Master

Dissertation

The Generalized Second Law in Euclidean Schwarzschild Black Hole

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Centro Brasileiro de Pesquisas Físicas - CBPF

Rio de Janeiro - RJ

Dezembro 2024

CBPF Centro Brasileiro de Pesquisas Físicas

MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E INOVAÇÃO

GOVERNO FEDERAL BRASIL UNIÃO E RECONSTRUÇÃO

"THE GENERALIZED SECOND LAW IN EUCLIDEAN SCHWARZSCHILD BLACK HOLE"

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Dissertação de Mestrado em Física apresentada no Centro Brasileiro de Pesquisas Físicas do Ministério da Ciência Tecnologia e Inovação. Fazendo parte da banca examinadora os seguintes professores:

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Rio de Janeiro, 25 de fevereiro de 2025.

Aos meus queridos pais,
Regina Lucia de Paiva Scorza da Silva e Ronaldo da Silva,
que me ensinaram a sonhar.

Agradecimentos

Agradeço aos meus pais, Regina Lucia de Paiva Scorza da Silva e Ronaldo da Silva por todo o amor e dedicação que tiveram por mim e por meus irmãos. Eles me mostraram que a educação abre portas, hoje abro mais uma.

Agradeço ao meu orientador e professor, Nami Fux Svaiter, pelos ensinamentos, pela paciência e por nossas conversas sobre tudo, em especial, sobre a beleza da física. É nítido sua paixão ao falar sobre física, o professor Nami é alguém brilhante e fora da curva.

Agradeço a minha avó por todo amor, atenção, paciência e carinho que sempre teve por mim.

Agradeço aos meus irmãos, Maiara Scorza da Silva e Tiago Scorza da Silva, por serem meus grandes amigos e entusiastas das minhas ideias.

Agradeço a minha namorada, Luiza Moura, por ser minha companheira, por ser o meu grande amor e por escutar minhas ideais loucas, e também por sua grande ajuda na criação das ilustrações.

Agradeço aos meus grandes amigos, Allan Lima, Gabriel Lobato e Guilherme Martins, por todos os anos de risadas e felicidades que compartilhamos.

Agradeço aos novos colegas do CBPF que fiz, por todas as discussões e ensinamentos, em especial aos colegas Gustavo O. H., C. D. Rodríguez-Camargo, Isaque F. e Felipe S. pela ajuda em todas as contas e pelas conversas que me inspiraram fora e dentro da física.

Agradeço aos grandes professores que tive durante minha formação, em especial aos professores José Hellayël, Raul Vallejos, Letícia Palhares e Sebastião Dias.

Gostaria de registrar um agradecimento especial aos servidores que compõem o CBPF, pois sem eles esse lugar incrível não funcionaria.

Por fim, agradeço ao CNPQ, pelo suporte financeiro, e a todos que me ajudaram de alguma forma nesta caminhada.

“Not only does God play dice, but... he sometimes throws them where they cannot be seen.”

- Stephen W. Hawking

Resumo

Utilizando integrais funcionais Euclidianas, nesta tese apresentamos a quantização do campo escalar. Em seguida, discutimos propriedades desse campo em uma variedade Riemanniana específica, a seção Euclidiana de Schwarzschild. Uma das principais consequências da teoria da relatividade geral é a existência de buracos negros. Fazendo-se uso da teoria quântica de campos na relatividade geral, questões importantes foram levantadas sobre propriedades termodinâmicas de buracos negros, como por exemplo, sua entropia. Como foi mostrado por Hawking que buracos negros evaporam, surge uma questão natural: neste processo de evaporação a segunda lei da termodinâmica seria violada? Este problema foi amplamente discutido por muitos autores. Bekenstein, introduziu a segunda lei generalizada da termodinâmica dos buracos negros. Essa generalização evita a violação da segunda lei. Nesta tese apresentamos um modelo que valida a, acima mencionada, segunda lei generalizada. Discutimos como a contribuição de um campo de matéria externo afetado pelos graus de liberdade internos ao horizonte de eventos pode contribuir para a que a entropia generalizada sempre aumente no tempo. Para considerar esse efeito na entropia generalizada, usamos métodos funcionais Euclidianos. Na seção Euclidiana da variedade de Schwarzschild, consideramos um modelo efetivo Euclidiano, uma teoria escalar na presença de uma desordem aditiva *quenched*. A energia livre média sobre o conjunto de configurações possíveis da desordem é obtida pelo método da função zeta distribucional. Na representação em série para a energia livre *quenched* média com as respectivas ações efetivas, aparece um operador de Schrödinger. Vale resaltar que o operador de Schrödinger em variedades Riemannians tem sido amplamente discutido pelos matemáticos. Finalmente, é apresentada a densidade de entropia generalizada com as contribuições da entropia geométrica do buraco negro e dos campos de matéria externa afetados pelos graus de liberdade internos. A validade da segunda lei generalizada da termodinâmica é apresentada.

Palavras-chave: Métodos funcionais Euclidianos, Solução de Schwarzschild Euclidiana, Segunda lei generalizada, Desordem aditiva.

Abstract

Using Euclidean functional integrals, this thesis presents the quantization of the scalar field. Next, we discuss the properties of this field on a specific Riemannian manifold, the Euclidean section of Schwarzschild spacetime. One of the main consequences of general relativity is the existence of black holes. Using quantum field theory within general relativity, important questions have been raised about the thermodynamic properties of black holes, such as their entropy. As Hawking demonstrated that black holes evaporate, a natural question arises: during this evaporation process, would the second law of thermodynamics be violated? This problem has been extensively discussed by many authors. Bekenstein introduced the generalized second law of black hole thermodynamics, which prevents the violation of the second law. In this thesis, we present a model that validates the aforementioned generalized second law. We discuss how the contribution of an external matter field, influenced by the internal degrees of freedom at the event horizon, can ensure that the generalized entropy always increases over time. To account for this effect on generalized entropy, we use Euclidean functional methods. In the Euclidean section of the Schwarzschild manifold, we consider an effective Euclidean model—a scalar theory in the presence of additive quenched disorder. The average free energy over the set of possible disorder configurations is obtained using the distributional zeta function method. In the series representation for the average quenched free energy and the respective effective actions, a Schrödinger operator appears. It is worth noting that the Schrödinger operator on Riemannian manifolds has been widely studied by mathematicians. Finally, the generalized entropy density is presented, including contributions from the geometric entropy of the black hole and external matter fields influenced by the internal degrees of freedom. The validity of the generalized second law of thermodynamics is demonstrated.

Keywords: Generalized second law, Euclidean functional methods, Euclidean Schwarzschild black hole.

Contents

Chapter 1

Introduction

The quantum fields are the mathematical fundamental objects to discuss the physical reality. Those objects can be understood by operator-valued distributions [1]. The connection between relativistic fields and measurable particles is given by asymptotic relations in the LSZ formalism, which establishes the connection between the time-ordered field product and the results of measurements in distant past and future characterizing states corresponding to non-interacting point-like particles [2]. In flat spacetime, the fundamental objects, the fields, can be represented by using creation and annihilation operators, and may be constructed from the idea of counting particles as an operator (the creation operator) acting in a vacuum state. As emphasized by Weinberg, Quantum Field Theory can be understood by the marriage of principles of quantum mechanics and special relativity. This theory is deeply build in fundamental principles as causality and the positivity of energy. The reason that Quantum Field Theory is the standard, and most accepted theory to understand quantum and local field phenomena is the reason that there is an extraordinary agreement between theory and experimental results of non-gravitational phenomena. Efforts to make Quantum Field Theory more rigorously, in a mathematical sense, were made [3]. The so-called Euclidean functional methods for QFT [4–7].

The Euclidean functional method for QFT uses elliptic partial differential operators to define the objects called Euclidean correlations functions, or just, Schwinger functions, those functions are going to be the fundamental objects of the theory. This formalism could be implemented, because Dyson and late Wick and others [8–12] discussed the analytic continuation of a Lorentzian manifold to an Euclidean space with a positive definite metric. Using the positive energy condition, the Schwinger functions of a scalar model are defined as the vacuum expectation values of products of the field operators analytically continued to the Euclidean region.

In contrast, it is known that spacetime has a proper structure, a globally hyperbolic, pseudo-Riemannian manifold. This fundamental characteristic of nature puts the limits of applicability of Quantum Field Theory in test by the formulation of quantum fields in curved spacetime, in such a way problems of different natures arise [13–16]. Nature for itself starts to behave, in a common view, strange in the presence of a curved spacetime, the non-uniqueness of the vacuum [17], which is not an exclusivity of this theory, establish to be the first problem for defining fields in this scenario and leads to interesting consequences, such as: the particle creation scenarios in expanding universes, the Unruh effect and Hawking radiation. Using Unruh's effect as an example, thermal properties of QFT emerges for observers in Minkowski spacetime with rectilinear, uniformly accelerated motion [18–23].

Since the birth of general relativity black holes intrigued physicists in general, but the bound between macro and micro physics, created by the development of QFT in curved spacetime, starts to get even tense, after Bekenstein introduced the concept of entropy for Black Holes [24, 25]. Hawking, using quantum fields in a fixed spacetime background, showed that a black hole of a mass M_0 should emit thermal radiation at a certain temperature β^{-1} proportional to the surface gravity κ of the black hole. This phenomenon is known in the literature as the Hawking effect. To derive this thermal radiation, Hartle and Hawking discussed the semi-classical propagator for a scalar field in the maximally extended Fronsdal-Kruskal manifold [26, 27] with an analytic continuation in the time variable [28]. Some others derivations were done to evaluate quantitatively this radiation, as an example: the thermofield dynamics, studied by Israel [29, 30]. Extending the concept, Hawking's effect is a particular manifestation of relativistic quantum fields that respect periodic conditions in time [31–33] and satisfy the Kubo-Martin-Schwinger condition [34, 35].

The derivation of Bekenstein's entropy for black holes and Hawking's proof of thermal radiation inversely proportional to the black hole mass, leads immediately to the necessity of a second kind of entropy to balance the entropy of the entire system, or in instance, should be reviewed the limits of the second law of thermodynamics. Bekenstein proposed that this second contribution for the entropy should be linked to matter and radiation corrections. This scheme is the so-called generalized second law of black hole thermodynamics, or just, generalized second law [36]. There are different derivations for the generalized second law, but they are not conclusive. This dissertation aims to propose a very simple approach to calculating this second contribution to guarantee the validity of the generalized second law.

This dissertation has the goal to calculate the second contribution of Bekenstein's generalized second law; to do it, it is proposed Euclidean functional methods [37, 38] to investigate how the influence of internal degrees of freedom, behind the event horizon of a Schwarzschild black hole, affects the external matter fields [39–42]. To develop this calculation, it is used functional integrals in Riemannian manifolds for an Euclidean Quantum Field Theory [43–45]. Inspired by Statistical Field Theory, where usually it is not known some particular interactions, an effective model with a "delta" correlated randomness, which varies along the radial coordinate, to mimic its effects is defined. To look for the influence of internal degrees of freedom, coming from the region near the event horizon, over the matter fields, it is proposed a scalar field theory defined in a Euclidean section of a Schwarzschild manifold. In the Euclidean theory defined in a compact domain, it is introduced an additive random field and the generating functional of connected correlations functions, which must be averaged over all the realization of the disorder.

In the literature, the standard way to calculate the average of the generating functional of connected correlations functions is the "replica trick", but in our description, an alternative method is used, the distributional zeta-function method [46–48]. For some specific values for the covariance of the disorder, the operator associated with the new effective action turns into a singular differential operator on a Riemannian manifold. In this new domain, the self-adjointness of the Schrödinger operator defined by the effective actions [49, 50] must be discussed. With countable sets of eigenvalues, it is possible to define a spectral entropy. And, using the zeta function regularization and a technique to calculate functional determinants, called the Gel'fand-Yaglom formalism for functional determinants, it is possible to discuss and calculate with some approximations the matter and radiation contribution of the generalized entropy density of black hole thermodynamics, and consequently, the full generalized second law of black holes. This dissertation presents the results of the Ref. [51].

This dissertation is organized as follows. In Chap. 2, the canonical quantization and Euclidean functional integral quantization for a neutral scalar field is presented. In the next chapter, Chap. 3, the fundamental aspects of the Quantum Field Theory in curved spacetime formalism are discussed. Then, in Chap. 4, the main features of black holes and their thermal properties are also discussed. To continue, Chap. 5 presents the key definitions of disordered systems and the distributional zeta-method, which is used to calculate the average of the generating functional of connected correlations functions. Chap. 6, the self-interacting scalar field in the Euclidean section of the Schwarzschild manifold in the presence of a disorder is discussed. In Chap. 7, it is calculated the generalized Boltzmann-Gibbs-Shannon entropy density, showing the validity of the generalized second law. Conclusions are provided in the last chapter, Chap. 8. Unless it is said in the text, the units $\hbar = c = k_b = 1$ are going to be used.

Chapter 2

Quantum Field Theory

2.1 Canonical Quantization of Spin-0 fields

The canonical quantization of fields attach the axioms of quantum mechanics to a classical theory of fields. In this section, we canonically quantize a real scalar field, which basically means: what are the commutation relations for the operators, what are the equations of motions that the fields respect and what is Hilbert space of the system. Some aspects of the Quantum Field Theory is going to emerge from this derivation, for example, divergences in the zero-point energy. A renormalization procedure is going to be introduced to deal with those divergences problems, called the normal-ordered product. The discussions of this section are based on the Refs. [52–55].

2.1.1 The quantization of the scalar field

To describe an infinite number of degrees of freedom, the classical dynamical variable, denoted by $\phi(x,t) \equiv \phi$, must be a field defined over a four-dimensional continuum, i.e., spacetime, is required. The Lagrangian density of a free scalar field of mass m is given by

$$\mathcal{L}(\phi) = \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - \frac{1}{2}m^2\phi^2, \tag{2.1}$$

where $\mu = (0,1,2,3)$.

From Euler-Lagrange equation $\dfrac{\partial}{\partial x_\mu}\dfrac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} - \dfrac{\partial\mathcal{L}}{\partial\phi} = 0$, leads to the Klein-Gordon equation

$$(\Box + m^2)\phi = 0, \tag{2.2}$$

where $\Box \equiv \eta_{\mu\nu}\partial^\mu\partial^\mu$ is the d'Alembertian operator and $\eta_{\mu\nu}$ is the four-dimensional Lorentztian metric tensor. The canonically conjugate momentum operator is defined by

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}(x), \tag{2.3}$$

where $\dot{\phi}(x)$ is the derivative of the field with respect to time coordinate. Having in mind the definition of $\phi(x)$ and $\pi(x)$, one can construct the Hamiltionian density of the system as

$$\mathcal{H}(x) = \pi(x)\phi(x) - \mathcal{L}(x) = \frac{1}{2}\Big(\pi(x)^2 + (\nabla\phi(x))^2 + m^2\phi(x)^2\Big). \tag{2.4}$$

Now, using the standard prescription of field quantization: the fields $\phi(x,t)$ and $\pi(x,t)$ are promoted to the operators $\hat{\phi}(x,t)$ and $\hat{\pi}(x,t)$, the commutation relations for a simultaneity surface Σ_t are

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\delta^3(x-x'), \tag{2.5}$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = [\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0. \tag{2.6}$$

Using the quantized Hamiltonian operator

$$\hat{\mathcal{H}}(x) = \frac{1}{2}\Big(\hat{\pi}(x)^2 + (\nabla\hat{\phi}(x))^2 + m^2\hat{\phi}(x)^2\Big), \tag{2.7}$$

and the commutation relations given by the equations (2.5) and (2.6), after some intermediate steps it is possible to calculate the Hamilton's equation of motion

$$\dot{\hat{\phi}}(x,t) = -i[\hat{\phi}(x,t), \hat{\mathcal{H}}] = \hat{\pi}(x,t), \tag{2.8}$$

and

$$\dot{\hat{\pi}}(x,t) = -i[\hat{\pi}(x,t), \hat{\mathcal{H}}] = (\nabla^2 - m^2)\hat{\phi}(x,t). \tag{2.9}$$

Visualizing the Eq. (2.9) it is understandable that the field operator of the quantized theory <u>still</u> satisfies the Klein-Gordon equation

$$\ddot{\hat{\phi}}(x,t) = (\nabla^2 - m^2)\hat{\phi}(x,t). \tag{2.10}$$

Now, to construct the Fock space of the theory, it is necessary to expand the field operator $\hat{\phi}(x,t)$ in Fourier modes (given by the set of plane waves solution $u_p(x) = N_p \exp\{ipx\}$), which is

$$\hat{\phi}(x,t) = \int d^3p\, N_p \exp\{ipx\}\hat{a}_p(t), \tag{2.11}$$

where N_p is a normalization constant.

Rewriting the Eq. (2.10) in terms of plane waves, a very similar equation of motion for the operators $\hat{a}_p(t)$ appears

$$\ddot{\hat{a}}_p(t) = -(p^2 + m^2)\hat{a}_p(t). \tag{2.12}$$

The solution of the Eq. (2.12) is given by

$$\hat{a}_p(t) = \hat{a}_p e^{-i\omega_p t} + \hat{a}^\dagger_{-p} e^{+i\omega_p t}, ^* \tag{2.13}$$

where the frequency ω_p is defined to be the relativistic dispersion relation

$$\omega_p = +\sqrt{p^2 + m^2}, \tag{2.14}$$

*Since it is started from a real-valued classical field, the corresponding operator have to be Hermitian. Following this ideia, it is possible to express $\hat{a}_p(t)$ as a function of the operators $\hat{a}_p$ and $\hat{a}^\dagger_{-p}$

and the operators $\hat{a}_p$ and $\hat{a}_p^\dagger$ are constant in time.

Now, the Eq. (2.11) as a function of the operators a_p and $a_p^\dagger$ can be rewritten as

$$\hat{\phi}(x,t) = \int d^3p\, N_p(\hat{a}_p e^{i(px-\omega_p t)} + \hat{a}_p^\dagger e^{-i(px-\omega_p t)}). \tag{2.15}$$

From Eq. (2.9) is is achievable to re-write the conjugate field as a function of the new basis expansion

$$\hat{\pi}(x,t) = \int d^3p\, N_p(-i\omega_p)(\hat{a}_p e^{i(px-\omega_p t)} - \hat{a}_p^\dagger e^{-i(px-\omega_p t)}). \tag{2.16}$$

Using the commutation relation of the fields $\hat{\phi}(x,t)$ and $\hat{\pi}(x,t)$ in equations (2.5) and (2.6), it is possible to find a commutation relation for the operators $\hat{a}_p$ and $\hat{a}_p^\dagger$

$$[\hat{a}_p, \hat{a}_{p'}^\dagger] = \delta^3(p-p'), \tag{2.17}$$

$$[\hat{a}_p, \hat{a}_{p'}] = [\hat{a}_p^\dagger, \hat{a}_{p'}^\dagger] = 0. \tag{2.18}$$

The operators $\hat{a}_p$ and $\hat{a}_p^\dagger$, similarly as done for harmonic oscillators in quantum mechanics, are interpreted as the annihilation and creation operators, and the Fock space of the system can be understood by the action of those field operators on the vacuum state

$$\hat{a}_p \left|0\right\rangle = 0, \tag{2.19}$$

and

$$\hat{a}_p^\dagger \left|0\right\rangle = \left|1\right\rangle, \tag{2.20}$$

which means that the annihilation operator "destroys" the vacuum and the creation operator excites the vacuum, or in other words, it creates a quantum of momentum p.

The Hamiltonian can be rewritten as a function of the operators $\hat{a}_p$ and $\hat{a}_p^\dagger$ as

$$\hat{H} = \frac{1}{2}\int d^3p\, \omega_p(\hat{a}_p^\dagger \hat{a}_p + \hat{a}_p \hat{a}_p^\dagger), \tag{2.21}$$

the expectation value of the Hamiltonian in the vacuum state diverges

$$\left\langle 0\right| \hat{H} \left|0\right\rangle \longrightarrow \infty. \tag{2.22}$$

A possible manner to take physical properties of this divergence is to work on the discretized version of the theory by confining the system to a finite box, with volume $V = L^3$. This modification leads to discrete values for the momenta p_l, which are given by $p_l = \dfrac{2\pi l}{L}$. The vacuum energy expectation value can be written by

$$\left\langle 0\right| \hat{H} \left|0\right\rangle = E_0 = \frac{\pi}{L}\sum_l l. \tag{2.23}$$

Since a physical observable usually involves differences of energy, the divergent zero-point energy E_0 is not a problem, in Minkowski spacetime. It can be understood by shifting the Hamiltonian in a way that the vacuum energy is removed

$$\hat{H}' = \hat{H} - E_0. \tag{2.24}$$

Divergences are very common in Quantum Field Theory, for this reason, there are effective procedures to take out those divergences. The method of treating those divergences is named as "regularization" and "renormalization". This process of renormalizing this problem consists in splitting the field operator in two parts, one containing positive frequencies and the other containing negative frequencies

$$\hat{\phi} = \hat{\phi}^{(-)} + \hat{\phi}^{(+)}. \tag{2.25}$$

Defining the first part as the information of the annihilation operators and the second part as the creation operators. Introducing the notion of "normal ordering" of two operators $\hat{\phi}$ and $\hat{\psi}$, which is defined as a product where the negative frequency part stands to the left of the positive frequency part

$$: \hat{\phi}\hat{\psi} := \hat{\phi}^{(-)}\hat{\psi}^{(-)} + \hat{\phi}^{(-)}\hat{\psi}^{(+)} + \hat{\psi}^{(-)}\hat{\phi}^{(+)} + \hat{\phi}^{(+)}\hat{\psi}^{(+)}. \tag{2.26}$$

Then, it should be defined the Hamiltonian as a normal-ordered product of the field operators

$$\hat{H}' = : \frac{1}{2}\int d^3x \Big(\hat{\pi}(x)^2 + (\nabla\hat{\phi}(x))^2 + m^2\hat{\phi}(x)^2\Big) : \tag{2.27}$$

$$= \int d^3p \, \hat{a}_p^\dagger \hat{a}_p \omega_p. \tag{2.28}$$

Using the normal-ordered product, the creation operators go to the left side of the annihilation operators and the problem of divergences does not appear at all.

