations above, we need to generate either divergences if we are talking about spatial contributions, or total time derivatives for temporal contributions. Let's look at each term separately

$$\begin{aligned}\bar{\theta}_3\mathbf{E}\cdot[\nabla^2(\nabla\times\mathbf{B})] - \bar{\theta}_3\mathbf{B}\cdot[\nabla^2(\nabla\times\mathbf{E})] &= \bar{\theta}_3[E_i\partial_j\partial_j(\nabla\times\mathbf{B})_i - B_i\partial_j\partial_j(\nabla\times\mathbf{E})_i] \\ &= \bar{\theta}_3\{\partial_j[E_i\partial_j(\nabla\times\mathbf{B})_i] - [\partial_j E_i]\cdot[\partial_j(\nabla\times\mathbf{B})_i] \\ &\quad - \partial_j[B_i\partial_j(\nabla\times\mathbf{E})_i] + [\partial_j B_i]\cdot[\partial_j(\nabla\times\mathbf{E})_i]\} \\ &= \bar{\theta}_3\partial_j\{[E_i\partial_j(\nabla\times\mathbf{B})_i] + [(\partial_k\partial_j E_i)\cdot\epsilon_{ikl}B_l] \\ &\quad - [B_i\partial_j(\nabla\times\mathbf{E})_i] - [(\partial_k\partial_j B_i)\cdot\epsilon_{ikl}E_l]\},\end{aligned} \tag{5.13}$$

or we can rewrite it as

$$\nabla\cdot\mathbf{S}_1 = \nabla\cdot\{\bar{\theta}_3[E_i\partial_j(\nabla\times\mathbf{B})_i] + [(\partial_k\partial_j E_i)\cdot\epsilon_{ikl}B_l] - [B_i\partial_j(\nabla\times\mathbf{E})_i] - [(\partial_k\partial_j B_i)\cdot\epsilon_{ikl}E_l]\}. \tag{5.14}$$

As for the other term

$$\begin{aligned}-\bar{\theta}_8[\mathbf{E}\cdot(\nabla^2\mathbf{B}) - \mathbf{B}\cdot(\nabla^2\mathbf{E})] &= -\bar{\theta}_8[E_i(\partial_j\partial_j B_i) - B_i(\partial_j\partial_j E_i)] \\ &= -\bar{\theta}_8[\partial_j(E_i\partial_j B_i) - (\partial_j E_i)(\partial_j B_i) - \partial_j(B_i\partial_j E_i) + (\partial_j B_i)(\partial_j E_i)] \\ &\quad - \bar{\theta}_8\partial_j[(E_i\partial_j B_i) - (B_i\partial_j E_i)],\end{aligned} \tag{5.15}$$

and we can rewrite it as

$$\nabla \cdot \mathbf{S}_2 = \nabla \cdot [\bar{\theta}_8(E_i\partial_j B_i - B_i\partial_j E_i)]. \tag{5.16}$$

Finally, we have

$$\begin{aligned}
\bar{\theta}_4\mathbf{E}\cdot[\nabla\times(B^2\cdot\mathbf{B})] &= \bar{\theta}_4 E_i\partial_j\epsilon_{ijk}B^2B_k \\
&= \bar{\theta}_4\{\partial_j[E_i\epsilon_{ijk}B^2B_k] - (\partial_j E_i)\epsilon_{ijk}B^2B_k\} \\
&= \bar{\theta}_4\left\{\partial_j[E_i\epsilon_{ijk}B^2B_k] - \left(\frac{\partial B_k}{\partial t}\right)B^2B_k\right\} \\
&= \bar{\theta}_4\left\{\partial_j[E_i\epsilon_{ijk}B^2B_k] - \frac{\partial}{\partial t}\left(\frac{B^4}{4}\right)\right\},
\end{aligned} \tag{5.17}$$

where in the second equality, we applied Eq. (5.6). It is possible to rewrite this last term as

$$\nabla\cdot\mathbf{S}_3 + \frac{\partial}{\partial t}u_1 = \bar{\theta}_4\left\{\partial_j(E_i\epsilon_{ijk}B^2B_k) + \frac{\partial}{\partial t}\left(\frac{B^4}{4}\right)\right\}. \tag{5.18}$$

Condensing all these expressions into just one, we obtain the famous continuity equation of electromagnetism

$$\nabla\cdot(\mathbf{S}-\mathbf{S}_1-\mathbf{S}_2+\mathbf{S}_3) + \frac{\partial}{\partial t}(u-u_1) = 0. \tag{5.19}$$

This equation have news terms that come from the LQG, more precisely, each rotation angle contributes with one term in the spatial form, as we can verify in terms $\mathbf{S}_1, \mathbf{S}_2$ and $\mathbf{S}_3$. An important term is obtained by the non linear magnetic field, who is the only term who modified the electromagnetic energy density. From the Poynting vector, is possible calculate the Stress tensor, we have

$$\begin{aligned}
\partial_t(\mathbf{S}-\mathbf{S}_1-\mathbf{S}_2+\mathbf{S}_3) = -\,\partial_k\Bigg\{ & A_\gamma^2\left[\delta_{ik}\left(\frac{B^2}{2}+\frac{E^2}{2}\right)-(B_iB_k+E_iE_k)\right] + \bar{\theta}_3[\delta_{ik}[(\partial_o\partial_o B_m)B_m- \\
& (\partial_o\partial_o E_m)E_m - (E_j\partial_m(\partial_m E_j)) + (E_m\partial_n(\partial_m E_n)) - (B_j\partial_m(\partial_m B_j))+ \\
& (B_m\partial_n(\partial_m B_n)) + \bar{\theta}_3[\partial_o\partial_o(E_m\partial_n(\partial_m E_n)) - \partial_o\partial_o(E_j\partial_m(\partial_m E_j))+ \\
& 2(\partial_o\partial_o(\partial_m\partial_m B_l)B_l) - \partial_o\partial_o(B_j\partial_m(\partial_m B_j)) + \partial_o\partial_o(B_m\partial_n(\partial_m B_n))+ \\
& 2(\partial_o\partial_o(\partial_m\partial_m E_l)E_l)]] + (\partial_o\partial_o B_i)\cdot B_k + (\partial_o\partial_o E_i)\cdot E_k] + \bar{\theta}_8\delta_{ik}\cdot \\
& [(\bar{\theta}_8\partial_o\partial_o B_j - \epsilon_{jmn}\partial_m B_n)\cdot B_j - (\bar{\theta}_8\partial_o\partial_o E_j - \epsilon_{jmn}\partial_m E_n)\cdot E_j]+ \\
& \bar{\theta}_4[\delta_{ik}[B^4 + \frac{\bar{\theta}_4}{2}(B^2B_m)^2] - 2(B^2B_iB_k) + \bar{\theta}_4[(B^2B_i)(B^2B_k)]- \\
& 3(E_kE_i)B^2]\Bigg\},
\end{aligned} \tag{5.20}$$

or we can rewrite in compact form, like this:

$$\partial_t(\mathbf{S}-\mathbf{S}_1-\mathbf{S}_2+\mathbf{S}_3) + \partial_k T_{ik} = 0. \tag{5.21}$$

The next step is to understand how the non-linear term in the magnetic field influences the theory, for this, we expand the magnetic field as

$$\mathbf{B} = \boldsymbol{\zeta} + \mathbf{b}_p, \tag{5.22}$$

where $\boldsymbol{\zeta}$ is a constant vector of magnetic field and $\mathbf{b}_p$ is the propagating magnetic field vector. So, we can rewrite the product

$$\vec{B}^2 \cdot \mathbf{B} = (\boldsymbol{\zeta} + \mathbf{b}_p)^2 \cdot (\boldsymbol{\zeta} + \mathbf{b}_p) \approx \zeta^2 \cdot \mathbf{b}_p + 2(\boldsymbol{\zeta} \cdot \mathbf{b}_p) \cdot \boldsymbol{\zeta}. \tag{5.23}$$

This method is called linearization of the magnetic field. With the result of this expansion, it is possible to rewrite Eq. (5.4) as follows

$$A_\gamma(\nabla \times (\boldsymbol{\zeta} + \mathbf{b}_p)) - \frac{\partial \mathbf{E}}{\partial t} + \bar{\theta}_3 \nabla^2 (\nabla \times (\boldsymbol{\zeta} + \mathbf{b}_p)) - \bar{\theta}_8 \nabla^2 (\boldsymbol{\zeta} + \mathbf{b}_p) + \bar{\theta}_4 \nabla \times (\zeta^2 \cdot \mathbf{b}_p + 2(\boldsymbol{\zeta} \cdot \mathbf{b}_p) \cdot \boldsymbol{\zeta}) = 0$$
$$\Rightarrow A_\gamma(\nabla \times \mathbf{b}_p) - \frac{\partial \mathbf{E}}{\partial t} + \bar{\theta}_3 \nabla^2 (\nabla \times \mathbf{b}_p) - \bar{\theta}_8 \nabla^2 \mathbf{b}_p + \bar{\theta}_4 \nabla \times [\zeta^2 \cdot \mathbf{b}_p + 2(\boldsymbol{\zeta} \cdot \mathbf{b}_p) \cdot \boldsymbol{\zeta}] = 0. \tag{5.24}$$

where the derivatives terms of ζ are $\nabla \times \zeta = \nabla \cdot \zeta = 0$, one of the important equations that can be obtained is the wave equation for both the electric field and the propagating magnetic field. The procedure is identical for both equations, starting with the electric field equation. Let's take the time derivative of Eq. (5.24)