It is valid to make a comment that the vacuum energy is not a pure and ignorable characteristic of the theory. For example, the Casimir effect consists of measurements of energy differences between two matter configurations

$$E^{(1)} - E^{(2)} \longrightarrow finite, \tag{2.29}$$

where $E^{(1)}$ refers to a specif geometry and $E^{(2)}$ refers to different boundary conditions over the field ϕ. The Casimir effect is an interesting feature in QFT, with multiples applications and discussions in literature, see, e.g. [56–59].

2.2 Functional integral quantization of the free scalar field

In the last section, it was discussed the canonical method for the quantization of the scalar field, where fields operators were introduced. Such operators act in the so-called Fock space and, it was also introduced by the canonical commutation rules of the theory. There is an alternative formalism to deal with fields quantization. The **Functional integral method** give up the idea of operators, instead the quantization is done by functional integration over classical fields. An heuristic idea to give a physical intuition of the situation was given by Feynman in last century. Feynman states that a motion of a particle between two points can move on an infinite variety of classical trajectories and each of these possible paths contributes to the transition amplitude of the particle. In the functional integral formalism, all properties of the system, in principle, can be deduced using functional methods.

It is important to emphasize that the functional integral formalism and the canonical quantization are completely equivalent. As usually happens when different formalisms that lead to the same prediction are available, one method seems more efficient than the other and vice-versa.

2.2.1 The Euclidean time

Let us discuss the Euclidean action of a point particle in a generic potential. Let's define the imaginary (Euclidean) time τ as

$$t = -i\tau, \tag{2.30}$$

for all $\tau > 0$. Using the Euclidean time, the time evolution operator is written as

$$e^{-H\tau}, \tag{2.31}$$

where H is the Hamiltonian of the system. The probability amplitude of a system to evolve from a point y to a point x is given by

$$\langle x| e^{-H\tau} |y\rangle = \int e^{-S_E} Dx, \tag{2.32}$$

where Dx is a functional measure and

$$S_E = \int_0^t \Big(\frac{m\dot{x}^2}{2} + v(x)\Big) d\tau', \tag{2.33}$$

is the Euclidean action. Note that the Euclidean action and the classical action are related by

$$S|_{t=-i\tau} = iS_E. \tag{2.34}$$

From Eq. (2.32) it is clear that the functional integral is weighted by e^{-S_E} and the exponential is real. There is a strong advantage of such a treatment instead of the Feynman path integrals. Since there is no convergence using path integrals, using euclidean methods the integrand naturally converges.

A convenient way to express the expectation values $\langle 0| A |0\rangle$, for a given operator A, in the ground state as Euclidean functional integrals. Let

$$Tr(e^{-H\tau}A) = \sum_{n=0}^{\infty} e^{-\tau E_n} \langle n| A |n\rangle \,, \tag{2.35}$$

and defining

$$Z(\tau) = Tr(e^{-H\tau}) = \sum_{n=0}^{\infty} e^{-\tau E_n}. \tag{2.36}$$

For $\tau \to \infty$, the ground state E_0 dominates and it follows that

$$\langle 0| A |0\rangle = \lim_{\tau\to\infty} \frac{Tr(e^{-H\tau}A)}{Tr(e^{-H\tau})}. \tag{2.37}$$

Notice that the Eq. (2.37) is equivalent as an average in canonical statistical ensemble. This is very important for Quantum Field Theory and for future discussions, because some methods of statistical field theory are going to be used. A possible case of Eq. (2.37) are the so-called correlation functions

$$\langle x(t_1)...x(t_n)\rangle \equiv \langle 0|\, x(t_1)...x(t_n)\, |0\rangle\,, \tag{2.38}$$

using analytic continuation for Euclidean time $\tau_k = it_k$, the correlation functions turns into

$$\langle x(t_1)...x(t_n)\rangle = \lim_{\tau\to\infty} \frac{1}{Z(\tau)} \int x(\tau_1)...x(\tau_n) e^{-S_E[x(\tau)]} Dx, \tag{2.39}$$

where

$$Z(\tau) = \int e^{-S_E[x(\tau)]} Dx. \tag{2.40}$$

Using the Euclidean time it is possible to turn the metric of space-time (Minkowski metric) into an Euclidean metric, setting the time coordinate as

$$x^0 = -ix^4. \tag{2.41}$$

Then, if the metric for the coordinates (x^0, x^1, x^2, x^3) was the Minkowski metric then the coordinates (x^1, x^2, x^3, x^4) becomes an Euclidean metric.

2.3 The Euclidean rotation

Defining the Wightman function as the n^{th} point correlation functions, in Minkowski spacetime, of the scalar field ϕ as

$$\mathcal{W}(x_1, ..., x_n) = \langle 0|\, \phi(x_1)...\phi(x_n)\, |0\rangle\,. \tag{2.42}$$

The Wightman functions are very important in Quantum Field Theory, because there is a connection between these functions and the time-ordered Green functions, which are the building blocks of all physical information. Writing the operator field as the following manner

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx}, \tag{2.43}$$

where P is the generator of translations, and if rewrite x_k as

$$x_k = u_k - iy_k, \tag{2.44}$$

using the spectrum condition, for the forward light cone, the Wightman functions can be extended analytically into this region. Then, it is possible to define the Schwinger functions as

$$\mathcal{S}(...; \vec{x}_k, x_k^4; ...) \equiv \mathcal{W}(...; -ix_k^4, \vec{x}_k; ...). \tag{2.45}$$

From the definition of the Wightman functions and from the appendix A, the 2-point function is

$$\mathcal{W}(x_1, x_2) \equiv \mathcal{W}(x_1 - x_2). \tag{2.46}$$

We have that $\mathcal{W}(x)$ is analytic in the lower half complex plane and analogously for $\mathcal{W}'(x) \equiv \mathcal{W}'(-x)$ in the upper half complex plane. Since $\mathcal{W}(x_1, x_2) = \mathcal{W}(x_2, x_1)$, we have that for $\mathcal{W}$ and $\mathcal{W}'$ are equal for a space-like coordinate. Consequently, they are equal on a open region, which implies that they form a single analytic function in the union of their domains of definition.

Following the 2-point function example, we know that we get the Wightman functions from the Schwinger functions if they approach the x^0 real axis. If we generalize it, we find for the n-point functions that

$$\mathcal{W}(x_1, ..., x_n) = \lim_{\epsilon_k \to 0, \epsilon_k - \epsilon_{k+1} > 0} \mathcal{S}(...; \vec{x}_k, ix_k^0 + \epsilon_k; ...), \tag{2.47}$$

for $x_k \in \mathbb{R}$.

Now, introducing the time-ordered Green functions as

$$\tau_n(x_1, ..., x_n) = \langle 0| T\phi(x_1)...\phi(x_n) |0\rangle , \tag{2.48}$$

Defined as

$$T\phi(x_1)\phi(x_2) = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1), \tag{2.49}$$

where θ is the Heaviside step function, and we can verify that τ is symmetric in its arguments and for the 2-point function, we have that

$$\tau(x) = \begin{cases} \mathcal{W}(x); & x^0 > 0, \\ \mathcal{W}(-x); & x^0 < 0. \end{cases} \tag{2.50}$$

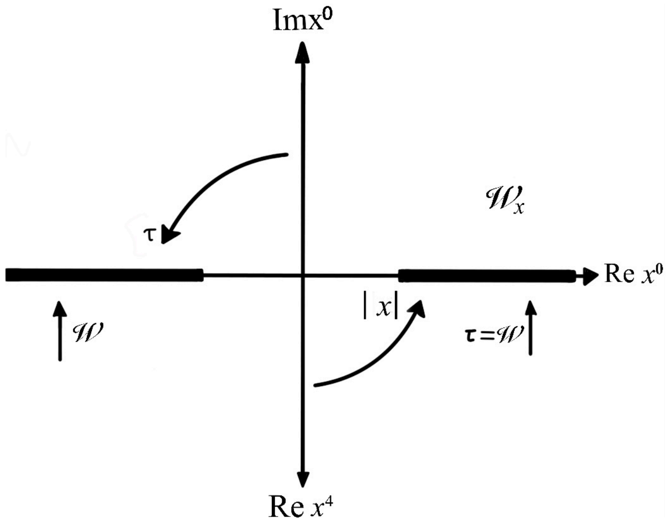

Figure 1 – Wick Rotation.

Concluding, knowing that $\mathcal{W}(x)$ is analytic in the lower half complex plane and that $\mathcal{W}(-x)$ is analytic in the upper half complex plane, we obtain the function τ by approaching the x^0 real axis in the complex plane through a counter-clockwise rotation, see fig.1. Generalizing this, we find the Wick Rotation

$$\tau(x_1, ..., x_n) = \lim_{\phi \to \frac{\pi}{2}} \mathcal{S}(...; \vec{x}_k, e^{i\phi}x_k^0; ...) \tag{2.51}$$

2.4 The functional integral formalism in QFT

We start this section by considering the scalar Euclidean field as random variables and not as operators, because by the property of reflection positivity of Schwinger functions, Euclidean fields commutes, which means that they have the same behavior as classical fields. The expectation value is defined as

$$\langle A[\phi]\rangle = \int A[\phi]d\mu, \tag{2.52}$$

where the Euclidean functional integral is given by the measure

$$d\mu = \frac{1}{Z}e^{-S[\phi]}\prod_x d\phi(x), \tag{2.53}$$

where $Z = \int \prod_x d\phi(x)e^{-S[\phi]}$, and $S[\phi]$ is the Euclidean action. We can combine the two equations above and write

$$\langle A[\phi]\rangle = \frac{1}{Z}\int \prod_x d\phi(x)A[\phi]e^{-S[\phi]}, \tag{2.54}$$

which is analogue to the Eq. (2.37) for the Euclidean functional integral formalism. Again, we see that there is a analogy with statistical mechanics, the term $e^{-S[\phi]}$ can be interpreted as the Boltzmann factor, which is going to be explored in the future sections and chapters.

2.4.1 The free scalar field

The main goal of this section is try to find the Euclidean correlations functions $\langle\phi(x_1)...\phi(x_n)\rangle$ for a scalar field. We may write the Euclidean free action for the scalar field as

$$S_0[\phi] = \int\Big(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2\Big)d^4x = \int \frac{1}{2}\phi(x)(-\Delta + m^2)\phi(x)d^4x, \tag{2.55}$$

where $-\Delta$ is is the Laplacian operator in 4 dimensions. The propagator

$$G(x, y) = \langle\phi(x)\phi(y)\rangle\,, \tag{2.56}$$

satisfies the following relation

$$(-\Delta + m^2)G(x, y) = \delta^4(x - y). \tag{2.57}$$

Hence, we find that the propagator is

$$G(x, y) = \int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{1}{(p^2 + m^2)}. \tag{2.58}$$

Defining a field $j(x)$, we have that the inner product of $j(x)$ with ϕ is

$$(j, \phi) = \int j(x)\phi(x)d^4x < \infty, \tag{2.59}$$

and so we can define the generating functional of Green's functions as

$$Z[j] \equiv \langle e^{(j,\phi)}\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1...d^4x_n j(x_1)...j(x_n)G(x_1, ..., x_n), \tag{2.60}$$

which is normalized to $Z[0] = 1$. By these definitions and with functional derivatives, we can write the n-point function as the n^{th} functional derivatives of the generating functional with respect of the sources $j(x_i)$, evaluated at $j = 0$.

$$G(x_1, ..., x_n) = \frac{\delta^n Z[j]}{\delta j(x_1)...\delta j(x_n)}\Big|_{j=0}. \tag{2.61}$$

We may note that for a free field, the generating functional is

$$Z[j] \equiv Z_0[j] = e^{\frac{1}{2}\int d^4x d^4y j(x)G(x,y)j(y)} = e^{\frac{1}{2}(j,G_0 j)}. \tag{2.62}$$

From Eq. (2.57) we find that, $G^{-1} = (-\Delta + m^2)$, and follows that

$$\frac{1}{2}(\phi, G^{-1}\phi) = \frac{1}{2}(\phi, (-\Delta + m^2)\phi) = \frac{1}{2}\int d^4x\, \phi(x)(-\Delta + m^2)\phi(x) = S_0[\phi]. \tag{2.63}$$

Using Eq. (2.62) and Eq. (2.63), we find that

$$Z_0[j] = \frac{1}{Z_0}\int \prod_x d\phi(x) \exp\Big\{(-\frac{1}{2}(\phi, (-\Delta + m^2)\phi) + (j, \phi))\Big\} = \frac{1}{Z_0}\int \prod_x d\phi(x) \exp\{-S_0 + (j, \phi)\}, \tag{2.64}$$

where

$$Z_0 = \int \prod_x d\phi(x) \exp\Big\{-\frac{1}{2}(\phi, (-\Delta + m^2)\phi)\Big\}. \tag{2.65}$$

There is a difference between the infinite-dimensional integral and the Gaussian integral. The infinite-dimensional integral is not well defined, but it is possible to solve the problem by defining the measure

$$d\mu_0(\phi) = \frac{1}{Z_0} D[\phi] e^{-S_0(\phi)}, \tag{2.66}$$

where

$$D[\phi] \equiv \prod_x d\phi(x). \tag{2.67}$$

Concluding, the n-point function is

$$\langle\phi(x_1)...\phi(x_n)\rangle = \int \phi(x_1)...\phi(x_n)\, d\mu_0 = \frac{1}{Z_0}\int \prod_x d\phi(x) e^{-S_0[\phi]}\phi(x_1)...\phi(x_n). \tag{2.68}$$

2.4.2 The interacting theory

In this subsection we are going to discuss the behavior of the functional integrals for an interacting field theory. Following the Euclidean action

$$S[\phi] = S_0[\phi] + S_I[\phi], \tag{2.69}$$

where the action $S_I[\phi]$ describes the interacting field part.

The Dyson's formula for the correlation functions in Euclidean space is given by is given by

$$\langle\phi(x(1))...\phi(x(n))\rangle = \frac{\langle 0|\, \phi_{in}(x(1))...\phi_{in}(x(n)) e^{-S_I[\phi_{in}]}\, |0\rangle}{\langle 0|\, e^{-S_I[\phi_{in}]}\, |0\rangle}. \tag{2.70}$$

We may rewrite the Dyson's formula defined above in terms of the generating functional

$$Z[j] = \frac{\langle 0| \, e^{-S_I[\phi_{in}]+(j,\phi_{in})} \, |0\rangle}{\langle 0| \, e^{-S_I[\phi_{in}]} \, |0\rangle}, \tag{2.71}$$

where ϕ_{in} is the free field, i.e., without any interaction. Comparing to the Gaussian functional integral we find that

$$Z[j] = \frac{1}{Z} \int \prod_x d\phi_{in}(x) e^{-S[\phi_{in}]+(j,\phi)}, \tag{2.72}$$

where

$$Z = \int \prod_x d\phi_{in}(x) e^{-S[\phi_{in}]}, \tag{2.73}$$

and the correlation functions are given by

$$\langle \phi(x_1)...\phi(x_n) \rangle = \frac{1}{Z} \int \prod_x d\phi_{in}(x) e^{-S[\phi_{in}]} \phi_{in}(x_1)...\phi_{in}(x_n). \tag{2.74}$$

2.4.3 The generating functional of connected correlation functions and the mean-field theory

The generating functional of connected correlation functions $W[j]$ is defined as

$$W[j] = \ln Z[j]. \tag{2.75}$$

The Green functions associated with $W[j]$ are leading to all connected diagrams, i.e, all the physical information is arising from this functional. By the definitions in Euclidean formalism, the order parameter associated with a statistical field theory can be defined as:

$$\langle \phi \rangle = \frac{\delta W[j]}{\delta j(x)}|_{j=0}. \tag{2.76}$$

To implement the mean-fied approximation, we need to define the so-called classical field,

$$\langle \phi \rangle_c (x)[j] \equiv \frac{\delta W[j]}{\delta j(x)}, \tag{2.77}$$

where $\langle \phi \rangle_c (x)[j]$ is the classical field.

Given these definitions, suppose that the generating functional of all correlation functions is written as

$$Z[j] = \int e^{-a^{-1}S[\phi,j]} D\phi, \tag{2.78}$$

where a is a positive constant. Let analyze the limit where the parameter a is very small. We may show that this integral can be approximated by the value of the integrand at its critical point. This leads us to the functional differential equation (in the zero-order approximation in a), which is given by

$$\frac{\delta S[\phi_c(x)[j], j]}{\delta \phi_c(x)[j]} - j(x) = 0, \tag{2.79}$$

which is the so-called Dyson-Schwinger equation of zeroth-order. For the particular case where $j(x) = 0$, we have a differential equation for the order parameter, which is just the classical equation of motion for the field. This equation is the so-called the mean-field equation, described as

$$\frac{\delta S[\phi_c(x)]}{\delta \phi_c(x)} = 0. \tag{2.80}$$

2.4.4 Spontaneous symmetry breaking

To construct a consistent theory, it is necessary to have a quantity that can be identified as the system's energy, and this quantity must have a global minimum. The quantum state corresponding to this minimum is called the quantum vacuum, and the entire theory is built upon it.

In the functional formalism, the Euclidean action is identified as the energy, which explicitly depends on the fields $\phi(x)$. A symmetry is said to be spontaneously broken if it remains a symmetry of the action but is no longer a symmetry of other physical objects calculated in the vacuum state. For example, the action of the $\lambda\phi^4$ theory has the symmetry $\phi \to -\phi$. In this case, spontaneous symmetry breaking occurs if

$$\langle 0|\,\phi\,|0\rangle \equiv \langle 0\rangle \neq 0, \tag{2.81}$$

where $|0\rangle$ is defined as the vacuum state of the theory.

This phenomenon is crucial because it can explain the emergence of distinct physical properties in systems that are symmetric at a fundamental level. For example, in particle physics, spontaneous symmetry breaking is essential for understanding the masses of elementary particles through the Higgs mechanism [60].

We define as a fundamental state of the theory a mean field $\langle\phi\rangle \equiv \varphi$, which satisfies the mean-field Eq. (2.80). Using the mean-field equation, and the scalar field theory with a interaction potential $V(x)$, we have that

$$-\Delta\varphi + \frac{\delta}{\delta\varphi}\Big(\frac{1}{2}m^2\varphi^2 + V(x)\Big) = 0. \tag{2.82}$$

We must conclude that

$$\frac{dU(\varphi)}{d\varphi} = 0, \tag{2.83}$$

where

$$U(\varphi) = \frac{1}{2}m^2\varphi^2 + V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4. \tag{2.84}$$

So, the ground state must be not only a critical point of the potential $U(\varphi)$ but a global minimum. For a theory with all coupling constants positive, the only extreme is $\varphi(x) = 0$, where this discussion is not relevant. However, we can analyze a more general case where the parameter λ can be negative and therefore it is possible that there are values of $\varphi \neq 0$ that correspond to global minima. The problem boils down to finding the global minima of the potential $U(\varphi)$, while maintaining the condition that $m^2 > 0$. The latter condition is fundamentally necessary for the energy to possess a global minimum.

For this problem, we must find the saddle-points of a particular potential, which are

$$\frac{dU(\varphi)}{d\varphi} = 0 \longrightarrow m^2\varphi + \frac{1}{3!}\lambda\varphi^3 = 0 \longrightarrow \varphi(m^2 + \frac{1}{3!}\lambda\varphi^2) = 0. \tag{2.85}$$

Clearly, $\varphi = 0$ is the trivial solution, as we were expecting. We may now investigate the non-trivial solution of this equation. They are given by,

$$\varphi_1 = +\sqrt{\frac{-6m^2}{\lambda}}, \tag{2.86}$$

and

$$\varphi_2 = -\sqrt{\frac{-6m^2}{\lambda}}. \tag{2.87}$$

The fact that $\lambda < 0$ and $m^2 > 0$ ensures that the potential vacuum is bounded from below. Having these new definitions of vacuum state, we can expand the field around the new vacuum, by a change of coordinates

$$\bar{\phi}(x) = \phi(x) + \varphi. \tag{2.88}$$

In this way we can conclude that,

$$\langle\bar{\phi}(x)\rangle = 0. \tag{2.89}$$

2.4.5 The perturbative expansion

The aim of this subsection is to derive the Feynman rules from functional integrals. Let's begin with a scalar field with a $\lambda\phi^4$ interaction

$$S_I[\phi] = \frac{\lambda}{4!}\int d^4x\phi(x)^4. \tag{2.90}$$

From Eq. (2.74) and expanding the exponential of the interaction, we find that the n-point function is given by

$$G(x_1, ..., x_n) = \frac{1}{Z}\int\prod_x d\phi(x)e^{-S_0[\phi]}\sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{\lambda}{4!}\int d^4x\phi(x)^4\Big)^n\phi(x_1)...\phi(x_n). \tag{2.91}$$

From the Eq. (2.73), we can write the generating functional as

$$\begin{aligned}
Z[j] =& \frac{1}{Z}\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_x d\phi(x)\Big(-\frac{\lambda}{4!}\int d^4x\phi(x)^4\Big)^n e^{-S_0[\phi]+(j,\phi)} = \\
=& \frac{1}{Z}\sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{\lambda}{4!}\int d^4x\Big(\frac{\delta}{\delta j(x)}\Big)^4\Big)^n\int\prod_x d\phi(x)e^{-S_0[\phi]+(j,\phi)} = \\
=& \frac{1}{Z}\exp\Big\{-\frac{\lambda}{4!}\int d^4x\Big(\frac{\delta}{\delta j(x)}\Big)^4\Big\}Z_0[j] = \\
=& \frac{1}{Z}\exp\Big\{\Big(S_I\Big[\frac{\delta}{\delta j(x)}\Big]\Big)\Big\}\exp\Big\{\Big[\frac{1}{2}(j, G_0 j)\Big]\Big\}\Big|_{j=0}.
\end{aligned} \tag{2.92}$$

In an analogous way, we can find that the Green functions are written as

$$G(x_1, ..., x_n) = \frac{1}{Z}\frac{\delta}{\delta j(x_1)}...\frac{\delta}{\delta j(x_n)}\exp\Big\{\Big(S_I\Big[\frac{\delta}{\delta j(x)}\Big]\Big)\Big\}\exp\Big\{\Big[\frac{1}{2}(j, G_0 j)\Big]\Big\}\Big|_{j=0}. \tag{2.93}$$

Those Green functions can be interpreted graphically. By expanding the exponential of $\Big(S_I\Big[\frac{\delta}{\delta j(x)}\Big]\Big)$, we have that:

1. each of the derivatives $\dfrac{\delta}{\delta j(x_k)}$ is represented as an external point from which a line emerges (the "external scalar term").
2. each factor $-\lambda\int d^4x\Big(\dfrac{\delta}{\delta j(x_k)}\Big)^4$ is indicated by an external vertex, from which the four lines emerges (the "vertex term").