$$\frac{\partial}{\partial t}\left[A_\gamma(\nabla \times \mathbf{b}_p) - \frac{\partial \mathbf{E}}{\partial t} + \bar{\theta}_3 \nabla^2 (\nabla \times \mathbf{b}_p) - \bar{\theta}_8 \nabla^2 \mathbf{b}_p + \bar{\theta}_4 \nabla \times [\zeta^2 \cdot \mathbf{b}_p + 2(\boldsymbol{\zeta} \cdot \mathbf{b}_p) \cdot \boldsymbol{\zeta}] = 0\right]$$
$$\Rightarrow A_\gamma \frac{\partial}{\partial t}(\nabla \times \mathbf{b}_p) - \frac{\partial^2 \mathbf{E}}{\partial t^2} + \bar{\theta}_3 \nabla^2 \frac{\partial}{\partial t}(\nabla \times \mathbf{b}_p) - \bar{\theta}_8 \frac{\partial}{\partial t} \nabla^2 \mathbf{b}_p + \bar{\theta}_4 \frac{\partial}{\partial t} \nabla \times [\zeta^2 \cdot \mathbf{b}_p + 2(\boldsymbol{\zeta} \cdot \mathbf{b}_p) \cdot \boldsymbol{\zeta}] = 0. \tag{5.25}$$

We take the curl of Eq. (5.6)

$$\nabla \times \left[A_\gamma(\nabla \times \mathbf{E}) + \frac{\partial \mathbf{b}_p}{\partial t} + \bar{\theta}_3 \nabla^2 (\nabla \times \mathbf{E}) - \bar{\theta}_8 \nabla^2 \mathbf{E} = 0\right]$$
$$\Rightarrow -A_\gamma \nabla \times (\nabla \times \mathbf{E}) + \bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) + \bar{\theta}_8 \nabla^2 (\nabla \times \mathbf{E}) = \frac{\partial}{\partial t}(\nabla \times \mathbf{b}_p), \tag{5.26}$$

where we used Eq. (5.3). Substituting the term $\frac{\partial}{\partial t}(\nabla \times \mathbf{b}_p)$ from Eq. (5.26) into each of the common terms from Eq. (5.25), we get

$$A_\gamma(-A_\gamma \nabla \times (\nabla \times \mathbf{E}) + \bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) + \bar{\theta}_8 \nabla^2 (\nabla \times \mathbf{E})) - \frac{\partial^2 \mathbf{E}}{\partial t^2} + \bar{\theta}_3 \nabla^2 (-A_\gamma \nabla \times (\nabla \times \mathbf{E}) +$$
$$\bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) + \bar{\theta}_8 \nabla^2 (\nabla \times \mathbf{E})) + \bar{\theta}_8 \nabla \times (-A_\gamma \nabla \times (\nabla \times \mathbf{E}) + \bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) + \bar{\theta}_8 \nabla^2 (\nabla \times \mathbf{E})) +$$
$$\bar{\theta}_4 [\zeta^2 (-A_\gamma \nabla \times (\nabla \times \mathbf{E}) + \bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) + \bar{\theta}_8 \nabla^2 (\nabla \times \mathbf{E})) - 2\boldsymbol{\zeta} \cdot (-A_\gamma \nabla \times (\nabla \times \mathbf{E}) +$$
$$\bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) + \bar{\theta}_8 \nabla^2 (\nabla \times \mathbf{E})) \cdot (\nabla \times \boldsymbol{\zeta})] = 0, \tag{5.27}$$

Simplifying using the system from Eq. (5.8), it is possible to obtain

$$A_\gamma^2 (\nabla^2 \mathbf{E}) - \frac{\partial^2 \mathbf{E}}{\partial t^2} = -2\bar{\theta}_3 \nabla^2 (\nabla^2 \mathbf{E}) - 2\bar{\theta}_8 \nabla \times (\nabla^2 \mathbf{E}) - \bar{\theta}_3^2 \nabla^2 (\nabla^2 (\nabla^2 \mathbf{E})) + \bar{\theta}_8^2 \nabla^2 (\nabla^2 \mathbf{E}) +$$
$$\bar{\theta}_4 \{\zeta^2 (\nabla^2 \mathbf{E}) - 2[\boldsymbol{\zeta} \cdot (\nabla \times \mathbf{E})] \cdot (\nabla \times \boldsymbol{\zeta})\}. \tag{5.28}$$

And for the magnetic field, the procedures are similar; however, the equations to be used change. It is necessary to take the time derivative of Eq. (5.6) and the curl of Eq. (5.24). After that, we will substitute the term $\frac{\partial}{\partial t}(\nabla \times \mathbf{E})$ and find

$$A_\gamma^2(\nabla^2\mathbf{b}_p) - \frac{\partial^2 \mathbf{b}_p}{\partial t^2} = -2\bar{\theta}_3\nabla^2(\nabla^2\mathbf{b}_p) - 2\bar{\theta}_8\nabla\times(\nabla^2\mathbf{b}_p) - \bar{\theta}_3^2\nabla^2(\nabla^2(\nabla^2\mathbf{b}_p)) + \bar{\theta}_8^2\nabla^2(\nabla^2\mathbf{b}_p) + \bar{\theta}_4\{\zeta^2(\nabla^2\mathbf{b}_p) - 2[\boldsymbol{\zeta}\cdot(\nabla\times\mathbf{b}_p)]\cdot(\nabla\times\boldsymbol{\zeta})\}. \tag{5.29}$$

From the waves equations we can choose one of this equations and calculate the dispersion relation from the theory. Firstly the solution for both equations are plane wave, more precisely an exponential form of functions like

$$\mathbf{E} = \mathbf{e}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad \mathbf{b}_p = \mathbf{b}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad k = |\mathbf{k}|. \tag{5.30}$$

We get

$$\mathbf{e}_0\cdot\mathbf{k} = 0, \qquad \mathbf{b}_0\cdot\mathbf{k} = 0, \tag{5.31}$$

from Eq. (5.6)

$$\begin{aligned}(\mathbf{k}\times\mathbf{e}_0)(A_\gamma - \bar{\theta}_3\mathbf{k}^2) - i\bar{\theta}_8\mathbf{k}^2\cdot\mathbf{e}_0 - w\mathbf{b}_0 = 0\\ \Rightarrow \mathbf{b}_0 = \frac{1}{w}\left[(\mathbf{k}\times\mathbf{e}_0)(A_\gamma - \bar{\theta}_3\mathbf{k}^2) - i\bar{\theta}_8\mathbf{k}^2\cdot\mathbf{e}_0\right],\end{aligned} \tag{5.32}$$

and last, from Eq. (5.4)

$$(\mathbf{k}\times\mathbf{b}_0)(A_\gamma - \bar{\theta}_3\mathbf{k}^2) - i\bar{\theta}_8\mathbf{k}^2\cdot\mathbf{b}_0 + w\cdot\mathbf{e}_0 + \bar{\theta}_4[\zeta^2(\mathbf{k}\times\mathbf{b}_0) + 2\bar{\theta}_4(\mathbf{k}\times\boldsymbol{\zeta})\cdot(\boldsymbol{\zeta}\cdot\mathbf{b}_0)] = 0. \tag{5.33}$$

So we can replace the Eq. (5.32) in Eq. (5.33) and obtain

$$\Big\{[\mathbf{k}^2(A_\gamma - \bar{\theta}_3\mathbf{k}^2)^2 - (i\bar{\theta}_8\mathbf{k}^2)^2 + w^2 - \bar{\theta}_4(\boldsymbol{\zeta}\cdot\mathbf{k})^2(A_\gamma - \bar{\theta}_3\mathbf{k}^2)]\delta_{ij} + [i\bar{\theta}_4\bar{\theta}_8(\boldsymbol{\zeta}\cdot\mathbf{k})^2 - 2i\bar{\theta}_8\mathbf{k}^2(A_\gamma - \bar{\theta}_3\mathbf{k}^2)]\cdot \epsilon_{ijk}k_k - 2\bar{\theta}_4(\mathbf{k}\times\boldsymbol{\zeta})_i\cdot(\mathbf{k}\times\boldsymbol{\zeta})_j(A_\gamma - \bar{\theta}_3\mathbf{k}^2) - 2i\bar{\theta}_4\bar{\theta}_8(\mathbf{k}\times\boldsymbol{\zeta})_i\cdot b_{0j}\mathbf{k}^2\Big\}\mathbf{e}_{0j} = 0, \tag{5.34}$$

making use of the previous equations, we obtain

$$M_{ij}e_{0j} = 0, \tag{5.35}$$

where M_{ij} is a matrix which from the Eq. (5.34) has the same form of the matrix equation:

$$M_{ij} = \alpha\delta_{ij} + \beta u_i\cdot u_j + c\epsilon_{ijk}v_k + \gamma u_i\cdot s_j, \tag{5.36}$$

whose determinant is given by

$$\det M = \alpha^3 + c^2(\mathbf{u}\cdot\mathbf{v})\cdot(\gamma\mathbf{s}\cdot\mathbf{v} + \beta(\mathbf{u}\cdot\mathbf{v})) + \alpha^2\beta\mathbf{u}^2 + \alpha c(c\mathbf{v}^2 + \gamma\mathbf{s}\cdot(\mathbf{k}\times(\mathbf{k}\times\mathbf{s}))). \tag{5.37}$$

Now, we can impose the condition det$M = 0$ what allows us to find the modified dispersion relation for the form

$$w_{\pm}^2 = k^2[A_\gamma - \bar{\theta}_3(\mathbf{k})^2]\Bigg\{[A_\gamma - \bar{\theta}_3(\mathbf{k})^2] + \bar{\theta}_4\zeta^2\Bigg\} - (\bar{\theta}_8 \cdot k^2)^2 - \psi \pm \Bigg(4k^4\Bigg\{k^2\Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \bar{\theta}_4\frac{\zeta^2}{2}\Bigg)^2 + \Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \frac{\zeta^2}{2}\Bigg)\bar{\theta}_4[\boldsymbol{\zeta}\cdot(\mathbf{k}\times(\mathbf{k}\times\boldsymbol{\zeta}))]\Bigg\}\bar{\theta}_8^2 + \psi^2\Bigg)^{1/2}. \tag{5.38}$$

The term $\psi = -\bar{\theta}_4[A_\gamma - \bar{\theta}_3(\mathbf{k})^2]\cdot|\mathbf{k}|^2\cdot|\boldsymbol{\zeta}|^2\sin^2\varphi$ is a scalar that corresponds to the anisotropy of the theory. So, photons have propagation speeds that are no longer constant on the Planck scale, and this is precisely due to the dependence that the ψ parameter has on $\sin\varphi$, where this is the angle between the vectors $\mathbf{k}\cdot\boldsymbol{\zeta}$. The $\pm$ sign in the dispersion relation is an indication of the possible phenomenon of birefringence in vacuum. The group velocity is necessarily the photon velocity, which in turn can be obtained as

$$v_\pm = \frac{dw}{dk} = \frac{1}{w_\pm}\cdot\Bigg[\mathbf{k}\cdot[A_\gamma - \bar{\theta}_3(\mathbf{k})^2]\Bigg\{[A_\gamma - \bar{\theta}_3(\mathbf{k})^2] + \bar{\theta}_4\zeta^2\Bigg\} - 2\bar{\theta}_3\mathbf{k}^3[A_\gamma - \bar{\theta}_3(\mathbf{k})^2] + 2i\bar{\theta}_8\mathbf{k}^3 - \bar{\theta}_4\mathbf{k}\cdot|\zeta|^2\sin^2\varphi \pm \frac{1}{4}\Bigg(4k^4\Bigg\{k^2\Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \bar{\theta}_4\frac{\zeta^2}{2}\Bigg)^2 + \Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \frac{\zeta^2}{2}\Bigg)\cdot\bar{\theta}_4\cdot[\boldsymbol{\zeta}\cdot(\mathbf{k}\times(\mathbf{k}\times\boldsymbol{\zeta}))]\Bigg\}\bar{\theta}_8^2 + \psi^2\Bigg)^{-1}\Bigg(12\mathbf{k}^5\Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \bar{\theta}_4\frac{\zeta^2}{2}\Bigg)^2\bar{\theta}_8^2 + 2\mathbf{k}^3\Bigg(\bar{\theta}_4[A_\gamma - \bar{\theta}_3(\mathbf{k})^2]|\zeta|^2\sin^2\varphi\Bigg)^2 - 8\bar{\theta}_3\mathbf{k}^7\Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \bar{\theta}_4\frac{\zeta^2}{2}\Bigg)^2\bar{\theta}_8^2\Bigg)\Bigg]. \tag{5.39}$$

The speed of the photon is no longer constant, this quantity can be greater than the usual speed of light or less. The index refraction can be obtained as

$$n_\pm = |k|\cdot\Bigg(k^2[A_\gamma - \bar{\theta}_3(\mathbf{k})^2]\Bigg\{[A_\gamma - \bar{\theta}_3(\mathbf{k})^2] + \bar{\theta}_4\zeta^2\Bigg\} - (\bar{\theta}_8\cdot k^2)^2 - \psi \pm \Bigg(4k^4\Bigg\{k^2\Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \bar{\theta}_4\frac{\zeta^2}{2}\Bigg)^2 + \Bigg([A_\gamma - \bar{\theta}_3(\mathbf{k})^2] - \frac{\zeta^2}{2}\Bigg)\bar{\theta}_4[\boldsymbol{\zeta}\cdot(\mathbf{k}\times(\mathbf{k}\times\boldsymbol{\zeta}))]\Bigg\}\bar{\theta}_8^2 + \psi^2\Bigg)\Bigg)^{-1/2}, \tag{5.40}$$

from this equation, we can interpret that the system is in a dispersive medium, since the refractive index depends on $|k|$. Furthermore, the refractive index includes square root terms, making it possible for the final value of this function to become negative. As a result, the refractive index may acquire both real and imaginary components. This suggests the possibility of dichroism in the theory, where both the birefringence discussed in the dispersion relation and the dichroism arising from the refractive index are optical phenomena of the vacuum. However, we cannot claim that these phenomena necessarily occur in this context, as a more detailed analysis using circularly polarized waves is required to confirm the existence of such optical effects.

5.2 Yang-Mills theories in LQG

Yang-Mills theories are widely used due to their effectiveness in describing fundamental interactions. In Loop Quantum Gravity, this approach is also explored, as evidenced by various works that present this formulation as an extension of the complete theory, for example, in [48, 85, 115]. In this section, our goal is not to develop a complete non-Abelian theory within the framework of Loop Quantum Gravity (LQG), but rather to investigate how the effects introduced by LQG in the electromagnetic sector influence a non-Abelian theory.

We will non-Abelianize the effects of LQG in the electromagnetic sector presented at the beginning of Chapter 5. The strategy used to perform such a procedure is the same as that presented in the classical electromagnetic sector shown in Chapter 2. Having done using the Maxwell limit has led to significant contributions to the development of LQG. This occurs because it is necessary to obtain the non-Abelian Maxwell equations in the limit where quantum effects become significant in LQG. Another important contribution of having first done it in Chapter 2 was the experience gained in developing the same procedure, now using much more involved expressions.

In this way, we will start from Eqs. (5.3), (5.4), (5.5), and (5.6), which form the set of modified Maxwell equations in our theory, and apply the Noether procedure as we have done before. This will lead to self-interactions among spin-1 fields, with our goal being to arrive at the non-Abelian extension of the Maxwell equations modified by LQG, resulting in the Yang-Mills field equations corrected by LQG. Once this is achieved, we aim to explore how quantum gravitational effects, as predicted by LQG, can be integrated into Yang-Mills theory. This allows us to study the physics of self-interacting massless vector bosons in the presence of small Quantum Gravity effects with an LQG signature. Quantum Gravity effects on photons and neutrinos have already been studied, as seen in [70]. Inspired by that work, our approach, by combining LQG and Yang-Mills, provides a framework for investigating how the electroweak and QCD sectors receive quantum gravitational corrections, contributing to interesting phenomenological analyses and raising the possibility of new hidden physics.

To carry out this approach, we will define the same quantities as we defined earlier in Chapter 2. We begin with an N-plet of vector fields (which will become our Yang-Mills gauge bosons at the end of the procedure) in an arbitrary representation of the symmetry group, as described below:

$$A_m, m = 1, 2, ..., N \qquad \phi_n, n = 1, 2, ..., N \tag{5.41}$$

this fields are represented in N-dimensions of the symmetry group, remember, LQG works in 4-dimension. We can define the electric- and magnetic-like fields in terms of these terms

$$\vec{E}_m = -\nabla \cdot \phi_m - \frac{\partial \vec{A}_m}{\partial t}, \tag{5.42}$$

and

$$\vec{B}_m = \nabla \times \vec{A}_m. \tag{5.43}$$

From the Lie group, that fields transforms like the rule

$$\phi'_m = R_{mn}\phi_n, \tag{5.44}$$

and

$$A'_m = R_{mn}A_n. \tag{5.45}$$

And in the same form showed at Eq. (2.12), the function of the Lie group in infinitesimal form is

$$R_{mn} = (e^{iw_h G_h})_{mn} \sim \delta_{mn} + iw_h(G_h)_{mn} + O(w^2), \tag{5.46}$$

So, similarly, the fields can also be expressed in a form

$$\delta A_m = iw_h(G_h)_{mn}A_n, \tag{5.47}$$

and

$$\delta\phi_m = iw_h(G_h)_{mn}\phi_n. \tag{5.48}$$

The same structure repeats, the $(G_h)_{mn}$ are the generators in the adjoint representation and they can be write in terms of the structure constants like Eq. (2.15). The parameters w_h stands for the parameters of the $SU(N)$ transformations, whereas the 3-index symbol f_{hij} represents the structure constants of $SU(N)$. The Noether procedure is initiate with the free Lagrangian of the theory. To get this Lagrangian, we perform a Legendre transformation by taking Eq. (5.2) and writing

$$L_{LQG} = \frac{1}{Q^2}\int d^3x \Bigg\{ A_\gamma \left(\frac{E^2}{2} - \frac{\vec{B}^2}{2} \right) + \theta_3 l_P^2 (\mathbf{B}\nabla^2\mathbf{B} + \mathbf{E}\nabla^2\mathbf{E}) + \theta_2 l_P^2 \mathbf{E}\cdot[\nabla\cdot(\nabla\mathbf{E})] + \theta_8 l_P[\mathbf{B}\cdot(\nabla\times\mathbf{B}) + \mathbf{E}\cdot(\nabla\times\mathbf{E})] + \theta_4\mathcal{L}^2 l_P^2 \left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}(B^2)^2 \Bigg\}. \tag{5.49}$$