We can also expand the exponential $(\frac{1}{2}(j,G_0 j))$, finding out that this term indicate the internal lines (the "propagator term").

Figure 2 – External scalar term.

Figure 3 – Vertex term.

Figure 4 – Propagator term.

When we have no external points, the graphs are called vacuum graphs. Integrals over internal loops, usually leads to divergences. This problem comes that since the beginning the $Z[j]$ functional is not well-defined. So, for example, if we have terms proportional to

$$\lambda\int d^4p\frac{1}{p^2+m^2}, \tag{2.94}$$

which diverge, we clearly need to find a method to evaluate this integral, in a way of separating the divergent part. In perturbation theory it is possible to find and separate the divergences and such a procedure is know as regularization, which can be achieved by different schemes. For one example, we can use dimensional regularization, where we evaluate the integral above not in the four dimensional space, but in $4+\epsilon$ dimensions

$$\lambda\int d^{4+\epsilon}p\frac{1}{p^2+m^2}, \tag{2.95}$$

where the terms with ϵ will lead to divergences, that are going to be treated; or, also, a cut-off method can be implemented, where we use a parameter Λ to avoid the loop integration to run up to arbitrarily large momenta. These regularization methods allow one to identify the divergent contributions in the loop integrals. Hence, those divergences can be formally reabsorbed by a redefinition of the parameters of the theory: this is called renormalization, and the renormalized values of the parameters are considered as the physical parameters (that need to be compared with the experiments). There are field theories for which, at every order in perturbation theory, only a finite number of parameters needs to be renormalized, they are called the renormalizable theories.

Chapter 3

Quantum Field Theory in curved spacetime

In Quantum Field Theory in curved spacetime formalism we study how quantum fields behave in fixed curved spacetime backgrounds. In this chapter we intend to introduce the necessary formalism to understand some important results of this theory, and the basic objects that we are going to discuss later on. This chapter is based on references [61, 62].

3.1 Spacetime structure

Here, we assume spacetime as a $\mathcal{C}^{\infty}$ n-dimensional, globally hyperbolic, pseudo-Riemannian manifold as discussed in [63].

The pseudo-Riemannian metric $g_{\mu\nu}$ is associated with the given line element by

$$ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\nu}. \tag{3.1}$$

where $\mu, \nu = 0, 1, ..., (n-1)$. We denote the determinant of the metric as

$$g = ||g_{\mu\nu}||. \tag{3.2}$$

To understand better the causal structure of spacetime, we make use of the Penrose conformal diagrams [64]. Those diagrams are constructed by applying a conformal transformation to the metric structure, see fig.**??**. The advantage is that we may represent the whole infinite spacetime in a finite diagram. The conformal transformation of the metric is defined as

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \tag{3.3}$$

for a continuous, real function $\Omega(x)$. For this transformation of the metric, we can show that the Christoffel symbol, Ricci tensor and Ricci scalar transform, respectively, as

$$\Gamma^{\rho}_{\mu\nu} \to \bar{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \Omega^{-1}(\delta^{\rho}_{\mu}\Omega_{;\nu} + \delta^{\rho}_{\nu}\Omega_{;\mu} - g_{\mu\nu}g^{\rho\alpha}\Omega_{;\alpha}), \tag{3.4}$$

$$R^{\nu}_{\mu} \to \bar{R}^{\nu}_{\mu} = \Omega^{-2} R^{\nu}_{\mu} + (n-2)\Omega^{-1}(\Omega^{-1})_{;\mu\rho} g^{\rho\nu} + (n-2)^{-1}\Omega^{-\mu}(\Omega^{\mu-2})_{;\rho\sigma} g^{\rho\sigma}\delta^{\nu}_{\mu}, \tag{3.5}$$

$$R \to \bar{R} = \Omega^{-2} R + 2(n-1)\Omega^{-3}\Omega_{;\mu\nu} g^{\mu\nu} + (n-1)(n-4)\Omega^{-4}\Omega_{;\mu}\Omega_{;\nu} g^{\mu\nu}, \tag{3.6}$$

The standard example of a Penrose Diagram is the Minkowski in 2-dimensional space, which has the line element

$$ds^2 = dt^2 - dx^2, \tag{3.7}$$

In terms of null coordinates u,v, defined by

$$u = t - x, \tag{3.8}$$

$$v = t + x, \tag{3.9}$$

the line element of Eq. (3.7) becomes

$$ds^2 = dudv, \tag{3.10}$$

so the metric tensor is given by

$$g_{\mu\nu} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.11}$$

Now, supposing that

$$u' = 2\tan^{-1} u, \tag{3.12}$$

$$v' = 2\tan^{-1} v, \tag{3.13}$$

where,

$$u' \geq -\pi, v' \leq \pi, \tag{3.14}$$

we can write Eq. (3.10) as

$$ds^2 = \frac{1}{4}\sec^2\left(\frac{1}{2}u'\right)\sec^2\left(\frac{1}{2}v'\right)du'dv', \tag{3.15}$$

so we re-write Eq. (3.11) as

$$g_{\mu\nu}(u', v') = \frac{1}{8}\sec^2\left(\frac{1}{2}u'\right)\sec^2\left(\frac{1}{2}v'\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.16}$$

Performing the following conformal transformation

$$\Omega^2(x) = \left(\frac{1}{4}\sec^2\left(\frac{1}{2}u'\right)\sec^2\left(\frac{1}{2}v'\right)\right)^{-1}, \tag{3.17}$$

then,

$$g_{\mu\nu}(u', v') \to \bar{g}_{\mu\nu}(u', v') = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.18}$$

and the conformally related line element is

$$\bar{ds}^2 = du'dv'. \tag{3.19}$$

The last equation has the same line element of the Eq. (3.10), but now we are just looking at the compact region (3.14). Effectively, the conformal transformation shrank the infinities to the boundary line on the diagram of Fig.5. Null rays remain at $\frac{\pi}{4}$ in the Penrose diagram, which is a feature of those diagrams. Penrose diagrams leave the null cones invariant, so any causal analysis can be proceed with the null rays as if it was in Minkowski space. The boundary lines $\mathcal{J}^+$ and $\mathcal{J}^-$ are called future and past null infinity, respectively. Also, asymptotically timelike lines converge on the points i^+ and i^- called the timelike future and timelike past infinity, respectively. In an analogous way, asymptotically spacelike lines converge on i^0 (spacelike infinity).

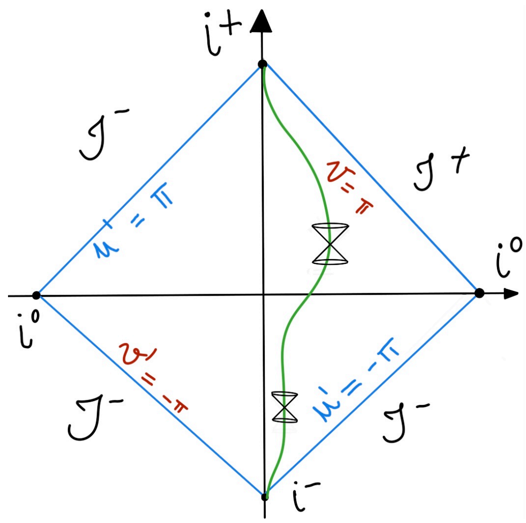

Figure 5 – Penrose diagram for the Minkowski space.

The development of the formalism of Quantum Field Theory in curves spacetimes often depends on the existence of symmetries of the underlying geometry. We can describe those symmetries using Killing vectors, denoted by ξ^μ, which are solutions of Killing's equation

$$\mathcal{L}_\xi g_{\mu\nu}(x) = 0, \tag{3.20}$$

where, $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ^μ. The Lie derivative of a vector field v along ξ is defined as

$$\mathcal{L}_\xi v^a = \xi^b \nabla_b v^a - v^b \nabla_b \xi^a, \tag{3.21}$$

where ∇_b is the covariant derivative. The last equation can also be written as

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{3.22}$$

The geometry is said to admit a conformal Killing vector field, if satisfy the conformal generalization of Eq. (3.20):

$$\mathcal{L}_\xi g_{\mu\nu}(x) = \lambda(x) g_{\mu\nu}(x), \tag{3.23}$$

where, $\lambda(x)$ is a scalar function.

3.2 Scalar field quantization

The scalar field quantization in curved spacetime is quite similar to the quantization in Minkowski space. The Lagrangian density for the scalar field is given by

$$\mathcal{L}(x) = \frac{1}{2}[-g]^{\frac{1}{2}}\Big[g^{\mu\nu}(x)\phi(x)_{,\nu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x)\Big], \tag{3.24}$$

where $\phi(x)$ is the scalar field and m the mass parameter. Notice that there exists a coupling between the scalar field $\phi(x)$ and the Ricci scalar curvature $R(x)$ given by the term $\xi R(x)\phi^2(x)$, where ξ is a real number. The action is

$$S = \int d^n x \mathcal{L}(x), \tag{3.25}$$

where n is the dimension of the spacetime. Using the action variational principle with respect to $\phi(x)$, leads to the equation of motion

$$\Big[\Box_x + m^2 + \xi R(x)\Big]\phi(x) = 0, \tag{3.26}$$

where $\Box_x$ is the so called Laplace-Beltrami operator, defined as

$$\Box_x = g^{\mu\nu}\nabla_\mu\nabla_\nu = (-g)^{-\frac{1}{2}}\partial_\mu[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu], \tag{3.27}$$

There are two interesting values for ξ: the $\xi = 0$ (called the minimally coupled case), and the conformally coupled case

$$\xi \equiv \xi(n) = \frac{1}{4}\frac{(n-2)}{(n-1)}, \tag{3.28}$$

in this case, if $m = 0$, the action and the field equations are invariant under conformal transformations.

Define a scalar product in this context wil bee useful in the future. We have that the scalar product is defined as

$$(\phi_1, \phi_2) = -i\int_\Sigma d\Sigma^\mu \phi_1(x)\overleftrightarrow{\partial}_\mu\phi_2^*(x)[-g_\Sigma(x)]^{\frac{1}{2}}, \tag{3.29}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, with n^μ a future-directed unit vector orthogonal to the hypersurface of simultaneity Σ and $d\Sigma$ the volume element in Σ.

Next to that, we assume the existence of a complete set of mode solutions $u_i(x)$ of the equation of motion (3.26) which are orthonormal with respect to the product defined in Eq. (3.29), satisfying

$$(u_i, u_j) = \delta_{ij}, \tag{3.30}$$

$$(u_i^*, u_j^*) = -\delta_{ij}, \tag{3.31}$$

$$(u_i, u_j^*) = 0. \tag{3.32}$$

The field ϕ can be expanded in terms of those orthonormal modes

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)]. \tag{3.33}$$

The quantization of the theory is implemented by fixing the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \tag{3.34}$$

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{3.35}$$

As in general procedure in Quantum Field Theory, we would like to construct a Fock space, but there is a big problem surrounding these definitions in Quantum Field Theory in curved spacetime. We have an intrinsic ambiguity in the formalism [65]. In Minkowski space the vacuum is an invariant under the action of the Poincaré group. However, in curved spacetime there is no preferred notion for a vacuum state as a generic feature. In general, there are no longer Killing vectors at all to define positive frequency modes. As represent the whole idea of general relativity, there are no privileged coordinates and no natural mode decomposition of ϕ in these privileged coordinates will emerge, i.e., coordinates systems are physically irrelevant.

We may consider now a second complete set of orthonormal modes $\bar{u}_j(x)$. The field ϕ can also be expanded in this set

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)]. \tag{3.36}$$

This decomposition of ϕ defines a new vacuum state $|\bar{0}\rangle$

$$\bar{a}_j\, |\bar{0}\rangle = 0, \forall j \tag{3.37}$$

consequently, a new Fock space.

Since those different modes are complete, we can expand the new modes $\bar{u}_j$ in terms of the old ones

$$\bar{u}_j(x) = \sum_i [\alpha_{ji} u_i(x) + \beta_{ji} u_i^*(x)], \tag{3.38}$$

and

$$u_i(x) = \sum_j [\alpha_{ji}^* \bar{u}_j(x) - \beta_{ji} \bar{u}_j^*(x)]. \tag{3.39}$$

The above relation between modes are known as the Bogolyubov transformations [66], and the matrices α_{ji} and β_{ji} are the so-called Bogolyubov coefficients. Using the Eq. (3.38) and the relations (3.30-3.32) leads to

$$\alpha_{ij} = (\bar{u}_i, u_j) \tag{3.40}$$

and

$$\beta_{ij} = -(\bar{u}_i, u_j^*). \tag{3.41}$$

From the expansions of the complete set modes in equations (3.33) and (3.36), making use of the equations (3.38) and (3.39) and using of the orthonormality of the modes, we can obtain the annihilation operators expanded in the two complete sets

$$a_i = \sum_j [\alpha_{ji}^* \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger], \tag{3.42}$$

and

$$\bar{a}_j = \sum_i [\alpha^*_{ji} a_i - \beta^*_{ji} a^\dagger_i]. \tag{3.43}$$

It is obvious that the two Fock spaces are different as long as $\beta_{ji} \neq 0$. This is a manifestation of the non-uniqueness of the vacuum. It follows that the expectation value of the number operator $N_i = a^\dagger_i a_i$ for the number of u_i-mode particle in state $|\bar{0}\rangle$ is

$$\langle \bar{0}| N_i |\bar{0}\rangle = \sum_j |\beta_{ij}|^2, \tag{3.44}$$

meaning that the vacuum of the $\bar{u}_j$ modes have $\sum_j |\beta_{ij}|^2$ particles in the u_i mode. Suppose that $\bar{u}_j$ are positive frequency modes associated with a timelike Killing vector ξ, see Fig.6:

$$\mathcal{L}_\xi \bar{u}_j = -i\omega \bar{u}_j, \tag{3.45}$$

for $\omega > 0$,

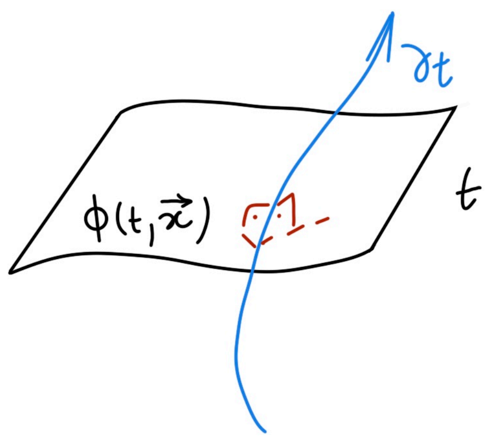

Figure 6 – Timelike Killing vector.

and $\partial_t \bar{u}_j = -i\omega \bar{u}_j$. If $u_i = \bar{u}_j$, that means that the Bogolyubov coefficient β_{ij} is equal to zero ($\beta_{ij} = 0$), we have that

$$a_i |\bar{0}\rangle = \sum_j \alpha^*_{ji} \bar{a}_j |\bar{0}\rangle = 0, \tag{3.46}$$

then $\bar{u}_j$ and u_i must share the same vacuum.

As in ordinary Quantum Field Theory, we can define Green functions. The only restriction is that we need to be clear on which choice of vacuum state $|0\rangle$ we are working on. For example, for the generalized Klein-Gordon equation, we can construct the generalized Feynman's Green function in curved spacetime

$$iG_F(x, x') = \langle 0| T\phi(x)\phi(x') |0\rangle \,, \tag{3.47}$$

then we obtain

$$\Big[\Box_x + m^2 + \xi R(x)\Big]\phi(x) G_F(x, x') = -(-g(x))^{\frac{1}{2}} \delta^n(x - x'). \tag{3.48}$$

It is important to note some features of the equation above. The first one is that we did not say anything about the vacuum state $|0\rangle$ chosen and the second one is that the equation also says nothing about the temporal ordering product. We can correct it with boundary conditions, but in contrast to the Minkowski space, for curved spacetimes the boundary conditions are not simple to choose.

3.3 The Rindler coordinates, an alternative vacuum

The beginning of this discussion is going to be over an accelerated frame in a flat spacetime. As discussed in [67], Fulling showed that even that the formalism of Lagrangian field theory is covariant, choosing a time coordinate systems give the meaning of what are positive and negative frequency, logically defining what is the vacuum state. Later was discussed that the field theory that is "natural" in Rindler coordinates are not unitary equivalent to the ordinary one, this fact is deeply connected with the existence of a horizon, as mentioned by Sciama in Ref. [67].

To understand better this statement, let us introduce the Rindler coordinates, represented in the literature by (ξ,τ), in the $x - t$ plane by

$$x = \frac{e^{a\xi}}{a} \cosh a\tau \tag{3.49}$$

$$t = \frac{e^{a\xi}}{a} \sinh a\tau, \tag{3.50}$$

where a is a nonzero positive constant and $-\infty < \tau, \xi < \infty$. The Minkowski spacetime line element in 1+1 on this new coordinates is

$$ds^2 = dt^2 - dx^2 = e^{2a\xi}(d\tau^2 - d\xi^2), \tag{3.51}$$

where, it configures a conformal transformation of the Minkowski spacetime. The curves where ξ =constant, are curves of proper acceleration $\alpha^{-1} = ae^{-a\xi}$.

The curves formed by ξ constant are asymptotic to the lines $x = \pm t$. Those lines turn out to be horizons to the uniformly accelerated observers, since the observes tracing the constant curves ξ cannot communicate with any spacetime point in region P and cannot receive any message from region F, see Fig.7. We see from this curious causal properties associated with uniformly accelerated observers in Minkowski spacetime.

Consider a massless scalar field ϕ, and let us quantize this field in the Rindler coordinate system. The wave equation in Minkowski spacetime is given by

$$\Box\phi = (\partial_t^2 - \partial_x^2)\phi = \frac{\partial^2\phi}{\partial u \partial v} = 0, \tag{3.52}$$

where the mode solution of this equation is

$$\bar{u}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ikx - i\omega t}, \tag{3.53}$$

with $\omega = |k| > 0$.

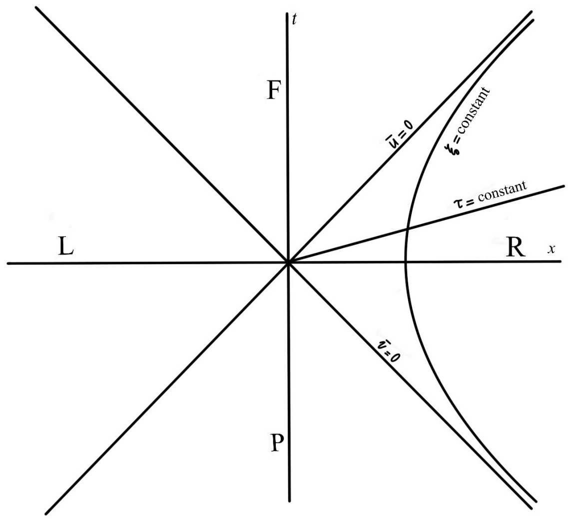

Figure 7 – Rindler coordinate spacetime.

The modes with positive frequency with respect to ∂_t are given by

$$\mathcal{L}_{\partial_t}\bar{u}_k = -i\omega\bar{u}_k, \tag{3.54}$$

where the "right move" is given by $\dfrac{e^{-i\omega\bar{u}}}{4\pi\omega}$ and the "left move" is given by $\dfrac{e^{-i\omega\bar{v}}}{4\pi\omega}$. We can define the Minkowski vacuum $|O_M\rangle$ since we have defined the positive frequency modes.

We want to quantize the field $\phi(\tau, \xi)$ in regions *R* and *L*. The field equations in Rindler coordinates are given by:

$$e^{2a\xi}\Box\phi = (d\tau^2 - d\xi^2)\phi(\tau, \xi) = \frac{\partial^2\phi}{\partial u \partial v} = 0, \tag{3.55}$$

using the solution modes of the Minkowski spacetime, we have that the solution modes for the regions *R* and *L* are, respectively, given by

$$R_k = \bar{u}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi - i\omega\tau} \tag{3.56}$$

and

$$L_k = \bar{u'}_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi + i\omega\tau}. \tag{3.57}$$

And we also define that $R_k = 0$ for region *L* and $L_k = 0$ for region *R*. In this way, we know that the sets R_k and L_k are not complete in Minkowski spacetime, but their joint set is.

Having this in mind, we can expand the fields as

$$\phi = \sum_{k=-\infty}^{\infty} (a_k\bar{u}_k + a_k^\dagger \bar{u}_k^*), \tag{3.58}$$

or also

$$\phi = \sum_{k=-\infty}^{\infty} (b_k L_k + b_k^\dagger L_k^* + c_k R_k + c_k^\dagger R_k^*), \tag{3.59}$$

with the vacuums $a_k |0_M\rangle$ and $b_k |0_R\rangle = c_k |0_R\rangle = 0$.

The vacuums can not be equivalents, because we have a change of sign of the mode passing from region *L* to *R*. The point $(u = v = 0)$, which is the change sign point from *L* to *R*, imply in a non analyticity of the modes R_k and L_k. In contrast, the Minkowski modes are analytic in all plane, which implies that the Rindler positive modes are a linear combination of the positive and negative Minkowski modes, therefore, the vacuums are different.