It is necessary to highlight an important point regarding the Hamiltonian and the types of terms that constitute it. Note that Eq. (5.2) only presents linear orders in the parameters θ_i, and to preserve the linearity of the quantum effects, we will also consider this. This idea implies that when we perform the Legendre transforms to obtain Eq. (5.49), only the classical term, that is, Maxwell's term $\frac{1}{2}(E^2 + B^2)$, undergoes a change. Fundamentally, what happens is that when we perform the Legendre transforms, we obtain new quadratic terms in θ_i, which go beyond the scope of the theory and can thus be neglected, retaining only the common terms that are unaffected by the transformation. Continuing the process, we will use the variational principle, we have

$$
\begin{aligned}
\delta S = 0 = & \int d^3x\ \delta L_{LQG} \\
= & \frac{1}{Q^2}\int d^3x \Bigg\{ \frac{A_\gamma}{2}\left(2\mathbf{E}_m\cdot\delta\mathbf{E}_m - 2\mathbf{B}_m\cdot\delta\mathbf{B}_m\right) + \theta_3 l_P^2[\delta\mathbf{B}_m\cdot(\nabla^2\mathbf{B}_m) + \mathbf{B}_m\cdot(\nabla^2(\delta\mathbf{B}_m)) + \\
& \delta\mathbf{E}_m(\nabla^2\mathbf{E}_m) + \mathbf{E}_m\cdot(\nabla^2(\delta\mathbf{E}_m))] + \theta_2 l_P^2[\delta\mathbf{E}_m\cdot(\nabla\cdot(\nabla\mathbf{E}_m)) + \mathbf{E}_m\cdot(\nabla\cdot(\nabla\delta\mathbf{E}_m))] + \\
& \theta_8 l_P[\delta\mathbf{B}_m\cdot(\nabla\times\mathbf{B}_m) + \mathbf{B}_m\cdot(\nabla\times\delta\mathbf{B}_m) + \delta\mathbf{E}_m\cdot(\nabla\times\mathbf{E}_m) + \mathbf{E}_m\cdot(\nabla\times\delta\mathbf{E}_m)] + \\
& 4\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}(\mathbf{B}_m^3\cdot\delta\mathbf{B}_m)\Bigg\} \\
= & \frac{1}{Q^2}\int d^3x \Bigg\{ A_\gamma\left(\mathbf{E}_m\cdot\left(\nabla\delta\phi_m + \frac{\partial\delta\mathbf{A}_m}{\partial t}\right) - \mathbf{B}_m\cdot(\nabla\times\delta\mathbf{A}_m)\right) + \theta_3 l_P^2\Big[(\nabla\times\delta\mathbf{A}_m)\cdot \\
& (\nabla^2\mathbf{B}_m) + \mathbf{B}_m\cdot(\nabla^2(\nabla\times\delta\mathbf{A}_m)) + \left(\nabla\delta\phi_m + \frac{\partial\delta\mathbf{A}_m}{\partial t}\right)(\nabla^2\mathbf{E}_m) + \mathbf{E}_m\cdot\Bigg(\nabla^2\Bigg(\nabla\delta\phi_m + \\
& \frac{\partial\delta\mathbf{A}_m}{\partial t}\Bigg)\Bigg)\Big] + \theta_2 l_P^2\Bigg[\left(\nabla\delta\phi_m + \frac{\partial\delta\mathbf{A}_m}{\partial t}\right)\cdot(\nabla\cdot(\nabla\mathbf{E}_m)) + \mathbf{E}_m\cdot\Bigg(\nabla\cdot\Bigg(\nabla\Bigg(\nabla\delta\phi_m + \\
& \frac{\partial\delta\mathbf{A}_m}{\partial t}\Bigg)\Bigg)\Bigg)\Bigg] + \theta_8 l_P[(\nabla\times\delta\mathbf{A}_m)\cdot(\nabla\times\mathbf{B}_m) + \mathbf{B}_m\cdot(\nabla\times(\nabla\times\delta\mathbf{A}_m)) + \Bigg(\nabla\delta\phi_m + \\
& \frac{\partial\delta\mathbf{A}_m}{\partial t}\Bigg)(\nabla\times\mathbf{E}_m) + \mathbf{E}_m\left(\nabla\times\left(\nabla\delta\phi_m + \frac{\partial\delta\mathbf{A}_m}{\partial t}\right)\right)\Bigg] + 4\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}(\mathbf{B}_m^3(\nabla\times\delta\mathbf{A}_m))\Bigg\},
\end{aligned}
\tag{5.50}
$$

from Eq. (5.50) we can obtain

$$
\begin{aligned}
0 = & A_\gamma\Bigg(\nabla\cdot(\mathbf{B}_i\times\delta\mathbf{A}_i) - \underbrace{(\nabla\times\mathbf{B}_i)\cdot\delta\mathbf{A}_i}_{Eq.\ (5.4)} - \nabla\cdot(\mathbf{E}_i\cdot\delta\phi_i) + \underbrace{(\nabla\mathbf{E}_i)\cdot\phi_i}_{Eq.\ (5.3)} - \frac{\partial}{\partial t}(\mathbf{E}_i\cdot\delta\mathbf{A}_i) + \underbrace{\frac{\partial\mathbf{E}_i}{\partial t}\cdot\delta\mathbf{A}_i}_{Eq.\ (5.4)}\Bigg) + \\
& \theta_3 l_P^2\Bigg\{\nabla\cdot[\delta\mathbf{A}_m\times(\nabla^2\mathbf{B}_m)] - \nabla\cdot[(\nabla^2\delta\mathbf{A}_m)\times\mathbf{B}_m] - \nabla\cdot(\delta\phi_m\cdot(\nabla^2\mathbf{E}_m)) - \frac{\partial}{\partial t}[\delta\mathbf{A}_m\cdot(\nabla^2\mathbf{E}_m)] - \\
& \nabla\cdot[\mathbf{E}_m\cdot(\nabla^2\delta\phi_m)] - \frac{\partial}{\partial t}[\mathbf{E}_m\cdot(\nabla^2\mathbf{A}_m)] + \underbrace{2[\delta\mathbf{A}_m\cdot\nabla^2(\nabla\times\mathbf{B}_m)]}_{Eq.\ (5.4)}\Bigg\} + \theta_2 l_P^2\Bigg\{2\nabla\cdot[\delta\mathbf{A}_m\times(\nabla\times\mathbf{B}_m)] - \\
& \nabla\cdot[\delta\phi_m\cdot(\nabla\times\mathbf{E}_m)] + \nabla\cdot\left[\mathbf{E}_m\times\left(\frac{\partial}{\partial t}\delta\mathbf{A}_m\right)\right] - 2\frac{\partial}{\partial t}[(\nabla\times\mathbf{E}_m)\cdot\delta\mathbf{A}_m] - \underbrace{2[\delta\mathbf{A}_m\cdot(\nabla^2\mathbf{B}_m)]}_{Eq.\ (2.4)}\Bigg\} - \\
& 4\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}[\nabla\cdot(\mathbf{B}_m^3\times\delta\mathbf{A}_m)] + \underbrace{4\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}[\nabla\times(B^2\cdot\mathbf{B}_m)]\cdot\delta\mathbf{A}_m}_{Eq.\ (5.4)}.
\end{aligned}
\tag{5.51}
$$

Where we set $Q^2 = 1$ to simplify the notation. All the terms with underbrace are equal to zero by the Eqs. (5.3) and (5.4) which are the modified Maxwell equations. The remaining terms are total spatial and temporal derivatives, which only need to be replaced by Eqs. (5.47) when they are of the type $\delta\mathbf{A}_m$ and by Eq. (5.48) when they are of the type $\delta\phi_m$. These total derivatives will enable to extract the (on-shell) conserved currents. As expected, just as in the construction of the classical formalism, the currents obtained from the variational principle will carry an index (h) of the adjoint representation of the symmetry group, which is, in turn, different from the index that the fields carry in an arbitrary representation (m). This allows us to proceed in the same way as was done previously. The invariant self-interaction can only occur if these indices coincide, i.e., $m = h$. To allow for self-interaction between fields and currents, it is necessary that the fields, which were previously in an arbitrary representation, be placed in the adjoint representation describe for Eq. (2.22), as the currents are. So, we have for the spatial current

$$\mathbf{j}_h = -f_{hij}\{A_\gamma[(\mathbf{E}_i\cdot\phi_j) - (\mathbf{B}_i\times\mathbf{A}_j)] - \theta_3 l_P^2[(\mathbf{B}_i\times\nabla^2\mathbf{A}_j) - (\nabla^2\mathbf{B}_i\times\mathbf{A}_j) - (\nabla^2\mathbf{E}_i\cdot\phi_j) -$$

$$(\mathbf{E}_i\cdot\nabla^2\phi_j)] - \theta_8 l_P[2(\nabla^2\mathbf{A}_i\times\mathbf{A}_j) - (\mathbf{E}_i\times\dot{\mathbf{A}}_j) - (\nabla\times\mathbf{E}_i)\phi_j] + 4\theta_4\mathcal{L}^2 l_P^2\left(\frac{\mathcal{L}}{l_p}\right)^{2\Upsilon}(B^2\cdot\mathbf{B}_i\times\mathbf{A}_j)\}, \tag{5.52}$$

and the temporal current as

$$j_h^0 = -f_{hij}\{A_\gamma(\mathbf{E}_i\cdot\mathbf{A}_j) + \theta_3 l_P^2[(\nabla^2\mathbf{E}_i\cdot\mathbf{A}_j) + (\mathbf{E}_i\cdot\nabla^2\mathbf{A}_j)] - 2\theta_8 l_P(\nabla\times\mathbf{E}_i)\cdot\mathbf{A}_j\}. \tag{5.53}$$