To establish how many Rindler particles are in Minkowski vacuum, we build the analytic modes from R_k and L_k, which are given by:

$$R_k + e^{\frac{-\pi\omega}{a}} L^*_{-k} \tag{3.60}$$

and

$$R^*_{-k} + e^{\frac{\pi\omega}{a}} L_k. \tag{3.61}$$

Now, we can expand the field as

$$\phi = \sum_{k=-\infty}^{k=\infty} \frac{1}{\sqrt{2sinh(\frac{\pi\omega}{a})}} \Big[d_k^{(1)} (e^{(\frac{\pi\omega}{2a})} R_k + e^{(\frac{-\pi\omega}{2a})} L^*_{-k}) + d_k^{(2)} (e^{(\frac{-\pi\omega}{2a})} R^*_{-k} + e^{(\frac{\pi\omega}{2a})} L_k) \Big] + h.c., \tag{3.62}$$

where $h.c$ is the Hermitian conjugate term.

In this manner, we may say that $d_k^{(1)} |0_M\rangle = d_k^{(2)} |0_M\rangle = 0$. And, from the expansion of the field above, we can show that b_k and c_k can be written, respectively as:

$$b_k = \frac{1}{\sqrt{2sinh(\frac{\pi\omega}{a})}} \Big[e^{(\frac{\pi\omega}{2a})} d_k^{(2)} + e^{(\frac{-\pi\omega}{2a})} d_{-k}^{(1)\dagger} \Big] \tag{3.63}$$

and

$$c_k = \frac{1}{\sqrt{2sinh(\frac{\pi\omega}{a})}} \Big[e^{(\frac{\pi\omega}{2a})} d_k^{(1)} + e^{(\frac{-\pi\omega}{2a})} d_{-k}^{(2)\dagger} \Big] \tag{3.64}$$

Finally, we can calculate the expectation value of the number of Rindler particles in Minkowski vacuum:

$$\langle 0_M | b_k^\dagger b_k + c_k^\dagger c_k | 0_M \rangle = \frac{1}{e^{(\frac{2\pi\omega}{a})} - 1} \tag{3.65}$$

Note that the special characteristic of this vacuum is that the Poincaré invariant vacuum state associated with the Minkowski space, have a thermal distribution with respect to the Rindler space. We can understand it as if the field is in Minkowski vacuum state, then the quantum expectation value of any observable on Fulling space is equal to its statistical ensemble average in thermal equilibrium with temperature $\left(\dfrac{a}{2\pi}\right)$.

Chapter 4

Black holes and their thermal properties

It is known that in general relativity, some kind of stars go through a gravitational collapse forming, what we call, a black hole. The most simple black is the Schwarzschild black hole. This chapter has the goal to introduce some aspects of the black holes most important dynamics and, consequently, their thermal properties [68–70].

4.1 The Schwarzschild black hole

The Birkhoff's theorem states that any spherically symmetric solution of the vacuum Einstein equation is **isometric** to the Schwarzschild solution. However, the Schwarzschild solution has an additional isometry (other than the spherical symmetry), the $\dfrac{\partial}{\partial t}$ is a Killing vector field. The solution is timelike for $r > 2M$, (where r is the radius coordinate and M is the black hole masss), so for this values of r, the Schwarzschild solution is static.

One of the solution of Einstein's equations of general relativity is the Schwarzschild solution. This solution was the first exact solution found for Einstein's equations. The assumption behind this solution is that $(T_{\mu\nu} = 0)$, spherical symmetry and asymptotic flatness of the Universe. The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega, \tag{4.1}$$

where M is the black hole mass, and $d\Omega = d\theta^2 + sin^2\theta d\varphi^2$. The radius $r \in (0, \infty)$, for $r > 2M$ the radius is the radial coordinate associated with the circumference; $t \in \mathbb{R}$ is the time that is measured by an stationary observer at $r \longrightarrow \infty$;the polar and azimutal angles, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. The Schwarzschild radius is, $r_s = 2M$, where the event horizon is located.

We can see that this metric appears to be singular at $r = 2M$ because of the divergence of some components, but we can show that this divergence is just a coordinate singularity. However we can show that at $r = 0$, there is a "physical" singularity in the sense that it cannot be removed by coordinate transformation. To understand better this coordinate singularity, we may look at the solution closer to the event horizon, i.e., close to $r = 2M$.

Let's suppose that $r \approx 2M$ and perform the change of coordinates $(r - 2M) = X$. The new line element is given by

$$ds^2 = -\frac{X}{2M}dt^2 + \frac{2M}{X}dX^2 + (2M)^2 d\Omega. \tag{4.2}$$

Introducing the coordinate $\rho^2 = 8MX$, which leads to $\frac{2M}{X}dX^2 = d\rho^2$, the new metric can be written as

$$ds^2 = -\frac{\rho^2}{16M^2}dt^2 + d\rho^2 + (2M)^2 d\Omega. \tag{4.3}$$

Basically, the metric can be viewed as a sum of two pieces, one corresponds to a 2-sphere of radius $2M$ and the other is a (ρ, t) space. Looking just for the (1+1) space,

$$ds_R^2 = -\frac{\rho^2}{16M^2}dt^2 + d\rho^2. \tag{4.4}$$

We notice that they are the Rindler coordinates in two dimensions. We can look at this line element and compare with the Rindler line element in Eq. (3.51), and we can interpret the factor $\kappa = \frac{1}{4M}$ as the surface gravity of the black hole (we can think it as the Newtonnian acceleration close to the event horizon). We can conclude that there is no singularity at all at $r = 2M$, because the geometry is just a Rindler metric times a sphere of radius $2M$.

4.2 The Kruskal-Szekeres coordinates

Let us first introduce the tortoise coordinates

$$r^*(r) = r - 2M + 2Mln\Big(\frac{2M}{r} - 1\Big), \tag{4.5}$$

which leads to

$$dr^* = \Big(1 - \frac{2M}{r}\Big)^{-1}dr. \tag{4.6}$$

The Schwarzschild element line, for simplicity, in two-dimensional space, can be rewritten as

$$ds^2 = \Big(1 - \frac{2M}{r}\Big)(-dt^2 + dr^{*2}). \tag{4.7}$$

The tortoise coordinates have the following properties:

- It is defined for $r > 2M$.
- $r^* \in (-\infty, \infty)$.
- r goes to $2M$, so r^* goes to $-\infty$.
- when r goes to ∞, r^* goes to r.

Defining the null coordinates

$$\bar{u} = t - r^* \tag{4.8}$$

$$\bar{v} = t + r^*, \tag{4.9}$$

the metric is given by

$$ds^2 = \Big(1 - \frac{2M}{r(\bar{u},\bar{u})}\Big) d\bar{u}d\bar{v}. \tag{4.10}$$

Having these definitions, we are able to construct the Kruskal-Szekeres coordinates as

$$u = -4Me^{-\frac{\bar{u}}{4M}} \tag{4.11}$$

$$v = 4Me^{-\frac{\bar{v}}{4M}}, \tag{4.12}$$

where, $-\infty < u < 0$ and $0 < v < +\infty$, for $r > 2M$.

It turns out that the metric can be written in this new Kruskal coordinates as

$$ds^2 = \frac{2M}{r(u,v)} e^{1-\frac{r(u,v)}{2M}} dudv. \tag{4.13}$$

One of the most strinking aspect of these new coordinates is that $r = 2M$ is not a singularity anymore. The Kruskal coordinates can be extended for $u > 0$ and for $v < 0$. In this way, $u \in (-\infty, \infty)$ and $v \in (-\infty, \infty)$. Note that for $r = 2M \to u, v = 0$, in this way, we have two horizons one for $v = 0$, where $t = -\infty$(past) and $u = 0$, where $t = \infty$(future).

Introducing two coordinates (T, R) as

$$u = T - R \tag{4.14}$$

$$v = T + R, \tag{4.15}$$

we can extract some properties. The first is that the null geodesics are given when u and v are constants. The second is that for a constant r we have that:

$$uv = T^2 - R^2 = l, \tag{4.16}$$

where l it is a constant. This relation gives an hyperbole. The third one is that for a constant t, we have that

$$\Big(\frac{u}{v}\Big)^2 = \Big(\frac{T+R}{T-R}\Big)^2 = q, \tag{4.17}$$

and

$$T + R = \alpha(T - R), \tag{4.18}$$

where q it is a constant. These relations gives straight lines in (T, R) space.

In Kruskal coordinates, we are also able to construct Penrose diagrams, we expect a kind of infinity in Kruskal spacetime to correspond to two copies of infinity in Minkowski spacetime. To construct the Penrose diagram for Kruskal coordinates we would like to define some coordinates $P = P(T, R)$ and $Q = Q(T, R)$ (so that lines of constant P or Q are radial null geodesics) such that

that the range of P, Q is finite, Let's say $(\frac{-\pi}{2}, \frac{\pi}{2})$, then we would need to find a conformal factor Ω so that the resulting nonphysical metric $\bar{g}$ can be extended smoothly onto a bigger manifold $\bar{M}$ (analogous to the Einstein static universe we used for Minkowski spacetime). Then, M is then a subset of $\bar{M}$ with a boundary that has four components, corresponding to places where either P or Q is $\frac{\pm\pi}{2}$. We identify these four components as future/past null infinity in region I, which we denote as $\mathcal{J}^{\pm}$ and future/past null infinity in region IV, which we denote as $\mathcal{J}^{\pm'}$.

The diagram shows radial null curves as straight lines at $45°$. The only important difference is that "infinity" corresponds to a boundary of the Penrose diagram. It is conventional to use the freedom in choosing Ω in a manner that the curvature singularity at $r = 0$ is a horizontal straight line in the Penrose diagram. The Penrose diagram is shown in Fig.8. In contrast to the conformal compactification of Minkowski spacetime, it happens that the nonphysical metric is singular at $i^{\pm}$ (and $i^{\pm'}$). This can be understood because lines of constant r meet at $i^{\pm}$, and includes the curvature singularity $r = 0$.

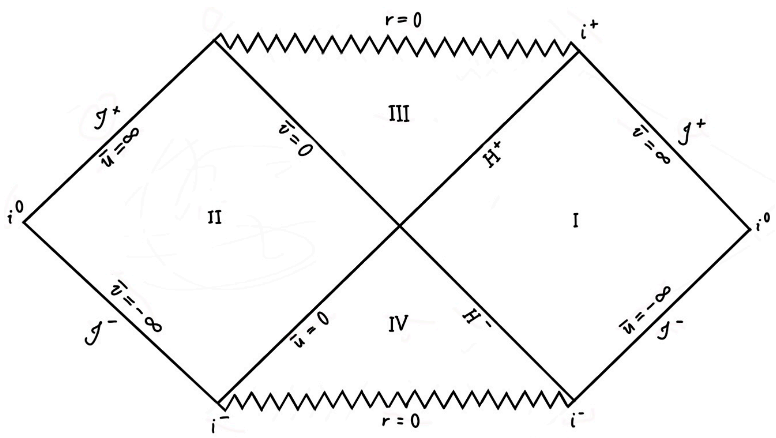

Figure 8 – Penrose diagram of the Kruskal spacetime.

4.2.1 The Boulware vacuum, the Hartle-Hawking vacuum and the Hawking effect

To construct the idea of the Hawking effect, we will discuss a scalar field in a two-dimensional Kruskal spacetime.

Similarly to the Rindler coordinates, discussed in the previous chapter, the equations of motion can be written as

$$\Box\phi = \frac{\partial^2\phi(u,v)}{\partial u \partial v} = \frac{\partial^2\phi(\bar{u},\bar{v})}{\partial\bar{u}\partial\bar{v}} = 0, \tag{4.19}$$

in tortoise coordinates the solution can be expanded in the form

$$\phi(\bar{u},\bar{v}) = A(\bar{u}) + B(\bar{v}), \tag{4.20}$$

in Kruskal coordinates the solution of the equation of motion can be also expanded as

$$\phi(u,v) = A(u) + B(v). \tag{4.21}$$

For the positive frequency modes moving to the "right", we have that $\phi \propto e^{-i\omega t}$. When we are far from the horizon, i.e., $r \to \infty$, the line element is given by

$$ds^2 = d\bar{u}d\bar{v} = -dt^2 + dr^{*2}, \tag{4.22}$$

therefore, we have positive frequency particle modes in relation to t.

With these definitions, we can expand the field as

$$\phi(\bar{u}, \bar{v}) = \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}(e^{-i\Omega\bar{u}}b_\Omega^- + e^{i\Omega\bar{u}}b_\Omega^+) + B(\bar{v}). \tag{4.23}$$

Note that we have defined two-coordinate systems, that one can be understood as a locally inertial observer (the Kruskal frame) and the other a locally accelerated observer (the tortoise frame). The question that again has to be asked is: Are the vacuum of these two coordinate systems equal? As we have a huge analogy of the Rindler case, the immediate answer for this question is no.

The Boulware vacuum is defined as

$$b_\Omega^- |0_B\rangle = 0, \tag{4.24}$$

namely, it is the state of a zero particles from the point of view of a distant observer, quite similar to the Rindler case. Here we are defining the quantization in the tortoise coordinate.

Now, for the Kruskal coordinate, the line element close to the horizon is given by

$$ds^2 = dudv = dT^2 - dR^2, \tag{4.25}$$

where the proper time T defines the modes.

Expanding the field in this coordinates system, we have that

$$\phi(u, v) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}(e^{-i\omega u}a_\omega^- + e^{i\omega u}a_\omega^+) + B(v). \tag{4.26}$$

In this system of coordinate, we define the Kruskal vacuum or most known as the Hartle-Hawking vacuum as

$$a_\omega^- |0_H\rangle = 0. \tag{4.27}$$

This vacuum have some properties as

- It is regular at the horizons and energy density is finite.
- The backreaction of the quantum fluctuations on the metric is negligible.

Have defined those vacuums we can ask the question: How many Boulware particles are in the Hartle-Hawking vacuum $|0_H\rangle$?

The answerr of this question is obtained calculating the expectation value

$$\langle 0_H | b_\Omega^+ b_\Omega^- | 0_H \rangle = \langle 0_H | N_\Omega | 0_H \rangle \,. \tag{4.28}$$

To perform it, we have to calculate the corresponding Bogolyubov coefficients. We can expand the annihilation operator of the Boulware vacuum as

$$b_\Omega^- = \int_0^\infty d\omega [\alpha_{\Omega\omega} a_\omega^- - \beta_{\Omega\omega} a_\omega^+], \tag{4.29}$$

as we have that $[b_\Omega^-, b_{\Omega'}^+] = \delta(\Omega - \Omega')$, the following normalization condition needs to be satisfied:

$$\int_0^\infty d\omega [\alpha_{\Omega\omega} \alpha_{\Omega'\omega}^* - \beta_{\Omega\omega} \beta_{\Omega'\omega}^*] = \delta(\Omega - \Omega'). \tag{4.30}$$

Relating the expansion of the field in those different coordinate systems, we get that

$$\begin{aligned} &\int_0^\infty \frac{d\omega}{\sqrt{\omega}} (e^{-i\omega u} a_\omega^- + e^{i\omega u} a_\omega^+) = \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} (e^{-i\Omega \bar{u}} b_\Omega^- + e^{i\Omega \bar{u}} b_\Omega^+) \\ &= \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \int_0^\infty d\omega (e^{-i\Omega \bar{u}} (\alpha_{\Omega\omega} a_\omega^- - \beta_{\Omega\omega} a_\omega^+) + e^{i\Omega \bar{u}} (\alpha_{\Omega\omega}^* a_\omega^+ - \beta_{\Omega\omega}^* a_\omega^-)), \end{aligned} \tag{4.31}$$

we find the correspondence

$$\frac{e^{-i\omega u \pm i\Omega' \bar{u}}}{\sqrt{u}} = \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} (e^{-i\Omega \bar{u} \pm i\Omega' \bar{u}} \alpha_{\Omega\omega} - e^{i\Omega \bar{u} \pm i\Omega' \bar{u}} \beta_{\Omega\omega}^*). \tag{4.32}$$

Integrating both sides on $\bar{u}$, we conclude that:

$$\alpha_{\Omega\omega} = +\frac{1}{2\pi} \frac{\sqrt{\Omega}}{\sqrt{\omega}} \int_{-\infty}^\infty d\bar{u} e^{+i\omega u - i\Omega \bar{u}} \tag{4.33}$$

and

$$\beta_{\Omega\omega} = -\frac{1}{2\pi} \frac{\sqrt{\Omega}}{\sqrt{\omega}} \int_{-\infty}^\infty d\bar{u} e^{-i\omega u - i\Omega \bar{u}}. \tag{4.34}$$

From Eq. (4.11), we have that $\bar{u} = -4M \ln\left(\frac{-u}{4M}\right)$. Then, we can compute the integral of Eq. (4.33) as

$$\begin{aligned} &\int_{-\infty}^\infty d\bar{u} e^{\pm i\omega u - i\Omega \bar{u}} = \int_{-\infty}^0 du \left(\frac{-u}{4M}\right)^{-1-4M\Omega i} e^{\mp i\omega u} \\ &= 4M e^{\pm 4M\pi\Omega + i4M\Omega ln(4M\omega)} \Gamma(-4M\Omega i). \end{aligned} \tag{4.35}$$

Given this relation, we can finally write the Bogolyubov coefficients as

$$\alpha_{\Omega\omega} = +\frac{4M}{2\pi} \frac{\sqrt{\Omega}}{\sqrt{\omega}} e^{4M\pi\Omega + i4M\Omega ln(4M\omega)} \Gamma(-4M\Omega i). \tag{4.36}$$

and

$$\beta_{\Omega\omega} = -\frac{4M}{2\pi} \frac{\sqrt{\Omega}}{\sqrt{\omega}} e^{-4M\pi\Omega + i4M\Omega ln(4M\omega)} \Gamma(-4M\Omega i), \tag{4.37}$$

and we can also obtain $|\alpha_{\Omega\omega}|^2$ and $|\beta_{\Omega\omega}|^2$, respectively as

$$|\alpha_{\Omega\omega}|^2 = +\frac{4M^2}{\pi}\frac{\Omega}{\omega}e^{4M\pi\Omega}\Gamma(-4M\Omega i)\Gamma(4M\Omega i) \tag{4.38}$$

and

$$|\beta_{\Omega\omega}|^2 = +\frac{4M^2}{\pi}\frac{\Omega}{\omega}e^{-4M\pi\Omega}\Gamma(-4M\Omega i)\Gamma(4M\Omega i). \tag{4.39}$$

With the equations above we can relate $|\alpha_{\Omega\omega}|^2$ and $|\beta_{\Omega\omega}|^2$ as

$$|\alpha_{\Omega\omega}|^2 = e^{8M\pi\Omega}|\beta_{\Omega\omega}|^2. \tag{4.40}$$

From the normalization condition given by Eq. (4.30), we have that

$$\int_0^\infty d\omega|\beta_{\Omega\omega}|^2(e^{8\pi M\Omega} - 1) = \delta(0). \tag{4.41}$$

Inserting the equation above in the relation (4.28), we conclude that

$$\langle 0_H|N_\Omega|0_H\rangle = \frac{\delta(0)}{e^{8\pi M\Omega} - 1}, \tag{4.42}$$

this relation gives a volume divergence in the expectation value of the number of particles, to deal with this problem we will look at the particle density, which leads to

$$n_\Omega \equiv \langle 0_H|N_\Omega|0_H\rangle \frac{1}{V} = \frac{1}{e^{8\pi M\Omega} - 1}. \tag{4.43}$$

We conclude that the Hartle-Hawking vacuum is populated with massless particles in a thermal bath with temperature:

$$T_H = \frac{1}{8\pi M}, \tag{4.44}$$

where T_H is the so-called Hawking temperature.

Having this thermal characteristic of black holes we should conclude some black hole properties:

- For an equilibrium situation black holes absorb particles, then they should emit the particles that we calculated above.
- Eternal black holes just have a physical sense if the system is also in the environment temperature.
- Black holes in a empty space should evaporate, emitting the thermal radiation T_H.

4.3 The laws of black hole thermodynamics

One of the most important properties of the black hole mechanics is that they appear to be very close to the thermodynamics laws. So one is tempted to call the laws of black hole mechanics as the black hole thermodynamics laws. A priori is interesting, because there is no reason to expect that the spacetime geometry of black holes has a relation between thermal physics.

The thermodynamics laws are the following:

1. The zeroth law: The surface gravity is constant on the event horizon for stationary black holes.

2. The first law: The energy of a black hole is conserved

$$dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ, \tag{4.45}$$

   where, M is the mass of the black hole, A is the area of the event horizon, Q is the charge with a associated potential μ and J is the angular momentum with an associated angular speed Ω. For a Schwarzschild black hole we have that $\mu = \Omega = 0$, by definition.

3. The second law: The second law is called the "Area theorem", which is states that the net area in any classical process of a black hole never decreases, $A \geq 0$.

The existence of black holes led to various physicists to question some fundamental ideas of physics. Bekenstein, asked a simple question, but very problematic one, is if black holes do not throw out anything, then it would violate the second law of thermodynamics? If we throw a hot water inside a black hole, then the net entropy of the world outside the black hole would decrease. The existence of black holes force us to give up one of the most fundamental laws of nature?

A simple idea to evade from this problem is that: if the hot water is falling in, the mass of the black hole increases accordingly to conserve its energy. This idea suggests that if a black hole has an entropy associated, the second law of thermodynamics could be preserved. Following this idea, Bekenstein proposed that a black hole must have an entropy proportional to its area.

This tentative to do not violate the second law leads to a contradiction to a classical level, because if the black hole has an energy E and entropy S, then it must have a temperature associated given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}. \tag{4.46}$$

So, if for a Schwarzschild black hole, the area and the entropy scale as $S \sim M^2$, we would have that the temperature is the inverse of its mass

$$\frac{1}{T} = \frac{\partial S}{\partial E} \sim \frac{\partial M^2}{\partial M} \sim M. \tag{4.47}$$

Hence, if the black hole has a temperature, it radiates. This affirmation is completely impossible for a classical black hole.

Hawking, showed that black holes could radiate by quantum effects. Hawking proved that a black hole has a temperature given by $T = \dfrac{1}{8\pi M}$, as derived in the previous section. Recovering now the International System Units and using the relation for the Hawking temperature and the first law of thermodynamics we have that

$$dM = tdS = \frac{\kappa}{8\pi G} dA, \tag{4.48}$$

and consequently,

$$S = \frac{Ac^3}{4G\hbar}. \tag{4.49}$$

4.4 The Hawking temperature

In quantum mechanics, we know that the thermal partition function, for a system with Hamiltonian H, can be written as

$$Z = \text{Tr}e^{-\beta H}, \tag{4.50}$$

where $\beta = \frac{1}{T}$. This expression can be related to a time evolution operator e^{-itH}, as mentioned in the Chap. (2), by an Euclidean analytic continuation (Wick rotation) $t = -i\tau$ if we identify $\tau = \beta$. Considering a single scalar field Φ, we may write the trace as

$$\text{Tr}e^{-\tau H} = \int \langle\phi|\, e^{-\tau H}\, |\phi\rangle\, d\phi = \int d\phi \int D\Phi e^{-S_E[\Phi]}, \tag{4.51}$$

where $S_E[\phi]$ is the Euclidean action over periodic field configurations, that satisfies the boundary conditions

$$\Phi(\beta) = \Phi(0) = \phi. \tag{4.52}$$

This KMS relation, discussed in Appendix B, leads us to a relation between the inverse temperature and the Euclidean time,

$$\beta = \tau = \frac{1}{T}. \tag{4.53}$$

If we study the Euclidean Schwarzschild metric by making a Wick rotation $t = -i\tau_E$, near the event horizon, the element line is

$$ds^2 = \rho^2\kappa^2 d\tau_E^2 + d\rho^2. \tag{4.54}$$

This line element is the just the Euclidean Rindler coordinates in two dimensions. The full Euclidean geometry looks like a cigar, see Fig.9. The tip of the cigar is at $\rho = 0$ and the geometry is asymptotically cylindrical (Minkowski metric far away from the horizon) far away from the tip.

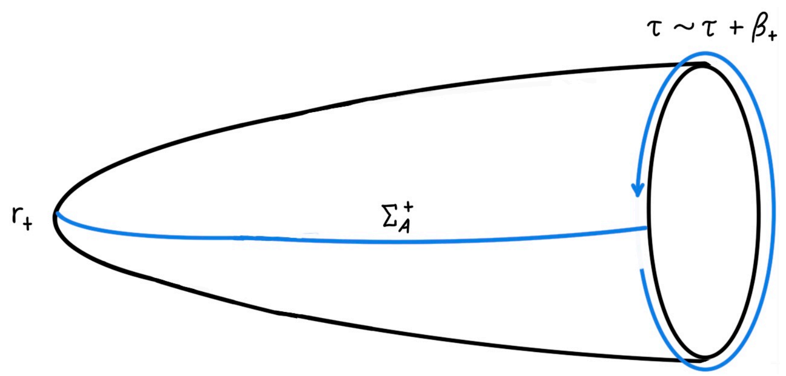

Figure 9 – The cigar diagram.

Using the relation between Euclidean periodicity and temperature expressed in Eq. (4.53), we must conclude that Hawking temperature of the black hole is written as

$$T = \frac{\kappa}{2\pi}. \tag{4.55}$$

In some way it is conjectured that after the black hole shrinks and the process of Hawking's radiation is over, it will happen a explosive disappearance, or a naked singularity, or, even, a some kind of Planck mass object. We may write a kind of Penrose diagram for this end event, see Fig.10.

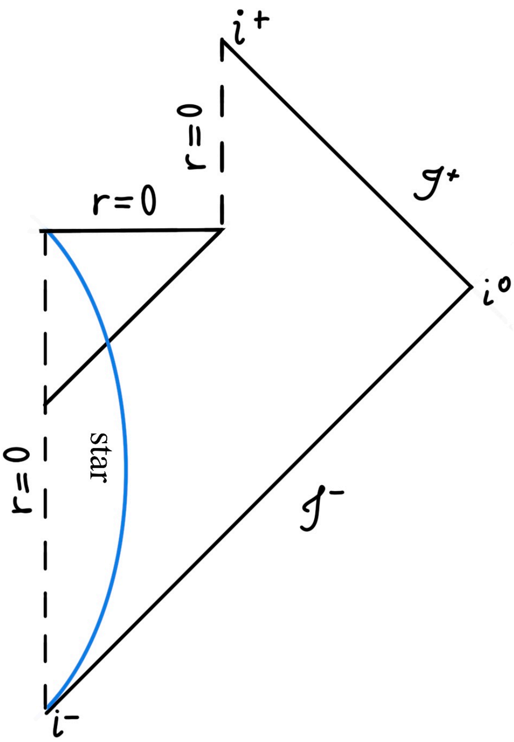

Figure 10 – Penrose diagram of an evaporating black hole.

As it is known in the literature, the calculation of the Hawking's radiation neglects its effect on the spacetime geometry. Probably, the calculation of backreaction effects would require a new theory, perhaps a quantum gravity. Still, we are able to calculate the rate of mass loss of a black hole using the Stefan–Boltzmann's law assuming that the black hole is a perfect spherical blackbody:

$$\frac{dE}{dt} \approx -AT^4, \tag{4.56}$$

where, A is the black hole are, i.e., the area of the event horizon. Performing a rough approximation, namely that $E = M$, $A \approx M^2$ and $T \approx M^{-1}$, the we may write the equation above as

$$\frac{dM}{dt} \approx -\frac{1}{M^2}, \tag{4.57}$$

consequently, the black hole should evaporate completely in a period of order

$$t \approx M^3 \approx 10^{70}\Big(\frac{M}{M_{sun}}\Big)^3, \tag{4.58}$$

where M_{sun} is the mass of the sun. An immediate consequence of this calculation is the decrease of the black hole's entropy and the information paradox. The information paradox can be described briefly in the following manner: Hawking's radiation seems to be a pure thermal radiation. Before the black hole collapse, it should be possible to order this matter in a pure state. The final system of a black hole is a mixed state, only being possible to be written in a density matrix form. This leads to a problem, because evolution from a pure state to a mixed state are not acceptable in the unitary time evolution scenario of quantum mechanics. In other words, we could say that the information is being lost in this process.

4.5 The Bekenstein-Hawking's black hole entropy

It is known that for all black holes with charge, spin and in any number of dimensions, the entropy and the temperature are given, respectively, by the relations:

$$S^1 = \frac{A}{4G\hbar} \tag{4.59}$$

and

$$T = \frac{\hbar\kappa}{2\pi}. \tag{4.60}$$

There is a deep curiosity about this entropy, because even though we are describing it by macroscopic thermodynamics properties, we have non trivial information about the microscopic structure of the theory by the Boltzmann equation

$$S = k\ln(\Omega), \tag{4.61}$$

where Ω is the total numbers of micro-states of the the system for a given energy, and k is the Boltzmann constant.

The Bekenstein-Hawking's black hole entropy is similar to the other ordinary thermodynamics entropy, so it is quite natural to ask what are the associated microstates. The Bekenstein-Hawking's black hole entropy given by

$$S^1 = \frac{Ac^3}{4G\hbar} = \frac{kA}{4l_p^2}, \tag{4.62}$$

where the Planck length $l_p^2 = \sqrt{\frac{\hbar G}{c^3}}$ is a function of the three fundamental constants of physics, so it is, again, natural to ask if there is any information of the degrees of freedom of some quantum theory of gravity hidden in this entropy. One of the possible interpretations for understanding this entropy in a "quantum gravity view", is that the black hole is constructed by finite pieces of those l_p's length.

4.6 The generalized second law

One of the main laws of nature is the second law of thermodynamics, which states that the entropy of a closed system should never decrease. Since the black hole is evaporating via Hawking radiation, the entropy of the black hole is clearly decreasing.

Bekenstein proposed in the $70's$, to solve the conflict between black hole evaporation and the second law of thermodynamics, that the sum of the Bekenstein-Hawking's black hole entropy and the entropy $S^{(2)}$ of the matter and radiation field in outside the black hole region cannot decrease. This conjecture is defined as

$$\Delta S_{BH} = \Delta(S^1 + S^{(2)}) = \Delta S^{(1)} + \Delta S^{(2)} \geq 0. \tag{4.63}$$

As emphasized by Wall [71], there are many alternatives to prove the Generalized Second Law in the literature, which are rich in ideas, but unfortunately most of those attempts have inconsistent assumptions. In this work we attempt to show the validity of this equation using semi-classical arguments.

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Chapter 5

Disordered systems and the distributional zeta function

In this chapter, we intend to present the key ideas that define a random system, in special the quenched disorder. The main goal of this chapter is to introduce random system and a method to calculate averages of a disordered field.

5.1 Random systems

In the 70's, condensate matter physicists started to study impurity on crystals. That was the beginning of the study of a random environment. Physicists realized that impurities break symmetries, and some calculations needed to be done to deal with this new problem [72].

In statistical field theory, the partition function is a function of variables of which the dynamics is well known. Let's introduce a variable δm^2, which is defined by a probability distribution $P(\delta m^2)$, meaning that δm^2 is a random variable.

There are two interesting cases for the nature of the random variable δm^2. The first one, the **annealed disorder**, consists when the fluctuations of δm^2 are of the same time order of the other variables of the system. The second case, the **quenched disorder**, happens when the fluctuations of the random variable δm^2 are slower than the others, in a way that we can define the generating functional of connected correlations functions for each value of the variable. The free energy as a function of the variable δm^2 is defined as

$$F(\delta m^2) = -\frac{1}{\beta}\ln\left(\sum_n e^{-\beta E_n(\delta m^2)}\right), \tag{5.1}$$

for the case of quenched disorder. We must perform the average over the ensemble of all realizations of the disorder

$$\mathbb{E}[F] = -\frac{1}{\beta}\int d(\delta m^2) P(\delta m^2)\ln\left(\sum_n e^{-\beta E_n(\delta m^2)}\right). \tag{5.2}$$

We can conclude that to find fundamental thermodynamics properties the average of the free energy is needed. In statistical field theory, the key to find the connected correlations functions is to calculate the average of the generating functional of connected correlations functions in the random variable

$$\mathbb{E}\Big[W(j,h)\Big] = \int [dh]P(h)\ln Z(j,h), \tag{5.3}$$

where $W(j,h)$ is the generating functional of connected correlations functions, $Z(j,h)$ is the generating functional of correlation functions in the presence of disorder, $[dh]$ is a functional measure and $[dh]\,P(h)$ is the probability distribution of the disorder field.

The first proposal to calculate the Eq. (5.3), which is very complicated, was a method to transform the logarithm of the partition function in a limit expression that depends on a power of the partition function [73–76]. This method is the so-called replica trick, which was improved by Parisi and used to solve the spin-glass problem in condensate matter [77]. The procedure consists, as mentioned above, in changing the logarithm of the partition function to the limit expression

$$\ln Z = \lim_{n\to 0}\frac{Z^n - 1}{n}. \tag{5.4}$$

In this present work, we are not going to use the replica trick. We work with the distributional zeta function method to calculate the generating functional of connected correlations functions, which we are going to discuss in the next section.

5.2 The distributional zeta-function method

The distributional zeta-function method mentioned in the last section, derived by Svaiter and Svaiter [78], is an alternative method to calculate the average of the generating functional of connected correlations functions.

First, Let's define the generalized Riemann zeta function from a Lebesgue integral representation

$$\zeta_{\mu,f}(s) = \int_\Omega f(x)^{-s} d\mu(x), \tag{5.5}$$

where $s \in \mathbb{C}$, $f^{-s} \in L^1(\mu)$, and which $(X, \mathcal{A}, \mu)$ a measure space, and $f :\to (0,\infty)$ a measurable function. For the calculation of the average quenched free-energy, Svaiter and Svaiter defined the distributional zeta function as

$$\Phi(s;j) = \int [d\eta]P(\eta)[Z(n;j)]^{-s}, \tag{5.6}$$

where for this case, $f = Z[\eta]$ and $d\mu = [d\eta]P(\eta)$. Using the ordinary identity $Z^{-s} = e^{-s\ln Z}$, we can conclude

$$\frac{d}{ds}\lim_{s\to 0^+} e^{-s\ln Z} = \lim_{s\to 0^+}[-\ln Z]e^{-s\ln Z} = -\ln Z. \tag{5.7}$$

We can calculate the average of the generating functional $W(j)$, which can be written as

$$W(j) = -\frac{d}{ds}\Phi(s;j)|_{s=0^+}, \tag{5.8}$$

for $Re(s) \geq 0$, where $\Phi(s;j)$ is well defined. To continue, we will use the general representation of the Euler integral for the gamma function

$$[Z(n;j)]^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-Z(n;j)t} dt, \tag{5.9}$$

for $Re(s) > 0$. The integral above converge only for $Re(s) > 0$, we may show that we can obtain the generating functional for $Re(s) \geq 0$. Using Eq. (5.9) in Eq. (5.6), we find that

$$\Phi(n;j) = \frac{1}{\Gamma(s)} \int [d\eta] P(\eta) \int_0^\infty t^{s-1} e^{-Z(n;j)t} dt. \tag{5.10}$$

We may write $\Phi = \Phi_1 + \Phi_2$ and assuming some a that $a > 0$, we can write

$$\Phi_1(n;j) = \frac{1}{\Gamma(s)} \int [d\eta] P(\eta) \int_0^a t^{s-1} e^{-Z(n;j)t} dt, \tag{5.11}$$

and

$$\Phi_2(n;j) = \frac{1}{\Gamma(s)} \int [d\eta] P(\eta) \int_a^\infty t^{s-1} e^{-Z(n;j)t} dt, \tag{5.12}$$

then the generating functional may be written as

$$W(j) = -\frac{d}{ds} \Phi_1(s;j)|_{s=0^+} - \frac{d}{ds} \Phi_2(s;j)|_{s=0}. \tag{5.13}$$

Defining the integer moment of the generating functional as

$$\mathbb{E}[(Z(n;j))^k] \equiv \mathbb{E}[Z^k], \tag{5.14}$$

where

$$\mathbb{E}[Z^k] = \int [d\eta] P(\eta) [Z(n;j)]^k. \tag{5.15}$$

The integral Φ_2 defines a analytic function in all complex plane. The integral Φ_1 is given by

$$\Phi_1(s;j) = \frac{a^s}{\Gamma(s+1)} + \frac{1}{\Gamma(s)} \sum_{k=1}^\infty \frac{(-1)^k a^{k+s}}{k!(k+s)} \mathbb{E}[Z^k], \tag{5.16}$$

where this expression is valid for $Re(s) \geq 0$ and the function $\Gamma(s)$ has a pole at $s = 0$, then

$$\frac{d}{ds} \Phi_1(s;j)|_{s=0^+} = \sum_{k=1}^\infty \frac{(-1)^k a^{k+s}}{k!(k+s)} \mathbb{E}[Z^k] + f(a), \tag{5.17}$$

where

$$f(a) = \frac{d}{ds} \Big(\frac{a^s}{\Gamma(s+1)} \Big)|_{s=0} = \ln a + \gamma, \tag{5.18}$$

and γ is the Euler constant. We should also find the derivative of Φ_2, which is given by

$$\frac{d}{ds} \Phi_2(s;j)|_{s=0} = \int [d\eta] P(\eta) \int_a^\infty e^{-Z(n;j)t} \frac{dt}{t} = -R(a,j). \tag{5.19}$$

Using Eq.(5.17) and Eq. (5.19) we have that the average generating functional of connected correlation functions, or the quenched free energy, can be represented as

$$\mathbb{E}\Big[W(j,h)\Big] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}a^k}{kk!}\,\mathbb{E}\left[(Z(j,h))^{\,k}\right] \\ - \ln(a) + \gamma + R(a,j). \tag{5.20}$$

For $a \to \infty$, $|R(a)|$ is quite small, therefore, the dominant contribution to the average generating functional of connected correlation functions is given by the moments of the generating functional of correlation functions of the model. Once that we are already assuming that a is large enough, we can write

$$\mathbb{E}\left[W(j,h)\right] = \sum_{k=1}^{\infty} c_k\,\mathbb{E}\Big[Z^k(j,h)\Big]\,, \tag{5.21}$$

where we defined $c_k = \dfrac{(-1)^{k+1}a^k}{kk!}$.

In the next chapter, we are going to discuss this method to calculate the generalized entropy for a $\lambda\phi^4$ scalar field for a Euclidean Schwarzschild black hole.

Chapter 6

Self-interacting scalar field in Euclidean section of the Schwarzschild manifold

The main point of this chapter is to discuss a self-interacting scalar field in the Euclidean section of the Schwarzschild manifold in the presence of a coloured disorder.

6.1 Introducing the problem

As discussed in the Chap. (4), the Birkhoff's theorem states that any vacuum spherical solution of the Einstein equation is isometric to a region in Schwarzschild spacetime, and starting the discussion from the pseudo-Riemannian manifold with the Schwarzschild metric in a d-dimensional spacetime [79]. The line element is given by

$$ds^2 = -\left(1 - \left(\frac{r_s}{r}\right)^{d-3}\right) dt^2 + \left(1 - \left(\frac{r_s}{r}\right)^{d-3}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2, \tag{6.1}$$

where the Schawrzschild radius r_s is proportional to the product of the d-dimensional Newton's constant and the black hole mass M_0,

$$r_s^{d-3} = \frac{8\Gamma(\frac{d-1}{2})}{(d-2)\pi^{\frac{(d-3)}{2}}} G^{(d)} M_0, \tag{6.2}$$

to simplify the calculations, we will define $G^{(d)} M_0 = M$, in this simplification this definition in four dimensions, the quantity M has unities of length.

As mentioned in Chap. (2), we can do a Wick rotation procedure in time coordinate, obtaining a positive definite Euclidean metric for $r > r_s$.

$$ds_E^2 = \left(1 - \left(\frac{r_s}{r}\right)^{d-3}\right) d\tau^2 + \left(1 - \left(\frac{r_s}{r}\right)^{d-3}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2. \tag{6.3}$$

Logically, this manifold has a conic singularity. To remove this problem in $r = r_s$ we assume the periodicity in the imaginary time coordinate, with period $4\pi r_s/(d-3)$. This Euclidean section of the Schwarzschild solution, with compactified imaginary time, is homeomorphic to $\mathbb{R}^2 \times S^2$. In this manifold, we define the Hartle-Hawking vacuum state. Any quantum field defined in such manifold behave as if they are being held at a temperature $\beta^{-1} = \dfrac{(d-3)}{4\pi r_s}$. The periodicy in imaginary time is associated to finite temperature states in relativistic field theory in the Matsubara formalism, where the topology of Euclidean space is $S^1 \times \mathbb{R}^3$, and in quantum many-body theory [80, 81]. Since in principle we do not have mathematical control of our expressions at the infinite volume limit, one need to enclose the black hole within a finite-volume box, imposing some boundary condition. From now on we impose Dirichlet boundary condition on the surface of the confining box. The total volume of the system is defined as $\text{Vol}_d(\Omega) = \beta\, V_{d-1}$. It is important to point out that in the case of Euclidean interacting field theories confined in compact domains it is necessary to introduce surface counterterms. This situation is quite different from the usual one, where a local action functional is renormalizable if it has only a finitely many local counterterms [82].

Given an arbitrary smooth connected d-dimensional Riemannian manifold $\mathcal{M}^d$, with a metric tensor g_{ij} in local coordinates, we should use the Laplace-Beltrami operator $-\Delta_g$ on scalar functions defined in Eq. (3.27)

$$-\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j}\right), \tag{6.4}$$

where (g^{ij}) is $(g_{ij})^{-1}$, and $g = \det\, g_{ij}$. We are working in a local arbitrary curvilinear coordinate system $x^\nu = (x^1, x^2, ..., x^d)$. Usually we are interested in the Hilbert space of square integrable functions defined on a compact domain, that is, $\mathcal{H} = L^2(\Omega, d\mu)$, where $\Omega \subseteq \mathcal{M}^d$ compact. The Schrödinger operator is defined as $-\Delta_g + V(x)$.

Using the fact that in the case of an interacting field theory, the black hole can remain in thermal equilibrium with a thermal bath [83], we should consider an Euclidean self-interacting scalar model. The action functional $S = S(\varphi)$ for a single self-interacting scalar field is given by

$$S = \frac{1}{2}\int_\beta d\mu \left[\varphi(x)\left(-\Delta_s + m_0^2\right)\varphi(x) + \frac{\lambda_0}{12}\varphi^4(x)\right], \tag{6.5}$$

where $d\mu$ the measure and the symbol $-\Delta_s$ denotes the Laplace-Beltrami operator in Euclidean section of the Schwarzschild manifold $\mathcal{M}_s^d$, λ_0 is the bare coupling constant and m_0^2 is the mass parameter of the model. Also, the notation $\int_\beta$ means that we have a periodic time coordinate $x^1 = it$, that is, $0 \le x^1 \le 4\pi r_s/(d-3)$. Therefore, $\varphi(x^1, x^2, x^3, ..., x^d) = \varphi(x^1 + \beta, x^2, x^3, ..., x^d)$. We define $x^2 = r$, as the radial coordinate. In such a manifold, the Laplace-Beltrami operator is explicitly given by

$$-\Delta_s\varphi = \Delta_\theta\varphi(x^3, ..., x^d) + \left(1 - \left(\frac{r_s}{x_2}\right)^{d-3}\right)^{-1} \frac{\partial^2\varphi}{\partial x_1^2} + \frac{1}{x_2^{d-2}}\frac{\partial}{\partial x_2}\left(x_2^{d-2}\left(1 - \left(\frac{r_s}{x_2}\right)^{d-3}\right)\frac{\partial\varphi}{\partial x_2}\right). \tag{6.6}$$

where Δ_θ denotes the Laplace-Beltrami operator in the S^{d-2}, the $(d-2)$-dimensional unit sphere, i.e., the contribution from the angular part. We are assuming Dirichlet boundary conditions. We write $\varphi(x)|_{\partial\mathcal{M}_s^d} = 0$, since we are considering the whole system inside a reflecting wall.