Notice that the first terms of each current, which are coupled by the constant A_γ, are the same current terms found in Eqs. (2.19) and (2.20). This happens because these terms carry classical electromagnetism, and, for example, if the quantum effects of LQG are negligible, these are the only terms that contribute to the currents. By coupling the Noether currents to the fields, we go over from the free regime to one that exhibits self-interaction among the massless vector fields. We then introduce the coupling between the currents and the fields to get a new Lagrangian, no more free. This yields:

$$L_{LQG}^1 = L_{LQG} - l\phi_h j_h^0 - l\mathbf{A}_h\mathbf{j}_h, \tag{5.54}$$

The parameter l is a coupling constant similar to the previous case, and we will adopt $l = \frac{1}{4}g$. At this point, the reader should notice that the procedures are identical to those in Chapter 2, and therefore, we will once again consider an infinitesimal transformation of the fields. This is because we aim to remove any dependence of the partial Lagrangian on space-time derivatives of the vector fields

$$\delta A_h = iw_k(G_k)_{hl}A_l = w_k f_{khl}A_l, \tag{5.55}$$

and

$$\delta\phi_h = iw_k(G_k)_{hl}\phi_l = w_k f_{khl}\phi_l. \tag{5.56}$$

And once again, we will apply the variational principle. The difference is that this time we will apply it to the expression in Eq. (5.54)

$$\delta S = \int d^4x\ \delta L_{LQG}^1 = 0. \tag{5.57}$$

For simplicity, we will not present Eq. (5.57), as it represents the coupling of the Lagrangian (5.49) with the conserved currents Eqs. (5.52) and (5.53), making it a rather extensive equation. The next step, before finding the new conserved currents from Eq. (5.54), is to derive the equations of motion for this Lagrangian. This is an essential point. From the Lagrangian, we can derive new Gauss' law of electricity and Ampère-Maxwell equations. This allows us to determine the currents to be coupled to the fields and then reinsert them into the Lagrangian. Each current is coupled to a field and depends on a coupling parameter l', where it is already known that l' satisfies $l' = \frac{1}{2}g$, which represents the coupling constant for this new term

$$L^2_{LQG} = L_{LQG} - l'\phi_k j^{0(1)}_k - l'\mathbf{A}_k\mathbf{j}^{(1)}_k, \tag{5.58}$$

The Eq. (5.58) is the last Lagrangian expression obtained, and it is no longer necessary to re-calculate other currents. The reason being that the expression of the new Lagrangian no longer depends on derivatives of the fields. Should the same steps be repeated, we would obtain the same expressions for the currents as before. From Eq. (5.58), we can obtain the field equations, namely, Gauss' electricity and Ampère-Maxwell. From these two equations of motion, it is possible to define the equations for the electric and magnetic field equations in terms of the potentials in the Yang-Mills formalism with LQG contributions:

$$\begin{aligned}\mathbf{E}_h = A_\gamma[-\nabla\cdot\phi_h - \dot{\mathbf{A}}_h + gf_{hij}\phi_i\mathbf{A}_j] - \frac{3g}{2}\theta_3 l_P^2 f_{hij}[\mathbf{A}_i(\nabla^2\phi_j) + (\nabla^2\mathbf{A}_i)\phi_j] + g\theta_3 l_P^2 f_{hij}[(\nabla\mathbf{A}_i)\cdot \\ (\nabla\phi_j)] + \frac{g}{2}\theta_8 l_P f_{hij}(\mathbf{A}_i\times\dot{\mathbf{A}}_j),\end{aligned} \tag{5.59}$$

and

$$\begin{aligned}\mathbf{B}_h = A_\gamma[(\nabla\times\mathbf{A}_h) + \frac{g}{2}f_{hij}(\mathbf{A}_i\times\mathbf{A}_j)] + 2l_P^2\theta_3\nabla^2\mathbf{B}_h - \frac{g}{2}\theta_3 l_P^2 f_{hij}[\nabla^2(\mathbf{A}_i\times\mathbf{A}_j) - [\mathbf{A}_i\times(\nabla^2\mathbf{A}_j)]] \\ + \frac{g}{2}\theta_8 l_P f_{hij}[(\dot{\mathbf{A}}_i)\cdot\phi_j] + 2\theta_4\mathcal{L}^2\left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon_\gamma} l_P^2[2(\vec{B}^2\mathbf{B}_h) + 3gf_{hij}(\mathbf{A}_i\times\mathbf{A}_j)\cdot B^2].\end{aligned} \tag{5.60}$$

Again, note that the terms coupled with the constant A_γ are the same as those appearing in Eqs. (2.37) and (2.38). However, observe the number of new terms arising from the contribution of LQG effects in the electromagnetic sector. These new contributions are independently generated by each θ_i term, being linear and not presenting cross terms, i.e., $\theta_i \cdot \theta_j$. As can be seen, in Chapter 2, the Noether procedure would be complete, and it would be possible to derive the four new non-Abelian Maxwell equations. It would only require taking the curl of Eq. (5.59) and the divergence of Eq. (5.60) would be sufficient. In electrodynamics with LQG effects, this is not possible, because the Faraday-Lenz equation is modified and, by doing this and taking the Abelian limit, we do not obtain again the Eqs. (5.5), (5.6) that should be obtained. As it can be consulted in the literature [124, 125], the non-Maxwellian extensions of electrodynamics does not modify the Faraday-Lenz equation, since together with Gauss'law for magnetism, these equations are obtained from the Bianchi Identity. The electrodynamics presented in this paper modifies Eq. (5.6), which allows us to conclude that the Bianchi Identities are modified. It is therefore necessary to follow another path to attain these two equations; one possible way consists in getting them from the Hamilton-Jacobi equations. By performing a Legendre transformation on the Lagrangian Eq. (5.58) to find its corresponding Hamiltonian, we can derive the equations for the electric field, E, and the vector potential, A:

$$\frac{\partial H^2_{LQG}}{\partial \mathbf{E}_h} = -\frac{\partial \mathbf{A}_h}{\partial t}, \tag{5.61}$$

and

$$\frac{\partial H^2_{LQG}}{\partial \mathbf{A}_h} + \frac{\partial H^2_{LQG}}{\partial (\nabla \times \mathbf{A}_h)} = \frac{\partial \mathbf{E}_h}{\partial t}. \tag{5.62}$$

This yields four Maxwell-type equations in the non-Abelian version with LQG contributions :

$$A_\gamma \Big[\nabla \cdot \mathbf{E}_h + g f_{hij} \mathbf{E}_i \cdot \mathbf{A}_j\Big] + \frac{g}{2}\theta_3 l_P^2 f_{hij}\Big[2(\nabla^2 \mathbf{E}_i)\cdot \mathbf{A}_j + \mathbf{E}_i \cdot (\nabla^2 \mathbf{A}_j) - 2\nabla[(\nabla \mathbf{A}_i)\cdot(\nabla \phi_j)] + \nabla^2(\mathbf{A}_i \cdot \mathbf{E}_j)\Big] + \frac{g}{2}\theta_8 l_P f_{hij}\Big[\nabla \cdot (\mathbf{A}_i \times \dot{\mathbf{A}}_j) + \mathbf{A}_i \cdot (\nabla \times \mathbf{E}_j)\Big] = 0, \tag{5.63}$$

$$A_\gamma \Big[\nabla \times \mathbf{B}_h + g f_{hij}\phi_i \mathbf{E}_j + g f_{hij}(\mathbf{A}_i \times \mathbf{B}_j)\Big] + \theta_3 l_P^2 \Big\{2\nabla^2(\nabla \times \mathbf{B}_h) - g f_{hij}[(\nabla^2 \mathbf{E}_i)\phi_j + \frac{1}{2}\nabla^2(\phi_i \mathbf{E}_j) + [(\nabla^2 \mathbf{B}_i)\times \mathbf{A}_j] - \frac{1}{2}\phi_i \nabla^2(\dot{\mathbf{A}}_j) + \frac{1}{2}\mathbf{E}_i(\nabla^2\phi_j) - \frac{1}{2}(\dot{\mathbf{A}}_i)\cdot(\nabla^2 \phi_j) - \frac{1}{2}\nabla^2(\mathbf{B}_i \times \mathbf{A}_j) - \frac{1}{2}[\mathbf{B}_i \times (\nabla^2 \mathbf{A}_j)] - \nabla^2[\phi_i(\dot{\mathbf{A}}_j)]]\Big\} - \theta_8 l_P \Big\{2\nabla^2 \mathbf{B}_h - \frac{g}{2} f_{hij}[\partial_t(\mathbf{A}_i \times \mathbf{A}_j) - [(\dot{\mathbf{E}}_i)\times \mathbf{A}_j] - 2\nabla^2(\mathbf{A}_i \times \mathbf{A}_j) - \nabla[\nabla\cdot(\mathbf{A}_i \times \mathbf{A}_j)] + (\nabla \times \mathbf{E}_i)\phi_j]\Big\} + 2\theta_4 \mathcal{L}^2 \left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon_\gamma} l_P^2 \nabla \times \Big[2(\vec{B}^2 \mathbf{B}_h) + 3g f_{hij}(\mathbf{A}_i \times \mathbf{A}_j)\cdot B^2\Big] = \frac{\partial \mathbf{E}_h}{\partial t}, \tag{5.64}$$