Introducing an external source $j(x)$, it is possible to define the generating functional of all n-point correlation functions $Z(j)$ as

$$Z(j) = \int [d\varphi]\ \exp\left(-S(\varphi) + \int_\beta d\mu\, j(x)\varphi(x)\right), \tag{6.7}$$

where $[d\varphi]$ is a functional measure, i.e., a formal measure, given by $[d\varphi] = \prod_x d\varphi(x)$. We have in mind that the functional integral is taken over a periodic field with respect to the imaginary time, with period $2\pi\beta$. The next step is to define the generating functional of connected correlation functions $W(j) = \ln Z(j)$ and the Gibbs free energy density.

We introduce a random source in which we are taking averages. In order to support such a construction and obtain a physical interpretation of such modeling we present some heuristic arguments. Working in the pseudo-Riemannian manifold, with functional integrals, the degrees of freedom defined inside the event horizon must be integrated out. In the semi-classical approximation this procedure is standard, since the effects of the quantum fields defined in the region behind the event horizon may not propagate outside the black hole. Notice that the black hole interior geometry can be described in $d = 4$ by the line element

$$ds^2 = -\left(\frac{2M}{T} - 1\right)^{-1} dT^2 + \left(\frac{2M}{T} - 1\right) dx^2 + T^2 d\Omega^2. \tag{6.8}$$

The spatial coordinate is defined for $-\infty < x < \infty$ while $0 \leq T < 2M$. The extended Schwarzschild manifold contains an anisotropic collapsing universe that describes the black hole interior. As discussed in the Chap. (4), this metric describes an anisotropic homogeneous cosmology. Near the singularity at $T = 0$ we may write the Schwarzschild metric as a Kasner universe, given by the line element

$$ds^2 = -d\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{-\frac{2}{3}} dx^2 + \left(\frac{\tau}{\tau_0}\right)^{\frac{4}{3}} (dy^2 + dz^2), \tag{6.9}$$

where $\tau_0 = \dfrac{4M}{3}$ and $\tau = \dfrac{\sqrt{2}}{3}\left(\dfrac{T^3}{M}\right)^{\frac{1}{2}}$. Close to the point $T = 2M$ the metric can be written as a flat Kasner solution. When $\tau \to 0$ we get

$$ds^2 = -d\tau^2 + \frac{\tau^2}{16M^2} dx^2 + (dy^2 + dz^2). \tag{6.10}$$

The existence of these internal degrees of freedom affecting the contribution of the Generalized entropy have not yet been correctly recognized. Many efforts have been done to understand and explain those unknowns contributions to the generalized entropy. For example, in Ref. [84] it is discussed that there is no necessity to introduce new degrees of freedom in order to understand correctly the origin of the black hole entropy, but just the knowledge of the quantum gravity effects. Another example is discussed in Refs. [85–87], Oppenheim argued that gravity is purely a classical background applied to quantum fields. Finally, there are approaches that take into account the degrees of freedom present in black hole's interior [88]. Maldacena and collaborators discussed that using the topological structure of replica wormholes, justify why the black hole interior should be included in the computation of radiation and matter entropy. The idea presented in this dissertation is quite similar to the idea of the "replica wormholes".

Recalling the argument given in Ref. [86], the proposed randomness nature of gravity can be understood as fundamental or an effective theory of a quantum gravity deeper theory, for instance, the structure given by the replica wormholes. Here we implement a connection between the random nature of the degrees of freedom and how they can explain the effects of the replica wormholes over the matter and radiation fields. Since there is no interior solution in the Euclidean section of Schwarzschild manifold, to model the influence of internal degrees of freedom of the pseudo-Riemannian manifold ($r < r_s$) over matter fields, we define a coarse grained variable (a reduced description of internal degrees of freedom), through an additive quenched disorder. The action functional for the scalar field in the presence of the disorder is given by

$$S(\varphi, h) = S(\varphi) + \int_\beta d\mu\, h(x)\varphi(x), \tag{6.11}$$

where $S(\varphi)$ is the Euclidean action functional for the standard self-interacting scalar field theory, and $h(x)$ is the additive disorder. At this point, let us introduce the functional $Z(j, h)$, the generating functional of correlation functions in the presence of disorder, where we use a auxiliary external source $j(x)$, to generate the n-point correlation functions of the model.

As discussed in Chap. (5), in the pure system case, one can define a generating functional of connected correlation functions in the presence of disorder, $i.e.$, the generating functional of connected correlation functions for one disorder realization, $W(j, h) = \ln Z(j, h)$. For the case of quenched disorder, one can define an average generating functional of connected correlation functions, performing the average over the ensemble of all realizations of the disorder. Since the entropy is an additive function, to obtain a physical (self-averaging) generating functional, we define the disorder-average of the generating functional $W(j, h)$. We have the average generating functional of connected correlation functions written as

$$\mathbb{E}\Big[W(j, h)\Big] = \int [dh]P(h) \ln Z(j, h), \tag{6.12}$$

where $[dh]$ is a functional measure, given by $[dh] = \prod_x dh(x)$, and the probability distribution of the disorder field is written as $[dh]\, P(h)$. The distribution of this generalized random variable will be called a generalized probability distribution. This procedure is similar to the one used in statistical field theory where the free energy must be self-averaged over all the realizations of the random interactions.

6.2 The distributional zeta-function method

To model the effects of the internal degrees of freedom of the black hole over the external matter fields we are interested to use the concept of quenched disorder. Such an effect will change the thermodynamical properties of the matter fields. Once the average is taken, the covariance of the disorder field must be chosen with some care. If we choose a simple constant σ^2 times a delta function, all the points of the Euclidean manifold will feel the effects in the same way, basically we would be doing a general white noise. However, to justify the black hole as the physical source of such a disorder, this choice is not the best one. Since that we are in the simplest case of a black hole, the Euclidean Schwarzschild black hole, we must expect that the effects increases in the vicinity of the black hole horizon, i.e., that they only depend on the distance from the black hole horizon. The physical argument is that since we know that in Euclidean quantum field there are divergences near the boundary due to fast modes [82], we are assuming in this model that the covariance of the disorder also diverges near the boundary. Therefore, respecting the symmetry of the system, the covariance of the disorder increases when $x_2 \to 0$. To model that we choose the the covariance of the disorder to be given by

$$\mathbb{E}[h(x)h(y)] = V(x_2)\delta^d(x-y), \tag{6.13}$$

where we are assuming that $V(x_2) = a^{\alpha-2}(x_2)^{-\alpha}$, for α positive definite, and a is a constant with dimension of length. Remember that $x_2 = r$, therefore V is spherically symmetric. From the definition of $\mathbb{E}[h(x)h(y)]$:

$$\mathbb{E}[h(x)h(y)] = \int [dh]P(h)h(x)h(y), \tag{6.14}$$

we have that $P(h)$ is given by

$$P(h) = e^{-\frac{1}{2}\int dx V^{-1}(x_2)h(x)^2}. \tag{6.15}$$

After integrating over all the realizations of the disorder,

$$\mathbb{E}\left[Z^{(k)}(j,h)\right] = \int [dh]P(h)Z^{(k)}(j,h) \tag{6.16}$$

we get that each moment of the generating functional of connected correlation functions $\mathbb{E}\left[Z^{(k)}(j,h)\right]$ can be written as

$$\mathbb{E}[Z^{(k)}(j,h)] = \int \prod_{i=1}^{k}[d\varphi_i^{(k)}]\, \exp\left(-S_{\text{eff}}(\varphi_i^{(k)}, j_i^{(k)})\right), \tag{6.17}$$

where the effective action $S_{\text{eff}}\left(\varphi_i^{(k)}\right)$ describing the field theory with k-field components is given by

$$S_{\text{eff}}\left(\varphi_i^{(k)}, j_i^{(k)}\right) = \int_\beta d\mu$$
$$\left[\sum_{i=1}^{k}\left(\frac{1}{2}\varphi_i^{(k)}(x)\left(-\Delta_s + m_0^2\right)\varphi_i^{(k)}(x) + \frac{\lambda_0}{4!}\left(\varphi_i^{(k)}(x)\right)^4\right) - \frac{V(x_2)}{2}\sum_{i,j=1}^{k}\varphi_i^{(k)}(x)\varphi_j^{(k)}(x) - \sum_{i=1}^{k}\varphi_i^{(k)}(x)j_i^{(k)}(x)\right]. \tag{6.18}$$

A remarkable aspect of the formalism is that after the averaged procedure with a reduced description of these degree of freedom, one gets new collective variables, i.e., multipLet's of fields in all moments. In some papers it was used the following configuration of the scalar fields $\varphi_i^{(k)}(x) = \varphi_j^{(k)}(x)$, in the function space and also $j_i^{(k)}(x) = j_j^{(k)}(x)\; \forall\, i,\, j$. All the terms of the series have the same structure and one minimizes each term of the series one by one, a scheme known as the "diagonal approximation" [89–93]. A different strategy was recently adopted. Instead of assuming that all the fields are equal for each moment, it is not imposed any relation over such fields [94]. Since we are interested only in the thermodynamical properties of the model, we have no need to generate the correlation functions. So, for simplicity, we set $j_i^{(k)}(x) = 0, \forall\, i$. From now on we omit the $j = 0$ from in. Let us discuss the Gaussian contribution of the action (6.18), since that is enough to access the thermodynamic properties. The free part of the effective action can be recast as

$$S_0\left(\varphi_i^{(k)}\right) = \frac{1}{2}\int_\beta d\mu \sum_{i,j=1}^{k} \varphi_i^{(k)}(x)\left[\left(-\Delta_s + m_0^2\right)\delta_{ij} - V(x_2)\right]\varphi_j^{(k)}(x) \tag{6.19}$$

The differential operator turns out to be non-diagonal in the (i,j)-space. Defining the $k \times k$ matrix, where

$$G \equiv [G_{ij}] \equiv \begin{bmatrix} G_{11} - V & -V & \cdots & -V \\ -V & G_{22} - V & \cdots & -V \\ \vdots & \cdots & \ddots & \vdots \\ -V & -V & \cdots & G_{kk} - V \end{bmatrix}, \tag{6.20}$$

where we have used the following definition

$$G_{ij} \equiv \left(-\Delta_s + m_0^2\right)\delta_{ij}. \tag{6.21}$$

As it can be readily checked, the matrix G is a symmetric matrix, since that $V(x_2)$ is a real-valued function. So G can be diagonalized by an orthogonal transformation, S. Where we are defining $A = \langle S, GS\rangle$ as the diagonal matrix. One can check explicitly that the matrix A is given by

$$A = \begin{bmatrix} G_{11} - kV & 0 & \cdots & 0 \\ 0 & G_{22} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & & G_{kk} \end{bmatrix}. \tag{6.22}$$

Using that Φ is the vector of components $\varphi_i^{(k)}$, we can use the matrix S to construct the vector $\tilde{\Phi} = S\Phi$. Denote the components of the vector $\tilde{\Phi}$ by $\phi_i^{(k)}$. Disregarding the interaction term, one can use the fact that the the matrix S is orthogonal and the Eq. (6.22) to write

$$\mathbb{E}\left[Z^{(k)}(h)\right] = \int \prod_{m=1}^{k-1}\left[d\phi_m^{(k)}\right] \exp\left(-S_0(\phi_m^{(k)})\right)\int [d\phi]\, \exp\left(-S_{\mathrm{V}}(\phi)\right), \tag{6.23}$$

where we are denoting $\phi_1^{(k)} = \phi$,

$$S_0(\phi_m^{(k)}) = \frac{1}{2}\int_\beta d\mu \sum_{m=1}^{k-1} \phi_m^{(k)}(x)\left(-\Delta_s + m_0^2\right)\phi_m^{(k)}(x), \tag{6.24}$$

and

$$S_{\mathrm{V}}(\phi) = \frac{1}{2}\int_\beta d\mu\, \phi(x)\left[-\Delta_s + m_0^2 - kV(x_2)\right]\phi(x). \tag{6.25}$$

Performing all the Gaussian integrations we can recast the quenched free energy, Eq. (5.21), in a more clarifying form

$$\mathbb{E}\left[W(h)\right] = \sum_{k=1}^{\infty} c_k \left[\det\left(-\Delta_s + m_0^2\right)\right]^{\frac{1-k}{2}}\left[\det\left(-\Delta_s + m_0^2 - kV(x_2)\right)\right]^{-\frac{1}{2}}. \tag{6.26}$$

Notice that the first determinant is an usual one, expected in the analyses of a scalar field on a Riemaniann manifold. The regularity and self-adjointness of such an operator follows from the regularity of the Laplace-Beltrami operator. However, the second determinant is involved. This operator is a Schrödinger operator on a Riemannian manifold. Some care must be taken in the analyses of such an operator. Before analyzing the Schrödinger operator on a Riemaniann manifold, it is important to discuss how to calculate functional determinants and their importance to physics.

6.3 Functional determinants

The calculation of functional determinants plays an important role in Quantum Field Theory as discussed by Dunne in Ref. [95]. There are multiple areas of physics that functional determinants are fundamental to a deep understanding of nature, for example, the calculation of Faddev-Popov determinants, lattice gauge theories and in the calculation of effective actions. Using the calculation of effective actions as an illustration, the functional action of a neutral scalar field φ subject to a potential $V(x)$, without the presence of sources is given by

$$S[\varphi] = \int dx\, \varphi(x)(-\Delta + V(x))\varphi(x). \tag{6.27}$$

The Euclidean generating functional is defined as

$$Z = \int d\varphi\, e^{-S[\varphi]}. \tag{6.28}$$

The one-loop contribution to the effective action is given in terms of functional determinants as

$$\Gamma^{(1)}[V] = -\ln(Z) = \frac{1}{2}\ln\det(-\Delta + V). \tag{6.29}$$

6.3.1 Zeta function regularization of functional determinants

In the present subsection, we would like to discuss one method to calculate functional determinants which is the zeta function regularization. This method play a key role later to discuss the Gel'fand-Yaglom formalism.

To introduce the Zeta regularization, Let's consider the following eigenvalue equation for an operator $\mathcal{D}$

$$\mathcal{D}\phi_n = \lambda_n\phi_n, \tag{6.30}$$

where the operator $\mathcal{D}$ could be, for example, the Klein-Gordon operator or the Dirac operator. We define the ζ-function as

$$\zeta(s) := \sum_n \frac{1}{\lambda_n^s}. \tag{6.31}$$

Having the ζ-function defined we calculate its derivative with respect to s and then evaluate it at $s \to 0$

$$\frac{d}{ds}\zeta(s) = -\sum_n \frac{\ln\lambda_n}{\lambda_n^s}, \tag{6.32}$$

and

$$\frac{d}{ds}\zeta(0) = -\ln\left(\prod_n \lambda_n\right). \tag{6.33}$$

Note that now we have a formal definition for the determinant of the operator $\mathcal{D}$ as

$$\det \mathcal{D} := \exp\left\{-\frac{d}{ds}\zeta(0)\right\}. \tag{6.34}$$

$$\det \mathcal{D} := \exp\left\{-\frac{d}{ds}\zeta(0)\right\}. \tag{6.35}$$

Usually, the convergence of this function is only given in the region that $\text{Re}(s) > \frac{d}{2}$, where d is the dimensionality of space. Since we need to evaluate the ζ-function at $s = 0$, we need to analytically continue the ζ-function at $s = 0$.

The most well-known and simplest representation of the Zeta function family is the Riemann ζ-function, defined by:

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \tag{6.36}$$

where here $\lambda_n = n$. Notice that the sum converge to $\text{Re}(s) > 1$.

To show another representation of the Riemann $\zeta_R(s)$ function, let us introduce the Gamma function:

$$\Gamma(s) := \int_0^\infty dt\, t^{s-1} e^{-t}, \tag{6.37}$$

that is only convergent to $\text{Re}(s) > 0$. It is well known in the literature, that using the above equation and properties of geometric series, the Riemann $\zeta_R(s)$ function integral representation is given

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-nt} = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \sum_{n=1}^{\infty} e^{-nt} = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-\frac{t}{2}}}{2\sinh\frac{t}{2}}. \tag{6.38}$$

Finally, the analytic continuation of the Riemann Zeta function to $\text{Re}(s) = 0$ is achieved when we take the small leading term of the equation above and add it back using the analytic continuation of the Gamma function:

$$\begin{aligned} \zeta_R(s) &= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \Big[\frac{e^{-\frac{t}{2}}}{2\sinh\frac{t}{2}} - \frac{1}{t}\Big] + \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-2} e^{-\frac{t}{2}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \Big[\frac{e^{-\frac{t}{2}}}{2\sinh\frac{t}{2}} - \frac{1}{t}\Big] + \frac{2^{s-1}}{(s-1)}. \end{aligned} \tag{6.39}$$

Notice now that when we evaluate it at $s = 0$ the first term is zero and we conclude that

$$\zeta_R(0) = -\frac{1}{2}. \tag{6.40}$$

And we similarly have that

$$\frac{d}{ds}\zeta_R(0) = -\frac{1}{2}\ln 2\pi. \tag{6.41}$$

The Zeta function regularization will be fundamental to the Gel'fand-Yaglom method to calculate functional determinants.

6.3.2 The Gel'fand-Yaglom method

The Zeta function regularization argues that when it is well-known the eigenvalues of the operator it is possible to calculate the functional determinant. Unfortunately, in most of the cases we do not know the spectrum of the operator. The Gel'fand-Yaglom [96] method consists in calculating a functional determinant using the eigenfunctions of the operator instead of its eigenvalues.

Suppose that the eigenvalues λ_n of some operator are not known, but we know that they are given by zeros for some function. Then, for all λ_n we have that:

$$P(\lambda_n) = 0. \tag{6.42}$$

In this particular case, the expression

$$\frac{d}{d\lambda_n} \ln P(\lambda_n) = \frac{P'(\lambda_n)}{P(\lambda_n)}, \tag{6.43}$$

has poles at λ_n. Expanding the denominator we may show that the residue of simple pole at (λ_n) is given by:

$$\text{Res}\left(\frac{P'(\lambda)}{P(\lambda)}, \lambda = \lambda_n\right) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) \cdot \frac{P'(\lambda)}{P(\lambda)}. \tag{6.44}$$

Taking the approximated expression of $\frac{P'(\lambda)}{P(\lambda)}$ close to λ_n:

$$\frac{P'(\lambda)}{P(\lambda)} \approx \frac{1}{\lambda - \lambda_n}.$$

Then, the residue is:

$$\text{Res}\left(\frac{P'(\lambda)}{P(\lambda)}, \lambda = \lambda_n\right) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) \cdot \frac{1}{\lambda - \lambda_n} = 1.$$

Using the Residue theorem, we can write the Zeta function as

$$\zeta(s) = \frac{1}{2\pi i} \int_C d\lambda_n \, \lambda_n^{-s} \frac{d}{d\lambda_n} \ln P(\lambda_n), \tag{6.45}$$

where C is the chosen contour. We have to choose a branch cut in the negative real axis to guarantee that the function is not multivalued.

We can now deform the contour $C \to -C$ to explore the symmetry of the problem and simplify calculations. When shifting the contour, the integrand gains a phase of $e^{-i\pi s}$ and $e^{-i\pi s}$, respectively, due to the branch cut. Then, the above equation becomes:

$$\begin{aligned} \zeta(s) &= \frac{1}{2\pi i}\left[e^{-i\pi s} \int_{-\infty}^{0} d\lambda_n \, \lambda_n^{-s} \frac{d}{d\lambda_n} \ln P(\lambda_n) + e^{i\pi s} \int_{-\infty}^{0} d\lambda_n \, \lambda_n^{-s} \frac{d}{d\lambda_n} \ln P(\lambda_n)\right] \\ &= \frac{\sin(\pi s)}{\pi} \int_0^{-\infty} d\lambda_n \, \lambda_n^{-s} \frac{d}{d\lambda_n} \ln P(\lambda_n). \end{aligned} \tag{6.46}$$

Differentiating the above equation and letting $s = 0$, we have that

$$-\frac{d}{ds}\zeta(0) = -\ln P(-\infty) + \ln P(0). \tag{6.47}$$

It is important to note two things: the first is that we can now make a relation between functional determinants and the eigenfunctions of the operator discussed. The second is that the term $P(-\infty)$ is independent of the potential $V(x)$. Then, the functional determinant normalized with respect to the free operator $(V(x) = 0)$ is given by

$$\det \mathcal{D} = \left(\frac{P(0)}{P(0)_{free}}\right). \tag{6.48}$$

The Gelf'and-Yaglom method to calculate functional determinants is going to be completely fundamental to give a quantitative result for the Generalized Second Law.

6.4 The Spectral Theory and Riemannian Geometry

Schrödinger operators on a Riemaniann manifold is an interesting object of study for mathematicians. In our discussion we need to guarantee the essential self-adjointness of this operator, or else, that an infinite set of self-adjoint extensions could exist. For singular potentials, for example, we cannot define the action of it on compactly supported smooth functions, since applying the differential operator to such functions results in functions outside L^2 [97–100]. The question that has to be done is in which conditions does the potential associated with this operator should satisfy to make sure that the underlying Schrödinger operator is essentially self-adjoint? The answer to this question was obtained by Oleikin [101]. Oleikin proved that in the absence of local singularities in the potential, the Schrödinger operator in a Riemannian manifold is essentially self-adjoint. This result may be used to define a generalized Boltzmann-Gibbs-Shannon entropy for matter fields in a Schwarzschild black hole. Note that $V(x_2)$ is a real-value measurable function which is locally in L^2 and globally semi-bounded, i.e., $V(x_2) \geq -C$ for $x_2 \in M_s^d$, with a constant $C \in \mathbb{R}$. Therefore we have a self-adjoint operator in the Hilbert space $L^2(M_s^d) = L^2(M_s^d, d\mu)$, where we defined the Riemannian d-volume. This result may helpful to define the generalized black hole entropy for matter fields in a Schwarzschild black hole. Let us discuss the operator carefully.

Consider the eigenfunctions and eigenvalues of the modified Laplace-Beltrami operator $-\Delta_s + m_0^2$ on a bounded (open connected) domain Ω in a d-dimensional Euclidean Schwarzschild manifold M_s^d. The spectrum of the this operator is real and discrete, and we assume that the eigenvalues λ_k are non-negative and using $k = 1, 2...$ are ordered as

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k) \to \infty, \tag{6.49}$$

when $k \to \infty$, with possible multiplicities. The eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ form a basis in the space $L^2(\Omega)$ of measurable and square integrable functions on Ω. Each ϕ_j is a eigenfunction with eigenvalue $\lambda_j(\Omega) \equiv \lambda_j$. The same may be discussed with the operator $-\Delta_s + m_0^2 - kV(x)$.

To proceed, let us discuss a well known situation. Now consider an evolution equation in $L^2(\Omega, d\mu)$ that can be formulated as the following initial-boundary problem in $(0,\infty)\times\Omega$.

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \Delta_s u \\ u(0,x) = f(x) \\ u(\tau,x)|_{x\in\partial\Omega} = 0. \end{cases}$$

The weak solution $u(\tau, x)$ is given by

$$u(\tau,x) = \int p_\Omega(\tau,x,y) f(y) d\mu(y), \tag{6.50}$$

where μ is the d-volume of M_s^d and $p_\Omega(\tau,x,y)$ is the diffusion kernel. The spectral decomposition of the heat kernel is written as

$$p_\Omega(\tau,x,y) = \sum_{j=1}^{\infty} e^{-\tau\lambda_j(\Omega)}\phi_j(x)\phi_j(y). \tag{6.51}$$

For $\Omega \subset M_s^d$, let us define the Minakshisundaram-Pleijel zeta-function $Z(s;x,y)$ [102–104], for $s \in \mathbb{C}$ as

$$Z(x,y;s) = \sum_{j=1}^{\infty} \frac{\phi_j(x)\phi_j(y)}{\lambda_j^s}. \tag{6.52}$$

The $Z(s;x,y)$ converges uniformly in x and y for $Re(s) > s_0$. Using a Mellin transform, we get

$$\Gamma(s)Z(x,y;s) = \int_0^\infty dt\, \tau^{s-1}\, p_\Omega(\tau,x,y). \tag{6.53}$$

For $x \neq y$, $\Gamma(s)Z(x,y;s)$ is an regular function of s in the entire s plane. For $x = y$ there is a pole at $s = 1$. Let us define the spectral zeta-function $Z(s)$, given by

$$\begin{aligned} Z(s) &= Tr(-\Delta_s)^{-s} \\ &= \sum_{j=1}^{\infty} \lambda_j^{-s} \;\; Re(s) > s_0. \end{aligned} \tag{6.54}$$

From the diffusion kernel, since we are interested in global issues, let us define the diffusion kernel trace,

$$\Theta(\tau) \;=\; Tr\, e^{\tau\Delta_s} = \sum_{j=1}^{\infty} e^{-\lambda_j\tau}. \tag{6.55}$$

It is clear that the spectral zeta-function and the diffusion kernel trace are connected by a Mellin transform. The classical spectral invariants, are the heat invariants, i.e., the coefficients of the expansion at $\tau = 0$ of the heat kernel trace. The asymptotic expansion of the heat trace of a d-dimensional Riemannian manifold without boundaries are integrals of curvature polynomials of various weights in the metric [105].

Let us define the spectral zeta-functions $Z_{m_0^2}(s)$ and $Z_v(s)$ [106, 107]. These spectral zeta-functions are given by

$$\begin{aligned} Z_{m_0^2}(s) &= Tr(-\Delta_s + m_0^2)^{-s} \\ &= \sum_{j=1}^{\infty} \lambda_j^{-s} \;\; Re(s) > s_0, \end{aligned} \tag{6.56}$$

and

$$Z_v(s) = Tr(-\Delta_s + m_0^2 - kV(x))^{-s}$$
$$= \sum_{j=1}^{\infty} \mu_j^{-s} \;\; Re(s) > s_1, \tag{6.57}$$

We define the zeta regularization products of these numbers as

$$\prod_{\lambda_j} \lambda_j = e^{-\frac{d}{ds} Z_{m_0^2}(s)|_{s=0}} \tag{6.58}$$

and

$$\prod_{\mu_j} \mu_j = e^{-\frac{d}{ds} Z_v(s)|_{s=0}}. \tag{6.59}$$

One can show that $Z_{m_0^2}(s)$ and $Z_v(s)$ process meromorphic analytic continuations into the complex s-plane. In particular, the point $s = 0$ is a regular point. Therefore we are able to regularize the functional determinants and the Schrödinger operator of the problem is essential self-adjoint. Here we are not interested to discuss the role of the normalization scale in the regularization procedure.

In the next chapter we define the generalized entropy density with the contribution of hidden degrees of freedom, near the event horizon. For non-compact Riemannian manifold, the spectrum of the generalized Schrödinger operators is continuous. In the case of a discrete spectrum, with a countably set of eigenvalues, we are able to define a spectral entropy. Using spectral zeta functions we are able to express the generalized entropy density as the ratio between two functional determinants.

Chapter 7

The generalized entropy density

Since black holes have thermodynamical properties [108], in order to preserve the universality of the second law of thermodynamics one defines the total entropy of the system that satisfy the generalized second law as

$$\Delta S_{BH} = \Delta S^{(1)} + \Delta S^{(2)} \geq 0, \tag{7.1}$$

where $S^{(1)}$ is the Bekenstein-Hawking entropy, proportional to the horizon area. Usually interpreted as the measure of the missing information of the black hole internal state for an outside observer. The second contribution, $S^{(2)}$, is the contribution from matter and radiation fields. The Euclidean functional integral of gravity coupled to matter, i.e., Einstein gravity and matter fields at finite temperature is written as

$$Z_{total}(\beta) = \int [dg_{\mu\nu}][d\varphi]e^{-I(g_{\mu\nu},\varphi)}, \tag{7.2}$$

where the functional integral is taken over matter fields and the gravitational field (enconded in the metric), which are assumed periodic with respect to the imaginary time, with period $2\pi\beta$. One can write the total action as a contribution from the gravitational action and matter fields as

$$I(\varphi, g_{\mu\nu}) = I_{matter}(\varphi, g_{\mu\nu}) + I_{grav}(g_{\mu\nu}). \tag{7.3}$$

The standard argument to obtain the one-loop correction is to work with metrics with conical singularities, discussing matter fields fluctuations in a conical singular background. In the semi-classical approximation, in the one-loop approximation one obtains the Bekenstein-Hawking entropy, and the entanglement entropy can be view as is the first quantum correction to the Bekenstein-Hawking entropy. Yet in the semi-classical approximation, we are proposing another path to go beyond the geometric contribution. Keeping these general considerations in mind, it is natural to investigate the consequences of modelling the influence of internal degrees of freedom over the matter fields, near the event horizon, introducing an additive quenched disorder. We now proceed to discuss the contribution given by $S^{(2)}$.

Since, in our case, we have a system with infinitely many degrees of freedom, we must use the concept of mean entropy, i.e., the entropy per unit $(d-1)-$volume $(\beta^{-1}\text{Vol}_d(\Omega))$ [109], i.e.,

$$s^{(2)} = \frac{\beta\, S^{(2)}}{\text{Vol}_d(\Omega)}. \tag{7.4}$$

Using the fact that $S = \ln Z + \beta E$, in a Euclidean Quantum Field Theory we can derive the generalized entropy density from the Gibbs free energy. In the case of a compact Riemannian manifold, the contribution of the quantum fields to the matter entropy contribution in the absence of the disorder is

$$s^{(2)} = \frac{1}{\text{Vol}_d(\Omega)}\left(\beta - \beta^2\frac{\partial}{\partial\beta}\right)\ln Z(j)\bigg|_{j=0}, \tag{7.5}$$

where $Z(j)$ is the partition function. Here we have a Gibbs entropy of a classical probability distribution.

In the presence of disorder, we defined the contribution of external matter fields, affected by the internal degrees of freedom, to the matter entropy density contribution, $s^{(2)}$, as

$$s^{(2)} = \frac{1}{\text{Vol}_d(\Omega)}\left(\beta - \beta^2\frac{\partial}{\partial\beta}\right)\mathbb{E}\big[W(h)\big]. \tag{7.6}$$

By the expression that we obtained for the quenched free energy, Eq. (6.26), and using the series of the distributional zeta-funtion, Eq (5.21), we obtain that

$$s^{(2)} = \sum_{k=1}^{\infty}\frac{c'_k}{\text{Vol}_d(\Omega)}\left(\beta - \beta^2\frac{\beta}{\partial\beta}\right)\left[\det(-\Delta_s + m_0^2)\right]^{-\frac{k}{2}}\left[\frac{\det\left(-\Delta_s + m_0^2\right)}{\det\left(-\Delta_s + m_0^2 - kV(x_2)\right)}\right]^{\frac{1}{2}}, \tag{7.7}$$

where $c'_k = \dfrac{(-1)^k}{kk!}$, that is, we absorb a^k into the total volume.

The entropy, in physical grounds, depends on the covariance of the disorder. One has to specify $V(x_2)$ to obtain $s^{(2)}$. As will become clear later, we will obtain the values of the functional determinants using they eigenfunctions. One can be verify that the operator Δ_s contains always the angular Laplace-Beltrami, $-\Delta_\theta$. Since $V(x_2)$ does not depends on the angular variables, we are going ignore such an angular operator. In practice, it is equivalent to work in $d = 2$. In the neighbourhood of the event horizon it is expected that the effects of the internal degrees of freedom be more relevant. Going to such a region, $x_2 = r \approx 2M$, we can define the radial coordinate $\rho = \sqrt{8M(r-2M)}$, the line element can be written as

$$ds^2 = \frac{\rho^2}{16M^2}d\tau^2 + d\rho^2, \tag{7.8}$$

where the horizon is located at $\rho = 0$. This line element is the Euclidean Rindler line element. The equation of motion for the free field in the Euclidean Rindler space is given by

$$(-\Delta_{\text{R}} + m_0^2)\phi = \left(\frac{16M^2}{\rho^2}\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + m_0^2\right)\phi = 0, \tag{7.9}$$

where $-\Delta_{\rm R}$ stands for the Laplace-Beltrami operator in the Rindler coordinates, Eq. (7.8). That is, it is $-\Delta_s$ near the horizon after one neglect the angular part. We can suppose that the solution of this equation is of the type

$$\phi = R(\rho)T(\tau). \tag{7.10}$$

Using the fact that τ is periodic, we can, without loss of generality, assume that the solution for $T(\tau)$ has the form:

$$T(\tau) = \frac{1}{\sqrt{\beta}}\exp\left\{\frac{2\pi n i\tau}{\beta}\right\}, \tag{7.11}$$

where n are the Matsubara modes. With this solution we can calculate $\frac{T''}{T}$

$$\frac{T''}{T} = -\frac{4\pi n^2}{\beta^2}, \tag{7.12}$$

where the prime is the derivative with respect to the τ variable. Inserting Eq. (7.12) in Eq. (7.9), we can express the equation of motion for the free field in the Euclidean Rindler space as

$$(-\Delta_{\rm R} + m_0^2)\phi = \left(\frac{n^2}{\rho^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + m_0^2\right)R(\rho) = 0. \tag{7.13}$$

In the near horizon approximation, that is $\rho \approx 0$, we can expand in Taylor series the Schrödinger's potential operator written in Eq. (6.13)

$$V(\rho, M) = \frac{a^{\alpha-2}}{(2M)^\alpha}\left(1 - \frac{\alpha\rho^2}{16M^2}\right). \tag{7.14}$$

Using the fact that the coordinate τ is periodic, the total entropy density will be a sum over all the Matsubara modes

$$s^{(2)} = \sum_{n=-\infty}^{\infty} s^{(2)}(n). \tag{7.15}$$

Where $s^{(2)}(n)$ is given by Eq. (7.7) in the near the horizon approximation with the angular part disregarded.

Note that for small ρ and defining $f(\alpha, M) = \alpha/2^{4+\alpha}M^{2+\alpha}$ we get that one determinant can be written as

$$\det\left[-\Delta_R + ka^{\alpha-2}\rho^2 f(\alpha, M) + m_0^2 - \frac{ka^{\alpha-2}}{(2M)^\alpha}\right]. \tag{7.16}$$

We have an effective mass for each effective action given by $m_{\rm eff}^2(k, M) = m_0^2 - ka^{\alpha-2}/(2M)^\alpha$. Disregarding the term $\rho^2 f(\alpha, M)$, because of its small contribution with respect to other terms and for computational simplicity, i.e., we are looking close to $\rho \to 0$, let us discuss the solution of the differential equation for each Matsubara mode. We have that $R_n(\rho)$ satisfies

$$\left[\rho^2\frac{{\rm d}^2}{d\rho^2} + \rho\frac{\rm d}{d\rho} + m_{\rm eff}^2\rho^2 - n^2\right]R_n(\rho) = 0. \tag{7.17}$$

The general solution of the above equation is written as

$$R_n(m_{eff}\rho) = AJ_n(m_{eff}\rho) + BY_n(m_{eff}\rho), \tag{7.18}$$

where $J_n(m_{eff}\rho)$ is the Bessel function of the first kind and $Y_n(m_{eff}\rho)$ is the Bessel function of the second kind. For Dirichlet boundary conditions, note that $Y_n(m_{eff}\rho)$ is not bounded when $R_n(0) = 0$, then, to have a consistent solution, we have to set $B = 0$. Though, the general solution can be written as

$$R_n(m_{eff}\rho) = AJ_n(m_{eff}\rho), \tag{7.19}$$

the boundary condition $R_n(l) = 0$ leads us to

$$AJ_n(m_{eff}l) = 0, \tag{7.20}$$

in order to have a non trivial solution, we demand

$$J_n(m_{eff}l) = 0. \tag{7.21}$$

The eigenvalues for the operator are those values of m_{eff} for which $J_n(m_{eff}l) = 0$, that is, for which $m_{eff}l = \lambda_m$, where λ_m is the m^{th} positive zero of J_n :

$$m_{eff} = \frac{\lambda_m}{l}, \tag{7.22}$$

with corresponding eigenfunction of

$$R_n(m_{eff}\rho) = J_n(\frac{\lambda_m}{l}\rho) = J_n(m_{eff}\rho). \tag{7.23}$$

For more details on the discussion of Bessel functions with Dirichlet boundary condition and other boundary conditions, see Ref. [110].

Using the fact that the large n Matsubara modes give the main contribution to the generalized entropy, we will write an asymptotic expansion for $J_n(m_{eff}\rho)$. Moving forward, since that $m^2_{\text{eff}}(k, M)$ can be negative for some k we write $s^{(2)}(n)$ as

$$s^{(2)}(n) = s^{(2)}_{k<k_c}(n) + s^{(2)}_{k\geq k_c}(n). \tag{7.24}$$

Denoting by $\lfloor m \rfloor$ the largest integer smaller than m, we define a critical k given by $k_c = \lfloor \frac{(2M)^\alpha m_0^2}{a^{\alpha-2}} \rfloor$. An interesting fact, about the critical k is that it ties in some way the mass parameter of the scalar field and the black hole mass, because we can rewrite the field mass parameter, for example, as a function of M, k_c and $a^{\alpha-2}$, i.e.,

$$m_0^2 = \frac{k_c a^{\alpha-2}}{(2M)^\alpha}. \tag{7.25}$$

Looking at the equation above, we note that the smallest value for the black hole mass M_{min} and for the mass parameter of the scalar field $m^2_{0_{min}}$ happens when $k_c = 1$, so we may write that

$$m^2_{0_{min}} = \frac{a^{\alpha-2}}{(2M_{min})^\alpha}. \tag{7.26}$$

For the particular case, where the disorder is $\approx r^{-2}$, which means that $\alpha = 2$, we have the following relation for the black hole mass and the mass parameter

$$M_{min} = \frac{1}{2m_{0_{min}}}. \tag{7.27}$$

We can conclude that, for this specific case, $\alpha = 2$, the minimum value for the black hole mass is inversely proportional to the mass parameter of the scalar field. We can make some great interpretations about this result in the limit that the distributional zeta function makes sense: the first point, is that the black hole mass and the mass parameter of the field are completely tied, so we may ask if we can think that the Mach's principle emerging in this situation. The second point, is that the zeta-distributional method in some way is giving us some lower bound for the black hole, and we may ask if it is a real bound or a semi-classical bound for black hole masses.

Continuing the calculation of the density entropy, using that $\beta = 8\pi M$, we have

$$s^{(2)}_{k<k_c}(n) = 8\pi \left(M - M^2 \frac{\partial}{\partial M}\right) \sum_{k=1}^{k_c-1} \frac{c'_k}{\text{Vol}_d(\Omega)} \left[\det(-\Delta_R + m_0^2)\right]^{\frac{-k}{2}} \left[\frac{\det(-\Delta_R + m_0^2)}{\det(-\Delta_R + m_{\text{eff}}^2)}\right]^{\frac{1}{2}}, \quad (7.28)$$

and

$$s^{(2)}_{k\geq k_c}(n) = 8\pi \left(M - M^2 \frac{\partial}{\partial M}\right) \sum_{k=k_c}^{\infty} \frac{c'_k}{\text{Vol}_d(\Omega)} \left[\det(-\Delta_R + m_0^2)\right]^{\frac{-k}{2}} \left[\frac{\det(-\Delta_R + m_0^2)}{\det\left(-\Delta_R + m'^2_{\text{eff}}\right)}\right]^{\frac{1}{2}}, \quad (7.29)$$

where $m'^2_{\text{eff}} = -2m^2_{eff}$ is the shifted effective mass.

Since the spectrum of the Schrödinger operator is unknown, we used an alternative procedure to calculate the above expression. Using the Gel'fand-Yaglom method to calculate functional determinants, we can show that the derivative of the spectral zeta function evaluated at $s = 0$ can be rewritten in terms of the eigenfunctions as

$$-\frac{d}{ds}\zeta(s)\bigg|_{s=0} = \ln\left[\frac{R(0)}{R(-\infty)}\right], \quad (7.30)$$

where R denotes the respective eigenfunctions. Using this procedure it is possible to evaluate the generalized entropy density. Using general solutions of the Eq. (7.18) and Dirichlet boundary conditions, we have that the ratio of the determinant $\left[\frac{\det(-\Delta_R + m_0^2)}{\det\left(-\Delta_R + m'^2_{\text{eff}}\right)}\right]^{\frac{1}{2}}$ can be calculated by

$$\left[\frac{\det(-\Delta_R + m_0^2)}{\det\left(-\Delta_R + m'^2_{\text{eff}}\right)}\right]^{\frac{1}{2}} = \left[\frac{R(m_0)}{R(m_{eff})}\right]^{\frac{1}{2}} = \left[\frac{J_n(m_0)}{J_n(m_{eff})}\right]^{\frac{1}{2}}. \quad (7.31)$$

Using the large n expansion for the Bessel function $J_n(x)$, we have that

$$J_n(x) \approx \frac{1}{\sqrt{2\pi n}}\left(\frac{ex}{2n}\right)^n, \quad (7.32)$$

we can find a approximate expression for the determinant of Eq. (7.31)

$$\left[\frac{J_n(m_0)}{J_n(m_{eff})}\right]^{\frac{1}{2}} \approx \left(\frac{m_0}{m_{eff}}\right)^n. \quad (7.33)$$

The calculation of the determinant $\left[\det(-\Delta_R + m_0^2)\right]^{\frac{-k}{2}}$ is completely analogue, so for a large n approximation

$$\left[\det(-\Delta_R + m_0^2)\right]^{\frac{-k}{2}} = \left[\frac{\det\left(-\Delta_R + m_0^2\right)}{\det\left(-\Delta_R\right)}\right]^{\frac{-k}{2}} = \left[\frac{J_n(x)}{x^n}\right]^{\frac{-k}{2}} \approx 1 \tag{7.34}$$

Notice that an eigenfunction which repeats in both limits will cancel out. This justifies the fact that we disregarded the angular Laplace-Beltrami in Eq. (7.8). Since the eigenfunctions of such operator are going to be spherical harmonics, which are ρ independent. For a general α, we have that

$$s^{(2)}_{k<k_c}(n) = \sum_{k=1}^{k_c-1} \frac{c'_k}{\text{Vol}_d}\left[\frac{2\pi a^{\alpha-2}\alpha k n}{2^{\alpha-2}M^{\alpha-1}m_{\text{eff}}^2} + 8\pi M\right]\left(\frac{m_0}{m_{\text{eff}}}\right)^n \tag{7.35}$$

A similar result can be find for $s^{(2)}_{k\geq k_c}(n)$

$$s^{(2)}_{k\geq k_c}(n) = \sum_{k=k_c}^{\infty} \frac{c'_k}{\text{Vol}_d}\left[\frac{2\pi a^{\alpha-2}\alpha k n}{2^{\alpha-2}M^{\alpha-1}m'^2_{\text{eff}}} + 8\pi M\right]\left(\frac{m_0}{m'_{\text{eff}}}\right)^n. \tag{7.36}$$

The first, and most simple, limit that we can study is the case where $\alpha = 0$, in this situation we interpret that the disorder is distributed in a isotropic way, which is not what we are looking for, but it seems interesting to see its behavior. Setting $\alpha = 0$ we find out that the entropy density it is given by:

$$s^{(2)}_{k<k_c}(n) = \sum_{k=1}^{k_c-1} \frac{c'_k}{\text{Vol}_d} 8\pi M\left(\frac{m_0}{m_{\text{eff}}}\right)^n \tag{7.37}$$

And for $s^{(2)}_{k\geq k_c}(n)$

$$s^{(2)}_{k\geq k_c}(n) = \sum_{k=k_c}^{\infty} \frac{c'_k}{\text{Vol}_d} 8\pi M\left(\frac{m_0}{m'_{\text{eff}}}\right)^n, \tag{7.38}$$

and the matter entropy contribution $S^{(2)}_{k<k_c}(n)$ and $S^{(2)}_{k\geq k_c}(n)$, we have that

$$S^{(2)}_{k<k_c}(n) = \sum_{k=1}^{k_c-1} c'_k\left(\frac{m_0}{m_{\text{eff}}}\right)^n \tag{7.39}$$

And for $S^{(2)}_{k\geq k_c}(n)$

$$S^{(2)}_{k\geq k_c}(n) = \sum_{k=k_c}^{\infty} c'_k\left(\frac{m_0}{m'_{\text{eff}}}\right)^n. \tag{7.40}$$

Then, the generalized entropy in this case will be given, in International System units, by

$$\begin{aligned} S_{BH} &= S^1 + S^{(2)}_{k<k_c}(n) + S^{(2)}_{k\geq k_c}(n) \\ &= \sum_{k=1}^{k_c-1} c_k\left(\frac{m_0}{m_{\text{eff}}}\right)^n + \sum_{k=k_c}^{\infty} c_k\left(\frac{m_0}{m'_{\text{eff}}}\right)^n + \frac{4\pi^2 M^2}{\hbar G} \end{aligned} \tag{7.41}$$

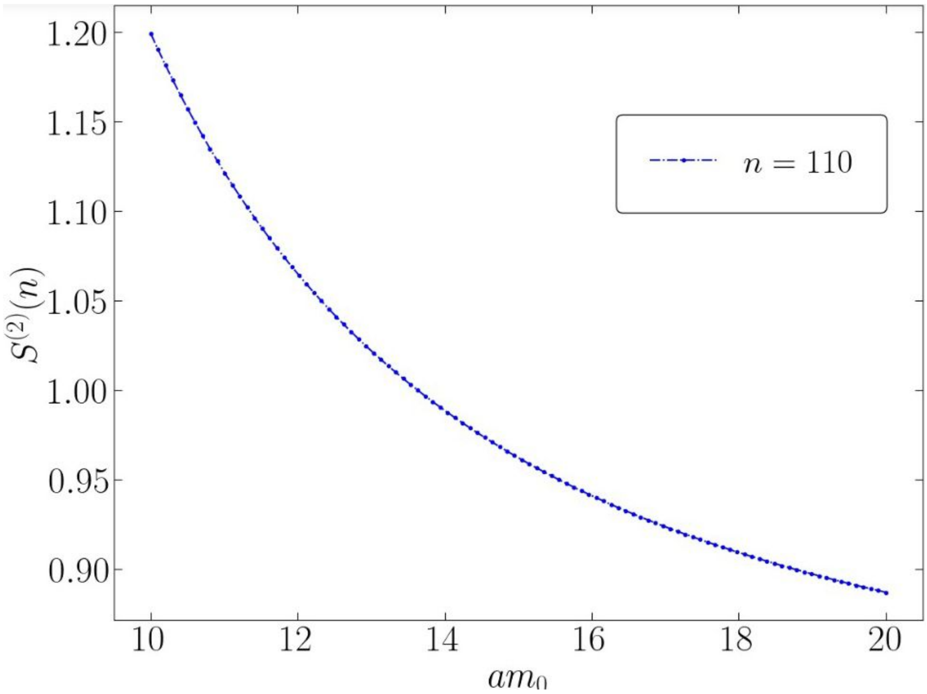

Figure 11 – behavior of the matter entropy as a function of the dimensionless parameter am_0 for a fixed Matsubara mode $n = 110$. We do remark the redefinition of M as $M = G^{(d)}M_0$.