$$A_\gamma \Big[(\nabla \cdot \mathbf{B}_h) + g f_{hij} \mathbf{A}_i \cdot \mathbf{B}_j\Big] - \frac{g}{2}\theta_3 l_P^2 f_{hij} \nabla \cdot \Big[(\mathbf{A}_i \times (\nabla^2 \mathbf{A}_j) - 3\nabla^2(\mathbf{B}_i \times \mathbf{A}_j)\Big] + \frac{g}{2}\theta_8 l_P f_{hij}\nabla \cdot \Big[(\dot{\mathbf{A}}_i)\cdot \phi_j\Big] + 9g\theta_4 \mathcal{L}^2 \left(\frac{\mathcal{L}}{l_P}\right)^{2\Upsilon_\gamma} l_P^2 f_{hij} \nabla \cdot \Big[(\mathbf{A}_i \times \mathbf{A}_j)\cdot B^2\Big] = 0, \tag{5.65}$$

$$A_\gamma \Big[(\nabla \times \mathbf{E}_h) - g f_{hij}\phi_i \mathbf{B}_j + g f_{hij}(\mathbf{A}_i \times \mathbf{E}_j)\Big] + \theta_3 l_P^2 \nabla^2(\nabla \times \mathbf{E}_h) - \frac{g}{2}\theta_3 l_P^2 f_{hij}\Big[\mathbf{B}_i \cdot (\nabla^2 \phi_j) - (\nabla^2 \mathbf{E}_i \times \mathbf{A}_j)\Big] - \theta_8 l_P \Big[\nabla^2 \mathbf{E}_h + g f_{hij}\nabla\cdot(\mathbf{E}_i \cdot \mathbf{A}_j) - \frac{g}{2} f_{hij} \nabla \times (\mathbf{A}_i \times (\dot{\mathbf{A}}_j))\Big] = -\frac{\partial \mathbf{B}_h}{\partial t}. \tag{5.66}$$

Having the four non-Abelian Maxwell-type equations for massless spin-1 vector bosons in the context of LQG effects, an important point is that when we take the Abelian limit, that is, the generators of the theory become trivial ($G_h = 0$). Which consequently leads the structure constants to become trivial $f_{hij} = 0$, we re-obtain the LQG-corrected Maxwell equations, (5.3), (5.4), (5.5), and (5.6). Note also that each term in θ_i contributes independently to the structure constants, and the term θ_4, coupled with the nonlinear magnetic field, modifies the new magnetic field equations. Previously, Eq. (5.65), known as Gauss's law for magnetism, did not include magnetic monopoles. However, this is no longer true in the non-Abelian formulation, as the terms θ_3, θ_8, and θ_4 independently contribute to this equation. It is conceivable, based on this interpretation, that magnetic monopoles might exist at the Planck scale, possibly associated with these parameters.

When the LQG terms become irrelevant, $\theta_i = 0$, we once again obtain Eqs. (2.43), (2.44), (2.45), and (2.46). Another important observation is that, unlike what occurs in classical electromagnetism, here there is no natural emergence of the covariant derivative in the theory. Recall that in classical electromagnetism this quantity appeared naturally in the theory, but here this does not occur due to the complexity of the theory.

Finally, as done in Chapter 2, we did not calculate the momentum and energy for the Y-M equations. The reason is purely practical, due to the complexity of the equations. However, we performed this calculation for the Abelian case, as shown in Section 5.1. Currently, our ambition to work with this model has not ended, as we aim to investigate how LQG effects in the electromagnetic sector influence the electroweak sector. Within this framework, we will seek to understand the influence of LQG on the anomalous vertices coupling the neutral gauge bosons, namely the photon and the Z^0-boson.

Chapter 6

Concluding comments

This Dissertation has been developed with the objective of exploiting the connection between nonlinear electromagnetism, Yang-Mills theories, and the effects of Loop Quantum Gravity (LQG), consolidating a bridge between classical and quantum concepts in modern physics. The work is organized in stages that progressively introduce the necessary theoretical foundations and develop new results. Below, we present a detailed summary of the contributions of each chapter.

Chapter 2 delved into classical electromagnetism, revisiting Maxwell's equations from a new perspective. By following the Noether's method, known for associating symmetries with conservation laws, a non-Abelian formulation for massless spin-1 particles was developed. This chapter detailed the construction of self-interacting currents, as well as modifications to the classical laws of electromagnetism, introducing new terms associated with SU(N) symmetries. This work revealed the possibility of magnetic monopoles arising in self-interacting fields, even in the absence of fermionic matter, a result that significantly distinguishes the Abelian and non-Abelian cases.

In Chapter 3, the foundations of General Relativity were presented, treating it as a gauge theory and emphasizing the use of the tetrad formalism. This chapter emphasized how Einstein's geometric description can be reformulated using local structures that facilitate connections with quantum field theories. Fundamental concepts, such as the ADM decomposition, which fragments spacetime into spatial and temporal hypersurfaces, enabling the Hamiltonian formulation of gravity, were also introduced. This approach lays the groundwork for introducing quantum formalisms like LQG.

Chapter 4 focused on Loop Quantum Gravity, exploring its motivations and formalism. Based on Ashtekar variables and the concept of Wilson loops, an introduction to the LQG framework was provided, highlighting its capability to quantize spacetime in a non-perturbative manner. Cosmological and black hole scenarios, as well as probabilistic effects associated with this approach, were discussed, illustrating how the granularity of spacetime naturally emerges.

Chapter 5 explored the effects of Loop Quantum Gravity on the electromagnetic sector. This chapter constituted the core of the work, connecting LQG effects to modified Maxwell equations and Yang-Mills theories. The main quantities obtained with the introduction of nonlinearities and non-Abelian interactions were highlighted, establishing analogies with fundamental models, such as the electroweak theory and quantum chromodynamics. This chapter also analyzed the application of Noether's procedure to achieve a complete non-Abelian scenario.

The results developed herein reinforce the importance of investigating the nonlinear and non-Abelian structure of fundamental interactions in the context of Quantum Gravity theories. In addition to offering a conceptual and mathematical bridge between different fields of theoretical physics, the work proposed new analytical tools to address complex problems at the interface of electromagnetic fields and spacetime geometry.

As a possible follow-up of this work, we aim to investigate the effects of LQG in connection with the electroweak sector of the Standard Model of Particle Physics. The goal is to study how these LQG effects modify the anomalous vertices coupled to the neutral gauge bosons, namely the photon and the Z^0. In this way, anomalous 3-point and 4-point vertices will arise from the coupling of the photon and the Z^0, respectively. This is particularly interesting since both types of vertices are being investigated in the ATLAS and CMS collaborations, which could consequently establish a new bridge connecting LQG to accelerator physics.

Appendix A

Gauss Equation and Codazzi-Mainardi Identities

Let us suppose we are in a $3D$ space, and therefore Eq. (3.53) holds. However, we must

$$\begin{aligned} D_\alpha D_\beta V^q = D_\alpha \left(D_\beta V^q\right) &= q^\mu{}_\alpha q^\nu{}_\beta q^q{}_\rho \nabla_\mu \left(D_\nu V^\rho\right) \\ &= q^\mu{}_\alpha q^v{}_\beta q^q{}_\rho \nabla_\mu \left(q^\sigma{}_\nu q^\rho{}_\lambda \nabla_\sigma V^\lambda\right) . \\ &= q^\mu_\alpha q^\nu_\beta q^q_\rho (n^\sigma \nabla_\mu n_\nu q^\rho_\lambda \nabla_\sigma V^\lambda + q^\sigma_\nu \nabla_\mu n^\rho \underbrace{n_\lambda \nabla_\sigma V^\lambda}_{-V^\lambda \nabla_\sigma n_\lambda} + q^\sigma_\nu q^\rho_\lambda \nabla_\mu \nabla_\sigma V^\lambda) \\ &= q^\mu_\alpha q^\nu_\beta q^q_\lambda \nabla_\mu n_\nu n^\sigma \nabla_\sigma V^\lambda - q^\mu_\alpha q^\sigma_\beta q^q_\rho V^\lambda \nabla_\mu n^\rho \nabla_\sigma n_\lambda + q^\mu_\alpha q^\sigma_\beta q^q_\lambda \nabla_\mu \nabla_\sigma V^\lambda \\ &= -K_{\alpha\beta} q^q_\lambda n^\sigma \nabla_\sigma V^\lambda - K^q_\alpha K_{\beta\lambda} V^\lambda + q^\mu_\alpha q^\sigma \beta q^q_\lambda \nabla_\sigma V^\lambda . \end{aligned} \tag{A.1}$$

Note, if we change the α por β index, we obtain

$$[D_\alpha, D_\beta] V^q = \left(K_{\alpha\mu} K^q_\beta - K_{\beta\mu} K^q_\alpha\right) V^\mu + q^\rho_\alpha q^\sigma \beta q^q_\lambda \underbrace{\left(\nabla_\rho \nabla_\sigma V^\lambda - \nabla_\sigma \nabla_\rho V^\lambda\right)}_{{}^{(4)}R^\lambda_{\mu\rho\sigma} V^\mu} , \tag{A.2}$$

so, we can rewrite the equation above

$$q^\mu_\alpha q^\nu_\beta q^q_\rho q^{\sigma}_\lambda {}^{(4)}R^\rho_{\sigma\mu\nu} V^\lambda = {}^{(3)}R^q_{\lambda\alpha\beta} V^\lambda + \left(K^q_\alpha K_{\lambda\beta} - K^q_\beta K_{\alpha\lambda}\right) V^\lambda , \tag{A.3}$$

we find the Gauss relation

$$q^\mu_\alpha q^\nu_\beta q^q_\rho q^{\sigma}_\delta {}^{(4)}R^\rho_{\sigma\mu\nu} = {}^{(3)}R^q_{\delta\alpha\beta} + K^q_\alpha K_{\delta\beta} - K^q_\beta K_{\alpha\delta} . \tag{A.4}$$

Contracted Gauss Relation

First, let's contracted the 3−metric, $q^\alpha_\beta q^\beta_\tau = q^\alpha_\tau = \delta^\alpha_\tau + n^\alpha n_\tau$. So, the contracted Gauss relation

$$q^\mu_\alpha q^{\nu}_\beta {}^{(4)}R_{\mu\nu} + q_{\alpha\mu} n^\nu q^\rho_\beta n^{\sigma} {}^{(4)}R^\mu_{\nu\rho\sigma} = {}^{(3)}R_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\mu} K^\mu_\beta . \tag{A.5}$$

To obtain the Ricci scalar from Eq. (A.5), we multiply the entire equation by $q^{\alpha\beta}$. Therefore, we have:

$$q^{\alpha\beta}\left(q^{\mu}_{\alpha}q^{\nu}_{\beta}{}^{(4)}R_{\mu\nu} + q_{\alpha\mu}n^{\nu}q^{\rho}_{\beta}n^{\sigma}{}^{(4)}R^{\mu}_{\nu\rho\sigma}\right) = q^{\alpha\beta}\left({}^{(3)}R_{\alpha\beta} + KK_{\alpha\beta} - K_{\alpha\mu}K^{\mu}_{\beta}\right) \tag{A.6}$$

$$\Rightarrow {}^{(4)}R + q^{\rho}_{\mu}n^{\nu}n^{\sigma}{}^{(4)}R^{\mu}_{\nu\rho\sigma} = {}^{(3)}R + K^2 - K_{ij}K^{ij}\,. \tag{A.7}$$

Using the metric decomposition $q^{\rho}_{\mu} = \delta^{\rho}_{\mu} + n^{\rho}n_{\mu}$ and the fact that ${}^{(4)}R^{\mu}_{\nu\rho\sigma}n^{\rho}n_{\mu}n^{\nu}n^{\sigma} = 0$, it is possible to derive:

$${}^{(4)}R + 2{}^{(4)}R_{\mu\nu}n^{\mu}n^{\nu} = {}^{(3)}R + K^2 - K_{ij}K^{ij}\,. \tag{A.8}$$

This expression is known as the Theorema Egregium, which was originally proposed for 2-D surfaces embedded in Euclidean space $\mathbb{R}^3$, where the curvature is 0.

Codazzi-Mainardi Identities

Still using Eq. (3.67), we multiply the entire equation by the metric terms $q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{r}$, obtaining

$$q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{r}\left[\nabla_{\rho},\nabla_{\tau}\right]n^{r} = q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{\sigma}{}^{(4)}R^{\sigma}_{r\rho\tau}n^{r}\,. \tag{A.9}$$

Let us expand the equation above:

$$q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{r}\nabla_{\rho}\nabla_{\tau}n^{r} - q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{r}\nabla_{\tau}\nabla_{\rho}n^{r} = q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{\sigma}{}^{(4)}R^{\sigma}_{r\rho\tau}n^{r}\,. \tag{A.10}$$

Next, we substitute the indices μ,ν by ρ,τ and ρ by r:

$$q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\mu}\nabla_{\nu}n^{\rho} - q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\nu}\nabla_{\mu}n^{\rho} = q^{\rho}_{\alpha}q^{\tau}_{\beta}q^{\mu}_{\sigma}{}^{(4)}R^{\sigma}_{r\rho\tau}n^{r}\,. \tag{A.11}$$

Now, we replace $\nabla_{\nu}n^{q}$ using Eq. (3.58) in the first commutator term, and recall that $a_{\mu} \equiv n^{\nu}\nabla_{\nu}n_{\mu}$. Thus, $K_{\mu\nu} = -\nabla_{\mu}n_{\nu} - n_{\mu}n^{\rho}\nabla_{\rho}n_{\nu}$. Therefore, we have:

$$\begin{aligned} q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\mu}\nabla_{\nu}n^{\rho} - q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\nu}\nabla_{\mu}n^{\rho} &= q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{q}_{\rho}\nabla_{\mu}\left(-K^{\rho}_{\nu} - a^{\rho}n_{\nu}\right) - q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\mu}\left(-K^{\rho}_{\mu} - a^{\rho}n_{\mu}\right) \\ &= -q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\Big[\left(\nabla_{\mu}K^{\rho}_{\nu} + \nabla_{\mu}a^{\rho}n_{\nu} + a^{\rho}\nabla_{\mu}n_{\nu}\right) - \left(\nabla_{\nu}K^{\rho}_{\mu} + \right. \\ &\qquad \left.\nabla_{\nu}a^{\rho}n_{\mu} + a^{\rho}\nabla_{\nu}n_{\mu}\right)\Big], \end{aligned} \tag{A.12}$$

To simplify this expression, we use the covariant derivative relation $D_{\mu}T^{\nu}_{\beta} = q^{\alpha}_{\mu}q^{\nu}_{\lambda}q^{\rho}_{\beta}\nabla_{\alpha}T^{\lambda}_{\rho}$, as well as $q^{\nu}_{\beta}n_{\nu} = 0$ (since n_{μ} is a timelike vector, so there is no projection on a spacelike hypersurface Σ_t), $q^{\nu}_{\beta}a^{\beta} = a^{\nu}$ (since a^{ν} is a spacelike vector, so the projection onto Σ_t gives the same vector) and the definition of the extrinsic curvature tensor in Eq. (3.55), to obtain:

$$q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\mu}\nabla_{\nu}n^{\rho} - q^{\mu}_{\alpha}q^{\nu}_{\beta}q^{r}_{\rho}\nabla_{\nu}\nabla_{\mu}n^{\rho} = -D_{\alpha}K^{r}_{\beta} + a^{r}K_{\alpha\beta} + D_{\beta}K^{r}_{\alpha} - a^{r}K_{\beta\alpha}, \tag{A.13}$$

However, as mentioned earlier, the extrinsic curvature tensor is symmetric, i.e., $K_{\mu\nu} = K_{\nu\mu}$. This allows us to simplify the equation above. Finally, since this term equals the right-hand side of Eq. (A.9), we have:

$$q^{r}_{\rho}n^{\sigma}q^{\mu}_{\alpha}q^{\nu}_{\beta}{}^{(4)}R^{\rho}_{\sigma\mu\nu} = D_{\beta}K^{r}_{\alpha} - D_{\alpha}K^{r}_{\beta}\,. \tag{A.14}$$

This is the Codazzi-Mainardi relation.

Contracted Codazzi Relation

In the Codazzi-Mainardi relation (eq. A.14), we contract the indices α and q to get:

$$q^\mu_\rho n^\sigma q^{\nu}_{\beta}{}^{(4)}R^\rho_{\sigma\mu\nu} = D_\beta K - D_\mu K^\mu_\beta \,. \tag{A.15}$$

Let's use the relation $q^\mu_\rho = \delta^\mu_\rho + n^\mu n_\rho$, we have

$$(\delta^\mu_\rho + n^\mu n_\rho) n^\sigma q^{\nu}_{\beta}{}^{(4)}R^\rho_{\sigma\mu\nu} \to n^\sigma q^{\nu}_{\beta}{}^{(4)}R_{\sigma\nu} + \underbrace{q^{\nu}_{\beta}{}^{(4)}R^\rho_{\sigma\mu\nu} n_\rho n^\sigma n^\mu}_{=0} = D_\beta K - D_\mu K^\mu_\beta \tag{A.16}$$

this term is zero because symmetric-antisymmetric indices $\{\rho, \sigma\}$ are contracted.