Note that in this case the S^2 entropy does not depend of the black hole mass, so we conclude that the generalized entropy can not be validated, because even though the matter entropy S^2 is positive, the process of Hawking's radiation is decreasing the total entropy. But, for pedagogical purposes we will evaluate the matter entropy S^2. Using a large value of the Matsubara mode n we have that the matter entropy S^2 is positive and increases for lower values of the mass field m_0, showed in figure 11.

The case of most interest for us, is the case that the disorder gets stronger closer to the black hole, we chose the case that $\alpha = 2$, i.e., $V(r) = \dfrac{1}{r^2}$ we find out that

$$s^{(2)}_{k<k_c}(n) = \sum_{k=1}^{k_c-1} \frac{c'_k}{\text{Vol}_d} \left[\frac{2\pi k n}{M m_{\text{eff}}^2} + 8\pi M \right] \left(\frac{m_0}{m_{\text{eff}}} \right)^n \tag{7.42}$$

For $s^{(2)}_{k\geq k_c}(n)$ we find that

$$s^{(2)}_{k\geq k_c}(n) = \sum_{k=k_c}^{\infty} \frac{c'_k}{\text{Vol}_d} \left[\frac{2\pi k n}{M {m'}_{\text{eff}}^2} + 8\pi M \right] \left(\frac{m_0}{{m'}_{\text{eff}}} \right)^n. \tag{7.43}$$

The generalized second law was introduced to ensure that the total entropy of the system also increase ($\Delta S^{(1)} + \Delta S^{(2)} \geq 0$). To show that the external matter field under influences of the internal degrees of freedom contributes to increases the entropy of the system, let us work again with the entropy $S^{(2)}$. We have that $S^{(2)}(n) = S^{(2)}_{k<k_c}(n) + S^{(2)}_{k\geq k_c}(n)$. We can write,

$$S^{(2)}_{k<k_c}(n) = \sum_{k=1}^{k_c-1} c_k \left[\frac{kn}{4M^2 m_{\text{eff}}^2} + 1 \right] \left(\frac{m_0}{m_{\text{eff}}} \right)^n, \tag{7.44}$$

and for $k \geq k_c$ the corresponding contribution yields,

$$S^{(2)}_{k \geq k_c}(n) = \sum_{k=k_c}^{\infty} c_k \left[\frac{kn}{4M^2 m'^2_{\text{eff}}} + 1 \right] \left(\frac{m_0}{m'_{\text{eff}}} \right)^n . \tag{7.45}$$

To conclude, the Bekenstein-Hawking entropy that satisfies the generalized law, will be given, in International System units, by

$$S_{BH} = S^1 + S^{(2)}_{k<k_c}(n) + S^{(2)}_{k \geq k_c}(n) =$$
$$\sum_{k=1}^{k_c - 1} c_k \left[\frac{kn}{4M^2 m^2_{\text{eff}}} + 1 \right] \left(\frac{m_0}{m_{\text{eff}}} \right)^n + \sum_{k=k_c}^{\infty} c_k \left[\frac{kn}{4M^2 m'^2_{\text{eff}}} + 1 \right] \left(\frac{m_0}{m'_{\text{eff}}} \right)^n + \frac{4\pi^2 M^2}{\hbar G} \tag{7.46}$$

If one considers the two angular variables that have been disregarded the result is preserved, as can be seem by Eq. (7.30). Further corrections must been analysed.

We depict the behavior of the entropy as a function of the dimensionless parameter Mm_0 in Fig.12. Since we have redefined the mass of the black hole as $M = G^{(d)} M_0$, we can observe that, for a fixed scalar-field mass, the matter contribution agrees with the generalized second law.

We also show the validity of the generalized second law of thermodynamics in black hole physics in Fig.13, where the total entropy of the system is the Bekenstein-Hawking entropy of the event horizon plus the contribution of the external matter fields corrected by the influence of internal degrees of freedom defined inside the event horizon.

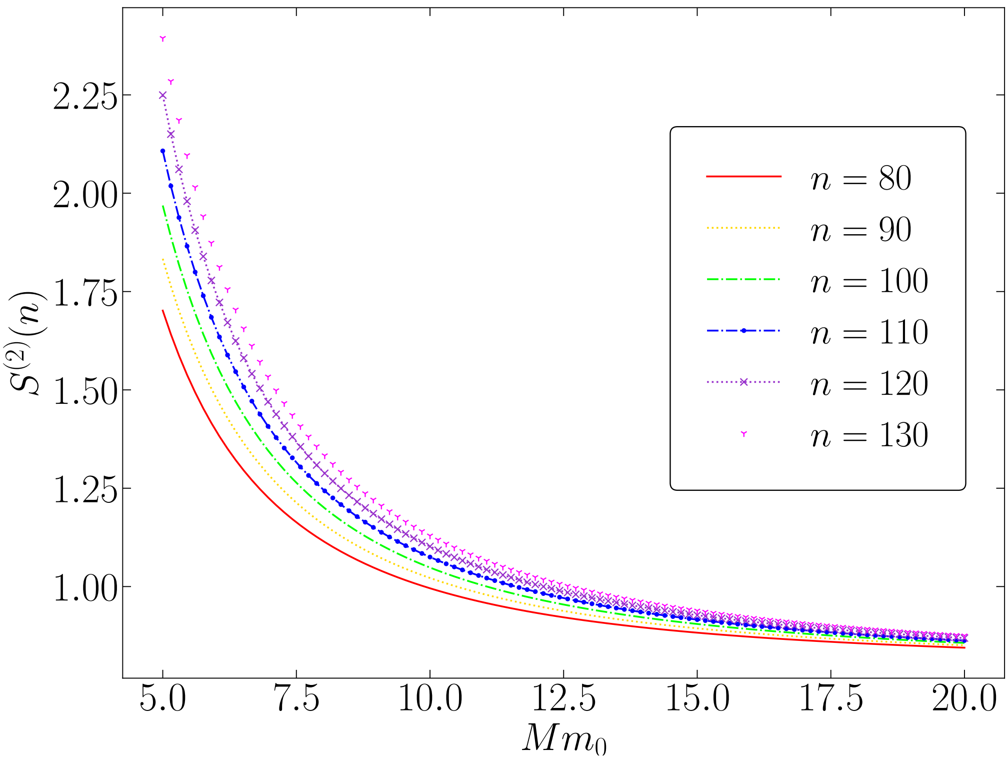

Figure 12 – behavior of the matter entropy as a function of the dimensionless parameter Mm_0 for different Matsubara modes n. We do remark the redefinition of M as $M = G^{(d)} M_0$.

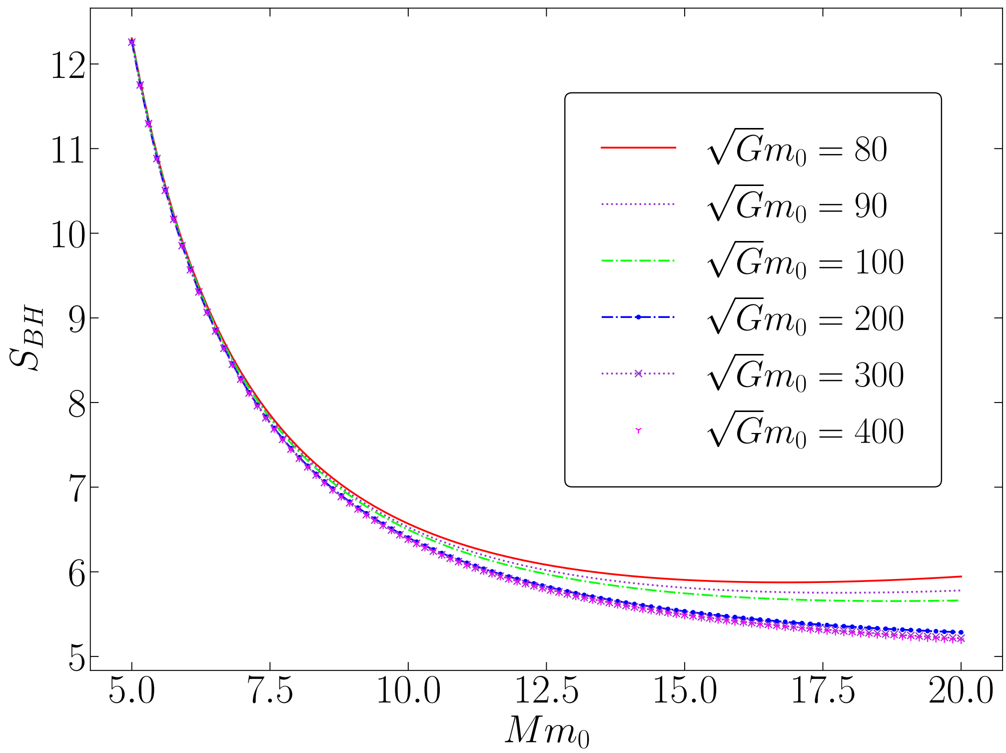

Figure 13 – behavior of the total black hole entropy S_{BH} as a function of the dimensionless parameter Mm_0 for different scaled field masses $\sqrt{G}m_0$. We do remark the redefinition of M as $M = G^{(d)}M_0$.

Chapter 8

Conclusions

The aim of this dissertation was to use tools of QFT in curved spacetime and functional integrals to verify the generalized second law in an Euclidean Schwarzschild metric. In this scenario, the concept of black hole entropy was introduced by Bekenstein. As we usually know, the concept of entropy can be discussed and introduced in different ways. For example, the thermodynamic or Boltzmann entropy which satisfies an additivity and non-decreasing condition which is related to observational states which are deterministic. On the other hand, the Gibbs entropy is defined in terms of ensembles, which is a function of the probabilities in a statistical ensemble. Both definitions are used for ordinary matter systems, i.e., usual thermodynamic systems. Since black holes are not such kind of systems, the literature has been emphasized that microscopic degrees of freedom are responsible for the Bekenstein-Hawking entropy. This idea of microscopic degrees of freedom of the internal configuration of the black hole have been used to discuss the definition of a statistical entropy. The total or generalized entropy of the black hole in four dimensions is given by the area of the event horizon $A_{hor}/4G^{(4)}$, which is proportional to the Bekenstein-Hawking entropy of the system, and the contribution from the quantized matter.

The constructive program claim that there is a unique formalism of Quantum Field Theory and statistical mechanics of classical fields, given by probability theory. We discuss a quantum scalar field in the Euclidean section of the Schwarzschild manifold, i.e., a Quantum Field Theory analytically continued to imaginary time. We use it to discussed a second contribution, defining the generalized entropy density of the black hole with contributions from external matter fields affected by the internal degrees of freedom inside the event horizon.

A conceptually simple way to model the influence of internal degrees of freedom over the matter fields, is to introduce a quenched disorder linearly coupled with the scalar field. To perform the integration over all the realizations of the disorder, the distributional zeta-function method was used. After integrating over all the realizations of the disorder, we obtain a series representation of the averaged generating functional of connected correlation functions, in terms of the moments of the generating functional of correlation functions. Effective actions are defined for each of these moments.

We show that this approach led us to the theory of Schrödinger operators in Riemannian manifolds. The relations between the black hole thermodynamics and Schrödinger operators in Riemannian manifolds are not uncovered in literature. The necessary and sufficient condition for essential self-adjointness of the generalized Schrödinger operator, constructed with the Laplace-Beltrami operator is discussed. If it is possible to define self-adjoint operators, the generating functional of connected correlation functions, can be defined. From the study of the effective action, we were able to define a critical k and study some of its aspects. From the structure of the critical k we were able to make a relation between the black hole mass, the mass parameter of the field and the effects of the disorder, but, even more, we were allowed to establish a minimum mass for the black hole as a function of the mass parameter of the field for specific conditions of the disorder. Finally, we present the generalized entropy density of the black hole with contributions from matter fields with the effective contribution of the internal degrees of freedom defined in the region inside the event horizon. We showed the validity of the generalized second law in the model considered.

Appendix A

Some formalism for the construction of the Euclidean Quantum Field Theory

A.1 Properties of Quantum Field Theory for a scalar field in Minkowski space

1. Exist a Hilbert space $\mathcal{H}$ of physical states, which contains the vacuum state $|0\rangle$.

2. On the Hilbert space $\mathcal{H}$ exist an unitary representation of the Poincaré group $U(a, b)$, where a is the spacetime translations and the b the rotation/boost. The Minkowski vacuum is invariant under these transformations.

3. The called "Spectrum condition": Given a generator P^μ of translations,

$$U(a, 1) = \exp\{iP^\mu a_\mu\}, \tag{A.1}$$

this generator have a spectrum which lies within the future light cone.

4. The vacuum state is the only vector invariant under the Poincaré transformation $U(a, b)$.

5. There is a field $\phi(x)$ that acts as an operator on $\mathcal{H}$.

6. Those fields have a covariant transformation, i.e.,

$$U(a, b)\phi(x)U^{-1}(a, b) = \phi(bx + a). \tag{A.2}$$

7. The fields commutes for space-like separations

$$[\phi(x), \phi(y)] = 0, \tag{A.3}$$

for $(x - y)^2 \leq 0$, i.e, we are defining that our theory is local.

A.2 The properties of the Schwinger functions

1. The property of Euclidean covariance: Schwinger functions are covariant under Euclidean transformations, i.e,
$$\mathcal{S}(x_1, ..., x_n) = \mathcal{S}(bx_1 + a, ..., bx_n + a), \tag{A.4}$$
where now $b \in SO(4)$.

2. The property of reflection positivity: Let the object
$$\theta(\vec{x}, x^4) = (\vec{x}, -x^4), \tag{A.5}$$
$$\Theta\phi(x) = \phi(\bar{\theta} x), \tag{A.6}$$
defines the Euclidean time reflection, where $\phi(\bar{\theta} x)$ is the complex conjugation of $\phi(\theta x)$, and K a function of the fields for positive times, we have that
$$\langle\langle \Theta K \rangle \, K \rangle \geq 0. \tag{A.7}$$

3. The property of symmetry: The Schwinger functions are symmetric in their argument.

Appendix B

General quantum statistical system: The Kubo-Martin-Schwinger condition

B.1 The KMS condition

Let us consider a arbitrary quantum system with a Hamiltonian $\hat{H}$ time-independent. If we have a observable $\hat{O}$, its time evolution in the Heisenberg picture is

$$\hat{O}_t = e^{it\hat{H}}\hat{O}e^{-it\hat{H}}. \tag{B.1}$$

We can define the equilibrium state of temperature $T = \dfrac{1}{\beta}$ as

$$\langle\hat{O}\rangle_\beta \equiv Z^{-1}Tr(e^{-\beta\hat{H}}\hat{O}), \tag{B.2}$$

where Z is the normalization factor $Z \equiv \text{Tr}(e^{-\beta\hat{H}})$. Using the cyclic property of trace, for two observables O and S, we define

$$G^\beta_+(t, O, S) \equiv \langle O_t S\rangle_\beta \tag{B.3}$$

$$= Z^{-1}Tr[e^{-\beta H}e^{itH}Oe^{-itH}S] = Z^{-1}Tr[e^{-\beta H}Oe^{-itH}Se^{itH}] \tag{B.4}$$

$$= \langle OS_{-t}\rangle_\beta\,. \tag{B.5}$$

For any t_2 and t_1 such that $t_2 - t_1 = t$ we have

$$\langle O_{t_2}S_{t_1}\rangle_\beta = G^\beta_+(t, O, S). \tag{B.6}$$

In a analogous ways, we can define

$$G^\beta_-(t, O, S) \equiv \langle OS_t\rangle_\beta \tag{B.7}$$

$$= Z^{-1}Tr[e^{-\beta H}Oe^{itH}Se^{-itH}] = \langle O_{-t}S\rangle_\beta \tag{B.8}$$

$$= \langle O_{t_1}S_{t_2}\rangle_\beta\,, \tag{B.9}$$

then, we conclude that

$$G_{-}^{\beta}(t,O,S) = G_{+}^{\beta}(-t,O,S). \tag{B.10}$$

If we now define a complex variable z, in a way that $t \to z$, we can rewrite the equations for G_+ and G_- as

$$G_{+}^{\beta}(z,O,S) = Z^{-1}Tr[e^{i(z+i\beta)H}Oe^{-izH}S], \tag{B.11}$$

$$G_{-}^{\beta}(z,O,S) = Z^{-1}Tr[Se^{izH}Oe^{-i(z-i\beta)H}]. \tag{B.12}$$

If we set $z = t + is$, consequently both exponents in Eq. (B.11) have negative real parts if $-\beta < s < 0$, for the Eq. (B.12) the condition is that $0 < s < \beta$.

If $0 \leq \text{Im}s \leq \beta$, we have that

$$G_{-}^{\beta}(z,O,S) = G_{+}^{\beta}(z-i\beta,O,S), \tag{B.13}$$

In fact, if we let $z \to z - i\beta$ in Eq. (B.11) and do cyclically permutations the trace leads to (B.12). For $z = t$, the Eq. (B.13) can be written as

$$\langle SO_t \rangle_{\beta} = \langle O_{t-i\beta}S \rangle_{\beta}. \tag{B.14}$$

The conditions (B.13) and (B.14) are the so-called KMS condition (or, the Kubo-Martin-Schwinger condition), for system where the $Tr[-\beta H]$ diverges, we can understand the KMS condition as a definition of "thermal equilibrium at temperature $\frac{1}{\beta}$.. The KMS condition is a boundary-value condition which emerge naturally in quantum statistical mechanics.

The KMS condition is related to the periodicity property of thermal two-point functions. Let's make it more clear. We can define a periodic function throughout the complex plane by

$$\mathcal{G}^{\beta}(z,O,S) = G_{-}^{\beta}(z,O,S), \tag{B.15}$$

for $0 < s < \beta$, and

$$\mathcal{G}^{\beta}(z,O,S) = G_{+}^{\beta}(z,O,S), \tag{B.16}$$

for $-\beta < s < 0$.

In general case,

$$\mathcal{G}^{\beta}(z, O, S) = G_{+}^{\beta}(z - iN\beta, O, S) = G_{-}^{\beta}(z - i(N-1)\beta, O, S), \tag{B.17}$$

for an appropriate N integer. Consequently, we may show that $\mathcal{G}$ satisfies the periodic condition

$$\mathcal{G}(z, O, S) = \mathcal{G}(z + iN\beta, O, S), \tag{B.18}$$

for all N integers.

Appendix C

The Kasner Universe

The Kasner universe is a exact solution of the Einstein's equations, that describes an anisotropic universe without matter. The Kasner metric, in a D dimensional spacetime for $D > 3$ is given by

$$ds^2 = dt^2 - \sum_{j=1}^{D-1} t^{2p_j}[dx^j]^2, \tag{C.1}$$

which contains $D - 1$ constants p_j that are the so-called, the Kasner constants. Basically, this metric is describing a space in which is spatially flat for equal-times, but it is expanding and/or contracting at different directions and rates, depending of the Kasner constants. The Kasner constants satisfies the following conditions

$$\sum_{j=1}^{D-1} p_j = 1, \tag{C.2}$$

and

$$\sum_{j=1}^{D-1} p_j^2 = 1, \tag{C.3}$$

which defines the Kasner plane and the Kasner sphere, respectively.

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