Thus we obtain contracted Codazzi relation

$$q^\mu_\alpha n^{\nu}{}^{(4)}R_{\mu\nu} = D_\alpha K - D_\mu K^\mu_\alpha \,. \tag{A.17}$$

Appendix B

Intrinsic Curvature Tensor

We start with the definition of the 4-Riemann tensor when applied to the normal vector n^μ, namely:

$$ {}^{(4)}R^{\rho}_{\sigma\mu\nu} n^{\sigma} = \left[\nabla_\mu, \nabla_\nu\right] n^\rho \,. \tag{B.1}$$

We multiply both sides by $(q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu})$:

$$ q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left({}^{(4)}R^{\rho}_{\sigma\mu\nu} n^{\sigma} \right) = q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left[\nabla_\mu, \nabla_\nu\right] n^\rho \,. \tag{B.2}$$

Note that the right-hand side of Eq. (B.2) is given by:

$$ q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left[\nabla_\mu, \nabla_\nu\right] n^\rho = q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left(\nabla_\mu \nabla_\nu n^\rho - \nabla_\nu \nabla_\mu n^\rho \right) \,. \tag{B.3}$$

We now have the terms $\nabla_\nu n^\rho$ and $\nabla_\mu n^\rho$, which can be replaced using Eqs. (3.57) and (3.58). Additionally, we use the normalization condition for the normal vector $n^\mu n_\mu = -1$, and, as shown earlier, $n^\mu \nabla_\nu n_\mu = \frac{1}{2} \nabla_\nu \left(n^\mu n_\mu \right) = 0$. Thus, we obtain:

$$\begin{aligned} q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left(\nabla_\mu \nabla_\nu n^\rho - \nabla_\nu \nabla_\mu n^\rho \right) = q^{\nu}_{\alpha} q^{\mu}_{\beta} \nabla_n K_{\mu\nu} & \underbrace{- q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \nabla_\mu K^{\rho}_{\nu}}_{\text{Term A}} \\ & \underbrace{+ q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left(\nabla_\nu n_\mu \right) D^\rho \ln(N)}_{\text{Term B}} \\ & \underbrace{+ q_{\rho\alpha} q^{\mu}_{\beta} \left(\nabla_\mu D^\rho \ln(N) \right)}_{\text{Term C}} \,. \end{aligned} \tag{B.4}$$

We simplify each term separately. Using Eq. (3.55), Term A can be rewritten as:

$$\begin{aligned} - q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \nabla_\mu K^{\rho}_{\nu} &= - q_{\rho\alpha} q^{\mu}_{\beta} \nabla_\mu \underbrace{\left(n^\nu K^{\rho}_{\nu} \right)}_{=0} + q_{\rho\alpha} q^{\mu}_{\beta} K^{\rho}_{\nu} \nabla_\mu n^\nu \\ &= q_{\rho\alpha} q^{\mu}_{\beta} K^{\rho}_{\sigma} q^{\sigma}_{\nu} \nabla_\mu n^\nu \\ &= - q_{\rho\alpha} K^{\rho}_{\sigma} K^{\sigma}_{\beta} \\ &= - K_{\alpha\sigma} K^{\sigma}_{\beta} \,. \end{aligned} \tag{B.5}$$

Similarly, Term B simplifies to:

$$
\begin{aligned}
q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left(\nabla_{\nu} n_{\mu}\right) D^{\rho} \ln(N) &= -q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} n_{\nu} \left(D_{\mu} \ln(N)\right) \left(D^{\rho} \ln(N)\right) \\
&= \frac{1}{N^2} \left(D_{\alpha} N\right) \left(D_{\beta} N\right) \ .
\end{aligned}
\tag{B.6}
$$

Finally, Term C can be rewritten using $q^{\mu}_{\beta} \nabla_{\mu} \left(D^{\rho} \ln(N)\right) = D_{\beta} \left(D^{\rho} \ln(N)\right)$ and $q_{\rho\alpha} D^{\rho} = D_{\alpha}$:

$$
\begin{aligned}
q_{\rho\alpha} q^{\mu}_{\beta} \left(\nabla_{\mu} D^{\rho} \ln(N)\right) &= D_{\beta} D_{\alpha} \ln(N) = D_{\alpha} D_{\beta} \ln(N) \\
&= D_{\alpha} \left(\frac{1}{N} D_{\beta} N\right) \\
&= \frac{1}{N} D_{\alpha} D_{\beta} N - \frac{1}{N^2} \left(D_{\alpha} N\right) \left(D_{\beta} N\right) \ .
\end{aligned}
\tag{B.7}
$$

Combining Eqs. (B.4), (B.5), (B.6), and (B.7) into Eq. (B.2), we have:

$$
\begin{aligned}
q_{\rho\alpha} q^{\mu}_{\beta} n^{\nu} \left({}^{(4)}R^{\rho}_{\sigma\mu\nu} n^{\sigma}\right) &= q^{\nu}_{\alpha} q^{\mu}_{\beta} \nabla_{n} K_{\mu\nu} - K_{\alpha\sigma} K^{\sigma}_{\beta} + \frac{1}{N^2} \left(D_{\alpha} N\right) \left(D_{\beta} N\right) \\
&\quad + \frac{1}{N} D_{\alpha} D_{\beta} N - \frac{1}{N^2} \left(D_{\alpha} N\right) \left(D_{\beta} N\right) .
\end{aligned}
\tag{B.8}
$$

Rearranging indices and applying simplifications:

$$
q_{\rho\alpha} q^{\mu (4)}_{\beta} R^{\rho}_{\sigma\mu\nu} n^{\sigma} n^{\nu} = -K_{\alpha\lambda} K^{\lambda}_{\beta} + q^{\mu}_{\alpha} q^{\nu}_{\beta} \nabla_{n} K_{\mu\nu} + \frac{1}{N} D_{\alpha} D_{\beta} N \ .
\tag{B.9}
$$

Appendix C

Connection to Thermodynamics and Entropy

In this appendix, we dedicated to calculation the Barbero-Immirzi Parameter and the Bekenstein-Hawking entropy.

Calculation of the Barbero-Immirzi Parameter

The Bekenstein-Hawking entropy is

$$S_{BH} = a\frac{k_B}{\hbar G}A. \tag{C.1}$$

Note the subindex BH is not associate to black hole but for Bekenstein-Hawking. We consider $c = 1$, a parameter is a constant of the order of unity k_B, where k_B is a Boltzmann constant, G is the Newton gravitational constant and A is the area of surface Schwarzschild black hole. The $\hbar$ constant is in Eq. (C.1) for two reasons, first for get dimensions right and the second is the connection to quantum world. The area of this black hole can be related with our energy like

$$E = \sqrt{\frac{A}{16\pi G^2}}, \tag{C.2}$$

from the Eq. (C.1), the black hole temperature can be obtained thought the thermodynamics relation

$$\frac{1}{T} = \frac{dS}{dE} \to T = \frac{\hbar}{32\pi a k_B G E}. \tag{C.3}$$

Until this moment every steps was made from Bekenstein. But this Hawking moment was come, first he postulate the black hole would emit thermal radiation at the temperature equation (C.3). And using quantum field theory in curved spacetime (which we no show here) he obtains a black hole thermal radiation, compute by the temperature

$$T = \frac{\hbar}{8\pi k_B G E}, \tag{C.4}$$

when we compare the Eqs. (C.3) and (C.4) they are differents from

$$a = \frac{1}{4}. \tag{C.5}$$

In 1973 Bekenstein find the constant of proportionality 0.2375, this value can be coupled to the a parameter, what we obtain

$$a \approx \frac{0.2375}{4\gamma}, \tag{C.6}$$

to restore the original expression Eq. (C.5), we can fix the γ value

$$\gamma = 0.2375. \tag{C.7}$$

The formulation of these equations have a several physics conceptituals problems. First, when Bekenstein propose the entropy he formulates the thermodynamics law can be extended in the presence of black holes and the ordinary entropy sum with black hole entropy is equal to total entropy. The $\hbar$ term leads to the questions that, when we work with entropy is necessary know the microscopical degrees of freedom responsible to that entropy, so here, which are the degrees of freedom for S_{BH}?, it is possible obtain the Eq. (C.1) for first principles? When Hawking obtain the Eq. (C.4) is a safety place to Bekenstein, the equations are almost identical, but Hawking, when deriving his equation, did not take into account Quantum Gravity effects, leaving a very important point out of his deduction.

In the next section, we show how to obtain the value of γ using the LQG approach.

Bekenstein-Hawking entropy

To construction the Bekenstein-Hawking entropy, let's start from the Boltzmann's entropy

$$S = k_B \ln(W) \tag{C.8}$$

where k_B is the Boltzmann constant and W is the number of microstates compatible with the macrostate of the system. The first question the reader may ask is, "How can we connect this to Loop Quantum Gravity (LQG)?" In fact, the connection can be made using a parameter $N(A)$, known as the number of states that the geometry of a surface with area A can assume. The area is described by Eq. (4.3). In that equation, as previously mentioned, the variable j_i corresponds to the spin network. As suggested by John Wheeler [126, 127], black holes should carry one bit of information per unit of area. The minimum possible value is $j_i = 1/2$, which corresponds to the area of a single link, of the form

$$A_0 = 4\pi\gamma\hbar G\sqrt{3} \tag{C.9}$$

Some geometric insights are neglected here, but can be seen in [44]. After the discovery of the Eq. (C.9), Domagala and Lewandowski find the entropy can not be dominated by the spin 1/2 and more details in [128].

$$n = \frac{A}{A_0} = \frac{A}{4\pi\gamma\hbar G\sqrt{3}} \tag{C.10}$$

the Hilbert space dimension of the spin 1/2 is $\mathcal{H}_{1/2} = 2$, so the number of microstates are

$$N = 2^n = 2^{A/4\pi\gamma\hbar G\sqrt{3}} \tag{C.11}$$

replacing in the Eq. (C.8), we obtain the entropy

$$S_{BH} = k_B \mathrm{ln}(N) = \frac{k_B \ A \ \mathrm{ln}2}{4\pi\sqrt{3}\gamma\hbar G}. \tag{C.12}$$

This equation needs to be equal to Eq. (C.1), which leads our free parameter, γ, to become

$$\gamma = \frac{\ln(2)}{\pi\sqrt{3}} \neq 0.2375 \tag{C.13}$$

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